Chapter - 3:
Functions of Random Variables

Aveek Dutta
Assistant Professor
Department of Electrical and Computer Engineering
University at Albany
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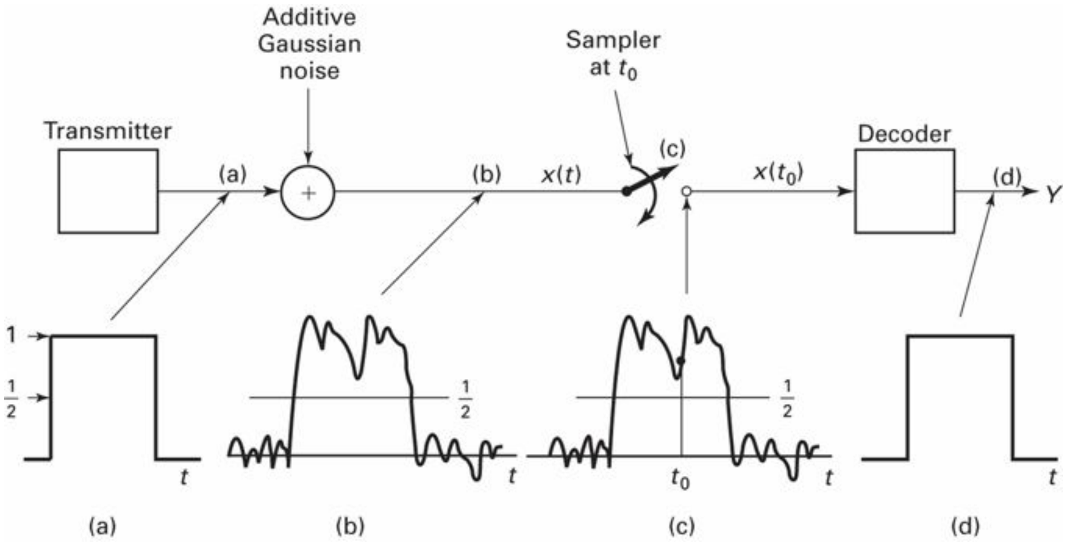
Example

Input	Output
$X(t_0) \geq 0.5$	Y=1
$X(t_0) < 0.5$	Y=0



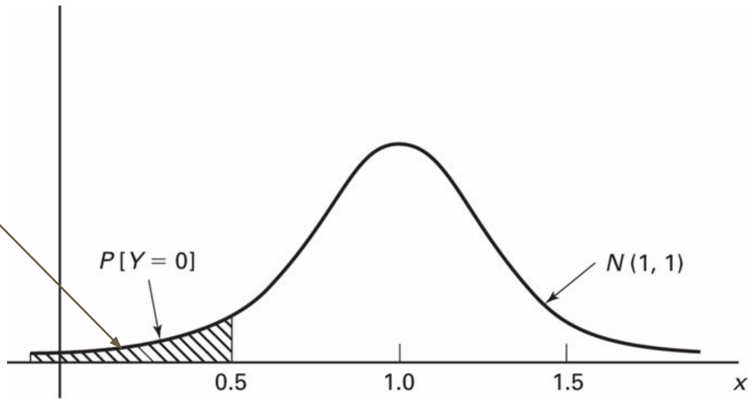
- If Y is a RV, we can write,

$$\{Y = 0\} = \{X < 0.5\}$$
$$\{Y = 1\} = \{X \geq 0.5\}$$

- If ***X: N(1,1)*** then we can calculate the probabilities as

$$P[Y=0] = P[X < 0.5] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.5} e^{-\frac{1}{2}(x-1)^2} dx$$
$$\simeq 0.31$$

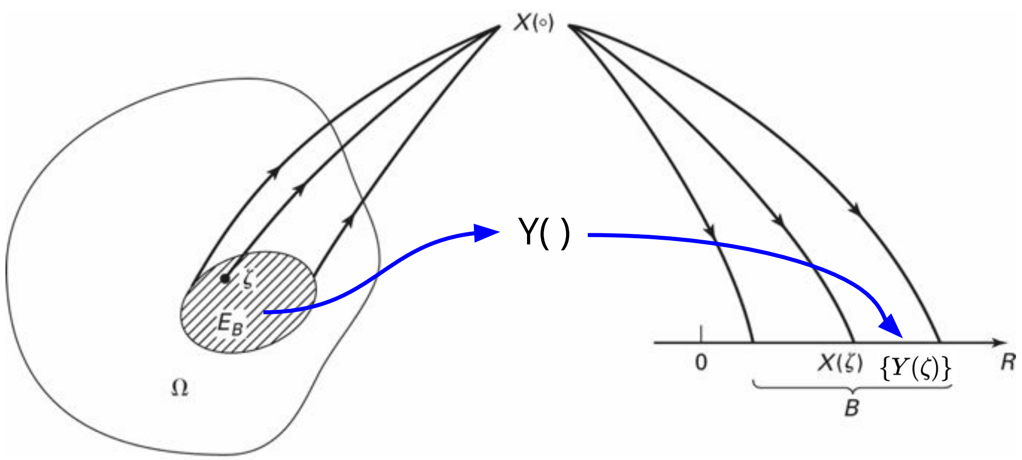
- $f_Y(y) = 0.31\delta(y) + 0.69\delta(y-1)$



Different Views of Functions of RV

First view ($Y : \Omega \to R_Y$) . For every $\zeta \in \Omega$, we generate a number $g[X(\zeta)] \triangleq Y(\zeta)$. The rule Y, which generates the numbers $\{Y(\zeta)\}$ for random outcomes $\{\zeta \in \Omega\}$, is a r.v. with domain Ω and range $R_Y \subset R$. Finally for every Borel set of real numbers B_Y, the set $\{\zeta : Y(\zeta) \in B_Y\}$ is an event. In particular the event $\{\zeta : Y(\zeta) \leq y\}$ is equal to the event $\{\zeta : g[X(\zeta)] \leq y\}$.

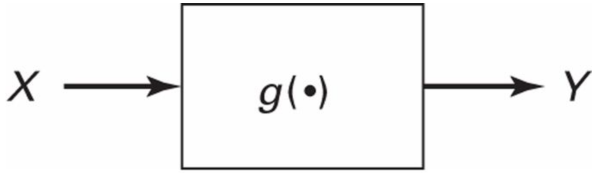
In this view, the stress is on Y as a mapping from Ω to R_Y. The intermediate role of X is suppressed.



