ATOMIC DECOMPOSITION OF CHARACTERS AND CRYSTALS

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Abstract. Lascoux stated that the type A Kostka-Foulkes polynomials $K_{\lambda,\mu}(t)$ expand positively in terms of so-called atomic polynomials. For any semisimple Lie algebra, the former polynomial is a t-analogue of the multiplicity of the dominant weight μ in the irreducible representation of highest weight λ. We formulate the atomic decomposition in arbitrary type, and view it as a strengthening of the monotonicity of $K_{\lambda,\mu}(t)$. We also define a combinatorial version of the atomic decomposition, as a decomposition of a modified crystal graph. We prove that this stronger version holds in type A (which provides a new, conceptual approach to Lascoux's statement), in types B, C, and D in a stable range for $t = 1$, as well as in some other cases, while we conjecture that it holds more generally. Another conjecture stemming from our work leads to an efficient computation of $K_{\lambda,\mu}(t)$. We also give a geometric interpretation.

1. Introduction

The starting point of this paper is a result of Lascoux on the (type A) Kostka-Foulkes polynomials $K_{\lambda,\mu}(t)$, which are well-known t-analogues of the Kostka numbers $K_{\lambda,\mu}$, i.e., the number of semistandard Young tableaux of shape λ and content μ. Lascoux [17] stated the decomposition of the Kostka-Foulkes polynomials into so-called *atomic polynomials*. Some arguments of the proof in [17] remained elusive, and it was not until the work of Shimozono [36] that the type A atomic decomposition was completely accepted, this time in larger generality (for the so-called *generalized Kostka-Foulkes polynomials*). However, the latter proof involves several intricate combinatorial arguments and related concepts, such as *plactic monoid*, *cyclage*, and *catabolism*. The main goal of this paper is to provide a simpler, more conceptual approach, which has the additional advantage of extending beyond type A.

Lusztig defined a remarkable t-analogue $K_{\lambda,\mu}(t)$ of the multiplicity of a weight μ in the irreducible representation with highest weight λ of a semisimple Lie algebra [26]. For dominant weights μ, these polynomials generalize the type A ones mentioned above, and are therefore also called *Kostka-Foulkes polynomials*. They have remarkable properties:

- they are essentially *affine Kazhdan-Lusztig polynomials* [15];
- they record the *Brylinski filtration* of weight spaces [5];
- they are the coefficients in the expansion of an irreducible character in terms of *Hall-Littlewood polynomials* [27];
- they are related to the *energy function* coming from solvable lattice models [24, 29].

In classical Lie types, when the rank increases, these polynomials exhibit a stabilization property, so they have *stable versions* [21]. We refer to [30, 38] for more information on Kostka-Foulkes polynomials.

There are two combinatorial formulas for type A Kostka-Foulkes polynomials: one due to Lascoux-Schützenberger, in terms of the *charge statistic* on semistandard Young tableaux [16], and one in terms of the corresponding Kashiwara *crystal graphs* [11, 13, 14], due to Lascoux-Leclerc-Thibon [18]. Similar partial combinatorial descriptions in types $B-D$, in terms of the corresponding

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Kashiwara-Nakashima tableaux [11], and a conjectured charge statistic in type C were constructed by the first author in [19, 20]. In our previous paper [22], we prove the first general formula beyond type A, namely a formula for $K_{\lambda,0}(t)$ of type C. This extends the Lascoux-Leclerc-Thibon formula by using a simpler approach, and can be expressed in terms of *King tableaux*.

In Section 2.2, following Lascoux [17], we formulate the *t-atomic decomposition* property in arbitrary Lie type as a nonnegative expansion for both a Kostka-Foulkes polynomial, and a t-analogue $\chi^+_\lambda(t)$ of the dominant part of an irreducible character (defined in terms of Kostka-Foulkes polynomials). Here a character expansion is in terms of so-called *layer sum polynomials*, which record all weights of some irreducible representation with multiplicity 1. The t-atomic decomposition property is a strengthening of the monotonicity of Kostka-Foulkes polynomials, which holds in the full generality of affine Kazhdan-Lusztig polynomials [1, Corollary 3.7].

For $t = 1$, the atomic decomposition was also considered from a purely algebraic point of view in [8, 32, 34]. Some partial results were given, for instance for the inverse expansion (of layer sum polynomials in terms of characters). However, the atomic decomposition itself is less understood, even for $t = 1$; for instance, it does not always exist, in the sense that the mentioned character expansion sometimes fails to be nonnegative, contrary to what was claimed in Theorem 2.2 of [34] (see Example 2.6). Nevertheless, these failures seem to be mild.

As opposed to the algebraic approach mentioned above, in Section 2.3 we define a stronger t-atomic decomposition property, at the combinatorial level of the highest weight crystal $B(\lambda)$. This property involves a partition of the dominant part $B(\lambda)^+$ of $B(\lambda)$, and a statistic $\mathrm{c}(\cdot)$ on $B(\lambda)^+$. The combinatorial t-atomic decomposition leads to combinatorial formulas for both the Kostka-Foulkes polynomials and the atomic polynomials (into which the former decompose).

In Section 3, we consider the case when λ goes to infinity, in types A_{n-1}, B_n, C_n, D_n, and G_2. We introduce the notion of a t-atomic decomposition of the crystal $B(\infty)$, and discuss how it can be realized.

Next we introduce the two main ingredients for constructing a t-atomic decomposition of a finite highest weight crystal: the *partial order on dominant weights*, and a *modified crystal graph* structure. The natural poset structure on dominant weights is discussed in Section 4, by recalling some important properties which hold in arbitrary Lie type [37]. We need extra information about this poset, namely the structure of its small intervals, which was only known in type A [4]. The second ingredient, namely a modified crystal graph structure on $B(\lambda)^+$, is discussed in Section 5.1; this structure can also be viewed as a poset, with covers corresponding to the modified crystal edges. The associated modified crystal operators are obtained mainly by conjugating the ordinary Kashiwara operator $\tilde{f}_1$ under the crystal action of the Weyl group. There are two main differences between the original and the modified crystal operators: (1) the latter are indexed by arbitrary positive roots; (2) the original $B(\lambda)$ is a connected graph/poset, whereas the modified crystal graph on $B(\lambda)^+$ is disconnected in general. An interesting question is whether the modified crystal operators for λ become those for $B(\infty)$ in Section 3 when λ goes to infinity.

In Section 6, we start by showing that type A crystals admit a t-atomic decomposition, thus realizing combinatorially the classical result, while also providing a relatively simple and conceptual proof of it. Here the desired partition of the modified crystal poset $B(\lambda)^+$ is given by its components, which are shown to admit unique minimal and maximal vertices. The relevant statistic $\mathrm{c}(\cdot)$ is the Lascoux-Schützenberger charge [16]. Similarly, given a fixed dominant weight λ of type B, C, or D, and assuming a large enough rank (depending on λ), we show that the corresponding modified crystal graph $B(\lambda)^+$ gives an atomic decomposition (i.e., in the case $t = 1$). Note that, in type B, we need to add to the modified crystal graph the operators obtained by conjugating the Kashiwara operator $\tilde{f}_n$.

In Section 7 we conjecture that the mentioned decompositions in types B, C, and D are, in fact, t-atomic decompositions, for appropriate choices of the statistic $\mathrm{c}(\cdot)$, which are related to the results in [19, 20] and [22]. We also discuss the atomic decomposition of the stable one-dimensional sums in classical types, and explain how to obtain a t-atomic decomposition of the crystal $B(\widetilde{\alpha})^+$ of the adjoint representation of $\mathfrak{g}$ in any type. The latter fact is useful because it clarifies the type and rank restrictions we considered in Section 6: for instance, in types C and D, they are needed to ensure that the covering relations in the poset $B(\lambda)^+$ make only appear roots in the orbit of the simple root α_1. Under such assumptions, we are indeed able to establish fine commutation relations satisfied by the modified crystal operators, which are required to derive the atomic decomposition of crystals (notably, the existence of unique maximal and minimal vertices). For the simply laced types D (with no large rank assumption) and E, the atomic decomposition of the characters can only hold with some restrictions on the dominant weights considered, but we expect that it can be derived similarly from the same modified crystal operators and a detailed analysis of the corresponding dominant weight poset. In the non-simply laced case of general rank, the situation becomes more complicated, but it is certainly possible to again derive relevant atomic decompositions of crystals by using modified crystal operators; this time, they would be defined using the Weyl group conjugation of two ordinary crystal operators, associated to one simple root of each length.

A geometric interpretation of Lascoux's atomic decomposition of the type A Kostka-Foulkes polynomial was given in [3] in terms of *nilpotent orbit varieties*. In Section 8, we provide a different, type-independent interpretation of an atomic decomposition of χ_λ^+, in terms of the *geometric Satake correspondence*.

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2. The atomic decomposition: background, definitions, and basic facts

2.1. **Characters and t-deformations.** Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of rank r with triangular decomposition

$$\mathfrak{g} = \bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in R^+} \mathfrak{g}_{-\alpha}\,,$$

so that $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$ and R^+ its set of positive roots. The root system $R = R^+ \sqcup (-R^+)$ of $\mathfrak{g}$ is realized in a real Euclidean space E of dimension r with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$. For any $\alpha \in R$, we write $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$ for its coroot. Let $S \subset R^+$ be the subset of simple roots and Q^+ the $\mathbb{Z}_+$-cone generated by S. The set P of integral weights for $\mathfrak{g}$ consists of elements λ satisfying $\langle\lambda,\alpha^\vee\rangle \in \mathbb{Z}$ for any $\alpha \in R$. We write $P^+ = \{\lambda \in P \mid \langle\lambda,\alpha^\vee\rangle \geq 0 \text{ for any } \alpha \in S\}$ for the cone of dominant weights of $\mathfrak{g}$, and denote by $\omega_1,\ldots,\omega_r$ its fundamental weights. We consider the group algebra of the weight lattice, denoted $\mathbb{Z}[P]$, which has a $\mathbb{Z}$-basis of formal exponentials e^λ, for $\lambda \in P$, with usual multiplication $e^\lambda e^\mu = e^{\lambda+\mu}$. Let W be the Weyl group of $\mathfrak{g}$ generated by the reflections s_α with $\alpha \in S$, and write $\ell(\cdot)$ for the corresponding length function. The *dominance order* $\leq$ on P^+ is defined by $\lambda < \mu$ if and only if $\mu - \lambda$ decomposes as a sum of positive roots (or equivalently, simple roots) with nonnegative integer coefficients. Many interesting properties of this order were studied in [37], and some of them will be used in this paper; for instance, each component of this poset is a lattice.

Let $\chi_\lambda^{\mathfrak{g}}$ be the *Weyl character* associated to the finite-dimensional irreducible representation $V(\lambda)$ of $\mathfrak{g}$ with highest weight $\lambda \in P^+$, namely

$$\chi_\lambda^{\mathfrak{g}} = \sum_{\gamma \in P} K_{\lambda,\gamma}^{\mathfrak{g}}\, e^\gamma\,,$$

where $K_{\lambda,\gamma}^{\mathfrak{g}}$ is the dimension of the weight space γ in $V(\lambda)$. For simplicity, we remove the superscript $\mathfrak{g}$ when the context makes it clear. By the *Weyl character formula* we have

$$\chi_\lambda = \frac{\sum_{w\in W}(-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in R^+}(1-e^{-\alpha})}\,. \tag{1}$$

This formula expresses the weight multiplicities $K_{\lambda,\gamma}$ as follows:

$$K_{\lambda,\gamma} = \sum_{w\in W} (-1)^{\ell(w)} P(w(\lambda+\rho)-(\gamma+\rho))\,,$$

where ρ is the half sum of positive roots, and $P(\cdot)$ is the *Kostant partition function*, defined by

$$\prod_{\alpha\in R^+} \frac{1}{1-e^\alpha} = \sum_{\beta\in Q^+} P(\beta)\, e^\beta\,.$$

When $\gamma = \mu$ is dominant, the multiplicity $K_{\lambda,\mu}$ has an interesting *t-analogue* $K_{\lambda,\mu}(t)$, also known as a *Kostka-Foulkes polynomial*. This was introduced by Lusztig [26], who defined

$$K_{\lambda,\mu}(t) := \sum_{w\in W} (-1)^{\ell(w)} P_t(w(\lambda+\rho)-(\mu+\rho))\,; \tag{2}$$

here the *t-analogue of the Kostant partition function* $P_t(\cdot)$ is given by

$$\prod_{\alpha\in R^+} \frac{1}{1-te^\alpha} = \sum_{\beta\in Q^+} P_t(\beta)\, e^\beta\,.$$

We have $K_{\lambda,\mu}(1) = K_{\lambda,\mu}$. Moreover, $K_{\lambda,\mu}(t)$ is essentially an *affine Kazhdan-Lusztig polynomial*, which implies that it has nonnegative integer coefficients. More precisely, we have

$$K_{\lambda,\mu}(t) = t^{\langle \lambda-\mu,\rho^\vee\rangle} P_{w_\mu,w_\lambda}(t^{-1})\,, \tag{3}$$

where w_λ denotes the longest element of $Wt_\lambda W$, and t_λ is the translation by λ in the extended affine Weyl group [15] (see also [38, Section 4]); note that $\langle \lambda-\mu,\rho^\vee\rangle$ is the number of simple roots in the decomposition of $\lambda-\mu$, counted with multiplicity. Based on (3), we let

$$\widetilde{K}_{\lambda,\mu}(t) := t^{\langle \lambda-\mu,\rho^\vee\rangle} K_{\lambda,\mu}(t^{-1})\,, \qquad \text{so } \widetilde{K}_{\lambda,\mu}(t) = P_{w_\mu,w_\lambda}(t)\,. \tag{4}$$

To each irreducible representation $V(\lambda)$ is associated an abstract *Kashiwara crystal* $B(\lambda)$ (see [11, 13, 14] for background on crystals), and we have

$$\chi_\lambda = \sum_{b\in B(\lambda)} e^{\mathrm{wt}(b)}\,, \tag{5}$$

where $\mathrm{wt}(b)$ is the weight of the vertex $b \in B(\lambda)$. The crystal $B(\infty)$ is defined as the direct limit of the crystal $B(\lambda)$ when λ goes to infinity in the interior of its Weyl chamber. It corresponds to the crystal of the positive part of the quantum group $U_q(\mathfrak{g})$ associated to $\mathfrak{g}$. One can then prove that

$$\mathrm{char}\, B(\infty) = \sum_{b\in B(\infty)} e^{\mathrm{wt}(b)} = \prod_{\alpha\in R^+} \frac{1}{1-e^{-\alpha}}\,. \tag{6}$$

2.2. **The definition of the atomic decomposition.** Let us denote by $P(\lambda)$ the set of weights of $V(\lambda)$, i.e., the set of γ such that $K_{\lambda,\gamma} > 0$. Also set $P^+(\lambda) = P(\lambda) \cap P^+$. Recall that we have $P^+(\lambda) = \{\mu \in P^+ \mid \mu \leq \lambda\}$. Since we have $K_{\lambda,\gamma} = K_{\lambda,w(\gamma)}$ for any $w \in W$, the character χ_λ is completely determined by its dominant part

$$\chi_\lambda^+ := \sum_{\mu \in P^+(\lambda)} K_{\lambda,\mu}\, e^\mu\,.$$

We also define the t-analogue of χ_λ^+ by

$$\chi_\lambda^+(t) := \sum_{\mu \in P^+(\lambda)} K_{\lambda,\mu}(t)\, e^\mu\,. \tag{7}$$

For any dominant weight μ, define the *layer sum polynomials* by

$$w_\mu := \sum_{\gamma \in P(\mu)} e^\gamma \quad \text{and} \quad w_\mu^+ := \sum_{\nu \in P^+(\mu)} e^\nu = \sum_{\nu \leq \mu} e^\nu\,. \tag{8}$$

Observe that we have

$$w_\mu = \sum_{\nu \in P^+(\mu)} m_\nu\,, \quad \text{where } m_\nu = \sum_{\gamma \in W\nu} e^\gamma\,.$$

Similarly, define

$$w_\mu^+(t) := \sum_{\nu \in P^+(\mu)} t^{\langle \mu-\nu, \rho^\vee\rangle} e^\nu = \sum_{\nu \leq \mu} t^{\langle \mu-\nu, \rho^\vee\rangle} e^\nu\,. \tag{9}$$

Remark 2.1. By the triangularity of the coresponding transition matrices, we have:

- w_μ is a $\mathbb{Z}$-basis of the W-invariants $\mathbb{Z}[P]^W$;
- w_μ^+ is a $\mathbb{Z}$-basis of $\mathbb{Z}[P^+]$;
- $w_\mu^+(t)$ is a $\mathbb{Z}[t]$-basis of $\mathbb{Z}[t][P^+]$.

