Chapter - 9:
Random Processes

Aveek Dutta
Assistant Professor
Department of Electrical and Computer Engineering
University at Albany
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Basic Concepts

- Analogous to random sequences except the time axis is uncountable

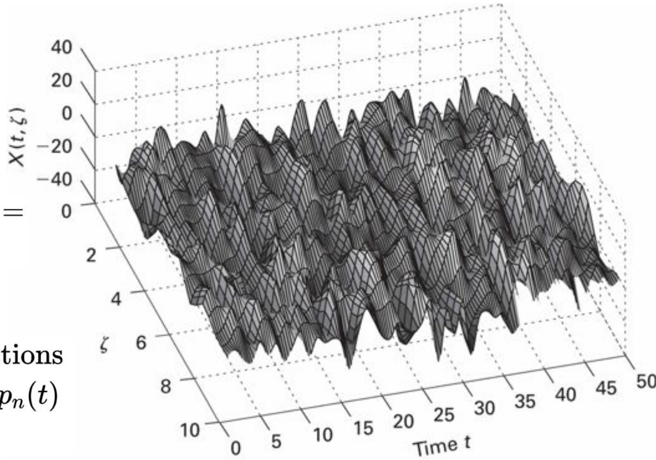
Definition 7.1 – 1 Let $(\Omega, \mathscr{F}, P)$ be a probability space. Then define a mapping X from the sample space Ω to a space of continuous time functions. The elements in this space will be called sample functions. This mapping is called a random process if at each fixed time the mapping is a random variable, that is †, $X(t, \zeta) \in \mathscr{F}$ for each fixed t on the real line $-\infty < t < +\infty$.

- Examples

$X(t, \zeta) = X(\zeta) f(t)$ where X is a random variable and f is a deterministic function of the parameter t. We also write $X(t) = X f(t)$.

$X(t, \zeta) = A(\zeta) \sin(\omega_0 t + \Theta(\zeta))$ where A and Θ are random variables. We also write $X(t) = A \sin(\omega_0 t + \Theta)$, suppressing the outcome ζ

$X(t) = \sum_n X[n] p_n(t - T[n])$ where $X[n]$ and $T[n]$ are random sequences and the functions $p_n(t)$ are deterministic waveforms that can take on various shapes. For example, the $p_n(t)$ could be an ideal step function.



- Definition: A random process $X(t)$ is statistically specified by its complete set of n th order PDFs (pdf's or PMFs) for all positive integers n, i.e., $F_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n)$ for all $x_1, x_2, \ldots, x_n$ and for all $t_1 < t_2 < \ldots < t_n$

Moment Functions

- Mean Function

$$\mu_X(t) \triangleq E[X(t)], \quad -\infty < t < +\infty$$

- Autocorrelation function

$$R_{XX}(t_1, t_2) \triangleq E[X(t_1)X^*(t_2)], \quad -\infty < t_1, t_2 < +\infty$$

- Autocovariance function

$$K_{XX}(t_1, t_2) \triangleq E[X_c(t_1)X_c^*(t_2)]$$
$$\triangleq E\left[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))^*\right]$$

$$K_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X^*(t_2)$$

- Variance Function

$$\sigma_X^2(t) \triangleq K_{XX}(t, t) = E\left[|X_c(t)|^2\right]$$

- Power Function

$$R_{XX}(t, t) = E\left[|X(t)|^2\right]$$

- Example 9.1-5

$$\begin{aligned}\mu_X(t) &= E[A\sin(\omega_0 t + \Theta)] \\ &= E[A]E[\sin(\omega_0 t + \Theta)] \\ &= \mu_A \cdot \frac{1}{2\pi}\int_{-\pi}^{+\pi} \sin(\omega_0 t + \theta)d\theta \\ &= \mu_A \cdot 0 = 0\end{aligned}$$

$$\begin{aligned}R_{XX}(t_1, t_2) &= E[X(t_1)X^*(t_2)] \\ &= E\left[A^2\sin(\omega_0 t_1 + \Theta)\sin(\omega_0 t_2 + \Theta)\right] \\ &= E\left[A^2\right]E[\sin(\omega_0 t_1 + \Theta)\sin(\omega_0 t_2 + \Theta)] = \frac{1}{2}E\left[A^2\right]\cos\omega_0(t_1 - t_2)\end{aligned}$$

$$R_{XX}(t_1, t_2) = R_{XX}^*(t_2, t_1)$$
$$K_{XX}(t_1, t_2) = K_{XX}^*(t_2, t_1)$$

Hermitian Symmetry

Example: Poisson Counting Process

- MGF of a exponential rv $\tau[n] \triangleq T[n] - T[n-1]$ is

$$M_X(t) = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} e^{(t-\lambda)x} dx = \frac{\lambda}{t - \lambda} \quad \text{provided that } |t| < \lambda$$

- Therefore MGF of *T[n]* is given by $\left(\frac{\lambda}{\lambda - t}\right)^n$ which is the MGF of Erlang distribution

