Chapter - 2: Random Variables

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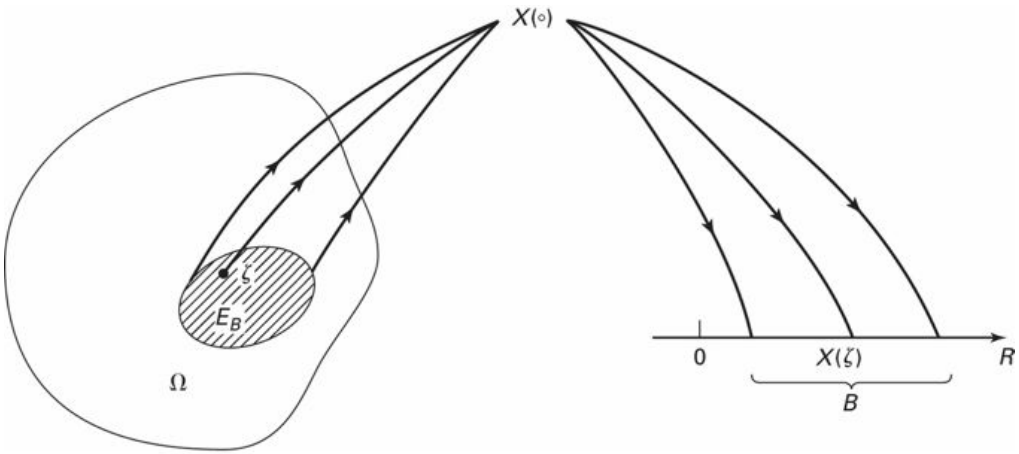
Definition

- The events are ***sets of numbers or a vector of numbers***
- An experiment $\mathscr{H}$ with sample space Ω, the elements ζ are random outcomes.
  - To every ζ a mapping, $X(\zeta)$ maps the outcome to the **R**^{1}
  - All subsets of <u>*Borel set*</u> of real numbers ***B*** and their union and intersection are events ($(-\infty, x]$)



- We are interested in assigning probability to an event $\{\zeta : X(\zeta) \leq x\}$
  - $P[X \leq x] \triangleq F_X(x)$ is called the ***cumulative distribution function*** of X
  - In practice we are interested to learn the ***behavior*** of X , even if the underlying experiment of probability space is unknown

CDF Example

- **Example 2.2-1:** Ask random people if they have a younger brother?
  - Two outcomes / events ζ = {Yes, No} and Mapping function $X(\zeta)$: **Yes** = 1, **No** = 0
  - Assign probabilities **P[Yes] = 1/4** and **P[N0] = 3/4** (assumption)
  - Now, we can ask the probability of the random variable X between any interval on $\mathbf{R}^1$, e,g (0,1] or [0,1)
- **To read**: Properties of CDF $F_X(x)$
  - *Note:* $F_X(x)$ is a non-decreasing function

$$P[x_1 < X \leq x_2] = F_X(x_2) - F_X(x_1) \geq 0 \text{ for } x_2 > x_1$$

$$\text{Calculate } P[a \leq X \leq b], \quad P[a < X < b], \quad P[a \leq X < b]$$

- **Example 2.3-2:** Bus arrive in **(0,T]. *X*** is the RV that denotes the time of arrival ***t***. If it is equally likely for the bus to arrive in interval (0,T], then what is $F_X(t)$?

PDF

- PDF is defined as

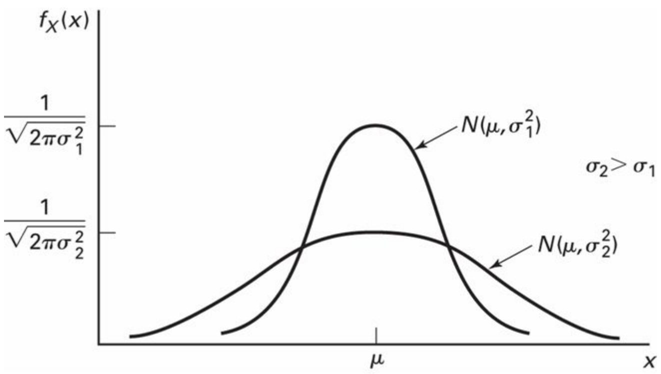
$$f_X(x) = \frac{dF_X(x)}{dx} \longrightarrow f_X(x) = \int_{-\infty}^{x} f_X(\xi)d\xi = P[X \leq x]$$

- Univariate Gaussian PDF $X : \mathcal{N}(\mu, \sigma^2)$

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dy$$

$$\mu \triangleq \int_{-\infty}^{\infty} x f_X(x)dx \quad \text{and} \quad \sigma^2 \triangleq \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)dx$$