There exists a Weyl-type formula for the polynomials w_μ, which follows from Brion's formula [2] counting the cardinality of the intersection between a convex polygon and a lattice, see [31, Theorem 4.3]; this formula was rederived in [34] from the axioms of root systems. It is stated as follows:

$$w_\mu = \sum_{w \in W} \frac{e^{w(\mu)}}{\prod_{\alpha \in S}(1 - e^{-w(\alpha)})}\,; \tag{10}$$

here, for any simple root α and any Weyl group element w, we set

$$\frac{1}{1 - e^{-w(\alpha)}} = \sum_{k=0}^{+\infty} e^{kw(\alpha)} \quad \text{if } w(\alpha) \in R^+\,, \text{ and}$$

$$\frac{1}{1 - e^{-w(\alpha)}} = -\frac{e^{w(\alpha)}}{1 - e^{w(\alpha)}} = -\sum_{k=0}^{+\infty} e^{(k+1)w(\alpha)} \quad \text{if } w(\alpha) \in -R^+\,.$$

Observe that $w_\mu^+(t)$ coincides with the dominant part in the expansion

$$\frac{e^\mu}{\prod_{\alpha \in S}(1 - te^{-\alpha})}\,.$$

Based on Remark 2.1, consider the expansion

$$\chi_\lambda = \sum_{\mu \in P^+(\lambda)} A_{\lambda,\mu}\, w_\mu\,, \quad \text{or equivalently } \chi_\lambda^+ = \sum_{\mu \in P^+(\lambda)} A_{\lambda,\mu}\, w_\mu^+\,. \tag{11}$$

Similarly, consider the polynomials $A_{\lambda,\mu}(t)$, called *atomic polynomials*, which are defined by

$$\chi^+_\lambda(t) = \sum_{\mu \in P^+(\lambda)} A_{\lambda,\mu}(t)\, w^+_\mu(t)\,. \tag{12}$$

We have $A_{\lambda,\lambda}(t) = 1$. The following property of the atomic polynomials gives an alternative definition of them, via Möbius inversion on the dominance order.

Proposition 2.2. *The atomic polynomials satisfy the following relation:*

$$K_{\lambda,\nu}(t) = \sum_{\nu \le \mu \le \lambda} t^{\langle \mu-\nu, \rho^\vee\rangle} A_{\lambda,\mu}(t)\,, \quad \text{for all } \nu \in P^+(\lambda)\,. \tag{13}$$

Proof. The expansion (12) can be written

$$\begin{aligned}\chi^+_\lambda(t) &= \sum_{\mu \in P^+(\lambda)} A_{\lambda,\mu}(t) \left(\sum_{\nu \le \mu} t^{\langle \mu-\nu, \rho^\vee\rangle} e^\nu \right) \\ &= \sum_{\nu \le \lambda} \left(\sum_{\nu \le \mu \le \lambda} t^{\langle \mu-\nu, \rho^\vee\rangle} A_{\lambda,\mu}(t) \right) e^\nu\,.\end{aligned}$$

By comparing with (7), the desired relation follows. □

Definition 2.3. *The character χ_λ admits an* atomic decomposition *if $A_{\lambda,\mu} \in \mathbb{Z}_{\ge 0}$ for any $\mu \in P^+(\lambda)$. Similarly, we say that $\chi^+_\lambda(t)$ admits a t-*atomic decomposition *if $A_{\lambda,\mu}(t) \in \mathbb{Z}_{\ge 0}[t]$ for any $\mu \in P^+(\lambda)$.*

By analogy with (7) and using (4), we define

$$\widetilde{\chi}^+_\lambda(t) := \sum_{\mu \in P^+(\lambda)} \widetilde{K}_{\lambda,\mu}(t)\, e^\mu\,.$$

Like in (12), consider the polynomials $\widetilde{A}_{\lambda,\mu}(t)$ defined by

$$\widetilde{\chi}^+_\lambda(t) = \sum_{\mu \in P^+(\lambda)} \widetilde{A}_{\lambda,\mu}(t)\, w^+_\mu\,, \tag{14}$$

where we recall that $w^+_\mu := w^+_\mu(1)$. When the context is clear, we will also refer to $\widetilde{A}_{\lambda,\mu}(t)$ as atomic polynomials. As shown below, the $\widetilde{A}_{\lambda,\mu}(t)$ are closely related to $A_{\lambda,\mu}(t)$, and are subject to an analogue of (13).

Proposition 2.4. *The polynomials $\widetilde{A}_{\lambda,\mu}(t)$ satisfy the following relation:*

$$\widetilde{K}_{\lambda,\nu}(t) = \sum_{\nu \le \mu \le \lambda} \widetilde{A}_{\lambda,\mu}(t)\,, \quad \text{for all } \nu \in P^+(\lambda)\,. \tag{15}$$

Moreover, we have

$$\widetilde{A}_{\lambda,\mu}(t) = t^{\langle \lambda-\mu, \rho^\vee\rangle} A_{\lambda,\mu}(t^{-1})\,. \tag{16}$$

Thus, the t-atomic decomposition is equivalent to the fact that $\widetilde{A}_{\lambda,\mu}(t) \in \mathbb{Z}_{\ge 0}[t]$.

Proof. The proof of (15) is completely similar to that of (13). As for (16), it can be seen in the following way:

$$\widetilde{K}_{\lambda,\nu}(t) = t^{\langle\lambda-\nu,\rho^\vee\rangle} K_{\lambda,\nu}(t^{-1}) = t^{\langle\lambda-\nu,\rho^\vee\rangle} \sum_{\nu\le\mu\le\lambda} t^{\langle\nu-\mu,\rho^\vee\rangle} A_{\lambda,\mu}(t^{-1})$$
$$= \sum_{\nu\le\mu\le\lambda} t^{\langle\lambda-\mu,\rho^\vee\rangle} A_{\lambda,\mu}(t^{-1}) = \sum_{\nu\le\mu\le\lambda} \widetilde{A}_{\lambda,\mu}(t)\,.$$

□

Remarks 2.5. (1) Lascoux [17] made a statement very closely related to (15), for type A. The slight difference consists in the definition of $\widetilde{K}_{\lambda,\mu}(t)$, for given partitions λ, μ, namely $\widetilde{K}_{\lambda,\mu}(t) := t^{n(\mu)}\, K_{\lambda,\mu}(t^{-1})$, where $n(\mu) := \sum_i (i-1)\mu_i$.

(2) The t-atomic decomposition, as stated in Proposition 2.4, implies the monotonicity of the Kostka-Foulkes polynomials, which holds in the full generality of Kazhdan-Lusztig polynomials for finite and affine Weyl groups [1, Corollary 3.7]. Indeed, the latter says that, for $x \le y \le z$ in such a Weyl group, the difference of Kazhdan-Lusztig polynomials $P_{x,z}(t) - P_{y,z}(t)$ is in $\mathbb{Z}_{\ge 0}[t]$. For $z = w_\lambda$, $y = w_\mu$, and $x = w_\nu$, with $\nu \le \mu \le \lambda$, this follows based on (4) and the fact that the atomic polynomials in the decomposition of $\widetilde{K}_{\lambda,\mu}(t)$ are among those in the decomposition of $\widetilde{K}_{\lambda,\nu}(t)$.

(3) We shall see that the t-atomic decomposition is always true in type A, as mentioned in Section 1. However, even the atomic decomposition (i.e., the positivity in (11)) might fail beyond type A, unlike it was claimed in [34, Theorem 2.2]. The shortest counterexample we found is in type D_4, and is given in Example 2.6 below. However, a slight increase in rank corrects this problem, and in fact we will see that this is a general phenomenon.

Example 2.6. Consider $\lambda := 2\omega_1 + 2\omega_2$ in type D_4. For simplicity, we let $w_{abcd} := w_\mu$ for $\mu := a\omega_1 + b\omega_2 + c\omega_3 + d\omega_4$. With this notation, we have:

$$\chi_\lambda = w_\lambda + w_{4000} + w_{1111} + w_{2002} + w_{0022} + w_{2020} + 2w_{2100} + w_{0200} +$$
$$+ 4w_{1011} + 5w_{0002} + 5w_{0020} + 11w_{2000} - 3w_{0100} + 17w_{0000}\,.$$

However, for the same λ we obtain a positive expansion in type D_5.

Set $t_i := e^{-\alpha_i} \in]0,1[$, for $i = 1, \dots, r$, and consider a sequence $(\lambda^{(k)})_{k\ge 0}$ of dominant weights such that $\lim_{k\to+\infty} \langle \lambda^{(k)}, \alpha_i\rangle = +\infty$, for any $i = 1, \dots, r$. We shall then write $\lambda \to +\infty$ for short.

Proposition 2.7. *Under the previous assumption we have*

$$\lim_{\lambda\to+\infty} e^{-\lambda}\chi_\lambda = \prod_{\alpha\in R^+} \frac{1}{1-e^{-\alpha}} \quad \text{and} \quad \lim_{\lambda\to+\infty} e^{-\lambda} w_\lambda = \prod_{\alpha\in S} \frac{1}{1-e^{-\alpha}}\,.$$

Proof. We have by the Weyl character formula

$$e^{-\lambda^{(k)}}\chi_{\lambda^{(k)}} = \frac{\sum_{w\in W} (-1)^{\ell(w)} e^{w(\lambda^{(k)}+\rho)-\rho-\lambda^{(k)}}}{\prod_{\alpha\in R^+}(1-e^{-\alpha})}\,.$$

Let $t = \max(t_1, \dots, t_r) \in]0,1[$. Since $w(\lambda^{(k)}+\rho) - \rho - \lambda^{(k)} \in -Q^+$, we can set

$$w(\lambda^{(k)}+\rho) - \rho - \lambda^{(k)} = -\sum_{i=1}^r a_i(\lambda^{(k)})\,\alpha_i\,,$$

where $a_i(\lambda^{(k)}) \in \mathbb{Z}_{\geq 0}$ for any $i = 1, \ldots, r$. The hypothesis $\lim_{k\to+\infty} \langle \lambda^{(k)}, \alpha_i \rangle = +\infty$ for any $i = 1, \ldots, r$ implies that $\lim_{k\to+\infty} \sum_{i=1}^{r} a_i(\lambda^{(k)}) = +\infty$ for any $w \in W$ such that $w \neq 1$. So we get for any such w

$$\lim_{k\to+\infty} e^{w(\lambda^{(k)}+\rho)-\rho-\lambda^{(k)}} \leq \lim_{k\to+\infty} t^{\sum_{i=1}^{r} a_i(\lambda^{(k)})} = 0\,.$$

Since for $w = 1$ we have $(-1)^{\ell(w)} e^{w(\lambda^{(k)}+\rho)-\rho-\lambda^{(k)}} = 1$, we get our first limit.

From (10), we can write

$$e^{-\lambda^{(k)}} w_{\lambda^{(k)}} = \sum_{w\in W} \frac{e^{w(\lambda^{(k)})-\lambda^{(k)}}}{\prod_{\alpha\in S}(1-e^{-w(\lambda)})}$$

and by using similar arguments, only $w = 1$ contributes when we consider the limit. □

Now set

$$\prod_{\alpha\in R^+\setminus S} \frac{1}{1-te^{-\alpha}} = \sum_{\beta\in Q^+} M_t(\beta)\, e^{-\beta}\,.$$

In particular, $M_1(\beta)$ is the number of decompositions of β as a sum of nonsimple positive roots. We get

$$\prod_{\alpha\in R^+} \frac{1}{1-te^{-\alpha}} = \sum_{\beta\in Q^+} M_t(\beta) \prod_{\alpha\in S} \frac{e^{-\beta}}{1-te^{-\alpha}}\,. \tag{17}$$

Proposition 2.7 and (10) suggest to consider (17) as a t-analogue of (11) when $\lambda \to +\infty$.

2.3. **Atomic decomposition of finite crystals.** Let $B(\lambda)^+$ be the subset of $B(\lambda)$ of vertices with dominant weights.

Definition 2.8. *An* atomic decomposition *of the crystal $B(\lambda)$ is a partition*

$$B(\lambda)^+ = \bigsqcup_{h\in H(\lambda)} \mathbb{B}(\lambda, h)\,, \tag{18}$$

where $H(\lambda) \subset B(\lambda)^+$, $h \in \mathbb{B}(\lambda, h)$ is a distinguished vertex, and each component $\mathbb{B}(\lambda, h)$ consists of exactly one vertex of dominant weight ν for each $\nu \leq \mathrm{wt}(h)$.

Observe that the cardinality of $\mathbb{B}(\lambda, h)$ is then independent of λ, and if $\mathrm{wt}(h) = \mu$ we have

$$w_\mu^+ = \sum_{b\in\mathbb{B}(\lambda,h)} e^{\mathrm{wt}(b)}\,. \tag{19}$$

If $B(\lambda)$ has an atomic decomposition, then clearly χ_λ^+ has the atomic decomposition (11), where $A_{\lambda,\mu}$ is the number of vertices of weight μ in $H(\lambda)$.