See example 8.1-11

$$f_T(t;n) = \frac{(\lambda t)^{n-1}}{(n-1)!} \lambda \exp(-\lambda t) u(t)$$

- Now, by construction (bottom rt. figure)

$$P\{N(t) = n\} = P\{T[n] \le t, T[n+1] > t\}$$

$$\Longrightarrow P\{N(t) = n\} = P\{T[n] \le t, \tau[n+1] > t - T[n]\}$$

- Using independence and definition of CDF

$$\int_0^t f_T(\alpha;n)\left[\int_{t-\alpha}^{\infty} f_\tau(\beta)d\beta\right]d\alpha = \int_0^t \frac{\lambda^n \alpha^{n-1} e^{-\lambda\alpha}}{(n-1)!}\left(\int_{t-\alpha}^{\infty} \lambda e^{-\lambda\beta} d\beta\right) d\alpha \cdot u(t)$$

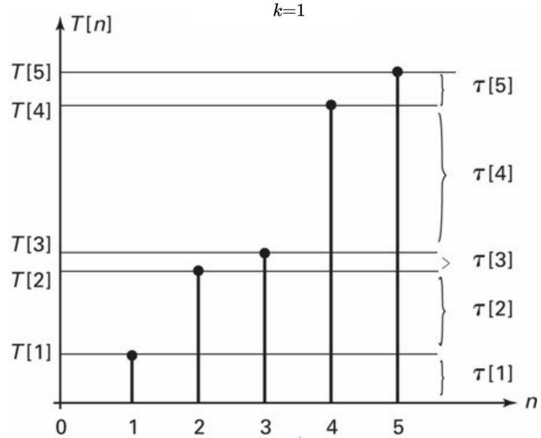
$$= \left(\int_0^t \alpha^{n-1} d\alpha\right) \lambda^n e^{-\lambda t}/(n-1)! u(t)$$

$$P_N(n;t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} u(t) \quad \text{for } t \ge 0, \quad n \ge 0$$

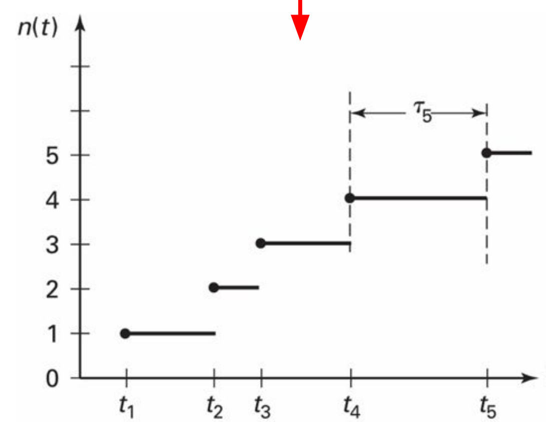
λ is the mean arrival rate and E[N(t)] = λt

- This yields the PMF of a Poisson counting process

$$T[n] \triangleq \sum_{k=1}^{n} \tau[k]$$



Exponential pdf of inter-arrival time τ[n] leads to Erlang pdf on total wait (arrival) time T[n] and a Poisson PMF for the count n



$$N(t) \triangleq \sum_{n=1}^{\infty} u(t - T[n])$$

Independent Increment

- The PMF of the increment in Poisson counting process in (t_a,t_b) is Poisson

$$P[N(t_b)-N(t_a)=n]=\frac{[\lambda(t_b-t_a)]^n}{n!}e^{-\lambda(t_b-t_a)}u(n)$$

- Definition 7.2 – 1 A random process has independent increments when the set of n random variables,

  $$X(t_1), X(t_2)-X(t_1), \ldots, X(t_n)-X(t_{n-1})$$

  are jointly independent for all $t_1 < t_2 < \ldots < t_n$ and for all $n \geq 1$

- It helps in calculating higher order distributions

$$\begin{aligned} P_N(n_1,n_2;t_1,t_2) &= P[N(t_1)=n_1]P[N(t_2)-N(t_1)=n_2-n_1] \\ &= \frac{(\lambda t_1)^{n_1}}{n_1!}e^{-\lambda t_1}\frac{[\lambda(t_2-t_1)]^{n_2-n_1}}{(n_2-n_1)!}e^{-\lambda(t_2-t_1)}u(n_1)u(n_2-n_1) \\ &= \frac{\lambda^{n_2}t_1^{n_1}(t_2-t_1)^{n_2-n_1}}{n_1!(n_2-n_1)!}e^{-\lambda t_2}u(n_1)u(n_2-n_1), \quad 0 \leq t_1 < t_2 \end{aligned}$$

- Autocorrelation and Autocovariance using independent increments

$$\begin{aligned} E[N(t_2)N(t_1)] &= E[(N(t_1)+[N(t_2)-N(t_1)])N(t_1)] \\ &= E[N^2(t_1)] + E[N(t_2)-N(t_1)]E[N(t_1)] \\ &= \lambda t_1 + \lambda^2 t_1^2 + \lambda(t_2-t_1)\lambda t_1 \\ &= \lambda t_1 + \lambda^2 t_1 t_2 = \lambda \min(t_1,t_2) + \lambda^2 t_1 t_2 \end{aligned}$$

$$K_{NN}(t_1,t_2) = \lambda \min(t_1,t_2)$$

Markov Random Process

- Continuous valued (first order) Markov process X(t) is satisfies the conditional pdf

$$f_X(x_n|x_{n-1}, x_{n-2}, \ldots, x_1; t_n, \ldots, t_1) = f_X(x_n|x_{n-1}; t_n, t_{n-1})$$

- Discrete valued Markov process X(t) satisfies the conditional PMF

$$P_X(x_n|x_{n-1}, \ldots, x_1; t_n, \ldots, t_1) = P_X(x_n|x_{n-1}; t_n, t_{n-1})$$