Similar definition exists for discrete RVs except using PMF instead of pdf



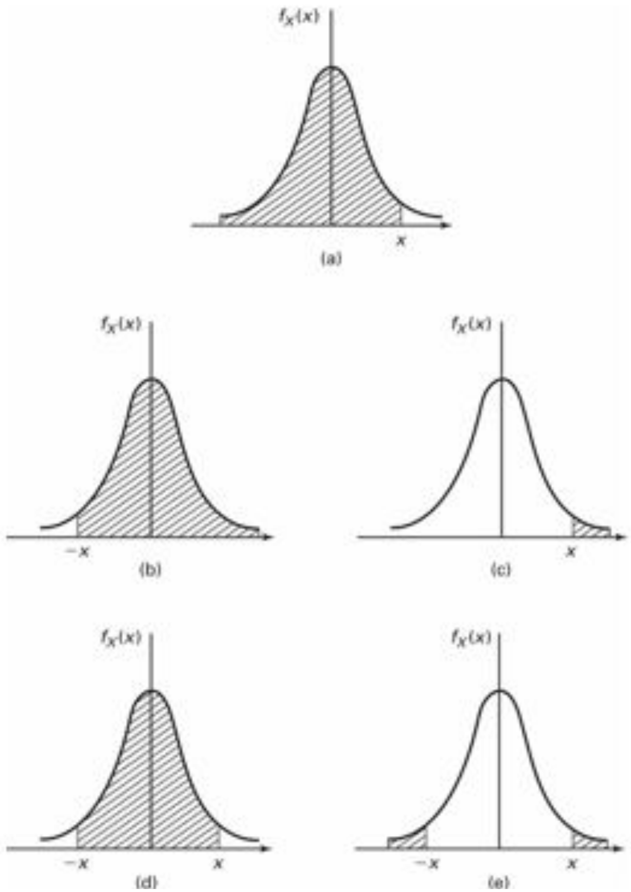
Convert Gaussian pdf to standard normal $X : \mathcal{N}(0, 1)$

$$P[a < X \leq b] = erf\left(\frac{b-\mu}{\sigma}\right) - erf\left(\frac{a-\mu}{\sigma}\right)$$

$$\text{erf}(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}} dt$$

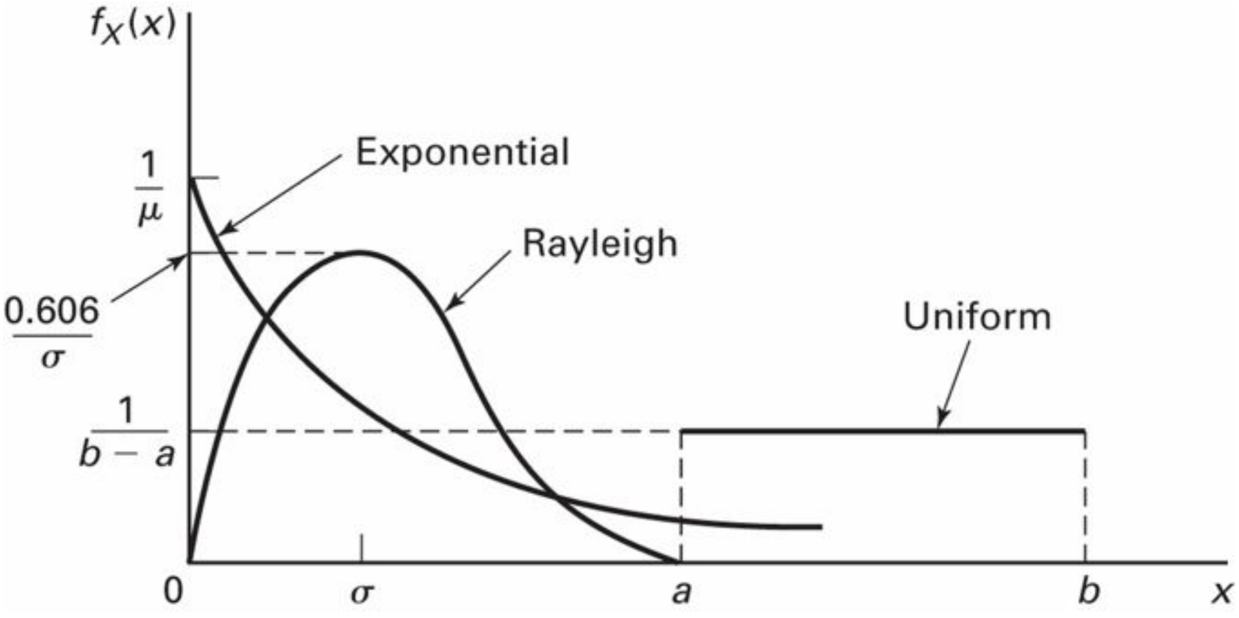
Example

Figure 2.4-3 The areas of the shaded region under curves are (a) $P[X \le x]$; (b) $P[X > -x]$; (c) $P[X > x]$; (d) $P[-x < X \le x]$; and (e) $P[|X| > x]$.