Definition 2.9. *A* t-atomic decomposition *of the crystal $B(\lambda)$ is an atomic decomposition together with a statistic $\mathrm{c} : H(\lambda) \to \mathbb{Z}_{\geq 0}$ such that the following polynomials satisfy* (12)*:*

$$A_{\lambda,\mu}(t) = \sum_{\substack{h\in H(\lambda)\\ \mathrm{wt}(h)=\mu}} t^{\mathrm{c}(h)}\,. \tag{20}$$

As we can see, the t-atomic decomposition property of $\chi_\lambda^+(t)$ is part of Definition 2.9. Assuming that $B(\lambda)$ has a t-atomic decomposition, one can extend the statistic c to $B(\lambda)^+$ by setting

$$\mathrm{c}(b) := \mathrm{c}(h) + \langle \mathrm{wt}(h) - \mathrm{wt}(b), \rho^\vee \rangle\,, \qquad \text{for any } b \in \mathbb{B}(\lambda, h)\,. \tag{21}$$

The t-analogues of the combinatorial formulas (19) and (7) immediately follow from Definition 2.9:

$$w_\mu^+(t) = \sum_{b \in \mathbb{B}(\lambda,h)} t^{\mathrm{c}(b)-\mathrm{c}(h)} e^{\mathrm{wt}(b)}\,, \tag{22}$$

$$\chi_\lambda^+(t) = \sum_{b \in B(\lambda)^+} t^{\mathrm{c}(b)} e^{\mathrm{wt}(b)}\,. \tag{23}$$

Moreover, by comparing (23) with (7), we obtain the following combinatorial formula for Kostka-Foulkes polynomials:

$$K_{\lambda,\mu}(t) = \sum_{\substack{b \in B(\lambda) \\ \mathrm{wt}(b)=\mu}} t^{\mathrm{c}(b)}\,. \tag{24}$$

To summarize, the existence of a t-atomic decomposition of a crystal is highly desirable because: (i) it implies the t-atomic decomposition of $\chi_\lambda^+(t)$ and of the Kostka-Foulkes polynomials $K_{\lambda,\mu}(t)$, which are now realized combinatorially; (ii) it leads to combinatorial formulas for both $K_{\lambda,\mu}(t)$ and the atomic polynomials $A_{\lambda,\mu}(t)$, namely (24) and (20), respectively.

3. Atomic decomposition of the crystal $B(\infty)$

In this section we assume that the Lie algebra $\mathfrak{g}$ is of type A_{n-1}, B_n, C_n, D_n, and G_2. Let r be the rank of $\mathfrak{g}$.

3.1. **Marginally large tableaux.** *Marginally large tableaux* were introduced by Hong and Lee [12] in order to describe the crystal $B(\infty)$ associated to $\mathfrak{g}$. Recall that they can be regarded as $\mathfrak{g}$-tableaux (that is, of type A_{n-1}, B_n, C_n, D_n, or G_2) with d rows such that

$$d = n \text{ in types } B_n, C_n\,, \qquad d = n-1 \text{ in types } A_{n-1}, D_n\,, \qquad \text{and } d = 2 \text{ in type } G_2\,;$$

furthermore, for any $i = 1, \ldots, d-1$, the number of boxes in row i containing i is equal to 1 plus the number of boxes in row $i+1$ (see the example below). Write $\mathcal{T}(\infty)$ for the set of marginally large tableaux associated to $\mathfrak{g}$. Since marginally large tableaux are special cases of tableaux for each type considered, the set $\mathcal{T}(\infty)$ comes with a crystal action which is essentially the same as in the finite crystal up to renormalization of rows, in order to insure that the obtained tableau is marginally large. This renormalization is defined as follows. Consider a marginally large $\mathfrak{g}$-tableau T, and let T' be a $\mathfrak{g}$-tableau obtained from T by modifying a letter in row i. If T' is not marginally large, this means that we have modified the rightmost letter i in row i. Then the renormalization of $\widehat{T'}$ is the marginally large tableau obtained from T' by adding a letter k in each row k between 1 and i, the others rows remaining unchanged. One can then define crystal operators $\tilde{F}_1, \ldots, \tilde{F}_r$ on $\mathcal{T}(\infty)$ by setting

$$\tilde{F}_i(T) = \widehat{\tilde{f}_i(T)}\,,$$

where $\tilde{f}_i$ is the ordinary Kashiwara crystal operator on the $\mathfrak{g}$-tableau T. It was established in [12] that the crystal structure on $\mathcal{T}(\infty)$ obtained in this way is isomorphic to $B(\infty)$.

Example 3.1. If $\mathfrak{g}$ is of type A_3, then

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&3&5&5\\\hline 2&2&2&3&3&4\\\cline{1-6} 3&5\\\cline{1-2}\end{array}$$

is a marginally large tableau, and

$$\tilde{f}_2(T) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&1&3&5&5\\\hline 2&2&2&3&3&3&4\\\cline{1-7} 3&5\\\cline{1-2}\end{array}\,.$$

We define a multisegment as a multiset of positive roots. We can write a multisegment $\mathfrak{m}$ as

$$\mathfrak{m} = \sum_{\alpha\in R^+} m_\alpha\,\alpha\,,$$

which means that the multiset $\mathfrak{m}$ contains m_α times the positive root α. Let $\mathfrak{M}$ be the set of $\mathfrak{g}$-multisegments. In [25, Proposition 3.7], a bijection Ξ[1] is given from $\mathcal{T}(\infty)$ to $\mathfrak{M}$. This bijection depends on the type considered. For example, in type A_{n-1}, for any positive root $\alpha_{ij} = \varepsilon_i - \varepsilon_j$ with $1 \le i < j \le n$ and any marginally large tableau T, the integer $m_{\alpha_{i,j}}$ is equal to the number of letters j in the i-th row of T. For any $T \in \mathcal{T}(\infty)$ such that $\Xi(T) = \sum_{\alpha\in R^+} m_\alpha\,\alpha$, write $|T| = \sum_{\alpha\in R^+} m_\alpha$.

Example 3.2. Resuming the previous example, we get

$$\Xi(T) = \alpha_{1,3} + 2\alpha_{1,5} + 2\alpha_{2,3} + \alpha_{2,4} + \alpha_{3,5}$$

and $|T| = 7$.

3.2. **Modified crystal operators and atomic decomposition of $B(\infty)$.** We refer to [25] for a complete description of the bijection Ξ in each case. In the sequel, we only need the following two properties of the map Ξ.

(1) Starting from any marginally large tableau T and any simple root $\alpha_i, i = 1,\dots,r$, there is a unique marginally large tableau T' such that

$$\Xi(T') = \Xi(T) + \alpha_i\,.$$

We then set $T' = \mathrm{F}_i(T)$.

(2) For any marginally large tableau T' and any $i = 1,\dots,n$ such that $\Xi(T') = \sum_{\alpha\in R^+} m_\alpha\,\alpha$ with $m_{\alpha_i} > 0$, there exists a (unique) marginally large tableau T such that $T' = \mathrm{F}_i(T)$.

More precisely, the construction of T' from T is as follows.

- For any $i = 1,\dots,r-1$, T' is obtained from T by replacing the rightmost letter i located in row i by a letter $i+1$ and then by renormalizing if needed.
- For $i = 2$ in type G_2, T' is obtained from T by replacing the rightmost letter 2 located in row 2 by a letter 3.
- For $i = n$ in type C_n, T' is obtained from T by replacing the rightmost letter n located in row n by a letter $\overline{n}$.
- For $i = n$ in type D_n, T' is obtained from T by replacing the rightmost letter $n-1$ located in row $n-1$ by a letter $\overline{n}$.
- For $i = n$ in type B_n, the rows of T can only contain one letter 0. Then we have:
  - if 0 belongs to row n, T' is obtained from T by replacing this letter 0 by a letter $\overline{n}$;
  - if 0 does not belong to row n, T' is obtained from T by replacing the rightmost letter n located in row n by a letter 0.

Observe that $\mathrm{F}_i(T)$ is a marginally large tableau for any $i = 1,\dots,n$ and any $T \in \mathcal{T}(\infty)$. We also define the operators E_i, $i = 1,\dots,n$, such that $\mathrm{E}_i(T_1) = T_2$ if there exists $T_2 \in \mathcal{T}(\infty)$ satisfying $\mathrm{F}_i(T_2) = T_1$, and $\mathrm{E}_i(T_1) = 0$ otherwise. The operators F_i and E_i, $i = 1,\dots,n$, are the modified crystal operators. We have $\mathrm{F}_i(T) \neq \tilde{F}_i(T)$ and $\mathrm{E}_i(T) \neq \tilde{F}_i(T)$ in general. Furthermore, by property (2), we have $\mathrm{E}_i(T) \neq 0$ for any T such that $m_{\alpha_i} > 0$. Now we can endow $\mathcal{T}(\infty)$ with a new colored

[1]As far as we are aware, such a bijection is not known in types E_6, E_7, E_8, and F_4.

directed graph structure $\mathbb{B}(\infty)$ such that $T \xrightarrow{i} T'$ if and only if $T' = \mathrm{F}_i(T)$. A source vertex for this structure is a marginally large tableau T such that $\mathrm{E}_i(T) = 0$ for any $i = 1, \dots, r$. Let us denote by $\mathcal{ST}(\infty)$ the set of source vertices in $\mathcal{T}(\infty)$.

Theorem 3.3.

(1) *We have $S \in \mathcal{ST}(\infty)$ if and only if $\Xi(T) = \sum_{\alpha \in R^+ \setminus S} m_\alpha \, \alpha$, that is, when $m_{\alpha_i} = 0$ for any $i = 1, \dots, r$.*
(2) *Each connected component $\mathbb{B}$ of the graph $\mathbb{B}(\infty)$ contains a unique source vertex S. We then write $\mathbb{B} = \mathbb{B}(S)$, and say that $\mathbb{B}(S)$ is an atom of $B(\infty)$.*
(3) *For any $S \in \mathcal{ST}(\infty)$, the vertices of $\mathbb{B}(S)$ have different weights, and*

$$\sum_{T \in \mathbb{B}(S)} t^{|T|} e^{\mathrm{wt}(T)} = \frac{t^{|S|} e^{\mathrm{wt}(S)}}{\prod_{\alpha \in S} (1 - te^{-\alpha})} \,. \tag{25}$$

Proof. The first part follows from the fact that

$$\Xi(\mathrm{E}_i(T)) = \Xi(T) - \alpha_i \,,$$

for any $i = 1, \dots, r$. For the second part, we observe that, for any marginally large tableau T and for any $i \neq j$ in $\{1, \dots, r\}$, we have $\mathrm{E}_i\mathrm{E}_j(T) \neq 0$ and $\mathrm{E}_i\mathrm{E}_j(T) \neq 0$ if and only if $m_{\alpha_i} > 0$ and $m_{\alpha_j} > 0$; in this case, we have $\mathrm{E}_i\mathrm{E}_j(T) = \mathrm{E}_i\mathrm{E}_j(T)$. Thus

$$S = \prod_{i=1}^{r} \mathrm{E}_i^{m_{\alpha_i}}(T)$$

is the unique source vertex of the connected component corresponding to T, and does not depend on the order in which the operators E_i are applied in the right hand side. For the third part, observe we have

$$\mathbb{B}(S) = \left\{ \prod_{i=1}^{r} \mathrm{F}_i^{m_{\alpha_i}}(S) \mid m_{\alpha_i} \in \mathbb{Z}_{\geq 0} \right\} ,$$

where the operators F_i, $i = 1, \dots, r$ commute. Since we have

$$\mathrm{wt}\left(\prod_{i=1}^{r} \mathrm{F}_i^{m_{\alpha_i}}(S) \right) = \mathrm{wt}(S) - \sum_{i=1}^{r} m_{\alpha_i} \alpha_i \,,$$

the weights of the vertices in $\mathbb{B}(S)$ are all distinct. This also yields the desired equality (25). □

Corollary 3.4. *The partition*

$$\mathbb{B}(\infty) = \bigsqcup_{S \in \mathcal{ST}(\infty)} \mathbb{B}(S)$$

is a t-atomic decomposition of $B(\infty)$; in other words, we have

$$\prod_{\alpha \in R^+} \frac{1}{1 - te^{-\alpha}} = \sum_{\beta \in Q^+} M_t(\beta) \prod_{\alpha \in S} \frac{e^{-\beta}}{1 - te^{-\alpha}} \,,$$

where

$$M_t(\beta) = \sum_{\substack{S \in \mathcal{ST}(\infty) \\ \mathrm{wt}(S) = \beta}} t^{|S|} \,.$$

4. The partial order on dominant weights

Before we consider the atomic decomposition of finite crystals, we need some information about the partial order on dominant weights that was defined in Section 2.1. In full generality, this poset was first studied in [37], so we will recall some results from this paper.

The components of the dominant weight poset are lattices. Each cocover is of the form $\mu \gtrdot \mu - \alpha$, where α is a positive root, so we can represent it as a downward edge in the Hasse diagram labeled by α. The cocovers were completely described in [37, Theorem 2.8]. Fixing a dominant weight λ, we will be interested in the lower order ideal determined by λ. This is known to be an interval $[\widehat{0}, \lambda]$, with $\widehat{0}$ a minimal element of the dominant weight poset.

4.1. **Type A_{n-1}.** Now λ is a partition ($\lambda_1 \geq \ldots \geq \lambda_{n-1} \geq 0$), and let $N = |\lambda| := \sum_i \lambda_i$. We identify partitions with their Young diagrams, and we denote a partition with p parts a, q parts b ($a \geq b$) etc. by $(a^p b^q \ldots)$. As explained below, the interval $[\widehat{0}, \lambda]$ mentioned above can be identified with the similar interval in the poset of partitions of N with the *dominance order*; the latter is defined by $(\mu_1, \mu_2, \ldots) \leq (\nu_1, \nu_2, \ldots)$ if and only if $\sum_{i=1}^{j} \mu_i \leq \sum_{i=1}^{j} \nu_j$, for any j. As a partition of N, the element $\widehat{0}$ is the partition with $\lfloor N/n \rfloor$ columns of height n, and the last column of height $p := N \bmod n$; as an element of the dominant weight poset, $\widehat{0}$ is the fundamental weight ω_p. We identify a partition of N with the partition obtained from it by removing all columns of height n (if a partition of N is greater or equal to $\widehat{0}$, then it has no columns of height larger than n).