View #2

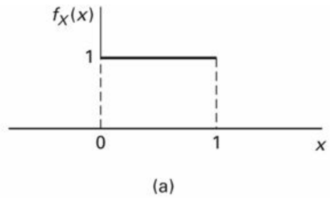
Second view (input/output systems view). For every value of $X(\zeta)$ in the range R_X we generate a new number $Y = g(X)$ whose range is R_Y. The rule Y whose domain is R_X and range is R_Y is a function of the random variable X. In this view the stress is on viewing Y as a mapping from one set of real numbers to another. A model for this view is to regard X as the input to a system with transmittance $g(\cdot)$. For such a system, an input x gets transformed to an output $y = g(x)$ and an input function X gets transformed to an output function $Y = g(X)$. (See Figure 3.1 – 4.)

$$\{\zeta : Y(\zeta) \leq y\} = \{\zeta : g[X(\zeta)] \leq y\}$$
$$= \{\zeta : X(\zeta) \in C_y\}$$



Example 3.2-1

- Let ***Y = 2X+3*** and ***X: U(0,1)***. Find $F_Y(y)$
- Solution: $\{Y \le y\} = \{2X + 3 \le y\} = \left\{X \le \frac{1}{2}(y-3)\right\}$

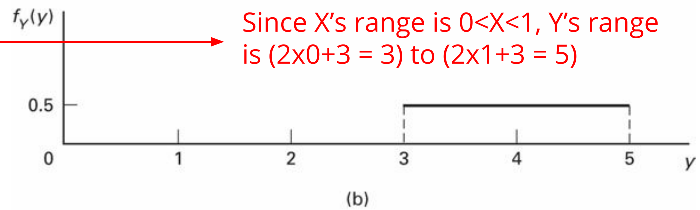


Hence C_y is the interval $\left(-\infty, \frac{1}{2}(y-3)\right)$ and

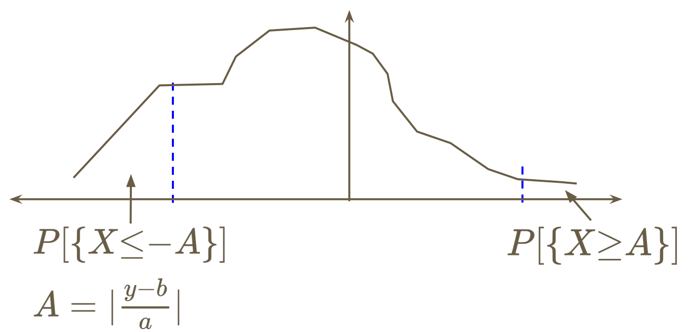
$$F_Y(y) = F_X\left(\frac{y-3}{2}\right)$$

The pdf of Y is

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy}\left[F_X\left(\frac{y-3}{2}\right)\right]$$
$$= \frac{1}{2} f_X\left(\frac{y-3}{2}\right)$$



Since X's range is 0<X<1, Y's range is (2x0+3 = 3) to (2x1+3 = 5)



- <u>**Generalization**</u>
  - <u>**NOTE: Read the text before ex 3.2-2 on discrete r.v.**</u>

$$\{Y \le y\} = \{aX + b \le y\} = \left\{X \le \frac{y-b}{a}\right\}$$

Since ***a*** is negative, the inequality is flipped. Whenever you multiply or divide an inequality by a negative number, you must flip the inequality sign.

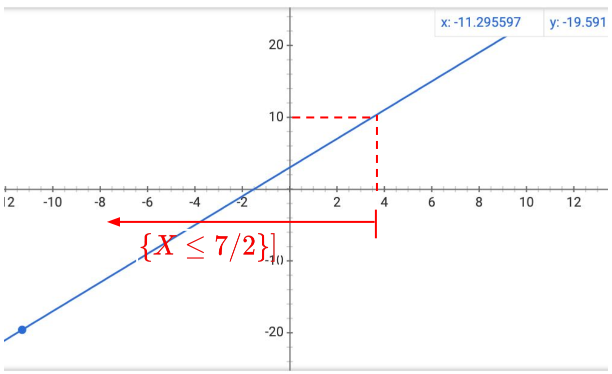
→ But for $a < 0$, $\{Y \le y\} = \{aX + b \le y\} = \{aX \le y - b\} = \left\{X \ge \frac{y-b}{a}\right\} = \{X \ge -A\}$

$$F_Y(y) = P[X \ge -A] = P[X > -A] = 1 - P[X \le -A] = 1 - F_X(-A)$$

This is only true for continuous r.v.

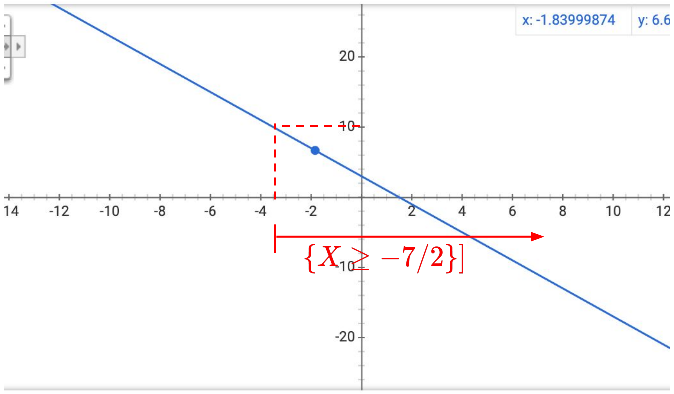
Example for a<0

Y = 2X+3 , a = 2, b = 3



$$F_Y(y) = P[\{Y \le 10\}] = P[\{X \le 7/2\}] = F_X(7/2)$$

Y = -2X+3, a = -2, b =3

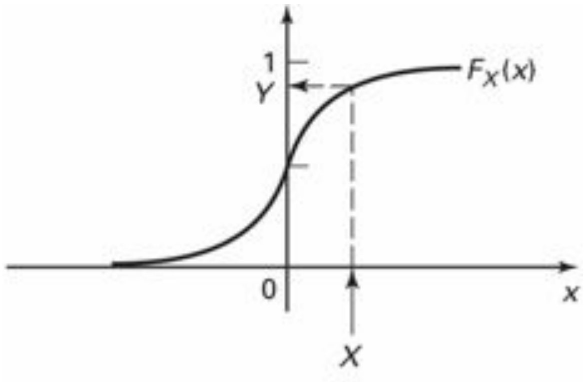


$$F_Y(y) = P[\{Y \le 10\}] = P[\{-2X \le y - 3\}]$$
$$= P[\{X \ge (10-3)/(-2)\}] = P[\{X \ge -7/2\}]$$

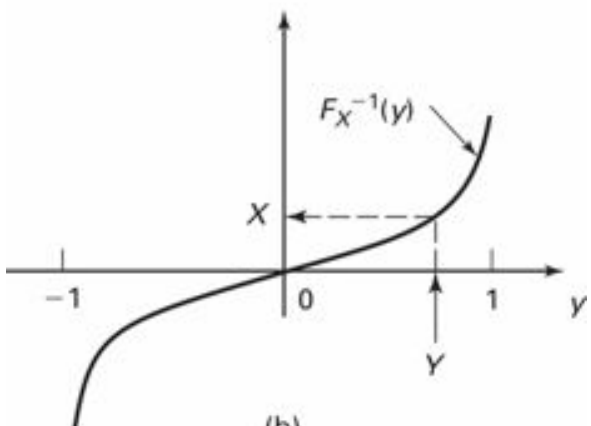
$$P[\{X \ge -7/2\}] = P[\{X > -7/2\}] = 1 - P[\{X \le -7/2\}] = 1 - F_X(-7/2)$$

This is only true for continuous r.v. because P[X=7/2] does not exist

Example 3.2-5



(a)



(b)

