Chapter 1: Probability Review

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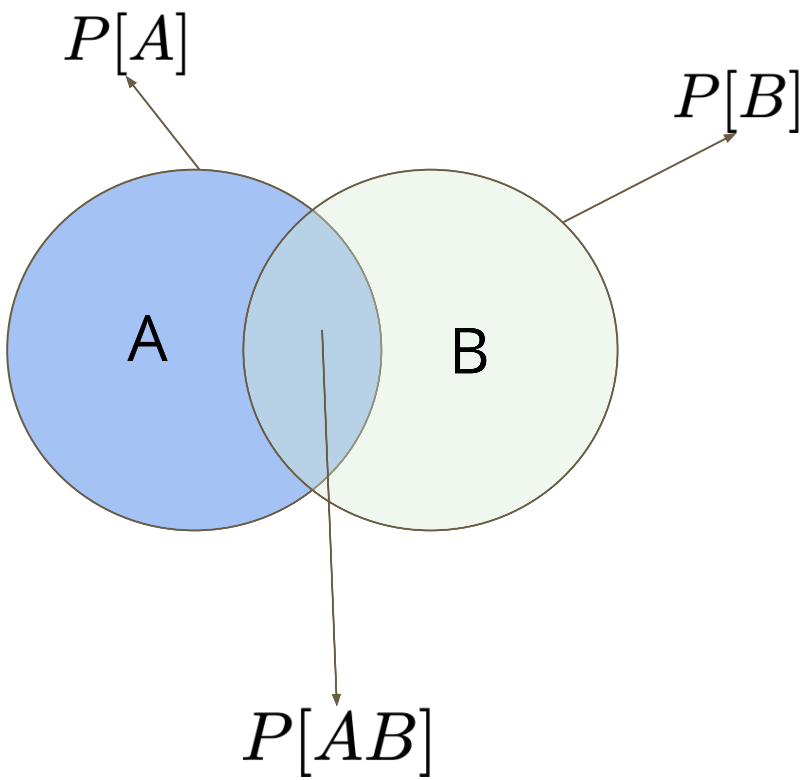
Fall 2019

Joint and Conditional Probability



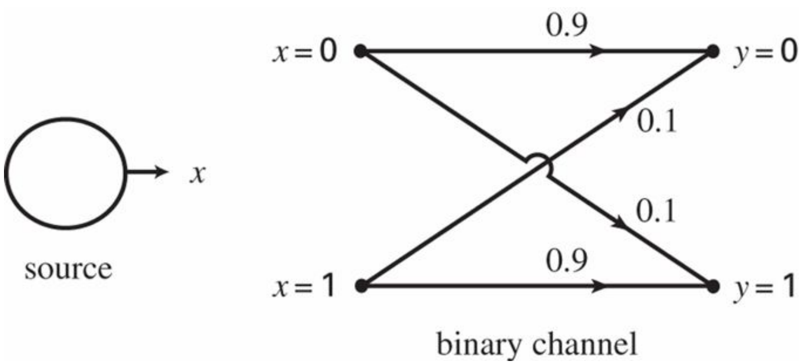
Joint Probability

Conditional Probability

$$P[B|A] = \frac{P[AB]}{P[A]}$$

$$P[A|B] = \frac{P[AB]}{P[B]}$$

Example



Transition probabilities due to noise

$$P[Y=1|X=1]=0.9 \quad P[Y=1|X=0]=0.1$$
$$P[Y=0|X=1]=0.1 \quad P[Y=0|X=0]=0.9$$

Assume the Priors to be

$$P[A]=P[B]=\frac{1}{2}$$

We may be interested to know the following probabilities:

$$P[X=0,Y=0]=P[Y=0|X=0]P[X=0]=0.45$$
$$P[X=0,Y=1]=P[Y=1|X=1|P[X=0]=0.05$$
$$P[X=1,Y=0]=P[Y=0|X=1]P[X=1]=0.05$$
$$P[X=1,Y=1]=P[Y=1|X=1|P[X=1]=0.45$$

Calculate

$$P[Y=0], P[Y=1]$$

Also, unconditional probability or total probability can be computed as

$$P[B]=P[B|A_1]P[A_1]+P[B|A_2]P[A_2]+\cdots+P[B|A_n]P[A_n]$$

Iff $\cup_{i=1}^{n} A_i=\Omega$, $A_iA_j=\phi$ and B is defined over Ω

Independence

If $A_i, i = 1, \ldots n$ are defined on a probability space $\mathscr{P}$, the $\{A_i\}$ are said to be independent iff,