The dominance order on partitions of N was first studied in [4], so we recall some results in this paper. Conjugation of partitions is an antiautomorphism. The cocovers $\mu \gtrdot \mu - \alpha_{ij}$, where $\alpha_{ij} = \varepsilon_i - \varepsilon_j$ is a positive root (i.e., $i < j$), are labeled by (i, j). It turns out that there are only two types of cocovers, namely:

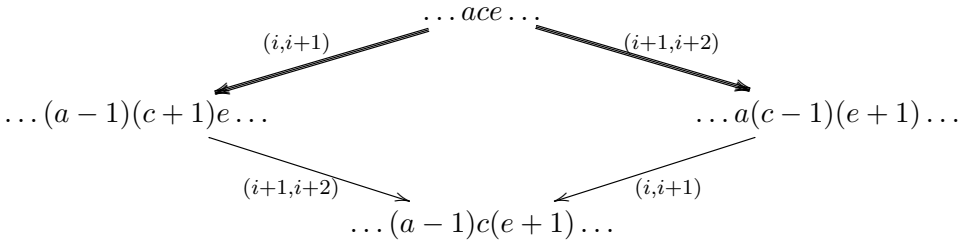
$$(\ldots ab \ldots) \gtrdot (\ldots (a-1)(b+1) \ldots), \quad \text{and} \quad (\ldots (a+1) a^p (a-1) \ldots) \gtrdot (\ldots a^{p+2} \ldots), \tag{26}$$

where in the first case $a \geq b + 2$ and the cocover is labeled by a simple root. These types are referred to as (*) and (**), respectively, while a cocover of type (**) which is not of type (*) is called *proper*.

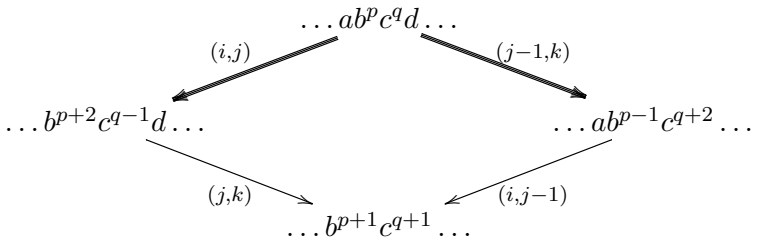
An important result in [4] concerns the structure of short intervals in the dominance order. To state it, we need some more definitions. Consider two distinct cocovers $\mu \gtrdot \nu$ and $\mu \gtrdot \pi$ of a partition μ, which are labeled (i, j) and (k, l), where we assume $i < k$. These cocovers can only have one of the following relative positions (in terms of their labels): (i) *nonoverlapping* if $j < k$; (ii) *partially overlapping* if $j = k$; (iii) *fully overlapping* if $k = j - 1$. By [4, Proposition 3.2], the interval $[\nu \wedge \pi, \mu]$ can only have one of the following structures; the two cocovers above are shown in the diagrams below in bold.

Case A1: cocovers which are (a) nonoverlapping; (b) partially overlapping and both of type (*); (c) fully overlapping and both proper of type (**). As subcase (a) is easy, only subcases (b) and (c) are represented in the diagrams below.

In subcase (b), we have $a \geq c + 2$ and $c \geq e + 2$, while i is the position of a in the partition μ.

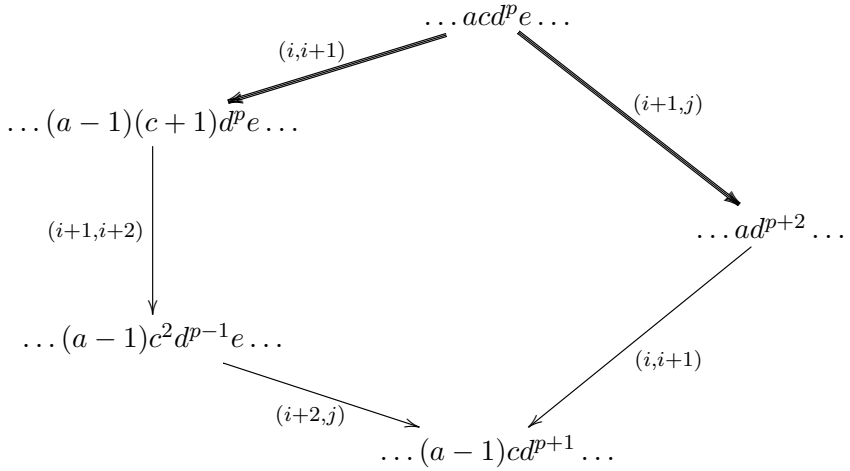


In subcase (c), we have $b = a - 1$, $c = b - 1$, $d = c - 1$, $p, q \geq 1$, while i is the position of a, $j = i + p + 1$ is the position of the first c, and $k = j + q$ is the position of d.

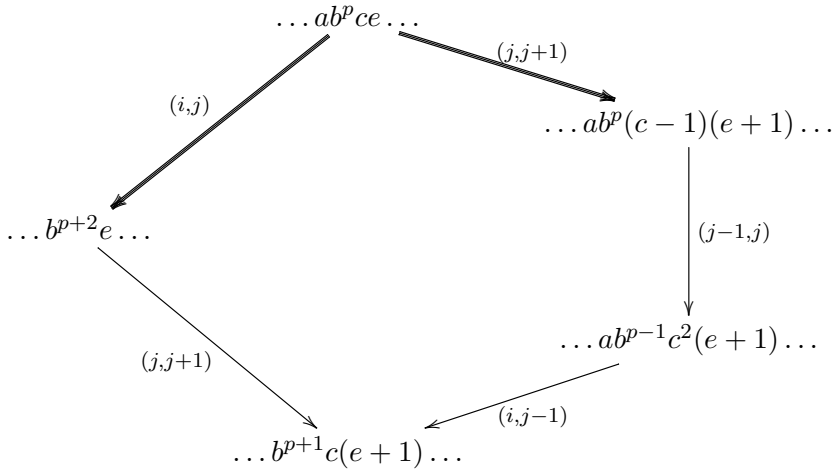


Case A2: partially overlapping cocovers, where (a) the first is of type (*) and the second proper of type (**); (b) vice versa.

In subcase (a), we have $a \geq c + 2$, $d = c - 1$, $e = d - 1$, $p \geq 1$, while i is the position of a in the partition μ and $j = i + p + 2$ is the position of e.

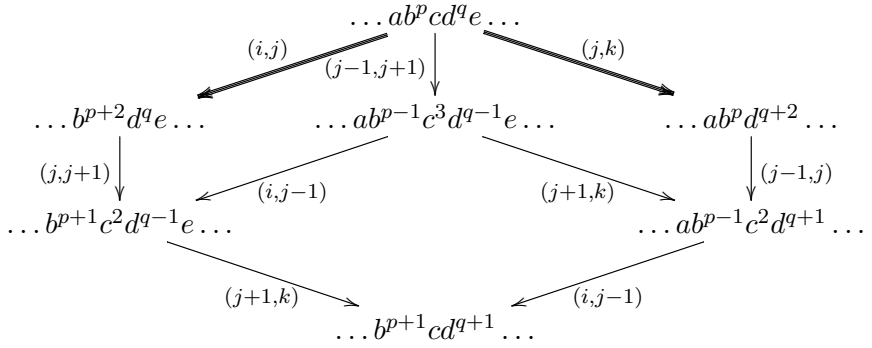


In subcase (b), we have $b = a - 1$, $c = b - 1$, $c \geq e + 2$, $p \geq 1$, while i is the position of a and $j = i + p + 1$ is the position of c.



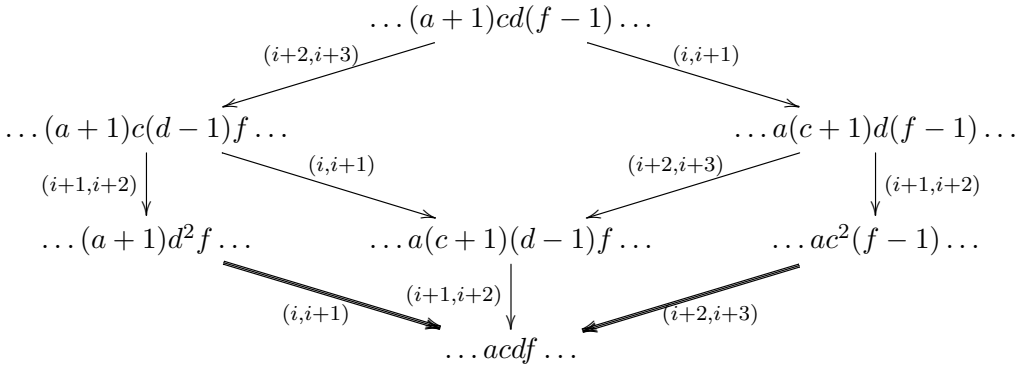
Case A3: partially overlapping cocovers, both proper of type (**). Here $b = a - 1$, $c = b - 1$, $d = c - 1$, $e = d - 1$, $p, q \geq 1$, while i is the position of a in the partition μ, $j = i + p + 1$ is the

position of c, and $k = j + q + 1$ is the position of e.



Due to the conjugation automorphism of the dominance order, given two distinct covers $\mu \lessdot \nu$ and $\mu \lessdot \pi$ of μ, the isomorphism type of the interval $[\mu, \nu \vee \pi]$ is always given by one of the above graphs turned upside down. Observe nevertheless the type (*) or (**) is not preserved by conjugation in general. So we have the corresponding Cases A1′−A3′. In fact, the structure of the intervals in Cases A1′ and A2′ is identical with that in Cases A1 and A2, respectively; however, Case A3 leads to the different structure shown below; the two covers above are again shown in bold.

Case A3′. Here $a \geq c+1$, $d = c-1$, $d \geq f+1$, while i is the position of a in the partition μ at the bottom.



4.2. **Type C_n.** Now λ is a partition ($\lambda_1 \geq \dots \geq \lambda_n \geq 0$), and we use the same notation as in Section 4.1. It is easy to see that the minimal element $\widehat{0}$ mentioned above (i.e., the unique minimal element below λ) is either 0 or $\omega_1 = (10^{n-1})$, depending on $|\lambda|$ being even or odd, respectively. By [37, Theorem 2.8], a cocover in the corresponding partial order on dominant weights is either of the same form (26) as in type A, or has one of the following three forms:

$$(\dots 1^2 0^{n-k-1}) \gtrdot (\dots 0^{n-k+1}), \quad (\dots 21) \gtrdot (\dots 10), \quad (\dots (a+2)) \gtrdot (\dots a), \tag{27}$$

where $1 \leq k \leq n-1$; these three cocovers are labeled by the roots $\varepsilon_k + \varepsilon_{k+1}$, $\varepsilon_{n-1} + \varepsilon_n$, and $2\varepsilon_n$, respectively. For simplicity, we denote a root $\varepsilon_i + \varepsilon_j$ by $\alpha_{i\bar{j}}$ or $(i, \bar{j})$. In the sequel, we will see that it is desirable for the last cover in (27) never to appear; the reason is that α_n is a long root, and hence does not appear in the W-orbit of α_1. In fact, there is an easy condition which simplifies the setup even more.

Proposition 4.1. *If $n > (|\lambda|+1)/2$, then the first cocover in* (27) *is the only one which can appear in the Hasse diagram of the interval $[\widehat{0}, \lambda]$ beside the type A cocovers in* (26)*.*

Proof. Assume that a partition $\mu \leq \lambda$ has the form $(\dots 21)$ or $(\dots 2)$. Then we must have $\mu_1 = \dots = \mu_{n-1} \geq 2$. Combining this with the fact that $|\lambda| \geq |\mu|$, we obtain $|\lambda| \geq 2n-1$ or $|\lambda| \geq 2n$, respectively. But this contradicts the condition in the proposition. □

From now on, we work under the assumption of Proposition 4.1, and we call this the *type C stable range*.

To the authors' knowledge, an analogue of the classification of short intervals that was described above in type A is not available beyond type A. In order to address this problem in type C_n, in the stable range, we focus on the new cases involving a pair of cocovers and covers.

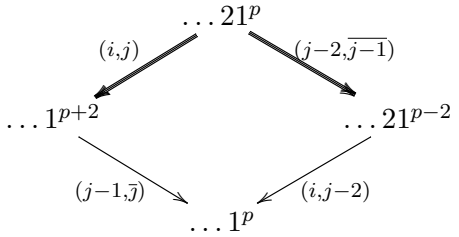
Given distinct cocovers $\mu \gtrdot \nu$ and $\mu \gtrdot \pi$ of a partition μ, we can assume that they are labeled (i, j) and $(k, \overline{k+1})$, where necessarily $i < k$. It is easy to see that these cocovers can only have one of the following relative positions (in terms of their labels): (i) nonoverlapping if $j < k$; (ii) partially overlapping if $j = k$; (iii) fully overlapping if $j = k + 2$. We are led to Cases C1−C3 below; the two cocovers above are shown in the diagrams below in bold, and for simplicity we omit the trailing 0's in a partition. Thus, we proved the following result.

Proposition 4.2. *Under the previous assumptions on ν and π, a lower bound of ν and π is always obtained as in one of the Cases* C1−C3.

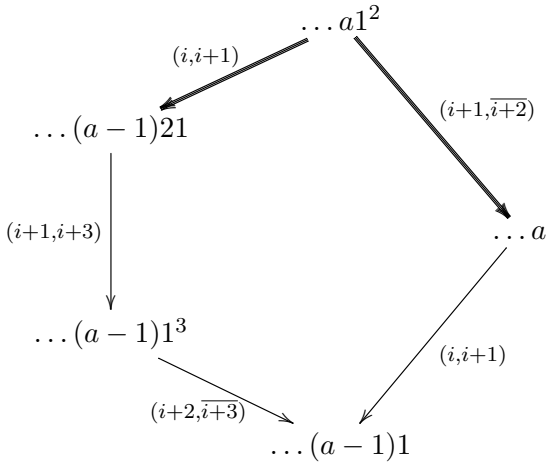
Furthermore, we claim that the structure of the interval $[\nu \wedge \pi, \mu]$ is always given by one of the diagrams below. The proof is completely similar to that of [4, Proposition 3.2], which was mentioned above. However, Proposition 4.2 suffices for our purposes.

Case C1: cocovers which are (a) nonoverlapping; (b) fully overlapping, with (i, j) proper of type (**). As subcase (a) is easy, only subcase (b) is represented in the diagram below.

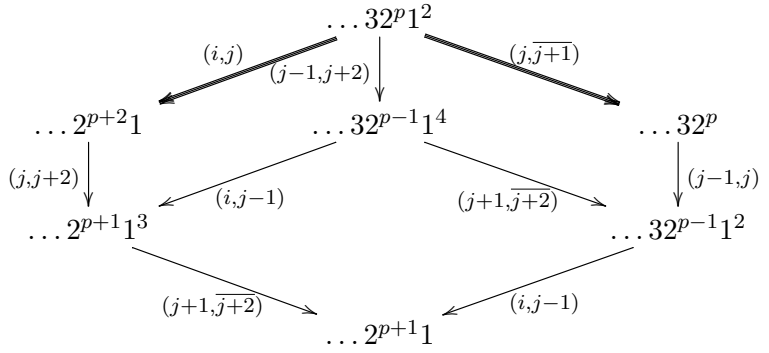
In subcase (b), we have $p \geq 2$, while i is the position of (the shown) 2 in the partition μ, and $j = i + p + 1$ is the position of the first 0.



Case C2: partially overlapping cocovers, with (i, j) of type (*), so $j = i + 1$. We have $a \geq 3$, while i is the position of a in the partition μ.