(Transformation of PDF's.) Let X have a continuous PDF $F_X(x)$, which is a strict monotone increasing function† of x. Let Y be an r.v. formed from X by the transformation

$$Y = F_X(X). \tag{3.2-12}$$

To compute $F_Y(y)$, we proceed as usual:

$$\begin{aligned}\{Y \le y\} &= \{F_X(X) \le y\} \\ &= \{X \le F_X^{-1}(y)\}.\end{aligned}$$

Hence

$$\begin{aligned}F_Y(y) &= P[F_X(X) \le y] \\ &= P[X \le F_X^{-1}(y)] \\ &= \int_{\{x: F_X(x) \le y\}} f_X(x)dx.\end{aligned}$$

1. Let $y < 0$. Then since $0 \le F_X(x) \le 1$ for all $x \in [-\infty, \infty]$, the set $\{x: F_X(x) \le y\} = \phi$ and $F_Y(y) = 0$.
2. Let $y > 1$. Then $\{x: F_X(x) \le y\} = [-\infty, \infty]$ and $F_Y(y) = 1$.
3. Let $0 \le y \le 1$. Then $\{x: F_X(x) \le y\} = \{x: x \le F_X^{-1}(y)\}$

and

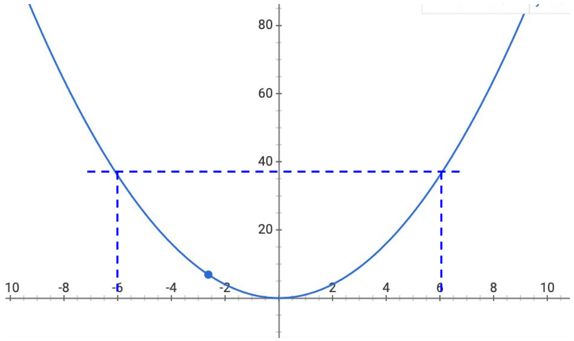
$$F_Y(y) = \int_{-\infty}^{F_X^{-1}(y)} f_X(x)dx = F_X(F_X^{-1}(y)) = y.$$

Hence

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ y, & 0 \le y \le 1 \\ 1, & y > 1. \end{cases} \tag{3.2-13}$$

Equation 3.2-13 says that whatever probability law X obeys, $Y \triangleq F_X(X)$ will be a *uniform* r.v. Conversely, given a uniform r.v. Y, the transformation $X \triangleq F_X^{-1}(Y)$ will generate a r.v. with PDF $F_X(x)$ (Figure 3.2-5). This technique is sometimes used in *simulation* to generate r.v.'s with specified distributions from a uniform r.v.

Example 3.2-2 - Multiple roots of Y = g(X)



- **Define Y = X^2**
- We can write the set of outcomes for the RV ***Y***

$$\{Y \le y\} = \{X^2 \le y\} = \{-\sqrt{y} \le X \le \sqrt{y}\}$$
$$= \{-\sqrt{y} < X \le \sqrt{y}\} \cup \{X = -\sqrt{y}\}$$

CDF → $F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) + P[X = -\sqrt{y}]$ **(Continuous RV)**

PDF → $f_Y(y) = \frac{d}{dy}[F_Y(y)]$

$$= \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

- If ***X: N(0,1)***

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{1}{2}y} u(y)$$ ← WHY?

Example 3.2-8 - Multiple roots of Y = g(X)

for $0 \le y \le 1$, the event $\{Y \le y\}$ satisfies

$$\{Y \le y\} = \{\sin X \le y\}$$
$$= \{\pi - \sin^{-1} y < X \le \pi\} \cup \{-\pi < X \le \sin^{-1} y\}.$$

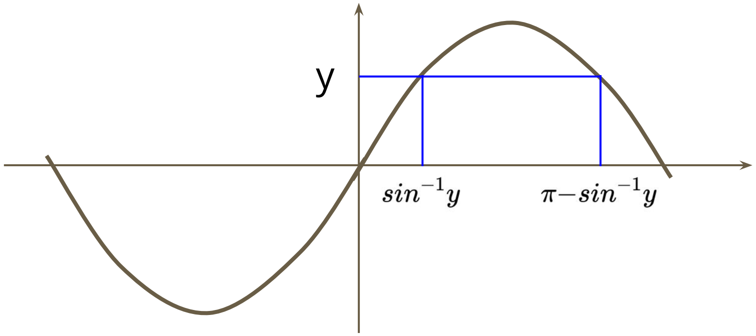
Since the two events on the last line are disjoint, we obtain

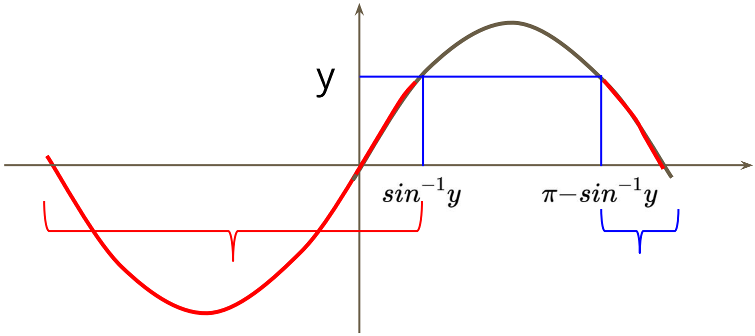
$$F_Y(y) = F_X(\pi) - F_X(\pi - \sin^{-1} y) + F_X(\sin^{-1} y) - F_X(-\pi).$$

Hence

$$f_Y(y) = \frac{dF_Y(y)}{dy} = f_X(\pi - \sin^{-1} y)\frac{1}{\sqrt{1-y^2}}$$
$$+ f_X(\sin^{-1} y)\frac{1}{\sqrt{1-y^2}}$$
$$= \frac{1}{\pi}\frac{1}{\sqrt{1-y^2}} \qquad 0 \le y < 1.$$