- **Problem:** Find CDF and pdf of Z = min(X,Y)

$$\begin{aligned} F_Z(z) &\triangleq P[Z \le z] \\ &= 1 - P[Z > z] \\ &= 1 - P[X > z]P[Y > z] \\ &= 1 - (1 - P[X \le z])(1 - P[Y \le z]) \\ &= 1 - (1 - F_X(z))(1 - F_Y(z)) \\ &= F_X(z) + F_Y(z) - F_X(z)F_Y(z). \end{aligned}$$

NOTE: For this condition to be true, both X>Z and Y>Z has to be true

$$\begin{aligned} f_Z(z) &\triangleq \frac{dF_Z(z)}{dz} \\ &= \frac{d\left(F_X(z) + F_Y(z) - F_X(z)F_Y(z)\right)}{dz} \\ &= f_X(z) + f_Y(z) - f_X(z)F_Y(z) - F_X(z)f_Y(z). \end{aligned}$$

If X and Y are iid exponential rv

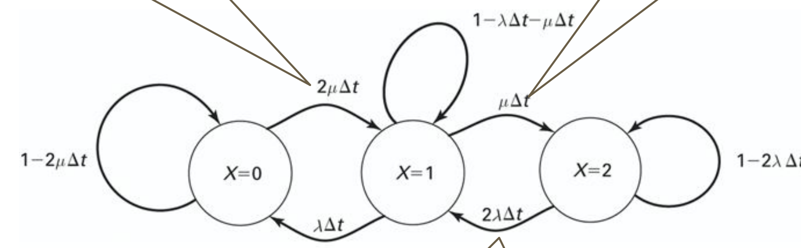
$$f_X(x) = f_Y(x) = \alpha \exp(-\alpha x)u(x).$$

We get

$$F_Z(z) = (1 - \exp(-2\alpha z))u(z)$$

$$f_Z(z) = 2\alpha \exp(-2\alpha z)u(z).$$

Multiprocessor Reliability



- State X=0 : Both processor down
- State X=1 : Either one is down
- State X=2 : Both are up

- Repair Time is exponential with parameter μ
  - Average time to repair is $1/\mu$
- Failure TIme is exponential with parameter λ
  - Average time to repair is 1/λ
  - Inter-transition times are exponentially distributed like the inter-arrival times in Poisson counting process
- Now, the probability of being in X=2 at (t+Δt) having been in X=1 at time t
  - This requires the service time T_s to be within interval (t,t+Δt] , conditioned on T_s >

$$P_2(t+\Delta t) = P_1(t)P[t < T_s \le t+\Delta t | T_s \ge t]$$

$$\text{where,} \quad P[t < T_s \le t+\Delta t | T_s \ge t] = \frac{F_{T_s}(t+\Delta t) - F_{T_s}(t)}{1-F_{T_s}(t)} = \mu\Delta t + o(\Delta t)$$

NOTE: Sum of probabilities leaving any state is always 1

  - The probability of staying in X=2 at t+Δt, having been in state 2

$$P_2(t+\Delta t) = P_2(t)P[T_Z = \min(T_{F_1}, T_{F_2}) > \Delta t] = P_2(t)\{1 - F_{T_z}(\Delta t)\} = P_2(t)(1-(1-e^{-2\lambda\Delta t})) = P_2(t)(1-2\lambda\Delta t)$$

...contd

- Continuing in similar fashion for other states,

$$\begin{bmatrix} P_0(t+\Delta t) \\ P_1(t+\Delta t) \\ P_2(t+\Delta t) \end{bmatrix} = \begin{bmatrix} 1-2\mu\Delta t & \lambda\Delta t & 0 \\ 2\mu\Delta t & 1-(\lambda+\mu)\Delta t & 2\lambda\Delta t \\ 0 & \mu\Delta t & 1-2\lambda\Delta t \end{bmatrix} \begin{bmatrix} P_0(t) \\ P_1(t) \\ P_2(t) \end{bmatrix} + \mathbf{o}(\Delta t)$$

- Rearranging the terms

$$\begin{bmatrix} P_0(t+\Delta t)-P_0(t) \\ P_1(t+\Delta t)-P_1(t) \\ P_2(t+\Delta t)-P_2(t) \end{bmatrix} = \begin{bmatrix} -2\mu & \lambda & 0 \\ 2\mu & -(\lambda+\mu) & 2\lambda \\ 0 & \mu & -2\lambda \end{bmatrix} \begin{bmatrix} P_0(t) \\ P_1(t) \\ P_2(t) \end{bmatrix} \Delta t + \mathrm{o}(\Delta t)$$

- Dividing both sides by Δt $\longrightarrow \dfrac{d\mathbf{P}(t)}{dt} = \mathbf{A}\mathbf{P}(t)$
- **A** is called the **generator matrix** of the Markov chain X
- The solution of the matrix differential equation is given by

$$\mathbf{P}(t) = e^{\mathbf{A}t}\mathbf{P}_0, \quad t \geq 0 \qquad\qquad \mathbf{P}(0) \triangleq \mathbf{P}_0$$