Other common distributions

- Rayleigh $f_X(x) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} u(x)$
- Exponential $f_X(x) = \frac{1}{\mu} e^{-x/\mu} u(x)$
- Uniform (b>a) $f_X(x) = \frac{1}{b-a} \qquad a < x < b$



Continuous vs Discrete RV

- If CDF is continuous, that is derivative exists for all x, then x is a continuous RV else discrete.
- Formally, for continuous RVs we can write the following

$$P[X \leq x] = F_X(x) = \int_{-\infty}^{x} f_X(\xi) d\xi$$

$$P[x_1 < X \leq x_2] = F_X((x_1, x_2]) = \int_{x_1}^{x_2} f_X(\xi) d\xi$$

Events are union of disjoint intervals in R^1

$$P[B] = \int_{a_1}^{b_1} f_X(\xi) d\xi + \int_{a_2}^{b_2} f_X(\xi) d\xi + \cdots + \int_{a_n}^{b_n} f_X(\xi) d\xi = \int_{\xi : \xi \in B} f_X(\xi) d\xi$$

$$\text{where} \quad B = \{\xi : \xi \in \cup_{i=1}^{n} I_i, I_i I_j = \phi \text{ for } i \neq j\} \text{ where } I_i = (a_i, b_i]$$

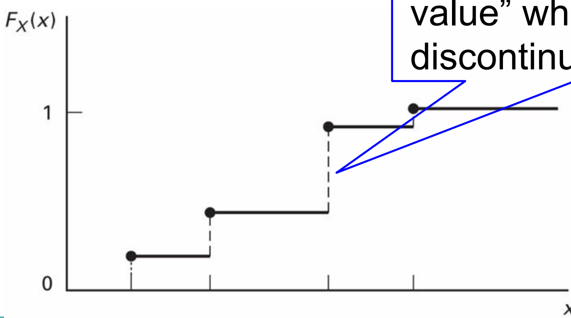
- Discrete RV **(see examples in sec 2.5)**

$$P_X(x) = P[X = x] = P[X \leq x] - P[X < x]$$

$$F_X(x) \triangleq P[X \leq x] = \sum_{\text{all } x_i \leq x} P_X[x_i]$$

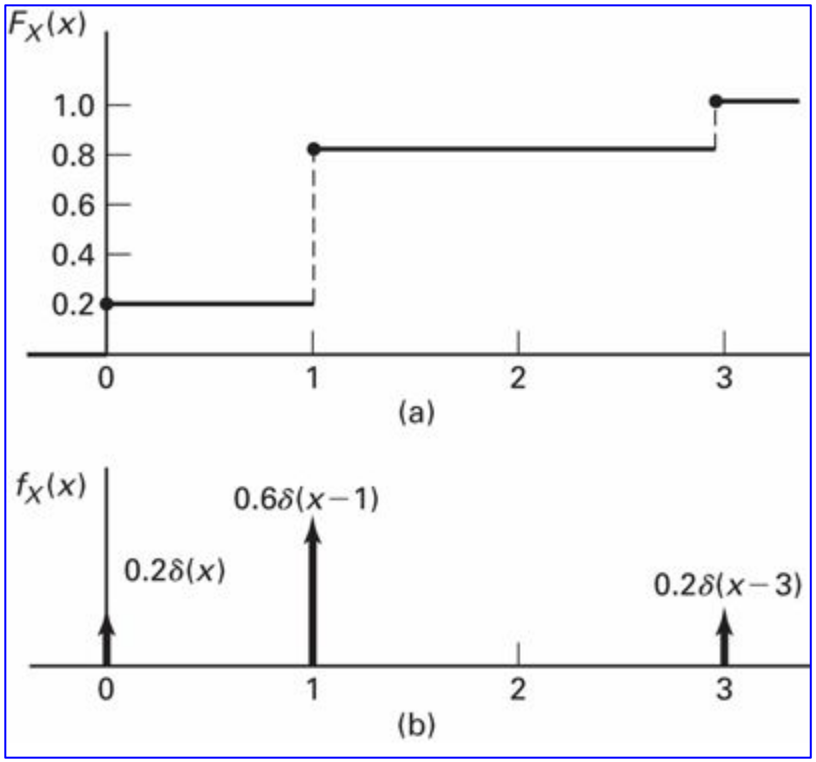
PMF for discrete variables

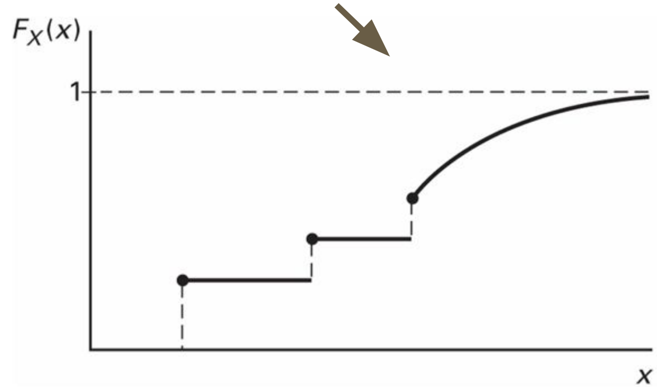
$P_X(x)$ = 0 where $F_X(x)$ is continuous and “has value” where discontinuous