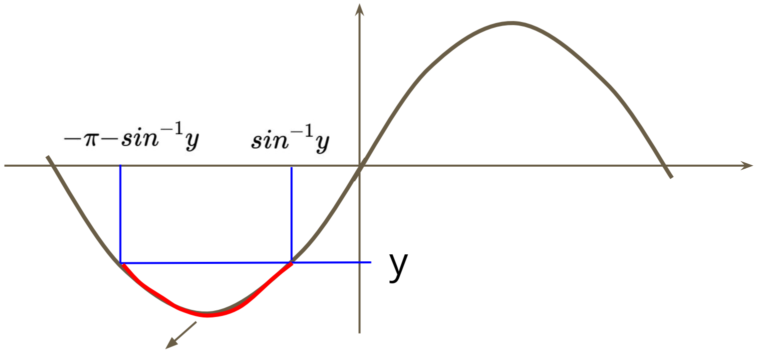
$$f_X(x) = \frac{1}{2\pi}$$





The red line defines the set $\{Y \le y\}$

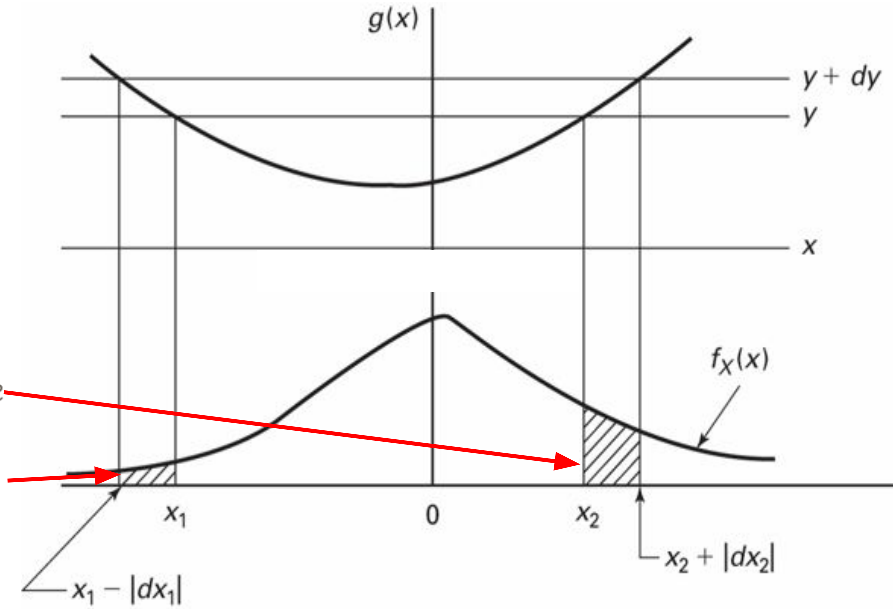
for $-1 < y < 0$



$$P[\{-\pi - sin^{-1} < X \le sin^{-1}\}] = F_X(sin^{-1}y) - F_X(-\pi - sin^{-1}y) \longrightarrow f_Y(y) = \frac{1}{\pi}\frac{1}{\sqrt{1-y^2}} \quad \text{for } -1 < y < 0$$

Generalization for roots of Y=g(X)

- The event $\{y < Y \leq y + dy\}$ can be written as union disjoint events of r.v. ***X***.
- If x_i are unique roots of $y = g(x)$ then the events of ***X***, has the form
  - $E_i = \{x_i < X \leq x_i + |dx_i|\}$ for $g'(x) + ve$
  - $E_i = \{x_i - |dx_i| < X \leq x_i\}$ for $g'(x) - ve$
  - And $P[E_i] = f_X(x_i)\,|dx_i|$



- $$P[y < Y \leq y + dy] = f_Y(y)|dy| = \sum_{i=1}^{n} f_X(x_i)\,|dx_i|$$ $P[\cup E_i]$
- $$f_Y(y) = \sum_{i=1}^{n} f_X(x_i)\left|\frac{dx_i}{dy}\right| = \sum_{i=1}^{n} f_X(x_i)\left|\frac{dy}{dx_i}\right|^{-1} = \sum_{i=1}^{n} f_X(x_i)\,/\,|g'(x_i)|$$

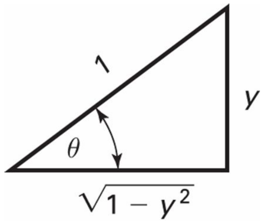
Note the modulus for $g'(x)$

Example 3.2-9

- The roots in example 3.2-8 were, for $y > 0$ are $x_1 = \sin^{-1} y, x_2 = \pi - \sin^{-1} y$

At $x_1 = \sin^{-1} y$ we get $dg/dx|_{x=x_1} = \cos(\sin^{-1} y)$

$$\left.\frac{dg}{dx}\right|_{x=x_2} = \cos\left(\pi - \sin^{-1} y\right) = \cos\pi\cos\left(\sin^{-1} y\right) + \sin\pi\sin\left(\sin^{-1} y\right)$$
$$= -\cos\left(\sin^{-1} y\right)$$



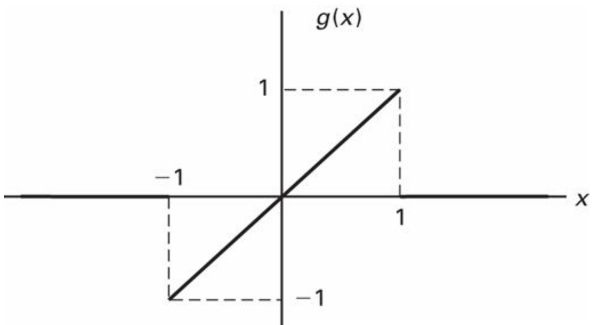
- $$\left|\frac{dg}{dx}\right|_{x_1} = \left|\frac{dg}{dx}\right|_{x_2} = \sqrt{1-y^2}$$

$$f_Y(y) = \tfrac{1}{\pi}\frac{1}{\sqrt{1-y^2}} \qquad 0 \le y < 1$$

Practice - Find pdf of ***Y = cos(X)*** using the same principle?

Example 3.2-11

- We know $f_Y(y) = \sum_{i=1}^{n} f_X(x_i) \,/\, |g'(x_i)|$



$g(x) = 0, \quad |x| \geq 1$

$g(x) = x, \quad -1 < x < 1.$

- Find roots of ***y=g(x)***
  - Thus, $g'(x) = 0$ for $|x| \geq 1$, and $g'(x) = 1$ for $-1 < x < 1$
  - For $y \geq 1$ and $y \leq -1$ there are no real roots to $y - g(x) = 0 \longrightarrow f_Y(y) = 0$
  - For $-1 < y < 1$, $|g'(x)| = 1 \longrightarrow f_Y(y) = f_X(y)$
- If $X : \mathcal{N}(0,1)$

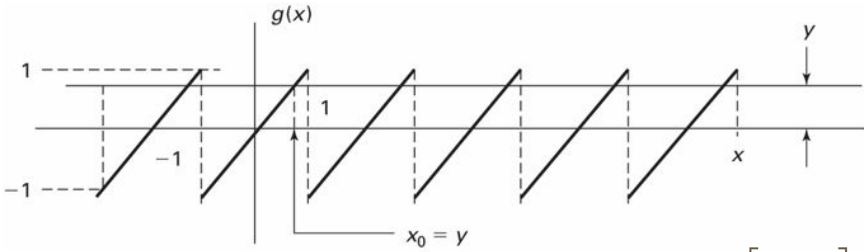
$$f_Y(y) = \begin{cases} 0, & |y| \geq 1 \\ (2\pi)^{-1/2} \exp\{-\frac{1}{2}y^2\} + 0.317\delta(y), & -1 < y < 1 \end{cases}$$

$$\begin{aligned} P[Y=0] &= P[X \geq 1] + P[X \leq -1] \\ &= (1 - P[X \leq 1]) + P[X \leq -1]] \\ &= (1 - \{1/2 + erf(1)\}) + (1/2 + erf(-1)) \\ &= 1/2 - erf(1) + 1/2 - erf(1) \\ &= 1 - 2.erf(1) \\ &= 1 - 2 \times 0.34134 \text{ (From Table 2.4-1 in text)} \end{aligned}$$