- We are interested in the steady state probabilities of MC or **AP = 0,** from first and last row

$$\begin{aligned} -2\mu P_0 + \lambda P_1 &= 0 \\ +\mu P_1 - 2\lambda P_2 &= 0 \end{aligned} \longrightarrow P_1 = (2\mu/\lambda)P_0 \text{ and } P_2 = (\mu/2\lambda)P_1 = (\mu/\lambda)^2 P_0$$

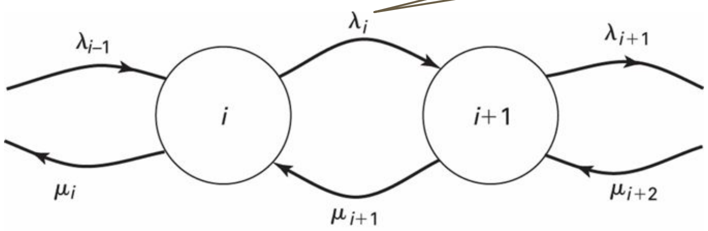
$P_0 + P_1 + P_2 = 1,$ we obtain $P_0 = \lambda^2/(\lambda^2 + 2\mu\lambda + \mu^2)$ and finally $\mathbf{P} = \dfrac{1}{\lambda^2 + 2\mu\lambda + \mu^2}\left[\lambda^2, 2\mu\lambda, \mu^2\right]^T$

To find a particular solution to the vector differential equation see this - http://people.math.gatech.edu/~xchen/teach/ode/ExpMatrix.pdf

Birth-Death Markov Chains

- In MC with only adjacent state transition is called a Birth-Death chain
  - Infinite no. of states and Finite number of states (M/M/1 Queue)
- The time between births and time between deaths are exponentially distributed with parameters μ and λ

NOTE: The rates are different than the previous example



This model is called M/M/1 queue. See Kendall's notation for queuing models

  - We can write $\mathbf{P}(t+\Delta t) = \mathbf{B}\mathbf{P}(t)$, where

$$\mathbf{B} = \begin{bmatrix} 1-\lambda_0\Delta t & \mu_1\Delta t & 0 & \cdots & \\ \lambda_0\Delta t & 1-(\lambda_1+\mu_1)\Delta t & \mu_2\Delta t & 0 & \cdots \\ 0 & \lambda_1\Delta t & 1-(\lambda_2+\mu_2)\Delta t & \mu_2\Delta t & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

- Rearranging and dividing by Δt $d\mathbf{P}(t)/dt = \mathbf{A}\mathbf{P}(t)$ and the steady state is given by **AP = 0**

$$\mathbf{A} = \begin{bmatrix} -\lambda_0 & \mu_1 & 0 & \cdots & \\ \lambda_0 & -(\lambda_1+\mu_1) & \mu_2 & 0 & \cdots \\ 0 & \lambda_1 & -(\lambda_2+\mu_2) & \mu_3 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \longrightarrow \begin{aligned} &P_1 = \rho_1 P_0 \\ &P_2 = \rho_2 P_1 = \rho_1\rho_2 P_0 \\ &\vdots \\ &P_j = \rho_j P_{j-1} = \rho_j \cdots \rho_2\rho_1 P_0 \end{aligned}$$

where $\rho_j \triangleq \lambda_{j-1}/\mu_j$, for $j \geq 1$

- Assuming that the series converges, we require that $\sum_{i=0}^{\infty} P_i = 1$. With the notation $r_j \triangleq \rho_j \cdots \rho_2\rho_1$, and $r_0 = 1$, this means $P_0 \sum_{i=0}^{\infty} r_i = 1$ or $P_0 = 1/\sum_{i=0}^{\infty} r_i$. Hence the steady-state probabilities for the birth-death Markov chain are given by

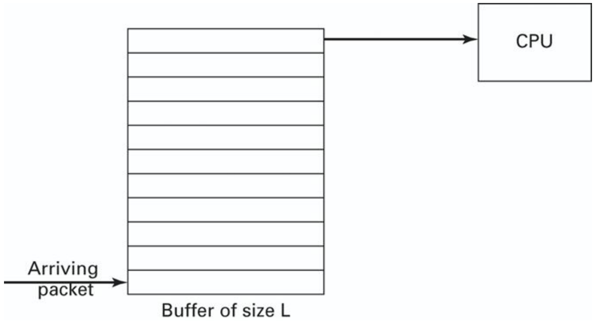
$$P_j = r_j / \sum_{i=0}^{\infty} r_i, \quad j \geq 0$$

⟵ Denominator does not converge there is no steady state.

Finite Capacity Buffer

- If we assume $\mu_i = \mu$ and $\lambda_i = \lambda$ for al i, and the queue length cannot exceed L
  - The dynamical equation of the MC is

$$\begin{aligned} dP_0(t)/dt &= -\lambda P_0(t) + \mu P_1(t) \\ dP_1(t)/dt &= +\lambda P_0(t) - (\lambda+\mu)P_1(t) + \mu P_2(t) \\ &\vdots \\ dP_L(t)/dt &= +\lambda P_{L-1}(t) - \mu P_L(t) \end{aligned}$$



  - The steady state solution is therefore,

$$P_i = \rho^i P_0, \text{ for } 0 \le i \le L \qquad \text{where } \rho \triangleq \lambda/\mu$$