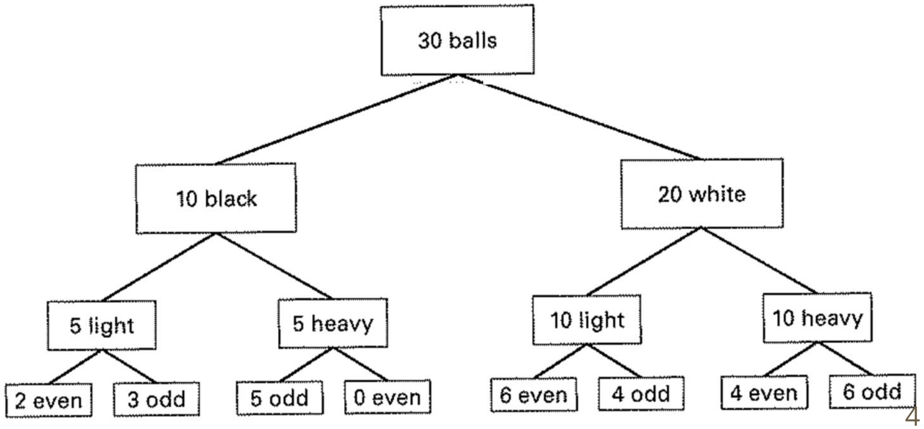
$$\begin{aligned} P[A_iA_j] &= P[A_i]P[A_j] \\ P[A_iA_jA_k] &= P[A_i]P[A_j]P[A_k] \\ &\vdots \\ P[A_1 \ldots A_n] &= P[A_1]P[A_2] \ldots P[A_n] \end{aligned}$$

For all combination of indices such that $1 \leq i \leq j \leq k \cdots \leq n$

Example: Test for Independence

Let
A= Event of picking a black ball
B= Picking one of the lighter balls
C = picking an even-numbered ball

Check P[ABC] = P[A]P[B]P[C]
And Pairwise independence



Bayes' Theorem and Applications

Prior Probability of Event A

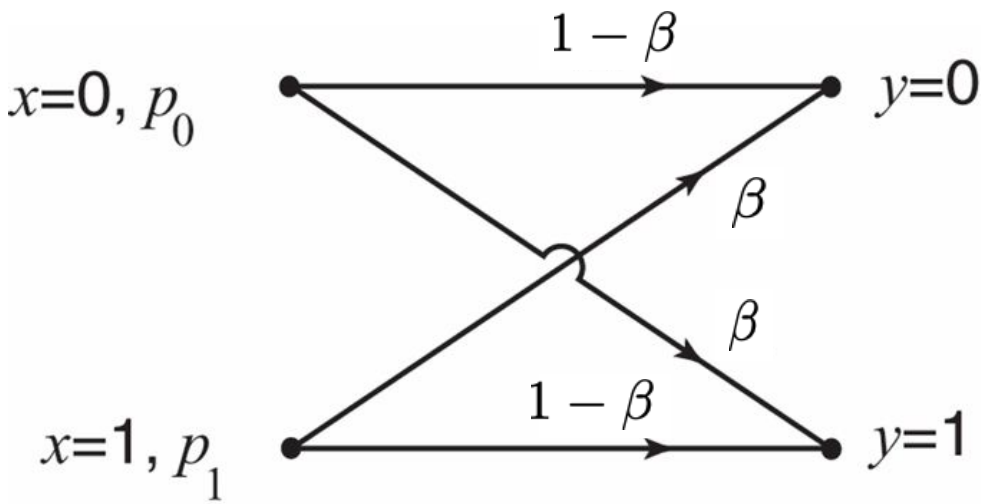
Likelihood

$$P[A_j|B] = \frac{P[B|A_j]P[A_j]}{\sum_{i=1}^{n} P[B|A_i]P[A_i]}$$

Posterior Probability

Prior Probability of Event B

Calculate Posterior Probability $P[X_i|Y_i]$ and its relation to β?



Bernoulli Trial

- An experiment that has two outcomes **P[success]=*p*** and **P[failure]=*q***
  - Since multiple events are independent, $P[z_{i_1}, \dots z_{i_n}] = P[z_{i_1}]. \dots .P[z_{i_n}]$
  - Example: **P[H] = p, P[T] = q**, what is the probability of the events
    - E_1 = {HTT}
    - E_2 = {THT}
    - E_3 = {TTH}
  - Probability of getting two T's regardless of order is ***$3pq^2$***
- Generalizing with ***n*** Bernoulli trials, consider an event

$$A_k \triangleq k \text{ success in } n \text{ trials} = \cup_{i=1}^{K} \{a_i^{'}\}$$

$\{a_i^{'}\}$ is the the set of all tuples with (exactly) ***k*** successes and ***n-k*** *failures*