Justification: This adjustment is required because when ***g'(x) = 0*** it is a flat region in ***y=g(x)***. So, for any ***x*** in that flat region the ***y*** values are identical. This will create a probability mass at that value of ***y***. The mass is equal to the probability of the event ***X*** falls in the flat area. In this case ***X≥1*** and ***X≤-1***

Infinite roots - Example 3.2-12

- The excursions of $y = g(x)$ suggests the same roots as before for $|y| > 1$
- But infinite roots for $-1 < y < 1$
  - $x_n = y + 2n$ With $|g'(x)| = 1$ at each root
- So, $f_Y(y) = \sum_{n=-\infty}^{\infty} f_X(y+2n)\mathrm{rect}\left[\frac{y}{2}\right]$
- If $X : \mathcal{N}(0,1)$, integrate to check if equal to 1



$g(x) = \sum_{n=-\infty}^{\infty}(x-2n)\mathrm{rect}\left[\frac{x-2n}{2}\right]$

$$\begin{aligned}\int_{-\infty}^{\infty} f_Y(y)dy &= \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\int_{-1}^{1}\exp\left\{-\frac{1}{2}(y+2n)^2\right\}dy \\ &= \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\int_{-1+2n}^{1+2n}\exp\left\{-\frac{1}{2}y^2\right\}dy \\ &= \sum_{n=-\infty}^{\infty}\{\mathrm{erf}(1+2n)-\mathrm{erf}(-1+2n)\}\end{aligned}$$

- All terms cancels, except the *second term of the first sum* and *the first term of the last sum*

$$\int_{-\infty}^{\infty} f_Y(y)dy = -\,\mathrm{erf}(-\infty)+\mathrm{erf}(\infty) = 2\times\mathrm{erf}(\infty) = 1$$

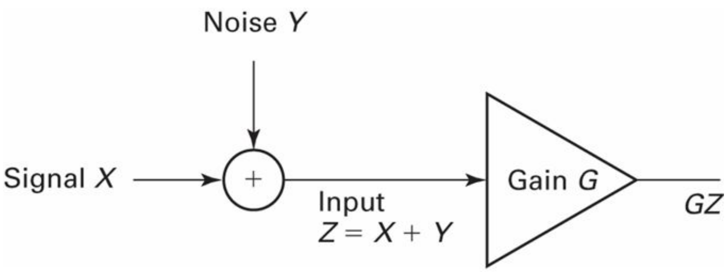
Two variable functions Z = g(X,Y)

- Definition

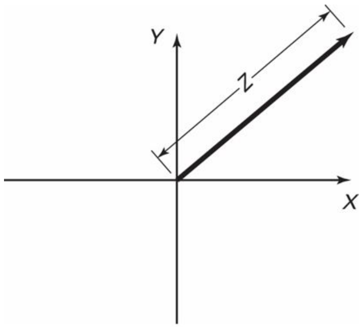
$$\{\zeta : Z(\zeta) \le z\} \text{ and } \{\zeta : X(\zeta), Y(\zeta) \in C_z\}$$

- Using shorter notations

$$\{Z \le z\} = \{(X, Y) \in C_z\}$$
$$F_Z(z) = \iint_{(x,y)\in C_z} f_{XY}(x, y)\,dx\,dy$$



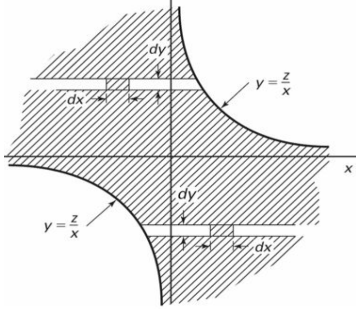
- Other functions include
  - Z = max(X,Y),
  - Z = X+Y, aX+bY
  - $Z = X^2+Y^2$
  - $Z = (X^2+Y^2)^{1/2}$



Example 3.3-1: Z=XY

Note: the limits on the second term for the integral over ***x***, is also from a smaller (-ve) number ***z/y*** to **+∞**

- For z>0 $F_Z(z) = \int_0^\infty \left(\int_{-\infty}^{z/y} f_{XY}(x,y)dx\right) dy + \int_{-\infty}^0 \left(\int_{z/y}^\infty f_{XY}(x,y)dx\right) dy$

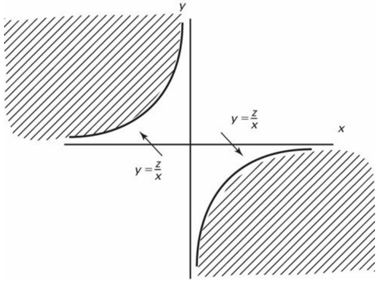


- The CDF is given by

$$F_Z(z) = \int_0^\infty [G_{XY}(z/y, y) - G_{XY}(-\infty, y)]\, dy$$
$$+ \int_{-\infty}^0 [G_{XY}(\infty, y) - G_{XY}(z/y, y)]\, dy$$

$$G_{XY}(x,y) \triangleq \int f_{XY}(x,y)dx$$

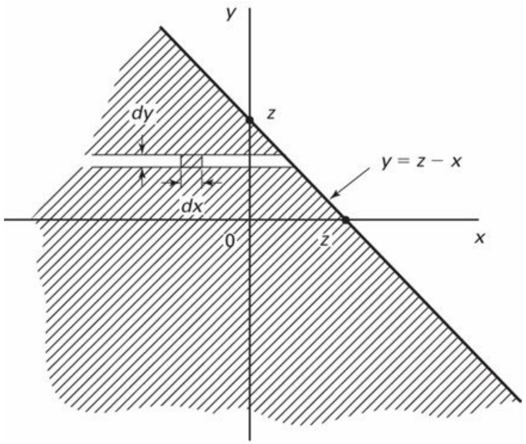
$$f_{Z(z)} = \frac{dF_Z(z)}{dz} = \int_{-\infty}^\infty \frac{1}{|y|} f_{XY}(z/y, y)dy$$



- The integral is same for z<0 as well, only the range of z is different
- Calculate $f_Z(z)$ if X and Y are iid Cauchy variables,

$$f_X(x) = f_Y(x) = \frac{\alpha/\pi}{\alpha^2 + x^2}$$

Sum of two variable: Z = X+Y



- Proceeding as per definition

$$F_Z(z) = \iint_{x+y\le z} f_{XY}(x,y)dxdy$$
$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{z-y} f_{XY}(x,y)dx\right)dy$$
$$= \int_{-\infty}^{\infty}\left[G_{XY}(z-y,y) - G_{XY}(-\infty,y)\right]dy$$