And since

$$\sum_{i=0}^{L} \rho^i P_0 = 1, \text{ or that } P_0 = (1-\rho)/\left(1-\rho^{L+1}\right)$$

$$P_L = \rho^L(1-\rho)/\left(1-\rho^{L+1}\right)$$

- **Example 9.2-6-** If the buffer is full and $\boldsymbol{\tau}_s$ and $\boldsymbol{\tau}_i$ is the service time and interarrival time, then prob. of packet loss is

$$\begin{aligned} P[\text{ “packet loss” }] &= P[\text{“saturation”} \cap \{\tau_s > \tau_i\}] \\ &= \rho^L(1-\rho)/\left(1-\rho^{L+1}\right) \times P[\tau_s - \tau_i > 0] = \rho^L(1-\rho)/\left(1-\rho^{L+1}\right) \times \rho/(1+\rho) \end{aligned}$$

$P[\tau_s - \tau_i > 0] = \lambda/(\lambda+\mu)$ is given by calculating the pdf of (**Z** = $\boldsymbol{\tau}_s$ - $\boldsymbol{\tau}_i$) , which is the difference of two exponential pdf $\boldsymbol{\tau}_s$ ~ exp(μ) and $\boldsymbol{\tau}_s$ ~ exp(λ), i.e. $f(x) = \dfrac{\lambda\mu}{\lambda+\mu}\begin{cases} e^{-\mu x} & \text{if } x > 0 \\ e^{\lambda x} & \text{if } x < 0 \end{cases}$

Chapman-Kolmogorov Equations

- A markov process random variable $X(t_1)$, $X(t_2)$, $X(t_3)$ at $t_3 > t_2 > t_1$, then C-K equations provide the **conditional pdf** of $X(t_3)$ given $X(t_1)$

$$f_X(x_3, x_1; t_3, t_1) = \int_{-\infty}^{+\infty} f_X(x_3|x_2, x_1; t_3, t_2, t_1) f_X(x_2, x_1; t_2, t_1) dx_2$$

Dividing both sides by $f(x_1; t_1)$, we obtain

$$f_X(x_3|x_1) = \int_{-\infty}^{+\infty} f_X(x_3|x_2, x_1) f_X(x_2|x_1) dx_2$$

Then using the Markov property the above becomes

$$f_X(x_3|x_1) = \int_{-\infty}^{+\infty} f_X(x_3|x_2) f_X(x_2|x_1) dx_2$$

- For discrete Markov chains
  - Given $X_0 = i, P_{i,k}^n = P\left(X_n = k | X_0 = i\right)$ is the probability that the state at time *n* is *k*
  - But given, $X_n = k$, the probability that the chain will be in state *j* at *m* time units later is $P_{k,j}^m$
  - Therefore, since transitions are independent, we get $P(X_n = k, X_{n+m} = j | X_0 = i) = P_{i,k}^n P_{k,j}^m$
  - Summing over *k* gives the C-K equations

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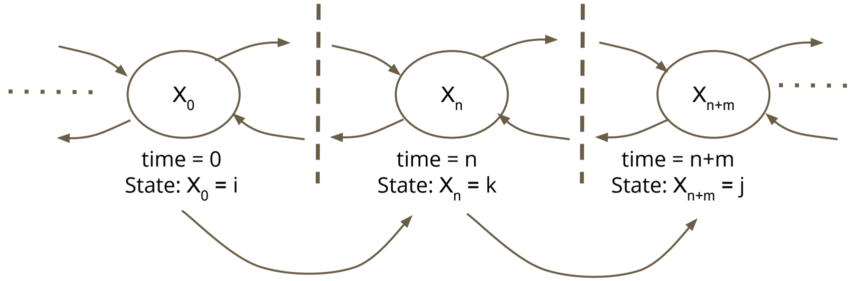


Transition Probability
$P^n_{ik} = P(X_n = k \mid X_0 = i)$

Transition Probability
$P^m_{kj} = P(X_{n+m} = j \mid X_n = k)$

$= P(X_m = j \mid X_0 = k)$
(memoryless property of MC set n = 0)

- Therefore, we get the following proof

$$
\begin{aligned}
P^{n+m}_{i,j} &= P(X_{n+m} = j | X_0 = i) \\
&= \sum_{k \in \mathcal{S}} P(X_{n+m} = j, X_n = k | X_0 = i) \\
&= \sum_{k \in \mathcal{S}} \frac{P(X_{n+m} = j, X_n = k, X_0 = i)}{P(X_0 = i)} \\
&= \sum_{k \in \mathcal{S}} \frac{P(X_{n+m} = j | X_n = k, X_0 = i) P(X_n = k, X_0 = i)}{P(X_0 = i)} \\
&= \sum_{k \in \mathcal{S}} \frac{P(X_n = k, X_0 = i) P^m_{k,j}}{P(X_0 = i)} \\
&= \sum_{k \in \mathcal{S}} P^n_{i,k} P^m_{k,j}
\end{aligned}
$$

(Using definition of Total Probability)

(Using definition of Conditional Probability)

(Using definition of Jointly Conditional on X_n and X_0)

(Using Markov Property)

$P(X_{n+m} = j | X_n = k, X_0 = i) = P(X_{n+m} = j | X_n = k) = \bar{P}(\bar{X}_m = j | X_0 = k) = P^m_{k,j}$

- When n = m = 1

$$P^2_{i,j} = \sum_{k \in \mathcal{S}} P_{i,k} P_{k,j}, i \in \mathcal{S}, j \in \mathcal{S}$$