Mixed RV

- Discontinuous at certain intervals but continuous at others





$$F_X(x) = \sum_{t=-\infty}^{\infty} P_X(x_i)u(x - x_i)$$

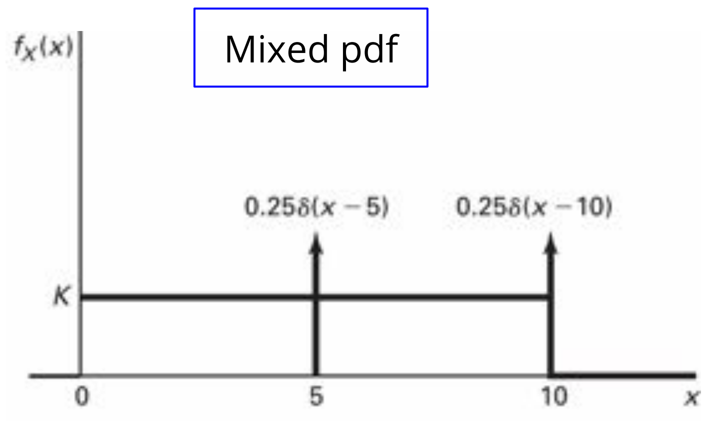
$$f_X(x) = \frac{dF_X(x)}{dx} \sum_{t=-\infty}^{\infty} P_X(x_i)\delta(x - x_i)$$

The ***unit step function*** and its derivative the ***Dirac Delta*** function allows us to express CDFs and PDFs of discrete RV in a continuous form

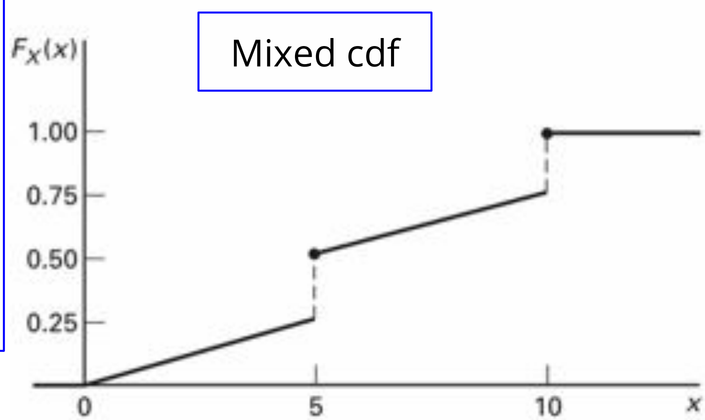
Example 2.5-5

In the pdf shown for a mixed RV:

1) What is the constant K?
2) Compute $P[X \leq 5], P[5 \leq X < 10]$
3) Draw the Distribution function



(a)



(b)

1. $$\int_0^{10} f_X(x)dx = \int_0^{10} Kdx + \int_0^{10} 0.25\delta(x-5)dx + \int_0^{10} 0.25\delta(x-10)dx = 1$$

2. $$\int_0^{5+} f_X(x)dx = \int_0^{5+} 0.05dx + \int_0^{5+} 0.25\delta(x-5)dx = 0.25 + 0.25 = 0.50$$

$$\int_{5-}^{10-} f_X(x)dx = \int_{5-}^{10-} 0.05dx + \int_{5-}^{10-} 0.25\delta(x-5)dx = 0.25 + 0.25 = 0.50$$

Conditional Distribution

- If event ***C*** with outcome ζ is at the intersection of $\{\zeta : X(\zeta) \leq x\}$ and $\{\zeta : \zeta \in B\}$

$$F_X(x|B) = \frac{P[C]}{P[B]} = \frac{P[X \leq x, B]}{P[B]} \xrightarrow{\text{PDF/PMF}} \begin{cases} f_X(x|B) = \frac{f_X(x,B)}{f_X(B)} \\ P_X(x|B) = \frac{P(X=x,B)}{P(B)} \end{cases}$$

- Similar to CDF for single events, if $x_2 \leq x_1$

$$\{X \leq x_2, B\} = \{X \leq x_1, B\} \cup \{x_1 < X \leq x_2, B\}$$

$$\implies P[X \leq x_2|B]\cancel{P[B]} = P[X \leq x_1|B]\cancel{P[B]} + P[x_1 < X \leq x_2|B]\cancel{P[B]}$$

$$\implies P[x_1 < X \leq x_2|B] = P[X \leq x_2|B] - P[X \leq x_1|B] = F_X(x_2|B) - F_X(x_1|B)$$

- Similar to *total probability* of events as sum of conditionals,

$$F_X(x) = \sum_i^n F_X(x|A_i)P[A_i]$$

- Bayes' rule for RVs

$$P[B|X=x] = \frac{P[X=x|B]P[B]}{P[X=x]}$$

Discrete RV

$$P[B|X=x] = \frac{f_X(x|B)P[B]}{f_X(x)}$$

Continuous RV

Example 2.6-2

X is Poisson rv with parameter μ. Find conditional PDF and CDF of {X|X is even}

Start with conditional PDF $P_X(x|X \text{ is even}) = \frac{P(X=x, X \text{ is even})}{P(X \text{ is even})}$