$$G_{XY}(x,y) \triangleq \int f_{XY}(x,y)dx$$

- Differentiating w.r.t ***z***

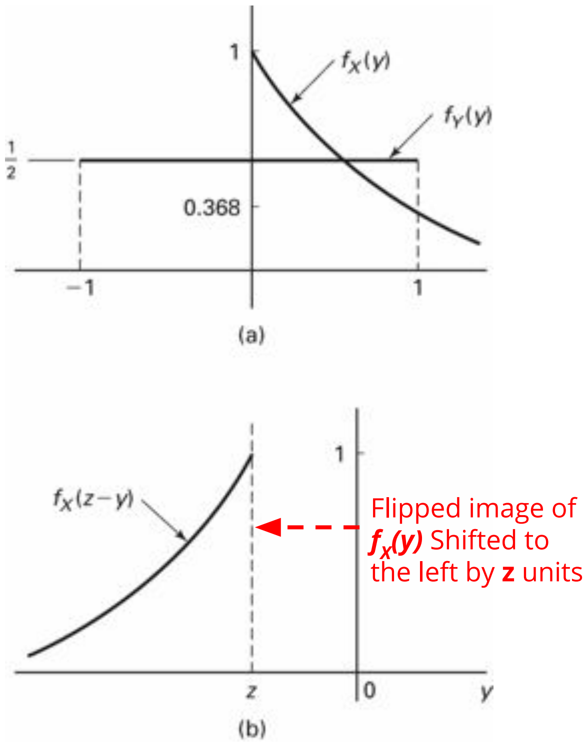
$$f_Z(z) = \frac{dF_Z(z)}{dz} = \int_{-\infty}^{\infty}\frac{d}{dz}\left[G_{XY}(z-y,y)\right]dy$$
$$= \int_{-\infty}^{\infty} f_{XY}(z-y,y)dy = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)dy \text{ (if X and Y are independent)}$$

This is called the convolution equation. Flipped image of $f_X(y)$ and shifted to the left by z units gives, $f_X(-y+z)$

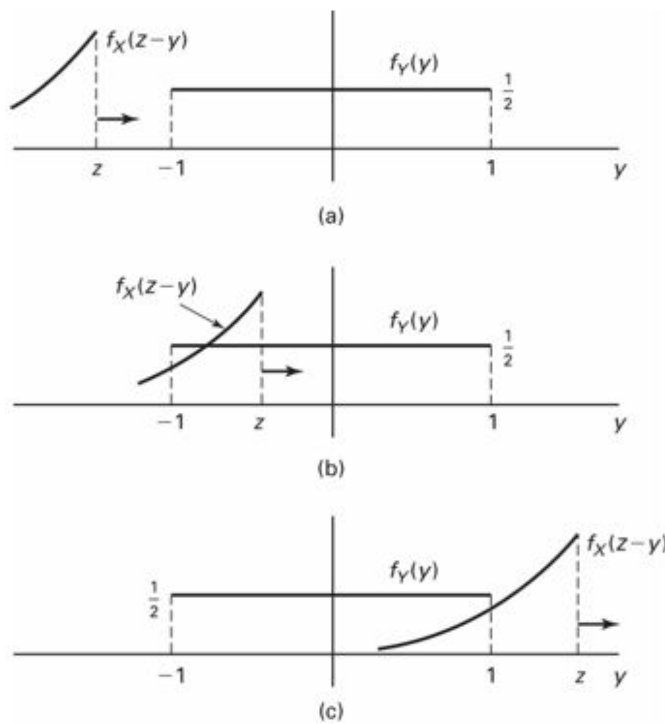
Example 3.3-4

- X and Y are r.v. with

$$f_X(x) = e^{-x}u(x) \qquad f_Y(y) = \frac{1}{2}[u(y+1) - u(y-1)] \qquad Z = X + Y$$



$$f_X(z-y) = e^{-(z-y)}u(z-y)$$



Region 1: $z < 1$

No overlap between $f_X(x)$ and $f_Y(y)$.
Hence $f_Z(z) = 0$

Region 2: $-1 \le z < 1$

$$f_Z(z) = \int_{-1}^{z} f_X(z-y).\, f_Y(y)dy$$
$$= \frac{1}{2}\int_{-1}^{z} e^{-(z-y)}\,dy = \frac{1}{2}[1 - e^{-(z+1)}]$$

Region 3: $z \ge 1$

$$f_Z(z) = \int_{-1}^{1} f_X(z-y).\, f_Y(y)dy$$
$$= \frac{1}{2}\int_{-1}^{1} e^{-(z-y)}\,dy = \frac{1}{2}[e^{-(z-1)} - e^{-(z+1)}]$$

Sanity check: $f_Z(z)$ is continuous since there are no delta functions involved in the integration. Check with ***z = 1*** in region 1 and region 2 to confirm equality, which should be the case for continuous functions.

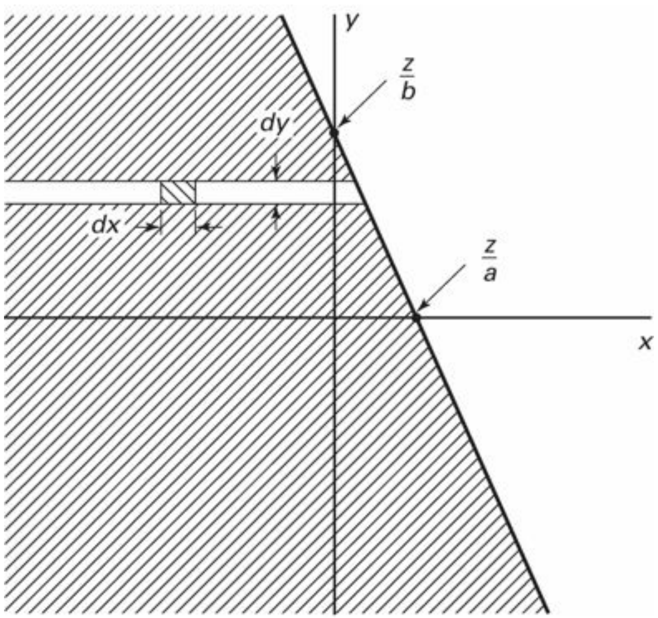
What about Z = aX+bY

- Let a>0, b>0, $y = \frac{z}{b} - \frac{ax}{b}$

$$F_Z(z) = \iint_{g(x,y)\le z} f_{XY}(x,y)\,dx\,dy$$
$$= \int_{-\infty}^{\infty} f_Y(y) \left(\int_{-\infty}^{z/a - by/a} f_X(x)\,dx \right) dy$$

Differentiating, w.r.t ***z***

$$f_Z(z) = \frac{1}{a} \int_{-\infty}^{\infty} f_X\left(\frac{z}{a} - \frac{by}{a}\right) f_Y(y)\,dy$$



- Another way to solve this is to define new r.v.,

$$V = aX, W = bY$$

Then apply $Z = V + W$ and convolve as before.