Case C3: partially overlapping cocovers, with (i,j) proper of type (**). Here $p \geq 1$, while i is the position of (the shown) 3 in the partition μ, and $j = i + p + 1$ is the position of the first 1.



Now consider two distinct covers $\mu \lessdot \nu$ and $\mu \lessdot \pi$ of μ, labeled (i,j) and $(k,\overline{k+1})$, respectively.

Proposition 4.3. *An upper bound of ν and π is always obtained as in one of the Cases* C1−C2*, now denoted* C1′−C2′*.*

Proof. We have $\mu_k = \mu_{k+1} = 0$ and $\pi_k = \pi_{k+1} = 1$. Clearly, we must have $j < k$ and $\mu_{k-1} > 0$. If $j < k-1$, or $j = k-1$ and $\mu_{k-1} > 1$, we are in Case C1 (a). Otherwise, when $j = k-1$ and $\mu_{k-1} = 1$, we have either $i < j-1$ or $i = j-1$; but these are precisely Cases C1 (b) and C2, respectively. □

Furthermore, we claim that the structure of the interval $[\mu, \nu \vee \pi]$ is always given by the diagrams in Cases C1 or C2. The proof is again completely similar to that of [4, Proposition 3.2]. However, Proposition 4.3 suffices for our purposes.

4.3. **Type D_n.** Now λ is a sequence $(\lambda_1 \geq \ldots \geq \lambda_n)$ with $\lambda_i \in \frac{1}{2}\mathbb{Z}$, all congruent mod $\mathbb{Z}$, such that $\lambda_{n-1} + \lambda_n \geq 0$. By [37, Theorem 2.8], a cocover in the corresponding partial order on dominant weights is either of the same form (26) as in type A (with entries in $\frac{1}{2}\mathbb{Z}$ now allowed), or has one of the following three forms:

$$(28) \qquad \begin{array}{l} (\ldots 1^2 0^{n-k-1}) \gtrdot (\ldots 0^{n-k+1}), \quad (\ldots (a+1)a^{n-l-1}(-a+1)) \gtrdot (\ldots a^{n-l}(-a)), \\ (\ldots (a+1)(b+1)) \gtrdot (\ldots ab), \end{array}$$

where $1 \leq k, l \leq n-1$ and $a \geq \frac{1}{2}$; these cocovers are labeled by the roots $\varepsilon_k + \varepsilon_{k+1}$, $\varepsilon_l + \varepsilon_n$, and $\varepsilon_{n-1} + \varepsilon_n$, respectively.

We will now assume that $\lambda_i \in \mathbb{Z}$. This implies that the interval $[\widehat{0}, \lambda]$ only contains weights $\mu = (\mu_1 \geq \ldots \geq \mu_n)$ with $\mu_i \in \mathbb{Z}$. Note that, in this case, there are the same possibilities for the minimal element $\widehat{0}$ as in type C, see Section 4.2.

Proposition 4.4. *If $n > |\lambda|$, then the first cocover in* (28) *is the only one which can appear in the Hasse diagram of the interval $[\widehat{0}, \lambda]$ beside the type A cocovers in* (26)*.*

Proof. Assume that the statement fails. Then there is $\mu \leq \lambda$ with $\mu_{n-2} \geq 1$, $\mu_1 \geq 2$, and $\mu_{n-1} + \mu_n \geq 1$. Combining this with the fact that $|\lambda| \geq |\mu|$, we obtain $|\lambda| \geq n$. But this contradicts the condition in the proposition. □

From now on, we work under the assumption of Proposition 4.4, and we call this the *type D stable range*. Clearly, all the results about the type C stable range in Section 4.2 apply to the type D one.

4.4. **Type B_n.** We use the same notation as in Section 4.2, and assume that λ is a partition $(\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$, where $\lambda_i \in \mathbb{Z}$. The unique minimal element below λ is clearly always 0. By [37, Theorem 2.8], a cocover in the corresponding partial order on dominant weights is either of the same form (26) as in type A, or has one of the following two forms:

$$(\ldots 10^{n-k}) \succcurlyeq (\ldots 0^{n-k+1}), \qquad (\ldots (a+1)) \succcurlyeq (\ldots a); \tag{29}$$

here the first is labeled by ε_k, and the second by ε_n.

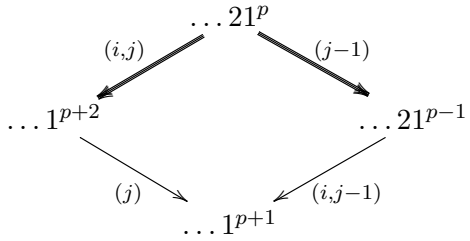
The following analogue of Proposition 4.1 is proved in the same way.

Proposition 4.5. *If $n > |\lambda|/2$, then the first cocover in* (29) *is the only one which can appear in the Hasse diagram of the interval $[\widehat{0}, \lambda]$ beside the type A cocovers in* (26).

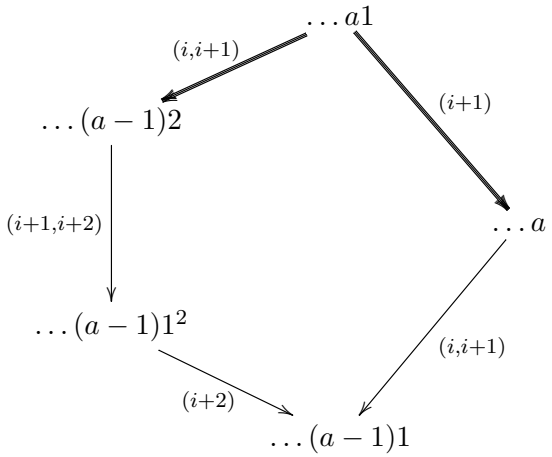
Let us now refer to the intervals $[\nu \wedge \pi, \mu]$ and $[\mu, \nu \vee \pi]$ which can appear in the poset $[\widehat{0}, \lambda]$. The intervals involving the new cocover/cover labeled by $(k) := \varepsilon_k$ are completely parallel to those in type C, as shown below. We start with distinct cocovers $\mu \succcurlyeq \nu$ and $\mu \succcurlyeq \pi$ of a partition μ, which are labeled (i, j) and (k), where necessarily $i < k$.

Case B1: cocovers which are (a) nonoverlapping; (b) fully overlapping, with (i, j) proper of type (**). As subcase (a) is easy, only subcase (b) is represented in the diagram below.

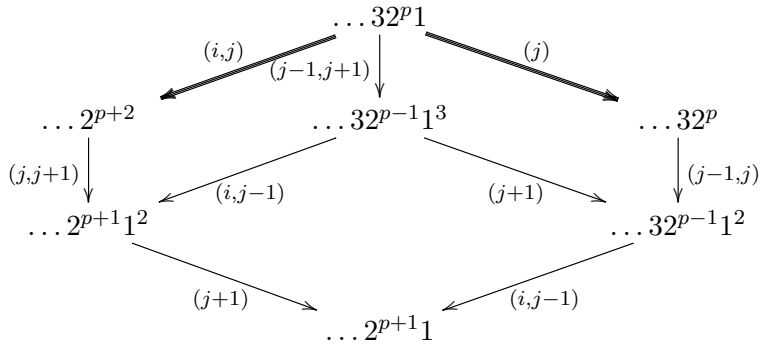
In subcase (b), we have $p \geq 1$, while i is the position of (the shown) 2 in the partition μ, and $j = i + p + 1$ is the position of the first 0.



Case B2: partially overlapping cocovers, with (i, j) of type (*), so $j = i + 1$. We have $a \geq 3$, while i is the position of a in the partition μ.



Case B3: partially overlapping cocovers, with (i,j) proper of type (**). Here $p \geq 1$, while i is the position of (the shown) 3 in the partition μ, and $j = i + p + 1$ is the position of 1.



The following result is proved in a completely similar way to the corresponding one in type C.

Proposition 4.6. *The lower bound $\nu \wedge \pi$ for $\mu \geqslant \nu, \pi$ is always obtained as in one of the Cases* B1−B3*. For similar covers $\mu \lessdot \nu, \pi$, the upper bound $\nu \vee \pi$ is always obtained as in one of the Cases* B1−B2*, now denoted* B1′−B2′*.*

5. Modified crystal operators

In this section we define the modified crystal operators and study several properties of them. The definitions and a few basic facts are stated in Section 5.1 for an arbitrary simple Lie algebra $\mathfrak{g}$ of rank r. For further properties, we restrict to a classical Lie algebra. Throughout, we use the standard labeling of the corresponding Dynkin diagrams.

5.1. **Definition of the modified crystal operators and basic facts.** These operators are indexed by arbitrary roots in the Weyl group orbit $W\alpha_1$ of the simple root α_1[2]. Given such a root α, consider the shortest length element in W satisfying $w(\alpha_1) = \alpha$. We define the modified crystal operators f_α and e_α as the conjugations

$$\mathrm{f}_\alpha := w\tilde{f}_1 w^{-1}\,, \qquad \mathrm{e}_\alpha := w\tilde{e}_1 w^{-1} \tag{30}$$

of the ordinary crystal operators $\tilde{f}_1$ and $\tilde{e}_1$ by the Kashiwara action of w on $B(\lambda)$ [13]. This means that $\mathrm{f}_\alpha(b) = 0$ precisely when $\tilde{f}_1$ applied to $w^{-1}(b)$ is 0. When $\alpha = \alpha_i$ is a simple root, we simply write $\mathrm{f}_i := \mathrm{f}_{\alpha_i}$ and $\mathrm{e}_i := \mathrm{e}_{\alpha_i}$. We clearly have $\mathrm{f}_1 = \tilde{f}_1$, but $\mathrm{f}_i \neq \tilde{f}_i$ in general, and similarly for e_i[3].

It is not hard to see that the above choice of w implies that ws_1 is the shortest length element mapping α_1 to $-\alpha$, so we have

$$\mathrm{f}_{-\alpha} = ws_1\tilde{f}_1 s_1 w^{-1} = w\tilde{e}_1 w^{-1} = \mathrm{e}_\alpha\,.$$

Furthermore, it is easy to check that $\mathrm{f}_\alpha(b) = b'$ if and only if $\mathrm{e}_\alpha(b') = b$, and that

$$\mathrm{wt}(\mathrm{f}_\alpha(b)) = \mathrm{wt}(b) - \alpha\,.$$

More generally, if we choose any $v \in W$ such that $v\alpha_1 = \alpha$, we can consider the operator $\mathrm{f}_v := v\tilde{f}_1 v^{-1}$, and ask if it coincides with $\mathrm{f}_\alpha := \mathrm{f}_w$. We address this question in Remark 5.1 below.

[2]Observe that the orbit $W\alpha_1$ coincides with the set of all roots only for simply laced root systems.

[3]The operators f_i were considered in [13, Remark 7.4.2], and it was observed there that they do not coincide with the ordinary ones, but to the authors' knowledge they were not studied further.

Remark 5.1. First note that $w^{-1}v$ belongs to the stabilizer W_{α_1} of α_1, which (as a parabolic subgroup) is generated by the reflections s_α with α orthogonal to α_1. In type A, we have $W_{\alpha_1} = \langle s_3, \dots, s_r \rangle$, so $w^{-1}v$ commutes with $\tilde{f}_1$, and thus $\mathrm{f}_v = \mathrm{f}_\alpha$. More generally, the same holds whenever $w^{-1}v$ belongs to $\langle s_3, \dots, s_r \rangle$. However, in general $\mathrm{f}_v \neq \mathrm{f}_\alpha$. For instance, in types B, C, and D we have $\alpha_1 = \varepsilon_1 - \varepsilon_2$, so $W_{\alpha_1} = \langle s_{\varepsilon_1+\varepsilon_2} \rangle \times \langle s_3, \dots, s_r \rangle$, where $s_{\varepsilon_1+\varepsilon_2}$ does not commute with $\tilde{f}_1$.

We endow the vertices of $B(\lambda)$ with the structure of a colored directed graph $\mathbb{B}(\lambda)$ with edges $b \overset{\alpha}{\dashrightarrow} b'$ when $b' = \mathrm{f}_\alpha(b)$ for a positive root $\alpha \in W\alpha_1$. As noted above, the graph $\mathbb{B}(\lambda)$ is different from the Kashiwara crystal $B(\lambda)$, and in fact, unlike the latter, the former is not connected in general.

Lemma 5.2. *Consider $b \in \mathbb{B}(\lambda)$ and a positive root $\alpha \in W\alpha_1$.*

1. *If $\langle \mathrm{wt}(b), \alpha \rangle > 0$, then $\mathrm{f}_\alpha(b) \neq 0$. In particular, if $\mathrm{wt}(b) - \alpha$ is dominant, where $\alpha \in R^+$, then $\mathrm{f}_\alpha(b) \neq 0$.*
2. *If $\langle \mathrm{wt}(b), \alpha \rangle < 0$, then $\mathrm{e}_\alpha(b) \neq 0$.*

Proof. Consider $w \in W$ of smallest length such that $w(\alpha_1) = \alpha$. If $\langle \mathrm{wt}(b), \alpha \rangle > 0$, then

$$\langle w^{-1}\mathrm{wt}(b), w^{-1}(\alpha) \rangle = \langle \mathrm{wt}(w^{-1}(b)), \alpha_1 \rangle > 0\,,$$

which implies that $\tilde{f}_1(w^{-1}(b)) \neq 0$. But this is equivalent to $\mathrm{f}_\alpha(b) \neq 0$. For the second part of (1), let $\mu := \mathrm{wt}(b)$, and observe that $\langle \mu, \alpha^\vee \rangle = \langle \mu - \alpha, \alpha^\vee \rangle + 2 \geq 2$; so we can apply the first part. The proof of (2) is completely similar. □

5.2. **Properties of the modified crystal operators in classical types.** From now on we assume that the underlying root system is of classical type.

Theorem 5.3. *Consider two positive roots α and β in $W\alpha_1$ and a vertex b in $\mathbb{B}(\lambda)$ such that $\langle \mathrm{wt}(b), \alpha \rangle > 0$ and $\langle \mathrm{wt}(b), \beta \rangle > 0$.*

1. *Assume that the pair (α, β) satisfies:* (i) *it is $(\varepsilon_i - \varepsilon_j, \varepsilon_j \pm \varepsilon_k)$ or $(\varepsilon_j \pm \varepsilon_k, \varepsilon_i - \varepsilon_j)$, for $i < j < k$;* (ii) *it is $(\varepsilon_{j-1} + \varepsilon_j, \varepsilon_i - \varepsilon_j)$ for $i < j - 1$, and $\langle \mathrm{wt}(b) - \beta, \varepsilon_{j-1} - \varepsilon_j \rangle = 0$. Then we have $\mathrm{f}_\alpha \mathrm{f}_\beta(b) = \mathrm{f}_{\alpha+\beta}(b) \neq 0$.*
2. *Assume that the pair (α, β) is in the W-orbit of (α_1, α_3), where $\alpha_3 = \varepsilon_3 - \varepsilon_4$. Then $\mathrm{f}_\alpha \mathrm{f}_\beta(b) = \mathrm{f}_\beta \mathrm{f}_\alpha(b) \neq 0$.*

We first reduce our theorem to an equivalent simpler statement. Its two parts will be proved in Sections 5.4 and 5.5, respectively.