$P(X = x, X \text{ is even})$ is the probability of the events in $\underbrace{\{X = x\} \cap \{X \text{ is even}\}}$

This is $P_X(x)$ when x is even and ϕ when x is odd

$P(X \text{ is even})$ Can be calculated as

$$\sum_{k=0,2,4,\ldots}^{\infty} \frac{\mu^k}{k!} e^{-\mu} + \sum_{k=1,3,5,\ldots}^{\infty} \frac{\mu^k}{k!} e^{-\mu} = 1$$

$$\sum_{k=0,2,4,\ldots}^{\infty} \frac{\mu^k}{k!} e^{-\mu} - \sum_{k=1,3,5,\ldots}^{\infty} \frac{\mu^k}{k!} e^{-\mu} = \sum_{k=0}^{\infty} \frac{\mu^k}{k!} (-1)^k e^{-\mu}$$

$$\longrightarrow \sum_{k=0,2,4,\ldots}^{\infty} \frac{\mu^k}{k!} e^{-\mu} = \frac{1}{2}(1 + e^{-2\mu})$$

$P_X(x|X \text{ is even}) = \frac{P(X=x, X \text{ is even})}{P(X \text{ is even})} = \frac{2}{1+e^{-2\mu}} \frac{\mu^k}{k!} e^{-\mu}$ When k≥0 and k is even, else 0

- **CHECK Examples 2.6-3**

Example 2.6-4

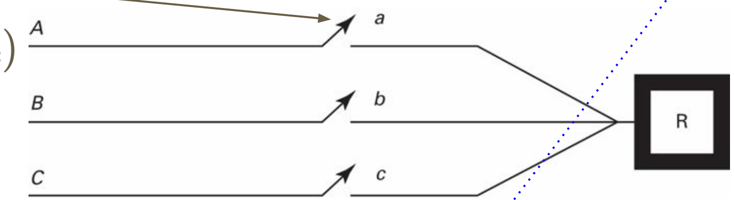
Given

$P[A] = 2P[B] = 4P[C] \rightarrow P[C]- = 1/7, P[B] = 2/7, P[A] = 4/7$

$X \sim \mathcal{N}(-1,4)$

$X \sim \mathcal{N}(0,1)$

$X \sim \mathcal{N}(1,4)$



- Compute $P[X \leq -1]$

Solution: Apply total probability rule as conditioned on disjoint events a,b,c

$$P[X \leq -1] = P[X \leq -1|A]P[A] + P[X \leq -1|B]P[B] + P[X \leq -1|C]P[C] = 0.354$$

$$P[X \leq -1|A] = 1/2$$

$$P[X \leq -1|B] = erf\left(\frac{-1-0}{1}\right) - erf(-\infty) = -erf(1) + 1/2 \quad (erf(-z) = -erf(z))$$

$$P[X \leq -1|C] = erf\left(\frac{-1-1}{2}\right) - erf(-\infty) = -erf(1) + 1/2$$

- Given we observe {X > -1}, from which one is the most likely source?

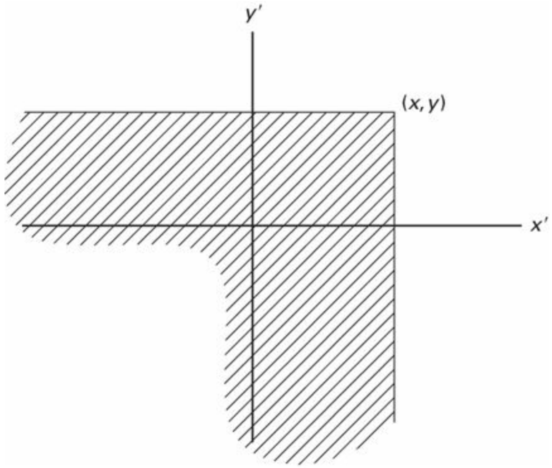
Solution: We want to compute $max\{P[A|X > -1], P[B|X > -1], P[C|X > -1]\}$

Note: $P[X > -1|A] = 1 - P[X \leq -1|A] = 1 - 1/2$ (from above)

Then use Bayes' rule for conditionals to compute posteriors,

$$P[A|X > -1] = \frac{P[X > -1|A].P[A]}{P[X > -1]} = \frac{(1-1/2).4/7}{1-0.354} = 0.44$$

Joint CDF/PDF



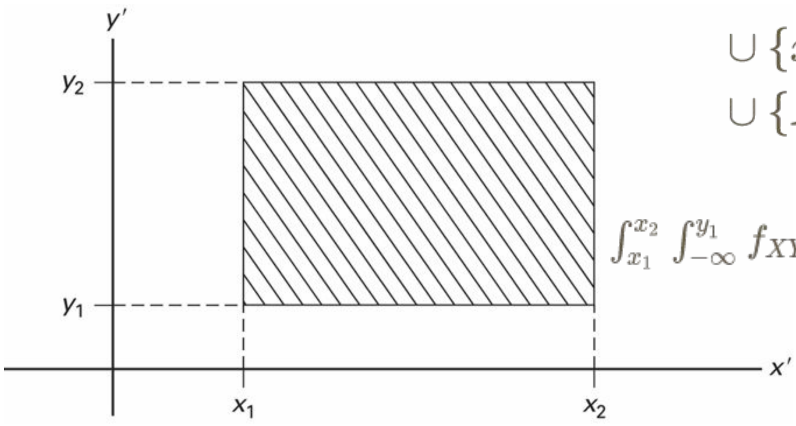
- Joint CDF specifies $F_{XY}(x,y) = P[X \leq x, Y \leq y]$
- Joint PDF is $f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$
- TO READ: Properties of joint CDF
- Proof of