$$f_Z(z) = \int_{-\infty}^{\infty} f_V(z-w) f_W(w)\,dw$$

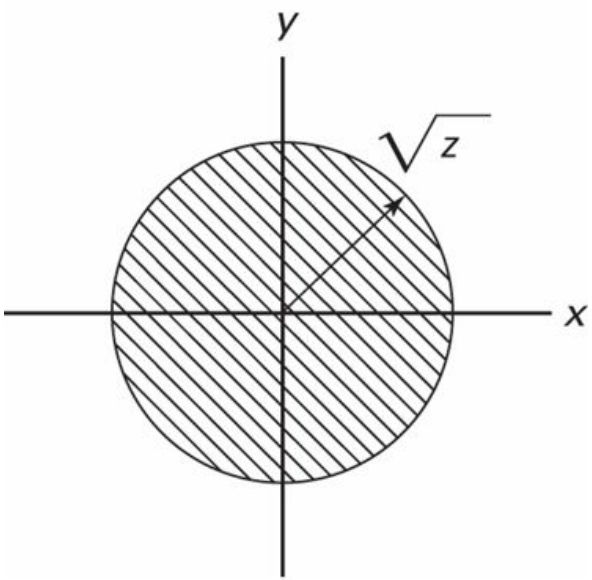
$$f_V(v) = \frac{1}{a} f_X\left(\frac{v}{a}\right)$$
$$f_W(w) = \frac{1}{b} f_Y\left(\frac{w}{b}\right)$$
$$f_Z(z) = \frac{1}{ab} \int_{-\infty}^{\infty} f_X\left(\frac{z-w}{a}\right) f_Y\left(\frac{w}{b}\right) dw$$

Substitute *w* = *by* to get the integral as above

$Z = X^2+Y^2$ and $Z = (X^2+Y^2)^{1/2}$

- If X,Y are iid with $\mathcal{N}(0,\sigma^2)$, proceed as before,

$$F_Z(z) = \iint_{(x,y)\in C_z} f_{XY}(x,y)dxdy \quad \text{for} \quad z \geq 0$$
$$= \frac{1}{2\pi\sigma^2} \iint_{x^2+y^2\leq z} e^{-(1/2\sigma^2)(x^2+y^2)} dxdy$$



- Converting to polar coordinates

$$x = r\cos\theta \quad y = r\sin\theta$$
$$dxdy \to rdrd\theta$$

Then $x^2+y^2 \leq z \to r \leq \sqrt{z}$ and the above equation becomes

$$F_Z(z) = \frac{1}{2\pi\sigma^2}\int_0^{2\pi} d\theta \int_0^{\sqrt{z}} r\exp\left[-\frac{1}{2\sigma^2}r^2\right]dr = \left[1 - e^{-z/2\sigma^2}\right]u(z)$$

- Differentiating w.r.t ***z***

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{1}{2\sigma^2}e^{-z/2\sigma^2}u(z)$$

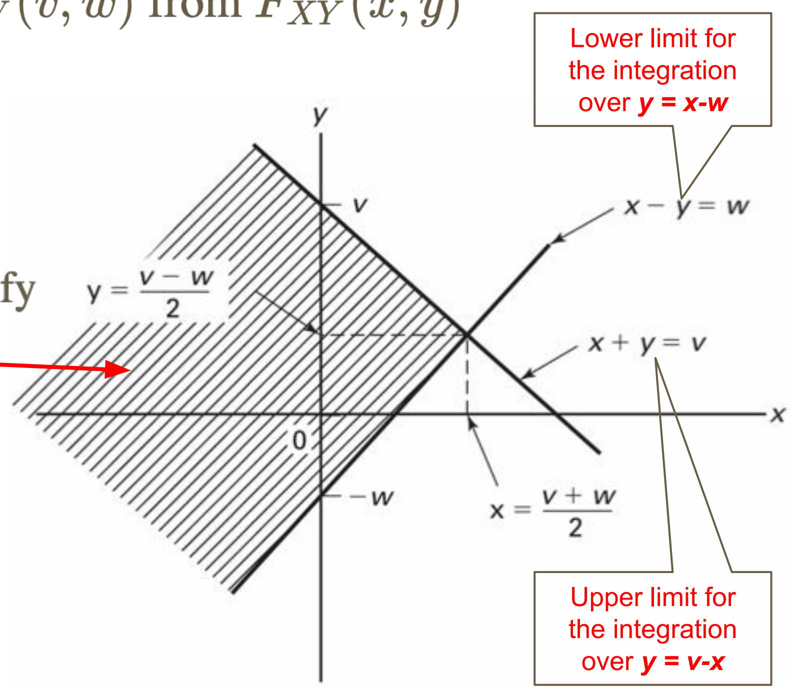
Multiple functions of R.V.s: *V=g(X,Y), W = h(X,Y)*

- **Problem:** Compute joint distribution $F_{VW}(v,w)$ from $F_{XY}(x,y)$

$$P[V \leq v, W \leq w] = F_{VW}(v,w) = \iint_{(x,y)\in C_{vw}} f_{XY}(x,y)dxdy$$

The region C_{vw} is given by the points x, y that satisfy

$$C_{vw} = \{(x,y) : g(x,y) \leq v, h(x,y) \leq w\}$$



- To integrate, express ***x*** and ***y*** in terms of ***v*** and ***w. (see example 3.4-1)***

$$F_{VW}(v,w) = \int_{-\infty}^{(v+w)/2} \left(\int_{x-w}^{v-x} f_{XY}(x,y)dy \right) dx$$

$$f_{VW}(v,w) = \frac{\partial^2 F_{VW}(v,w)}{\partial v \partial w}$$

Study the integration from the text

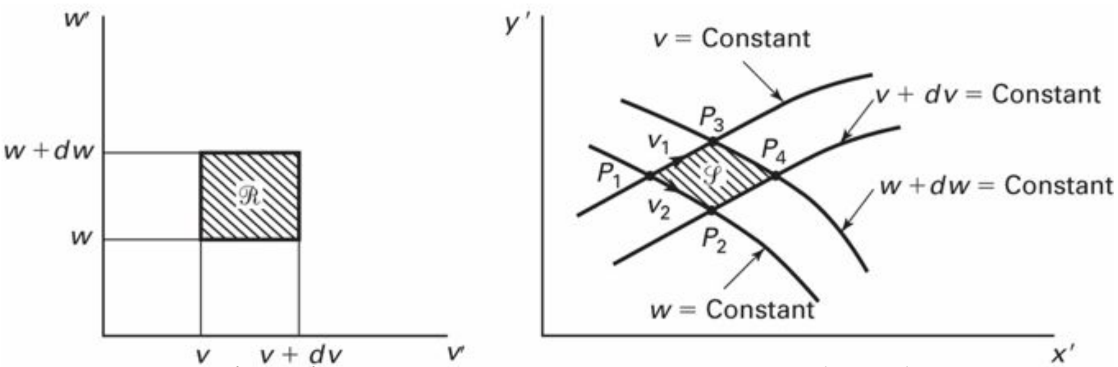
Simpler Approach

- In the infinitesimal small region, $\{v < V \leq v + dv, w < W \leq w + dw\}$
- Now, $P[\{v < V \leq v + dv, w < W \leq w + dw\}]$ is the probability that V and W lie in the infinitesimal rectangle of area $\partial v \partial w$.
- The image of this area in the x'-y' plane

$$P_1 = (x, y)$$
$$P_2 = \left(x + \frac{\partial \phi}{\partial v} dv, y + \frac{\partial \psi}{\partial v} dv\right)$$
$$P_3 = \left(x + \frac{\partial \phi}{\partial w} dw, y + \frac{\partial \psi}{\partial w} dw\right)$$
$$P_4 = \left(x + \frac{\partial \phi}{\partial v} dv + \frac{\partial \phi}{\partial w} dw, y + \frac{\partial \psi}{\partial v} dv + \frac{\partial \psi}{\partial w} dw\right)$$

The change in x (*dx*) is given by:
Rate of change of ***x= ϕ(v,w)*** along ***v*** axis ***(∂ϕ/∂v)*** multiplied by the change in ***v*** *(dv)*

The change in ***x= ϕ(v,w)*** because of change in ***v*** plus the change in ***x= ϕ(v,w)*** because of change in ***w***.



$$v = g(x, y) \qquad x = \phi(v, w)$$
$$w = h(x, y) \qquad y = \psi(v, w)$$

Any transformation from ***v'-w'*** plane to ***x'-y'*** plane distorts the infinitesimal region from a rectangle to a parallelogram.