Lemma 5.4. *The two parts of Theorem 5.3 follow from the two statements below, respectively.*

1. *For any b such that $\langle \mathrm{wt}(b), \alpha_1 \rangle > 0$ and $\langle \mathrm{wt}(b), \alpha_2 \rangle > 0$, we have*

$$\mathrm{f}_1 \mathrm{f}_2(b) = \mathrm{f}_2 \mathrm{f}_1(b) = \mathrm{f}_{\alpha_1+\alpha_2}(b) \neq 0\,,$$

   where $\alpha_2 = \varepsilon_2 - \varepsilon_3$.
2. *For any b such that $\langle \mathrm{wt}(b), \alpha_1 \rangle > 0$ and $\langle \mathrm{wt}(b), \alpha_3 \rangle > 0$, we have*

$$\mathrm{f}_1 \mathrm{f}_3(b) = \mathrm{f}_3 \mathrm{f}_1(b) \neq 0\,,$$

   where $\alpha_3 = \varepsilon_3 - \varepsilon_4$.

Proof. Given a signed permutation w, we use the notation $w[a, b, \dots] := [w(a), w(b), \dots]$.

We first address Theorem 5.3 (1), and start with case (i), where $(\alpha, \beta) = (\varepsilon_i - \varepsilon_j, \varepsilon_j \pm \varepsilon_k)$. Consider the shortest length element $w \in W$ with $w[1,2,3] = [i, j, \pm k]$, as well as $u := s_1 s_2$ and $v := s_2$. Assertion (1) of the lemma implies

$$\begin{aligned} &\mathrm{f}_1\mathrm{f}_u(b) = \mathrm{f}_u\mathrm{f}_1(b) = \mathrm{f}_v(b) \neq 0 \implies \\ &(w\mathrm{f}_1 w^{-1})(w\mathrm{f}_u w^{-1})(b') = (w\mathrm{f}_u w^{-1})(w\mathrm{f}_1 w^{-1})(b') = w\mathrm{f}_v w^{-1}(b') \neq 0 \implies \\ &\mathrm{f}_w\mathrm{f}_{wu}(b') = \mathrm{f}_{wu}\mathrm{f}_w(b') = \mathrm{f}_{wv}(b') \neq 0\,, \end{aligned} \tag{31}$$

where $b' := w(b)$. The condition on weights in Assertion 1 implies $\langle \mathrm{wt}(b'), \alpha\rangle > 0$ and $\langle \mathrm{wt}(b'), \beta\rangle > 0$. On another hand, note that $wu[1,2] = [j, \pm k]$ and $wv[1,2] = [i, \pm k]$. Moreover, one can check by using Remark 5.1 that we are in the situation where $\mathrm{f}_w = \mathrm{f}_\alpha$, $\mathrm{f}_{wu} = \mathrm{f}_\beta$, and $\mathrm{f}_{wv} = \mathrm{f}_{\alpha+\beta}$. By plugging into (31), the proof in case (i) is concluded.

The statement in Theorem 5.3 (2) is proved in a completely similar way, based on Assertion (2) of the lemma.

We now turn to case (ii) in Theorem 5.3 (1). By a similar reasoning as above, the statement follows from the special case $(\alpha, \beta) = (\varepsilon_2 + \varepsilon_3, \varepsilon_1 - \varepsilon_3)$. Consider the shortest length element $w \in W$ with $w[1,2,3] = [1, 3, \overline{2}]$, as well as $u := s_1 s_2$ and $v := s_2$, like before. By using conjugation as above, Assertion (1) of the lemma implies

$$\mathrm{f}_{wu}\mathrm{f}_w(b') = \mathrm{f}_{wv}(b') \neq 0\,. \tag{32}$$

Note that $wu[1,2] = [3, \overline{2}]$ and $wv[1,2] = [1, \overline{2}]$. Therefore, by Remark 5.1, we have $\mathrm{f}_w = \mathrm{f}_\beta$ and $\mathrm{f}_{wv} = \mathrm{f}_{\alpha+\beta}$, but $\mathrm{f}_{wu} \neq \mathrm{f}_\alpha$. In fact, the shortest coset representative in wuW_{α_1} is $wus_{\varepsilon_1+\varepsilon_2} = s_2 wu$, so $\mathrm{f}_\alpha = s_2\mathrm{f}_{wu}s_2$. However, letting $b'' := \mathrm{f}_\beta(b')$, where $\mathrm{wt(b'')} = \mathrm{wt(b')} - \beta$, we can see that the condition $\langle \mathrm{wt}(b''), \varepsilon_2 - \varepsilon_3\rangle = 0$ implies

$$\mathrm{f}_\alpha(b'') = s_2\mathrm{f}_{wu}s_2(b'') = \mathrm{f}_{wu}(b'')\,.$$

The proof is concluded by plugging into (32). □

We have an analogous result to Theorem 5.3 for the e. operators.

Theorem 5.5. *Consider two positive roots α and β in $W\alpha_1$ and a vertex b in $\mathbb{B}(\lambda)$ such that $\langle \mathrm{wt}(b), \alpha\rangle \geq 0$ and $\langle \mathrm{wt}(b), \beta\rangle \geq 0$. Assume also that $\mathrm{e}_\alpha(b) \neq 0$ and $\mathrm{e}_\beta(b) \neq 0$.*

1. *Assume that the pair (α, β) satisfies:* (i) *it is $(\varepsilon_i - \varepsilon_j, \varepsilon_j \pm \varepsilon_k)$ or $(\varepsilon_j \pm \varepsilon_k, \varepsilon_i - \varepsilon_j)$, for $i < j < k$;* (ii) *it is $(\varepsilon_i - \varepsilon_j, \varepsilon_{j-1} + \varepsilon_j)$ for $i < j - 1$, and $\langle \mathrm{wt}(b), \varepsilon_{j-1} - \varepsilon_j\rangle = 0$. Then we have $\mathrm{e}_\alpha\mathrm{e}_\beta(b) = \mathrm{e}_{\alpha+\beta}(b) \neq 0$.*
2. *Assume that the pair (α, β) is in the W-orbit of (α_1, α_3), where $\alpha_3 = \varepsilon_3 - \varepsilon_4$, and w is a shortest length element satisfying $w(\alpha_1, \alpha_3) = (\alpha, \beta)$. Let $\gamma := w(\alpha_2)$, and also assume that $\langle \mathrm{wt}(b), \gamma\rangle > 0$. Then $\mathrm{e}_\alpha\mathrm{e}_\beta(b) = \mathrm{e}_\beta\mathrm{e}_\alpha(b) \neq 0$.*

By analogy with Theorem 5.3, the above result is proved based on the following reduction. In turn, the two parts of Lemma 5.6 are proved in the same way as those of Lemma 5.4, also in Sections 5.4 and 5.5, respectively.

Lemma 5.6. *The two parts of Theorem 5.5 follow from the two statements below, respectively.*

1. *For any b such that $\langle \mathrm{wt}(b), \alpha_1\rangle \geq 0$, $\langle \mathrm{wt}(b), \alpha_2\rangle \geq 0$, $\mathrm{e}_1(b) \neq 0$, and $\mathrm{e}_2(b) \neq 0$, we have*
$$\mathrm{e}_1\mathrm{e}_2(b) = \mathrm{e}_2\mathrm{e}_1(b) = \mathrm{e}_{\alpha_1+\alpha_2}(b) \neq 0\,,$$
where $\alpha_2 = \varepsilon_2 - \varepsilon_3$.
2. *For any b such that $\langle \mathrm{wt}(b), \alpha_1\rangle \geq 0$, $\langle \mathrm{wt}(b), \alpha_2\rangle > 0$, $\langle \mathrm{wt}(b), \alpha_3\rangle \geq 0$, $\mathrm{e}_1(b) \neq 0$ and $\mathrm{e}_3(b) \neq 0$, we have*
$$\mathrm{e}_1\mathrm{e}_3(b) = \mathrm{e}_3\mathrm{e}_1(b) \neq 0\,,$$

where $\alpha_3 = \varepsilon_3 - \varepsilon_4$.

Proof. The proof is completely similar to that of Lemma 5.4. □

As mentioned, we start with a direct proof of the first parts of Lemmas 5.4 and 5.6; these are based on the description of the actions of f_2 and $\mathrm{f}_{\alpha_1+\alpha_2}$ on semistandard tableaux. We then prove the second parts by using the *cyclage* of Lascoux and Schützenberger (which can also be used to reprove the first parts) on type A_3-tableaux.

5.3. **Cyclage, charge and the modified crystal operators.** Given a semistandard tableau T, write $C(T) = x \hookrightarrow T^\flat$ where $x \hookrightarrow T^\flat$ is the semistandard tableau obtained after the row-insertion of the south-west letter x of T in the tableau $T \setminus \{x\}$. The combinatorial procedure $T \to C(T)$ is called the cyclage of the tableau T. It is known (see [18]) that the sequence of cyclages applied to T will eventually lead to the unique row-tableau R_μ where $\mu = \mathrm{wt}(T)$ is the weight of T. The number $\mathrm{co}(T)$ of cyclage operations used in this sequence is called the cocharge of T. The charge of T is then defined as $\mathrm{c}(T) = \|\mu\| - \mathrm{co}(T)$ where $\|\mu\| = \sum_{i=1}^{n-1}(i-1)\mu_i$.

Example 5.7. For $T = \begin{array}{|c|c|c|}\hline 1&1&4\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}$ we get

$$T_1 = C(T) = \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2&4\\\hline\end{array}\ , T_2 = C^2(T) = \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3\\\cline{1-2} 4\\\cline{1-1}\end{array}\ , T_3 = C^3(T) = \begin{array}{|c|c|c|c|}\hline 1&1&2&4\\\hline 2&3\\\cline{1-2}\end{array}$$

$$T_4 = C^4(T) = \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 3&4\\\cline{1-2}\end{array}\ , T_5 = C^5(T) = \begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&3\\\hline 4\\\cline{1-1}\end{array}$$

$$T_6 = C^6(T) = \begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&2&3&4\\\hline\end{array}\ .$$

Therefore $\mathrm{co}(T) = 6$ and $\mathrm{c}(T) = 7 - 6 = 1$.

We can endow the set Tab_μ of semistandard tableaux of weight μ with the structure of an oriented graph with an oriented edge $T \rightsquigarrow T'$ when $T' = C(T)$. We then have a unique sink vertex in Tab_μ which is the row tableau of evaluation μ. Given μ and ν two weights, we write $\nu \leq \mu$ when $\mu - \nu$ can be written as a linear combination of simple roots $\varepsilon_i - \varepsilon_{i+1}, i = 1, \ldots, n-1$ with nonnegative integral coefficients. We refer to [18] for a proof of the following theorem.

Theorem 5.8.

(1) *For any* $\sigma \in \mathfrak{S}_n$ *the Kashiwara action* $T \longmapsto \sigma(T)$ *gives an isomorphism of oriented graphs from* Tab_μ *to* $\mathrm{Tab}_{\sigma(\mu)}$.
(2) *Assume* $\mu_1 > \mu_2$. *Then the action of the Kashiwara crystal operator* $\tilde{f}_1$ *yields an embedding of oriented graphs from* Tab_μ *to* $\mathrm{Tab}_{\mu-\alpha_1}$.
(3) *If* $C(T) = C(T')$ *where* T *and* T' *are two tableaux with the same shape, then* $T = T'$. *Thus the embedding in* (2) *is the unique which preserves the shape of the tableaux.*

Corollary 5.9. *Consider a tableau* T *and a positive root* α *such that* $\mathrm{f}_\alpha(T) \neq 0$. *Then* $\mathrm{f}_\alpha(C(T)) = C(\mathrm{f}_\alpha(T)) \neq 0$.[4]

It is well-know that the charge statistic yields a combinatorial description of the Kostka polynomials in type A.

[4]Nevertheless, there are tableaux T such that $\mathrm{f}_\alpha(C(T)) \neq 0$ but $\mathrm{f}_\alpha(T) = 0$.

Theorem 5.10. *For any partitions λ and μ we have*

$$K_{\lambda,\mu}(t) = \sum_{T \in \mathrm{Tab}(\lambda)_\mu} t^{\mathrm{c}(T)}$$

where $\mathrm{Tab}(\lambda)_\mu$ *is the set of semistandard tableaux of shape* λ *and evaluation* μ.

The following proposition is a consequence of Theorems 5.8 and 5.10. It shows how the charge and cocharge statistics are modified when a modified crystal operator is applied to a tableau T of shape λ (regarded as a vertex of the crystal $B(\lambda)$). For any positive root α, let $|\alpha|$ be the height of α, that is the number of simple roots appearing in the decomposition of α as a sum of simple roots.

Proposition 5.11. *Let T be a vertex of $B(\lambda)$ of weight μ and α a positive root such that $\mathrm{f}_\alpha(T) \neq 0$. Then*

$$\mathrm{co}(\mathrm{f}_\alpha(T)) = \mathrm{co}(T) \text{ and } \mathrm{c}(\mathrm{f}_\alpha(T)) = \mathrm{c}(T) + |\alpha|\,.$$

Proof. Since f_α is obtained by conjugation of the action of $\tilde{f}_1$, we get the equality $\mathrm{co}(\mathrm{f}_\alpha(T)) = \mathrm{co}(T)$ from Assertion 1 and 2 of Theorem 5.8. Write $\alpha = \varepsilon_i - \varepsilon_i$ with $1 < i < j < n$. Then $\mathrm{f}_\alpha(T)$ has weight $\mu - \alpha$. Since $\mathrm{co}(\mathrm{f}_\alpha(T)) = \mathrm{co}(T)$, we have

$$\begin{aligned}\mathrm{c}(\mathrm{f}_\alpha(T)) - \mathrm{c}(T) &= \|\mu - \alpha\| - \|\mu\| \\ &= (i-1)((\mu_i - 1) - \mu_i) + (j-1)((\mu_j + 1) - \mu_j) = j - i = |\alpha|\,.\end{aligned}$$

□

5.4. **Proof of Lemmas 5.4 (1) and 5.6 (1).** Observe first that it suffices to prove Lemmas 5.4 (1) and 5.6 (1) in type A_2. To do this, we will use the tableau realization of the crystal $B(\lambda)$. As we restrict to type A_2, we view λ as a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq 0)$, and we index the vertices of $B(\lambda)$ by semistandard tableaux of shape λ with the alphabet $\{1 < 2 < 3\}$. We adopt the English notation for semistandard tableaux. The row reading of a semistandard tableau T is the word $\mathrm{w}(T)$ obtained by reading its rows from right to left and top to bottom. The actions of f_1, s_{α_1}, and s_{α_2} on $B(\lambda)$ are computed by using the following classical procedures.