  for all $x_2 \geq x_1, y_2 \geq y_1,$

  $F_{XY}(x_2,y_2) - F_{XY}(x_2,y_1) - F_{XY}(x_1,y_2) + F_{XY}(x_1,y_1) \geq 0$
- Start with defining the set

$$\{X \leq x_2, Y \leq y_2\} = \{x_1 < X \leq x_2, y_1 < Y \leq y_2\}$$
$$\cup \{x_1 < X \leq x_2, Y \leq y_1\} \cup \{X \leq x_1, y_1 < Y \leq y_2\}$$
$$\cup \{X \leq x_1, Y \leq y_1\}$$

P[ ]

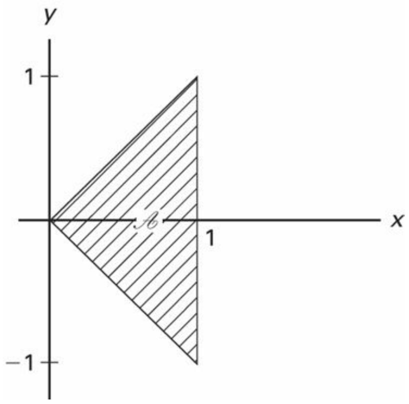


$$\int_{x_1}^{x_2} \int_{-\infty}^{y_1} f_{XY}(\xi,\eta) d\xi d\eta = \int_{-\infty}^{x_2} \int_{-\infty}^{y_1} f_{XY}(\xi,\eta) d\xi d\eta - \int_{-\infty}^{x_1} \int_{-\infty}^{y_1} f_{XY}(\xi,\eta) d\xi d\eta$$
$$= F_{XY}(x_2,y_1) - F_{XY}(x_1,y_1)$$

Example 2.6-6

$$f_{XY}(x, y = e^{-(x+y)}u(x)u(y)$$

Calculate $P[(X, Y)] \in \mathscr{A}$, where $\mathscr{A} = \{(x, y) : 0 \le x \le 1, |y| \le x\}$



$$P[(X, Y)] = \int_{x=0}^{1} \int_{y=-x}^{x} e^{-(x+y)} u(x)u(y) dx dy$$

Only positive limits

$$= \int_{x=0}^{1} (e^{-x} - e^{-2x}) dx = -e^{-1} + 1 - (-\tfrac{1}{2}e^{-2} + \tfrac{1}{2})$$

Marginal Distribution and Independence

- Marginal Distributions and densities

$$F_X(x) = F_{XY}(x, \infty) = \int_{-\infty}^{x} d\xi \int_{-\infty}^{\infty} dy f(\xi, y) \rightarrow f_X(x) = \frac{dF_X(x)}{dx}$$

$$F_Y(y) = F_{XY}(\infty, y) = \int_{-\infty}^{y} d\eta \int_{-\infty}^{\infty} dx f(x, \eta) \rightarrow f_Y(y) = \frac{dF_Y(y)}{dy}$$

- Marginal densities are also integrals of the joint density function

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx$$

$$P_X(x_i) = \sum_{\text{all } y_k} P_{XY}(x_i, y_k)$$

$$P_Y(y_i) = \sum_{\text{all } x_i} P_{XY}(x_i, y_k)$$

- Independence

$$F_{XY}(x, y) = F_X(x). F_Y(y)$$

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y} = \frac{\partial F_X(x)}{\partial x} \cdot \frac{\partial F_Y(y)}{\partial y} = f_X(x) f_Y(y)$$