..contd

Recall in single variable case in chapter 2

$$P[\{x < X \le x + \Delta x\}] \simeq f_X(x).\Delta x$$

- Therefore, we can write,

$$\begin{aligned} P[v < V \le v + dv, w < W \le w + dw] &= \iint_{\mathscr{R}} f_{VW}(\xi, \eta) d\xi d\eta \\ &= f_{VW}(v, w) A(\mathscr{R}) \\ &= \iint_{\mathscr{S}} f_{XY}(\xi, \eta) d\xi d\eta \\ &= f_{XY}(x, y) A(\mathscr{S}) \end{aligned}$$

Note: $P(B) = \iint_{\mathscr{R}} f_{XY}(x, y) dx dy \neq \iint_{\mathscr{R}} f_{XY}(\phi(v, w), \varphi(v, w)) dv dw$

$\rightarrow$ Because the area (or volume), $dxdy \neq dvdw$

- The area of the parallelogram $P_1P_2P_3P_4$ in vector notation is given by

$$\begin{aligned} A(\mathscr{S}) = \left|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\right| &= \left|\left(\frac{\partial\phi}{\partial v} dv\mathbf{i} + \frac{\partial\varphi}{\partial v}\mathbf{j}dv\right) \times \left(\frac{\partial\phi}{\partial w}\mathbf{i}dw + \frac{\partial\varphi}{\partial w}\mathbf{j}dw\right)\right| \\ &= \left|\frac{\partial\phi}{\partial v}\frac{\partial\varphi}{\partial w} - \frac{\partial\phi}{\partial w}\frac{\partial\varphi}{\partial v}\right| dvdw \end{aligned}$$

- Therefore, the Jacobian of functions $x=\phi(v,w)$ *and* $y=\psi(v,w)$ is

$$|\tilde{J}_i| = \text{mag}\begin{vmatrix} \partial\phi_i/\partial v & \partial\phi_i/\partial w \\ \partial\varphi_i/\partial v & \partial\varphi_i/\partial w \end{vmatrix} = |\partial\phi_i/\partial v \times \partial\varphi_i/\partial w - \partial\varphi_i/\partial v \times \partial\phi_i/\partial w|$$

.. contd

- From above we can write

$$f_{VW}(v,w) = \frac{A(\mathscr{S})}{A(\mathscr{B})} f_{XY}(x,y)$$

$$f_{VW}(v,w) = \sum_{i=1}^{n} f_{XY}(x_i, y_i) \left| \tilde{J}_i \right|$$

Jacobian of the transformation $x=\phi(v,w)$ and $y=\psi(v,w)$

Subscript, *i*, denotes multiple values of (x_i, y_i) maps to the same (v,w)

Also, Note the definition of $\left| \tilde{J}_i^{-1} \right|$

- Using the forward functions v= g(X,Y) and w= H(X,Y)

$$J = \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix} = \tilde{J}_i^{-1}$$

Example 3.5-2

We are given two functions

$$v \triangleq g(x, y) = 3x + 5y$$

$$w \triangleq h(x, y) = x + 2y$$

and the joint pdf f_{XY} of two r.v.'s X, Y. What is the joint pdf of two new random variables $V = g(X, Y)$, $W = h(X, Y)$?

Solution The inverse mappings are computed from Equation 3.4-13 to be

$$x = \phi(v, w) = 2v - 5w$$

$$y = \Phi(v, w) = -v + 3w.$$

Then

$$\frac{\partial \phi}{\partial v} = 2, \frac{\partial \phi}{\partial w} = -5, \frac{\partial \Phi}{\partial v} = -1, \frac{\partial \Phi}{\partial w} = 3$$

and

$$|\tilde{J}| = \text{mag}\begin{vmatrix} 2 & -5 \\ -1 & 3 \end{vmatrix} = 1.$$

Assume $f_{XY}(x, y) = (2\pi)^{-1} \exp[-\frac{1}{2}(x^2 + y^2)]$. Then, from Equation 3.4-11

$$f_{VW}(v, w) = \frac{1}{2\pi} \exp\left[-\frac{1}{2}[(2v - 5w)^2 + (-v + 3w)^2]\right]$$

$$= \frac{1}{2\pi} \exp\left[-\frac{1}{2}(5v^2 - 26vw + 34w^2)\right].$$

Example 3.5-5

$$V \triangleq X + Y$$
$$W \triangleq X - Y.$$

The only root to

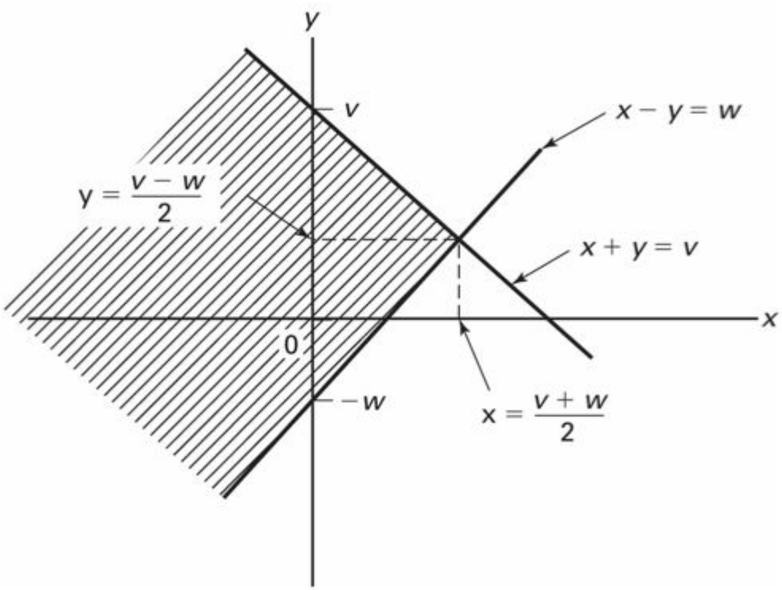
$$v - (x + y) = 0$$
$$w - (x - y) = 0$$

is

$$x = \frac{v + w}{2}$$
$$y = \frac{v - w}{2}$$

and $|\tilde{J}| = \frac{1}{2}$. Hence

$$f_{VW}(v, w) = \frac{1}{2} f_{XY}\left(\frac{v + w}{2}, \frac{v - w}{2}\right).$$



TO DO: Compute marginal density ***$f_V(v)$*** to obtain the convolution integral under independence of v and w

Practice examples at the end of Chapter 3