- What is the value of ***K*** ?
- Therefore using independence of $\{a_i^{'}\}$,

$$P[A_k] = P\left[\bigcup_{i=1}^{K}\{a_i'\}\right] = \sum_{i=1}^{K} P[\{a_i'\}] \longrightarrow P[A_k] = \binom{n}{k} p^k q^{n-k}$$

Binomial Probability Law

.. contd

- If ***k*** or fewer success instead of exactly the the binomial law is

$$B(k : n, p) = \sum_{i=0}^{k} \left(\begin{array}{c} n \\ i \end{array} \right) p^i q^{n-i}$$

Binomial Distribution Function

- Consequently, the probability of **at least *k* success** ***(or k or more)*** is

$$1 - B(k-1 : n, p) = \sum_{i=k}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) p^i q^{n-i}$$

Example 1.9-7: 5 missile strike, Need at least **2 hits** to destroy. P[miss] = 0.9. Calculate **P[target still active]**?

P[target destroyed] = P[at east 2 hits out of 5] = $\sum_{i=2}^{5} \left(\begin{array}{c} 5 \\ i \end{array} \right) (0.1)^i (0.9)^{5-i}$

P[target still active] = 1 - P[target destroyed]

Multinomials

- Bernoulli trials outcome with **multiple** (l) **outcomes** ζ_i with probabilities p_i
- For a particular ordered set of outcomes, let $l = 3, n = 7$

$$P[\zeta_1\zeta_2\zeta_3\zeta_3\zeta_1\zeta_2\zeta_3] = p_1^2 p_2^2 p_3^3$$

- For unordered outcomes count all combination of the outcomes

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\dots\binom{n-r_1-r_2-\dots r_{k-1}}{r_k} = \frac{n!}{r_1!r_2!r_3!\dots r_k!}$$

Multinomial coefficient

- The generalized probability of a set of (l) outcomes

$$P(r:n,p) = \frac{n!}{r_1!r_2!r_3!\dots r_k!} p_1^{r_2} p_2^{r_1} \dots p_l^{r_l}$$

- For what value of (l), we get Binomial law?

Poisson Law

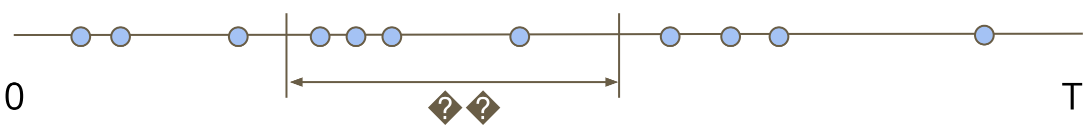
- If ***n>>k*** and ***p<<1***, but ***np = μ*** remain constant, the binomial law reduces to

$$P[A_k] = \binom{n}{k} p^k q^{n-k} \xrightarrow{n \to \infty} \frac{\mu^k}{k!} e^{-\mu}$$

Derivation ?

Poisson's probability law. It has only one parameter *μ*

- **Application : n** points arrive in **T** sec. What is the probability that exactly ***k*** points are presents in interval **τ**?



Probability of ***k*** events in **τ**

$$b(k, t, t+\tau) = \left(\frac{n\tau}{T}\right)^k \frac{e^{-(n\tau/T)}}{k!} = e^{(-\lambda\tau)} \frac{(\lambda\tau)^k}{k!}$$

Where, $p = \tau/T$ $\mu = n.p$ $\lambda = n/T$ is the average arrival rate

What happens when **λ** is a function of time? See example 1.10-5 in textbook

Normal Approx. of Binomial Law

- Binomial and Poisson distributions are discrete distributions
- A standard normal distribution has the cdf

$$F_{SN}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-y^2}{2}} dy$$

What is the pdf?

- Therefore a bernoulli can be approximated using the normal distribution

$$b(n;k,p) \approx \frac{1}{\sqrt{npq}} f_{SN}\left(\frac{k-np}{\sqrt{npq}}\right) = \frac{1}{\sqrt{2\pi}\sqrt{npq}} e^{-\frac{1}{2}\left(\frac{k-np}{\sqrt{npq}}\right)^2}$$

- Error function ***erfc(x)***

$$erf(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{\frac{-y^2}{2}} dy$$

- Therefore,

$$F_{SN} = \frac{1}{2} + erf(x) \quad for \quad x > 0$$

$$F_{SN} = \frac{1}{2} - erf(|x|) \quad for \quad x < 0$$

P[k_1< ***success*** <k_2] =

$$F_{SN}\left[\frac{k_2 - np + 0.5}{\sqrt{npq}}\right] - F_{SN}\left[\frac{k_1 - np - 0.5}{\sqrt{npq}}\right]$$

The 0.5 appears for continuity correction from discrete Binomial to a continuous Normal approximation