$$F_X(x|Y \leq y) = \frac{F_{XY}(x, y)}{F_Y(y)} = F_X(x)$$

Conditionals really doesn't mean anything if the RVs are independent

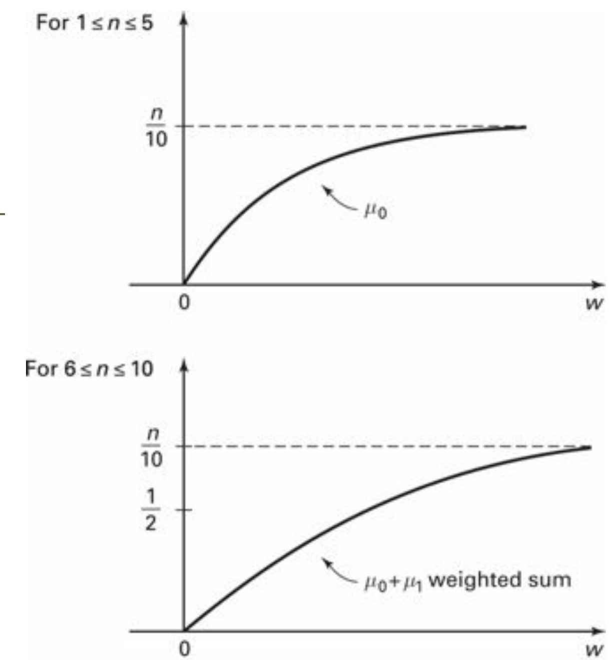
Differentiating

$$f_X(x|Y \leq y) = f_X(x)$$

Example 2.6-8

A certain restaurant has been found to have the following joint distribution for the waiting time for the service for a newly arriving customer and the total number of customers including the new arrival. Let ***W*** be a RV ***representing the continuous waiting time for a newly arriving customer*** & Let ***N*** be a discrete RV ***representing the total number of customers***. Find the joint density function?

$$F_{W,N}(w,n)=\begin{cases} 0, n<0 \text{ or } w<0 \\ \left(1-e^{-w/\mu_0}\right)\dfrac{n}{10}, 0\le n<5, w\ge 0 \\ \left(1-e^{-w/\mu_0}\right)\dfrac{5}{10}+\left(1-e^{-w/\mu_1}\right)\left(\dfrac{n-5}{10}\right), 5\le n<10, w\ge 0 \\ \left(1-e^{-w/\mu_0}\right)\dfrac{5}{10}+\left(1-e^{-w/\mu_1}\right)\left(\dfrac{5}{10}\right), 10\le n, w\ge 0 \end{cases}$$



The mixed pdf is given by,

$$\begin{aligned} f_{W,N}(w,n) &\triangleq \frac{\partial}{\partial w}\nabla_n F_{W,N}(w,n) \\ &= \frac{\partial}{\partial w}\{F_{W,N}(w,n)-F_{W,N}(w,n-1)\} \\ &= \frac{\partial}{\partial w}F_{W,N}(w,n)-\frac{\partial}{\partial w}F_{W,N}(w,n-1) \end{aligned}$$

$$\nabla_n F_{W,N}(w,n)=F_{W,N}(w,n)-F_{W,N}(w,n-1)=u(w)\begin{cases} \left(1-e^{-w/\mu_0}\right)\frac{1}{10}, & 0<n\le 5 \\ \left(1-e^{-w/\mu_1}\right)\frac{1}{10}, & 5<n\le 10 \\ 0, & \text{else.} \end{cases}$$

$$\begin{aligned} f_{W,N}(w,n) &= \frac{\partial}{\partial w}\nabla_n F_{W,N}(w,n) \\ &= u(w)\begin{cases} \frac{1}{10}\frac{1}{\mu_0}e^{-w/\mu_0}, & 0<n\le 5 \\ \frac{1}{10}\frac{1}{\mu_1}e^{-w/\mu_1}, & 5<n\le 10 \\ 0, & \text{else.} \end{cases} \end{aligned}$$

Example 2.6-12

$$f_{XY}(x,y) = A(x+y) \quad 0 < x \le 1, \quad 0 < y \le 1$$
$$= 0, \qquad Otherwise$$

What is A?

$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y) dx dy = 1$

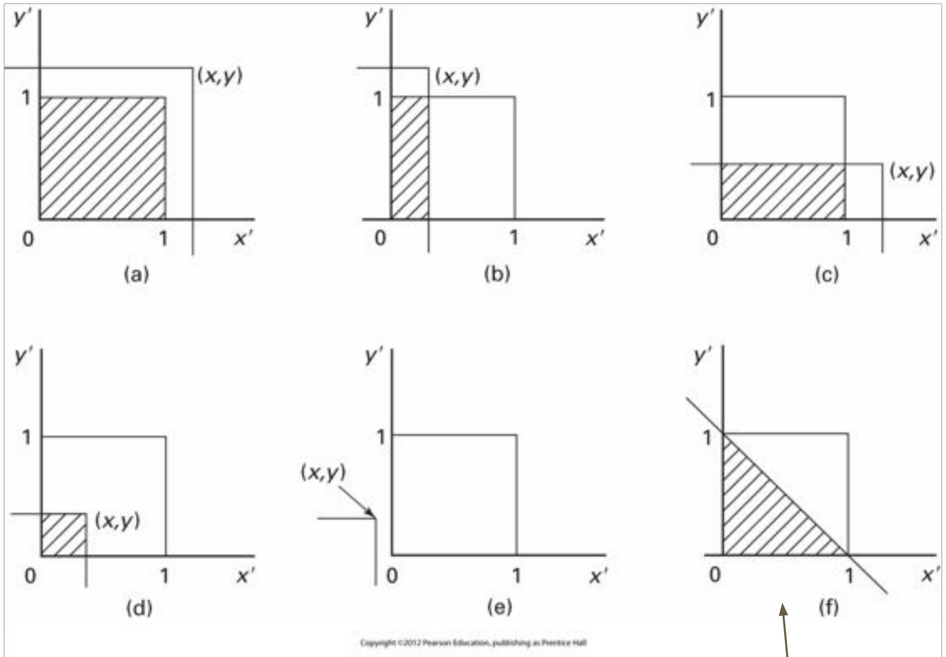
What are the marginal pdfs?