For any $i = 1, 2$, write w_i for the subword of $\mathrm{w}(T)$ formed by the letters in $\{i, i+1\}$. Let $w_i^{\mathrm{red}} = (i+1)^r i^s$ be the subword of w_i obtained by recursively deleting factors $i(i+1)$. Now consider w_i^{red} as a subword of $\mathrm{w}(T)$. If $r > s$, then s_{α_i} is obtained by replacing in $\mathrm{w}(T)$ the $r - s$ rightmost letters $i+1$ of w_i^{red} with i. Otherwise, s_{α_i} is obtained by replacing in $\mathrm{w}(T)$ the $s - r$ leftmost letters i of w_i^{red} with $i+1$. Observe here that for each factorization $w_i = (i+1)^a u_i i^a$, we have $s_i(w_i) = (i+1)^a s_i(u_i) i^a$.

The action of f_1 on a tableau T is straightforward. If T does not contain any letter 1, then $\mathrm{f}_1(T) = 0$. Otherwise $\mathrm{f}_1(T)$ is obtained by replacing its rightmost letter 1 (always in its first row) with 2.

Let us now describe the actions of $\mathrm{f}_{\alpha_1+\alpha_2}$ and f_2 on a tableau T. Let $m_i^{(k)} = m_i^{(k)}(T)$ be the number of letters i in the k-th row of T. Let $\mu = (\mu_1, \mu_2, \mu_3)$ be the *content* of T, i.e., μ_i is the number of letters i in T.

Lemma 5.12. *Assume $\mu_1 > \mu_3$.*

(1) *If T does not contains any letter 2, then $\mathrm{f}_{\alpha_1+\alpha_2}(T)$ is obtained by changing in T the rightmost letter 1 into a letter 3.*
(2) *Otherwise $\mathrm{f}_{\alpha_1+\alpha_2}(T)$ is obtained by changing in T the rightmost letter 1 into a letter 2 and*
   - *the rightmost letter 2 in its first row into a letter 3 when $m_2^{(2)} \leq m_3^{(1)}$,*
   - *the rightmost letter 2 in its second row into a letter 3 when $m_2^{(2)} > m_3^{(1)}$.*

Proof. Recall that $\mathrm{f}_{\alpha_1+\alpha_2}(T) = s_{\alpha_2}\mathrm{f}_1 s_{\alpha_2}(T)$.

Let $\mathrm{w}(T) = 3^a 2^b 1^c 3^d 2^e$. The assumption $\mu_1 > \mu_3$ is written $c > a+d \geq 0$. Let $m = \min(a, e)$. By the definition of the actions of s_{α_2} and f_1, the contributions of the m leftmost letters 2 in the second row and that of the m rightmost letters 3 in the first row cancel. So, without loss of generality, we have the following two cases, in which we show the successive actions of s_{α_2}, f_1, and s_{α_2}; the paired entries when applying s_{α_2} are underlined.

Case 1: $a \geq e = 0$.

$$\begin{aligned} 3^a\,2^b\,1^c\,3^d &= 3^a\,2^{\max(b-d,0)}\,\underline{2^{\min(b,d)}}\,1^c\,\underline{3^{\min(b,d)}}\,3^{\max(d-b,0)} \xrightarrow{s_{\alpha_2}} \\ &\xrightarrow{s_{\alpha_2}} 3^{\max(b-d,0)}\,2^a\,2^{\min(b,d)}\,1^c\,3^{\min(b,d)}\,2^{\max(d-b,0)} \xrightarrow{\mathrm{f}_1} \\ &\xrightarrow{\mathrm{f}_1} 3^{\max(b-d,0)}\,2^{a+1}\,\underline{2^{\min(b,d)}}\,1^{c-1}\,\underline{3^{\min(b,d)}}\,2^{\max(d-b,0)} \xrightarrow{s_{\alpha_2}} \\ &\xrightarrow{s_{\alpha_2}} 3^{a+1}\,2^{\max(b-d,0)}\,2^{\min(b,d)}\,1^{c-1}\,3^{\min(b,d)}\,3^{\max(d-b,0)} = 3^{a+1}\,2^b\,1^{c-1}\,3^d\,. \end{aligned}$$

Case 2: $e > a = 0$.

$$\begin{aligned} 2^b\,1^c\,3^d\,2^e &= 2^{\max(b-d,0)}\,\underline{2^{\min(b,d)}}\,1^c\,\underline{3^{\min(b,d)}}\,3^{\max(d-b,0)}\,2^e \xrightarrow{s_{\alpha_2}} \\ &\xrightarrow{s_{\alpha_2}} 3^{\max(b-d,0)}\,2^{\min(b,d)}\,1^c\,3^{\min(b,d)}\,3^e\,2^{\max(d-b,0)} \xrightarrow{\mathrm{f}_1} \\ &\xrightarrow{\mathrm{f}_1} 3^{\max(b-d,0)}\,\underline{2^{\min(b,d)+1}}\,1^{c-1}\,\underline{3^{\min(b,d)+1}}\,3^{e-1}\,2^{\max(d-b,0)} \xrightarrow{s_{\alpha_2}} \\ &\xrightarrow{s_{\alpha_2}} 2^{\max(b-d,0)}\,2^{\min(b,d)+1}\,1^{c-1}\,3^{\min(b,d)+1}\,3^{\max(d-b,0)}2^{e-1} = 2^{b+1}\,1^{c-1}\,3^{d+1}\,2^{e-1}\,. \end{aligned}$$

□

Lemma 5.13. *Assume $\mu_2 > \mu_3$. If $m_2^{(2)} \leq m_3^{(1)}$ (respectively $m_2^{(2)} > m_3^{(1)}$), the tableau $\mathrm{f}_2(T)$ is obtained by changing in T the rightmost letter 2 in the first (respectively second) row into a letter 3.*

Proof. Recall that $\mathrm{f}_2(T) = s_1\mathrm{f}_{\alpha_1+\alpha_2}s_1(T)$.

Let $\mathrm{w}(T) = 3^a 2^b 1^c 3^d 2^e$, so we have $c \geq e$. The assumption $\mu_2 > \mu_3$ is written $b+e > a+d$. Based on Lemma 5.12, we consider the following two cases, in which we show the successive actions of s_{α_1}, $\mathrm{f}_{\alpha_1+\alpha_2}$, and s_{α_1}; the paired entries when applying s_{α_1} are underlined.

Case 1: $a \geq e$. We have $b+e > a+d \geq d+e$, so $b > d \geq 0$.

$$\begin{aligned} 3^a\,2^b\,1^c\,3^d\,2^e &= 3^a\,2^b\,1^{c-e}\,\underline{1^e}\,3^d\,\underline{2^e} \xrightarrow{s_{\alpha_1}} 3^a\,2^{c-e}\,1^{b+e}\,3^d\,2^e \xrightarrow{\mathrm{f}_{\alpha_1+\alpha_2}} 3^{a+1}\,2^{c-e}\,1^{b+e-1}\,3^d\,2^e = \\ &= 3^{a+1}\,2^{c-e}\,1^{b-1}\,\underline{1^e}\,3^d\,\underline{2^e} \xrightarrow{s_{\alpha_1}} 3^{a+1}\,2^{b-1}\,1^c\,3^d\,2^e\,. \end{aligned}$$

Case 2: $e > a \geq 0$.

$$\begin{aligned} 3^a\,2^b\,1^c\,3^d\,2^e &\xrightarrow{s_{\alpha_1}} 3^a\,2^{c-e}\,1^{b+e}\,3^d\,2^e \xrightarrow{\mathrm{f}_{\alpha_1+\alpha_2}} 3^a\,2^{c-e+1}\,1^{b+e-1}\,3^{d+1}\,2^{e-1} = \\ &= 3^a\,2^{c-e+1}\,1^b\,\underline{1^{e-1}}\,3^{d+1}\,\underline{2^{e-1}} \xrightarrow{s_{\alpha_1}} 3^a\,2^b\,1^c\,3^{d+1}\,2^{e-1}\,. \end{aligned}$$

□

Corollary 5.14. *Consider a tableau T of weight (μ_1, μ_2, μ_3) such that $\mu_2 \geq \mu_3$. Then $\mathrm{e}_2(T) = 0$ if and only if $m_2^{(2)} \geq m_3^{(1)}$ and $m_3^{(2)} = 0$.*

Proof. Assume $m_2^{(2)} < m_3^{(1)}$. Since $m_3^{(1)} > 0$, we can consider the tableau T' obtained by changing the leftmost letter 3 in 2 in the first row of T. By the previous lemma, we have then $\mathrm{e}_2(T) = T' \neq 0$. Similarly, when $m_2^{(2)} \geq m_3^{(1)}$ and $m_3^{(2)} > 0$, we also get $\mathrm{e}_2(T) \neq T'$ where T' is then obtained from

T by changing the leftmost letter 3 in 2 in the second row. Conversely, when $m_2^{(2)} \geq m_3^{(1)}$ and $m_3^{(2)} = 0$, the tableau T' obtained from T by changing the leftmost letter 3 in 2 in the first row (if any) does not satisfy $\mathrm{f}_2(T') = T$. Thus, we have $\mathrm{e}_2(T) = 0$. □

The following lemma is a rephrasing of the statement in Lemma 5.4 (1). It is an easy consequence of Lemmas 5.12 and 5.13.

Lemma 5.15. *If the tableau T is such that $\mu_1 > \mu_2 > \mu_3$, then we have*

$$\mathrm{f}_1\mathrm{f}_2(T) = \mathrm{f}_2\mathrm{f}_1(T) = \mathrm{f}_{\alpha_1+\alpha_2}(T) \neq 0\,.$$

Lemma 5.6 (1) is rephrased as follows.

Lemma 5.16. *If the tableau T is such that $\mu_1 \geq \mu_2 \geq \mu_3$, $\mathrm{e}_1(T) \neq 0$, and $\mathrm{e}_2(T) \neq 0$, then we have*

$$\mathrm{e}_1\mathrm{e}_2(T) = \mathrm{e}_2\mathrm{e}_1(T) = \mathrm{e}_{\alpha_1+\alpha_2}(T) \neq 0\,.$$

Proof. Let $T' := \mathrm{e}_2(T)$. We have $\langle \mathrm{wt}(T'), \alpha_2 \rangle > 0$. Therefore, the action of f_2 on T', which produces T, is described by Lemma 5.13. Thus, we have the following two cases, where we use the notation $m_i^{(k)}(\cdot)$ introduced above.

Case 1: $m_2^{(2)}(T') \leq m_3^{(1)}(T')$. In this case, we have $m_2^{(1)}(T) = m_2^{(1)}(T') - 1$. Since $\mathrm{e}_1(T) \neq 0$, we have $m_2^{(1)}(T) > 0$, which implies $m_2^{(1)}(T') > 0$. It means that $T'' := \mathrm{e}_1(T') \neq 0$.

Case 2: $m_2^{(2)}(T') > m_3^{(1)}(T')$. In this case, we have $m_2^{(1)}(T) = m_2^{(1)}(T')$. In the same way as in Case 1, we deduce $T'' := \mathrm{e}_1(T') \neq 0$.

In both cases, we have $\mathrm{wt}(T'') = \mathrm{wt}(T) + \alpha_1 + \alpha_2$. Therefore, the hypothesis of Lemma 5.15 is satisfied for T'', and the proof is completed by applying this lemma. □

5.5. **Proof of Lemmas 5.4 (2) and 5.6 (2).** The difficulty is that the second parts of Lemmas 5.4 and 5.6 reduce to type A_3, rather than type A_2, like the first parts. The action of the modified crystal operator f_3 can also be described in the same spirit as in Lemma 5.13, but this requires the enumeration of numerous configurations. Fortunately, in order to prove Lemmas 5.4 (2) and 5.6 (2), this can be avoided by using the cyclage on semistandard tableaux.

Lemma 5.4 (2) is now rephrased as follows.

Lemma 5.17. *If the tableau T is such that $\mu_1 > \mu_2$ and $\mu_3 > \mu_4$, then we have*

$$\mathrm{f}_1\mathrm{f}_3(T) = \mathrm{f}_3\mathrm{f}_1(T) \neq 0\,.$$

Proof. By Lemma 5.2, we have $\mathrm{f}_1\mathrm{f}_3(T) \neq 0$ and $\mathrm{f}_3\mathrm{f}_1(T) \neq 0$. We argue by induction on the cocharge. When $\mathrm{co}(T) = 0$, the tableau T has row shape, and the equality is clear since $\mathrm{f}_1\mathrm{f}_3(T) = \mathrm{f}_3\mathrm{f}_1(T)$ is the unique row of weight $\mathrm{wt}(T) - \alpha_1 - \alpha_3$. Assume that the lemma holds for any tableau with cocharge $k-1$, and consider T such that $\mathrm{co}(T) = k$. Set $\mathrm{f}_1\mathrm{f}_3(T) = U$ and $\mathrm{f}_3\mathrm{f}_1(T) = U'$. By Theorem 5.8 (2), we get $\mathrm{f}_1(C(T)) = C(\mathrm{f}_1(T))$. Then $\mathrm{f}_3\mathrm{f}_1(C(T)) = \mathrm{f}_3 C(\mathrm{f}_1(T))$. But the cyclage operation also commutes with f_3 (in fact with any modified crystal operator), because it commutes with the actions of W and f_1, by Theorem 5.8 (1), (2). So we have $\mathrm{f}_3\mathrm{f}_1(C(T)) = C(\mathrm{f}_3\mathrm{f}_1(T)) = C(U)$. We obtain similarly the equality $\mathrm{f}_1\mathrm{f}_3(C(T)) = C(\mathrm{f}_1\mathrm{f}_3(T)) = C(U')$. By our induction hypothesis, we thus deduce that $C(U) = C(U')$. But since $C(U)$ and $C(U')$ have the same shape (i.e., that of $C(T)$), Theorem 5.8 (3) implies the desired equality $U = U'$. □

Lemma 5.6 (2) is rephrased as follows.

Lemma 5.18. *If the tableau T is such that $\mu_1 \geq \mu_2 > \mu_3 \geq \mu_4$, $\mathrm{e}_1(T) \neq 0$ and $\mathrm{e}_3(T) \neq 0$, then we have*

$$\mathrm{e}_1\mathrm{e}_3(T) = \mathrm{e}_3\mathrm{e}_1(T) \neq 0\,.$$

We first prove a weaker version for two-row tableaux.