- Which in the matrix form yields $\mathbf{P}^{(2)} = \mathbf{P}^2$, where $\mathbf{P}^{(n)} = \left(p^n_{ij}\right), n \geq 1$
- Similarly, when n = 1 and m = 2 (3 time steps in future)

$$P^3_{i,j} = \sum_{k \in \mathcal{S}} P_{i,k} P^2_{k,j} \longrightarrow \mathbf{P}^{(3)} = \mathbf{P} \times \mathbf{P}^{(2)} = \mathbf{P} \times \mathbf{P}^2 = \mathbf{P}^3$$

In general, for *n=1, m = l*

$\mathbf{P} \times \mathbf{P}^l = \mathbf{P}^{l+1}$

Example

- Example: Let $X_i = 0$ if it rains on day i; otherwise, $X_i = 1$
- Suppose $P_{00} = 0.7$ and $P_{10} = 0.4$. Then $\mathbf{P} = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$
- Suppose it rains on Monday. Then the prob that it rains on Friday is $\mathbf{P_{00}^{(4)}}$

$$\mathbf{P}^{(4)} = \mathbf{P}^4 = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}^4 = \begin{pmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{pmatrix}$$

so that $\mathbf{P_{00}^{(4)} = 0.5749}$

- **NOTE:** To compute power of a matrix - Diagonalize and raise to the power
  - Diagonalize $A = S^{-1}\Lambda S$, where S = matrix with columns are eigenvectors and Λ is a matrix with eigenvalues as diagonals.
  - Then, $A^n = S^{-1}\Lambda^n S$, where $\Lambda^n = \text{diag}(\Lambda_1^n, \Lambda_2^n, \ldots\ldots\Lambda_N^n)$

Continuous Time Linear Systems

- **SELF STUDY Section 9.3**
- **It follows the same methodology as random sequences**
- **Pay attention to the conjugate operator**

Useful Classifications

- If X and Y are random processes

(a) Uncorrelated if $R_{XY}(t_1, t_2) = \mu_X(t_1)\mu_Y^*(t_2)$, for all t_1 and t_2
(b) Orthogonal if $R_{XY}(t_1, t_2) = 0$ for all t_1 and t_2;
(c) Independent if for all positive integers n, the n th order PDF of X and Y factors, that is,

$$F_{XY}(x_1, y_1, x_2, y_2, \ldots, x_n, y_n; t_1, \ldots, t_n) = F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) F_Y(y_1, \ldots, y_n; t_1, \ldots, t_n) \qquad \text{for all } x_i, y_i \text{ and for all } t_1, \ldots, t_n$$

- Note: Two processes are orthogonal if they are uncorrelated and at least one has zero mean

- If $R_{XX}(t_1,t_2) = 0$ then it is called a orthogonal random process

- X(t) is **Stationarity** if it has the same n^{th} order CDF/PDF as X(t+T)

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = F_X(x_1, \ldots, x_n; t_1 + T, \ldots, t_n + T)$$

If differentiable then

$$f_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = f_X(x_1, \ldots, x_n; t_1 + T, \ldots, t_n + T)$$

- Stationary random process implies

$f(x;t) = f(x;t+T)$ for all T implies $f(x;t) = f(x;0)$ by taking $T = -t$ which in turn implies $E[x(t)] = \mu_X(t) = \mu_X(0)$

Stationarity

- Since the second order density is also shift invariant

$$f(x_1, x_2; t_1, t_2) = f(x_1, x_2; t_1 + T, t_2 + T) \xrightarrow{T=-t_2} f(x_1, x_2; t_1, t_2) = f(x_1, x_2; t_1 - t_2, 0) \longrightarrow E[X(t_1)X^*(t_2)] = R_{XX}(t_1 - t_2, 0)$$

- Therefore, we can write the one-parameter correlation function

$$\begin{aligned} R_{XX}(\tau) &\triangleq R_{XX}(\tau, 0) \\ &= E[X(t+\tau)X^*(t)] \end{aligned}$$

- Definition 7.4 − 3 A random process X is wide-sense stationary (WSS) if $E[X(t)] = \mu_X$, a constant, and $E[X(t+\tau)X^*(t)] = R_{XX}(\tau)$ for all $-\infty < \tau + \infty$, independent of the time parameter t.

- **SELF STUDY Section 9.5 on interaction of WSS random process with linear systems. The discussion follows the same reasoning as in random sequences**

Power Spectral Density

- PSD is defined for WSS and hence for stationary random process
- Definition — Let $R_{XX}(\tau)$ be an autocorrelation function. Then we define the power spectral density $S_{XX}(\omega)$ to be its Fourier transform (if it exists), that is,

$$S_{XX}(\omega) \triangleq \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega\tau} d\omega$$

- We can also define Fourier transform of the cross-correlation function, which is called cross power spectral density

$$S_{XY}(\omega) \triangleq \int_{-\infty}^{+\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau$$

- Properties of PSD (Also see table 9.5-1)

1. $S_{XX}(\omega)$ is real-valued since $R_{XX}(\tau)$ is conjugate symmetric.
2. If $X(t)$ is a real-valued WSS process, then $S_{XX}(\omega)$ is an even function since $R_{XX}(\tau)$ is real and even. Otherwise $S_{XX}(\omega)$ may not be an even function of ω.
3. $S_{XX}(\omega) \geq 0$ (to be shown in Section in Theorem $9.5-1$).