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_0^1 (x+y)dy = (xy + y^2/2)\Big|_0^1$$
$$= \begin{cases} x + \frac{1}{2} & 0 < x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

What is the cdf?

Since, $F_{XY} = P[X \le x, Y \le y]$ we will have to integrate over the infinite rectangle with vertices (x,y), (-∞,-∞), (-∞,y), (x,-∞)

Figure 2.6-9 Shaded region in (a) to (e) is the intersection of supp(f_{XY}) with the point set associated with the event $\{-\infty < X \le x, -\infty < Y \le y\}$. In (f), the shaded region is the intersection of supp(f_{XY}) with $\{X + Y \le 1\}$.



$F_{XY}(x,y) = \int_0^1 \int_0^1 f_{XY}(x',y')\, dx'\, dy' = 1 \quad \text{for } x \ge 1, y \ge 1$ ← Figure (a)

$F_{XY}(x,y) = \int_{y'=0}^{1} dy' \left(\int_{x'=0}^{x} dx' \,(x'+y')\right) = \frac{x}{2}(x+1) \quad \text{for } 1 < x \le 1, y \ge 1$ ← Figure (b)

$F_{XY}(x,y) = \int_{y'=0}^{y} dx' \left(\int_{x'=0}^{1} dx' \,(x'+y')\right) = \frac{y}{2}(y+1) \quad \text{for } x \ge 1, 0 < y \le 1$ ← Figure (c)

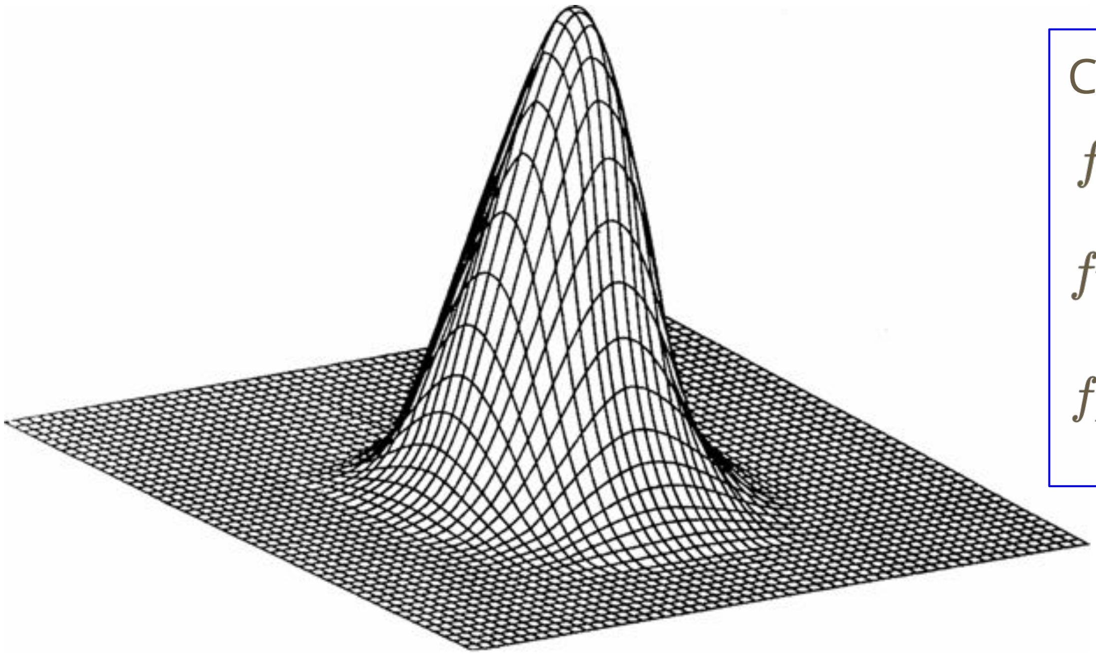
$P[X+Y \le 1]$

$$P[X+Y \le 1] = \int_{x'=0}^{1} \int_{y'=0}^{1-x'} (x'+y')\, dy'\, dx'$$
$$= \int_{x'=0}^{1} x'(1-x')\, dx' + \int_{x'=0}^{1} \frac{(1-x')^2}{2} dx'$$
$$= \frac{1}{3}$$

Non-independent RVs

$$f_{XY}(x,y) = (2\pi)^{-1}\exp\left[-\frac{1}{2}\left(x^2+y^2\right)\right]$$

$$f_{XY}(x,y) = \left[2\pi\sqrt{1-\rho^2}\right]^{-1}\exp\left[-\frac{1}{2(1-\rho^2)}\left(x^2+y^2-2\rho xy\right)\right]$$



Conditional pdfs

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}, \quad f_X(x) \neq 0$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x)dx$$

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)}$$

Conditional pdf is the probability of RV X given the event {Y=y}

Practice Example 2.6-13