Interpretation of PSD

- For a WSS process X(t), consider the finite support segment (to ensure validity and existence of FT) ⟶ $X_T(t) \triangleq X(t)I_{[-T,+T]}(t)$
  - $I_{[-T,+T]}$ is an indicator function equal to 1 if -T ≤ t ≤ T
  - Therefore, FT of $X_T(t)$ is given by

$$FT\{X_T(t)\} = \int_{-T}^{+T} X(t)e^{-j\omega t}dt$$

- The magnitude squared of this random variable is

$$|FT\{X_T(t)\}|^2 = \int_{-T}^{+T}\int_{-T}^{+T} X(t_1)X^*(t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2$$

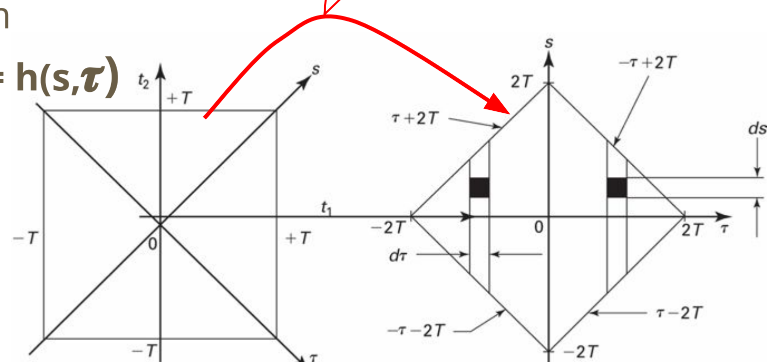
- Dividing both sides by *2T* and taking expectation,

$$\frac{1}{2T}E\Big[|FT\{X_T(t)\}|^2\Big] = \frac{1}{2T}\int_{-T}^{+T}\int_{-T}^{+T} R_{XX}(t_1-t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2$$

The area of integration in the t_1-t_2 plane changes to the diamond shape in the **s-τ** plane.
Then, the line t_2 = T becomes (s-τ)/2 = T -> s = τ + 2T (red arrow)

- Define coordinated $\mathbf{s = t_1 + t_2}$ and $\boldsymbol{\tau} \mathbf{= t_1 - t_2}$ -> $\mathbf{t_1 = (s+\tau)/2}$ and $\mathbf{t_2 = (s-\tau)/2}$
  - Obtained by 45° rotation of t_1-t_2 coordinate system
  - The Jacobian for the transform $\mathbf{t_1 = g(s,\tau)}$ and $\mathbf{t_2 = h(s,\tau)}$

$$\frac{\partial(t_1,t_2)}{\partial(s,\tau)} = \begin{vmatrix} \frac{\partial t_1}{\partial s} & \frac{\partial t_1}{\partial \tau} \\ \frac{\partial t_2}{\partial s} & \frac{\partial t_2}{\partial \tau} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \frac{1}{2}$$



Example and tutorial on variable changes for multivariate integrals - http://tutorial.math.lamar.edu/Classes/CalcIII/ChangeOfVariables.aspx

...contd

- The integral then becomes

$$\frac{1}{4T}\iint_{\wp} R_{XX}(\tau)e^{-j\omega\tau}d\tau ds = \frac{1}{4T}\left\{\int_{-2T}^{0} R_{XX}(\tau)e^{-j\omega\tau}\left[\int_{-(2T+\tau)}^{2T+\tau} ds\right]d\tau\right\} + \frac{1}{4T}\left\{\int_{0}^{2T} R_{XX}(\tau)e^{-j\omega\tau}\left[\int_{-(2T-\tau)}^{2T-\tau} ds\right]d\tau\right\} = \int_{-2T}^{+2T}\left[1-\frac{|\tau|}{2T}\right]R_{XX}(\tau)e^{-j\omega\tau}d\tau$$

- In the limit $T \to +\infty$ the integral tends to the def. of PSD

$$S_{XX}(\omega) = \lim_{T\to\infty}\frac{1}{2T}E\left[|FT\{X_T(t)\}|^2\right]$$

- Therefore, $S_{XX}(\omega)$ is real and non-negative and specifies average power at frequency ω. **See examples 9..5-3, 9.5-4**
- **Theorem:** If X(t) is stationary with $R_{xx}(\tau)$ and psd $S_{XX}(\omega)$, then $S_{XX}(\omega) \geq 0$ and for all $\omega_2 > \omega_1$, $\frac{1}{2\pi}\int_{\omega_1}^{\omega_2} S_{XX}(\omega)d\omega$ the average power in the band (ω_1, ω_2)
  - Define a filter with $H(\omega) \triangleq \begin{cases} 1, & \omega \in (\omega_1, \omega_2) \\ 0, & \text{else} \end{cases}$
  - Then when X(t) passes through the filter $S_{YY}(\omega) = \begin{cases} S_{XX}(\omega), & \omega \in (\omega_1, \omega_2) \\ 0, & \text{else.} \end{cases}$
  - Then the average output power of Y(t) is $E\left[|Y(t)|^2\right] = R_{YY}(0)$ (This is autocorrelation at 0 shift)

$$R_{YY}(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{YY}(\omega)d\omega = \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} S_{XX}(\omega)d\omega \geq 0$$

Periodic Process

Definition — A random process $X(t)$ is wide-sense periodic if there is a $T > 0$ such that

$$\mu_X(t) = \mu_X(t+T) \text{ for all } t$$

and

$$K_{XX}(t_1, t_2) = K_{XX}(t_1 + T, t_2) = K_{XX}(t_1, t_2 + T) \quad \text{for all } t_1, t_2$$

The smallest such T is called the period. Note that $K_{XX}(t_1, t_2)$ is then periodic with period T along both axes.

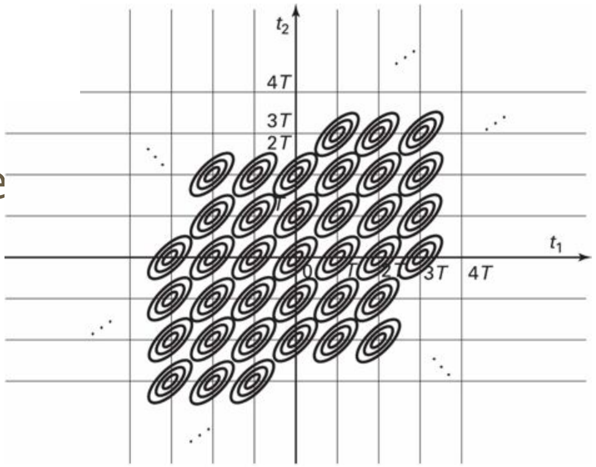
- The random complex exponential is an example of Periodic process

(A random complex exponential.) Let $X(t) \triangleq A\exp(j2\pi ft)$ with f a known real constant and A a real-valued random variable with mean $E[A] = 0$ and finite average power $E[A^2]$. Calculating the mean and correlation of $X(t)$, we obtain

$$E[X(t)] = E[A\exp(j2\pi ft)] = E[A]\exp(j2\pi ft) = 0$$

and

$$\begin{aligned} E[X(t+\tau)X^*(t)] &= E[A\exp(j2\pi f(t+\tau))A\exp(-j2\pi ft)] \\ &= E\left[A^2\right]\exp(j2\pi f\tau) = R_{XX}(\tau) \end{aligned}$$



- A periodic process can also be WSS (like above) are called ***wide-sense periodic stationary.***
- $K_{XX}(t_1,t_2)$ is doubly periodic with a two dimensional period of (T,T)

Cyclostationarity Process

Definition — A random process $X(t)$ is wide-sense cyclostationary if there exists a positive value T such that

$$\mu_X(t) = \mu_X(t+T) \text{ for all } t$$

$$K_{XX}(t_1, t_2) = K_{XX}(t_1 + T, t_2 + T) \text{ for all } t_1 \text{ and } t_2$$

- The covariance function is shift invariant both of its arguments
- **It means statistics are periodic but not the process itself**
- Typically modulated communication signals possess this property.

Consider a random amplitude sinusoid with period T:

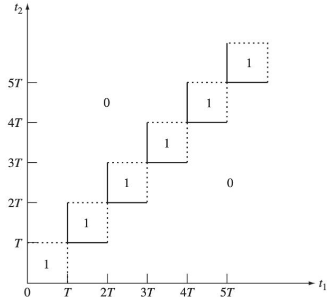
$$X(t) = A\cos(2\pi t/T).$$

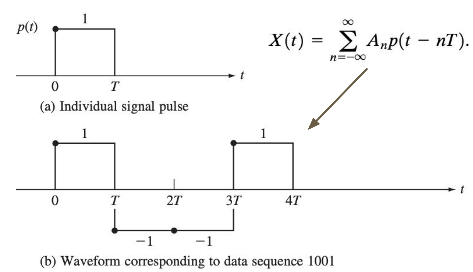
Is $X(t)$ cyclostationary? wide-sense cyclostationary?

Consider the joint cdf for the time samples $t_1, \dots, t_k$:

$$\begin{aligned} P[X(t_1) \le x_1, X(t_2) \le x_2, \dots, X(t_k) \le x_k)] &= P[A\cos(2\pi t_1/T) \le x_1, \dots, A\cos(2\pi t_k/T) \le x_k] \\ &= P[A\cos(2\pi(t_1 + mT)/T) \le x_1, \dots, A\cos(2\pi(t_k + mT)/T) \le x_k] \\ &= P[X(t_1 + mT) \le x_1, X(t_2 + mT) \le x_2, \dots, X(t_k + mT) \le x_k]. \end{aligned}$$

Thus $X(t)$ is a cyclostationary random process and hence also a wide-sense cyclostationary process.





The mean of $X(t)$ is

$$m_X(t) = E\left[\sum_{n=-\infty}^{\infty} A_n p(t - nT)\right] = \sum_{n=-\infty}^{\infty} E[A_n] p(t - nT) = 0$$

since $E[A_n] = 0$. The autocovariance function is

$$\begin{aligned} C_X(t_1, t_2) &= E[X(t_1)X(t_2)] - 0 \\ &= \begin{cases} E[X(t_1)^2] = 1 & \text{if } nT \le t_1, t_2 < (n+1)T \\ E[X(t_1)]E[X(t_2)] = 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Figure shows the autocovariance function in terms of t_1 and t_2. It is clear that $C_X(t_1 + mT, t_2 + mT) = C_X(t_1, t_2)$ for all integers m. Therefore the process is wide-sense cyclostationary.