Lemma 5.19. *Lemma* 5.18 *is true for any two-row tableau T.*

Proof. Observe first we must have $\mu_3 > 0$. Otherwise $\mu_3 = \mu_4 = 0$ and we cannot have $\mathrm{e}_3(T) \neq 0$ for this would give a tableau of evaluation $(\mu_1, \mu_2, 1, -1)$. Set $T' = \mathrm{e}_1(T)$. When $\mathrm{e}_3(T') \neq 0$, the tableau $T'' = \mathrm{e}_3(T')$ has weight $(\mu_1 + 1, \mu_2 - 1, \mu_3 + 1, \mu_4 - 1)$ and by applying Lemma 5.17 to T'' we get

$$T = \mathrm{f}_1\mathrm{f}_3(T'') = \mathrm{f}_3\mathrm{f}_1(T'') \neq 0\,,$$

which is equivalent to the desired equality.

We claim we cannot have $\mathrm{e}_3(T') = 0$. Indeed, assume $\mathrm{e}_3(T') = 0$. Recall we have $\mathrm{e}_3 = s_2s_3\mathrm{e}_2s_3s_2$. We obtain $\mathrm{e}_2s_3s_2(T') = 0$ and $\mathrm{e}_2s_3s_2(T) \neq 0$ because $\mathrm{e}_3(T) \neq 0$. Set $U = s_3s_2(T)$ and $U' = s_3s_2(T')$. Then $\mathrm{e}_2(U') = 0$ and $\mathrm{e}_2(U) \neq 0$. Since $T = \mathrm{f}_1(T')$ we can write

$$U = s_3s_2\mathrm{f}_1s_2s_3(U') \Longleftrightarrow U = s_3\mathrm{f}_{\varepsilon_1-\varepsilon_3}s_3(U')\,.$$

Moreover

$$\mathrm{wt}(U') = (\mu_1 + 1, \mu_3, \mu_4, \mu_2 - 1) \text{ and } \mathrm{wt}(s_3(U')) = (\mu_1 + 1, \mu_3, \mu_2 - 1, \mu_4)\,.$$

We shall need the two integers $a = m_2^{(2)}(U')$ (thus $a \leq \mu_3$ for U' contains μ_3 letters 2) and $b = m_4^{(1)}(T)$. We get $m_3^{(1)}(U') = \mu_4$ and $a \geq \mu_4$ by Corollary 5.14. In particular, there is no letter 3 in the second row of U'. We shall discuss three cases.

Case 1: $m_4^{(2)}(U') \leq m_3^{(1)}(U') = \mu_4$. During the computation of $s_3(U')$, all the letters 4 in the second row of U' are paired with letters 3 of its first row. Since $\mu_2 - 1 - \mu_4 \geq 0$ because $\mu_2 > \mu_3 \geq \mu_4$, exactly $\mu_2 - 1 - \mu_4$ letters 4 of the first row are changed in letters 3. We get $m_3^{(1)}(s_3(U')) = \mu_2 - 1$ and $m_2^{(2)}(s_3(U')) = m_2^{(2)}(U') = a$. Thus $m_2^{(2)}(s_3(U')) \leq m_3^{(1)}(s_3(U'))$ (otherwise $\mu_3 \geq a \geq \mu_2$). By Lemma 5.12, $\mathrm{f}_{\varepsilon_1-\varepsilon_3}(s_3(U'))$ is thus obtained by changing a letter 1 in a letter 3 in the first row of $s_3(U')$. We get U by applying s_3 to $\mathrm{f}_{\varepsilon_1-\varepsilon_3}(s_3(U'))$, that is by changing $\mu_2 - \mu_4$ letters 3 in letters 4 in its first row. We see that U is obtained from U' by changing a letter 1 in a letter 4 in its first row (up to reordering). Therefore U has no letter 2 in its second row and $m_2^{(2)}(U) = a \geq \mu_4 = m_3^{(1)}(U)$. By Corollary 5.14 we derive the contradiction $\mathrm{e}_2(U) = 0$.

Case 2: $m_4^{(2)}(U') > m_3^{(1)}(U') = \mu_4$. During the computation of $s_3(U')$, all the letters 3 in the first row of U' are paired with letters 4 of its second row. Then, all the remaining $\mu_2 - 1 - \mu_4$ letters 4 are changed into letters 3 in both rows. In particular, we get $m_3^{(1)}(s_3(U')) = \mu_4 + b$. We shall consider two subcases.

Case 2.a: $a = m_2^{(2)}(U') \leq m_3^{(1)}(s_3(U')) = \mu_4 + b$. By Lemma 5.12, $\mathrm{f}_{\varepsilon_1-\varepsilon_3}(s_3(U'))$ is then obtained by changing a letter 1 in a letter 3 in the first row of $s_3(U')$ and we get a contraction exactly as in Case 1.

Case 2.b: $a = m_2^{(2)}(U') > m_3^{(1)}(s_3(U')) = \mu_4 + b$. By Lemma 5.12, $\mathrm{f}_{\varepsilon_1-\varepsilon_3}(s_3(U'))$ is then obtained from $s_3(U')$ by changing a letter 1 in a letter 2 in its first row and a letter 2 in a letter 3 in its second row. We get U by applying s_3 to $\mathrm{f}_{\varepsilon_1-\varepsilon_3}(s_3(U'))$, that is by changing in letters 4 all the letters 3 unpaired with the letters 4 located in the second row. We see that U is obtained from U' by changing a letter 1 in a letter 2 in its first row and a letter 2 in a letter 4 in its second row. Thus, there is no letter 3 in the second row of U. Moreover $m_2^{(2)}(U) = m_2^{(2)}(U') - 1 = a - 1$. But by

hypothesis, $a > \mu_4 + b$, thus $a - 1 \geq \mu_4$ and $m_2^{(2)}(U) \geq \mu_4 = m_3^{(1)}(U)$. We yet get the contradiction $\mathrm{e}_2(U) = 0$ by Corollary 5.14. □

We can now prove Lemma 5.18 by induction on the cocharge.

Proof. Observe first Lemma 5.18 is clearly true for row tableaux, that is for tableaux of cocharge 0. Assume it holds for any tableau of cocharge k and the tableau T considered as cocharge $k+1$. If T has two rows we are done by the previous lemma. So we can assume that T has 3 rows. Set $T' = \mathrm{e}_3(T)$. By Corollary 5.9, we can apply our induction hypothesis to the tableau $C(T)$, we must have $\mathrm{e}_1(C(T')) \neq 0$. We claim this implies that $\mathrm{e}_1(T') \neq 0$. Indeed, the conditions $\mathrm{e}_1(C(T')) \neq 0$ but $\mathrm{e}_1(T') = 0$ would imply that the letter of T' which is used in the cyclage operation is a 2 (any other letter would not modify the locations of the letters 1 and 2 in T'). But by definition of the cyclage, 2 is then the leftmost letter of the shortest row of T', which is only possible if T' has two rows contrary to the assumption we made. To terminate the proof it suffices to apply Lemma 5.17 to $T'' = \mathrm{e}_1(T') = \mathrm{e}_1\mathrm{e}_3(T)$. □

Remarks 5.20. (1) One can use a similar method to give alternative proofs of Lemmas 5.15 and 5.16, that is, without making explicit the action of f_2 and $\mathrm{f}_{\alpha_1+\alpha_2}$. Note that only the basic properties of cyclage mentioned in Section 5.3 are used to carry out the induction step, and no related combinatorics.

(2) On another hand, it would be interesting to prove Lemmas 5.17 and 5.18 in a similar way to Lemmas 5.15 and 5.16, that is, by making explicit the action of f_3. While such a proof would only use the crystal structure (without referring to cyclage), we found it challenging because the Weyl group action is not easy to express in any of the combinatorial models we considered, so we were led to an unmanageable number of cases. Nevertheless, the special case considered in the proof of Lemma 5.19 is manageable with the tableau model.

5.6. **Additional modified crystal operators in type B_n.** In type B_n, in addition to the operators f_α for $\alpha \in W\alpha_1$, i.e., a long root, we need such operators indexed by short roots. They are defined completely similarly to (30). Namely, given a short root α, consider *any* $w \in W$ satisfying $w(\alpha_n) = \alpha$. We then define

$$\mathrm{f}_\alpha := w\tilde{f}_n w^{-1}\,, \qquad \mathrm{e}_\alpha := w\tilde{e}_n w^{-1}\,. \tag{33}$$

Note that, in this case, the definition does not depend on the choice of w, because the stabilizer of α_n is $W_{\alpha_n} = \langle s_1, \ldots, s_{n-2}, s_{\varepsilon_{n-1}}\rangle$, so all its elements commute with $\tilde{f}_n$; see Remark 5.1.

All the basic properties of the modified crystal operators in Section 5.1 extend to the additional operators. Moreover, to the modified crystal graph $\mathbb{B}(\lambda)$ constructed before, we add the extra edges $b \overset{\alpha}{\dashrightarrow} b'$ when $b' = \mathrm{f}_\alpha(b)$ and α is a short positive root.

The following properties are needed for the additional operators.

Theorem 5.21.

(1) *Given $i < j < k$, assume that $\mathrm{f}_{\varepsilon_i-\varepsilon_j}(b) \neq 0$ and $\mathrm{f}_{\varepsilon_k}(b) \neq 0$. Then we have*

$$\mathrm{f}_{\varepsilon_i-\varepsilon_j}\mathrm{f}_{\varepsilon_k}(b) = \mathrm{f}_{\varepsilon_k}\mathrm{f}_{\varepsilon_i-\varepsilon_j}(b) \neq 0\,.$$

The same is true with the f. *operators replaced by the* e. *operators.*

(2) *Consider a vertex b with $\langle \mathrm{wt}(b), \varepsilon_{j-1}\rangle = 1$ and $\langle \mathrm{wt}(b), \varepsilon_j\rangle = 0$. Then we have*

$$\mathrm{f}_{\varepsilon_j}\mathrm{f}_{\varepsilon_{j-1}-\varepsilon_j}(b) = \mathrm{f}_{\varepsilon_{j-1}}(b) \neq 0\,.$$

(3) *Assume that, in addition to the conditions in* (1)*, we have* $i < j-1$ *and* $\langle \mathrm{wt}(b), \varepsilon_i \rangle > 1$*. Then we have*

$$\mathrm{f}_{\varepsilon_j} \mathrm{f}_{\varepsilon_i - \varepsilon_j}(b) = \mathrm{f}_{\varepsilon_i - \varepsilon_{j-1}} \mathrm{f}_{\varepsilon_{j-1}}(b) \neq 0\,.^{5}$$

Proof. The first commutation in (1) is easy to check based on the fact that all Kashiwara crystal operators used in the definition of $\mathrm{f}_{\varepsilon_i - \varepsilon_j}$ commute with those corresponding to $\mathrm{f}_{\varepsilon_k}$. This is because the first operator is defined via conjugation based on $\tilde{f}_1$, whereas $\tilde{f}_n$ is used for the second. The situation is identical for the e. operators.

The fact that none of the expressions in (2) and (3) is 0 holds by Lemma 5.2 (1).

Let us now turn to (2). By the usual reduction procedure based on conjugation, which was applied several times in the proof of Lemma 5.4, the relation can be reduced to the special case corresponding to $j = 2$. (In fact, the reduction procedure is simpler because the definition of the additional operators is independent of the corresponding elements of W.) Now observe that, since $s_i \dots s_{n-1}(\alpha_n) = \alpha_i$, and due to the weight condition on b, we have

$$\begin{aligned} \mathrm{f}_{\varepsilon_2}(b') &= s_2 \dots s_{n-1} \tilde{f}_n s_{n-1} \dots s_2(b')\,, \\ \mathrm{f}_{\varepsilon_1}(b) &= s_1 \dots s_{n-1} \tilde{f}_n s_{n-1} \dots s_1(b) = s_2 \dots s_{n-1} \tilde{f}_n s_{n-1} \dots s_2 \tilde{f}_1(b)\,, \end{aligned}$$

where $b' := \mathrm{f}_{\varepsilon_1 - \varepsilon_2}(b)$. Therefore, the relation to prove is reduced to $\tilde{f}_1(b) = \mathrm{f}_{\varepsilon_1 - \varepsilon_2}(b)$, which is true by definition.

In order to prove (3), we plug the following identity (based on Lemma 5.15) into the relation to prove: $\mathrm{f}_{\varepsilon_i - \varepsilon_j}(b) = \mathrm{f}_{\varepsilon_i - \varepsilon_{j-1}} \mathrm{f}_{\varepsilon_{j-1} - \varepsilon_j}(b)$. Using the commutation of $\mathrm{f}_{\varepsilon_j}$ and $\mathrm{f}_{\varepsilon_i - \varepsilon_{j-1}}$, which is true by (1), the relation to prove becomes precisely the one in (2). □

6. Proof of the atomic decomposition

Fix a dominant weight λ for a classical Lie algebra. Consider the subgraph of $\mathbb{B}(\lambda)$ consisting of the vertices of dominant weight, and the edges $b \overset{\alpha}{\dashrightarrow} \mathrm{f}_\alpha(b)$ for which $\mathrm{wt}(b) \succcurlyeq \mathrm{wt}(\mathrm{f}_\alpha(b))$ is a cocover in the dominant weight poset. This new colored directed graph on the vertices of $B(\lambda)^+$ will be denoted by $\mathbb{B}(\lambda)^+$. It can also be viewed as a poset (with cocovers given by the above edges), and the weight function is a poset projection to the interval $[\widehat{0}, \lambda]$ in the dominant weight poset. The two points of view will be used interchangeably.

The main goal is to identify situations in which the components of the poset $\mathbb{B}(\lambda)^+$ define an atomic, respectively t-atomic decomposition, cf. Definitions 2.8 and 2.9.

Remark 6.1. It is important to use the setup mentioned above, as we found that several other variations fail, as explained below.

- If we consider all vertices of $B(\lambda)$, rather than just those of dominant weight, then Lemma 5.17 fails, for instance, in type A_3, for $\lambda = (4, 1, 1)$.
- In type A_{n-1}, we obtain the same results by defining the modified crystal operators based on f_{n-1} rather than f_1 (see Section 5.1), due to the symmetry of the Dynkin diagram. However, if any other node of the Dynkin diagram is used, the connected components of the corresponding $\mathbb{B}(\lambda)$ do not satisfy the properties in Theorem 6.5, which are needed for the atomic decomposition. An example is $\lambda = (4, 3)$ in type A_3.
- The same complication arises if we do not remove the modified crystal edges which do not correspond to cocovers in the dominant weight poset. An example is $\lambda = (3, 1, 1)$ in type C_3.

[5] Observe we will not need analogues of assertions (2) and (3) for the e. operators in the sequel.

Lemma 5.2 (1) immediately gives the following result, which is a converse of the fact that every edge in $\mathbb{B}(\lambda)^+$ projects to a cocover in the dominant weight poset (by definition).

Lemma 6.2. *Given a vertex b in $\mathbb{B}(\lambda)^+$ of weight μ and a cocover $\mu \gtrdot \mu - \alpha$ in the dominant weight poset (with $\alpha \in R^+$), we have an edge $b \overset{\alpha}{\dashrightarrow} \mathrm{f}_\alpha(b)$ in $\mathbb{B}(\lambda)^+$.*

6.1. **Type A_{n-1}.** We refer freely to Section 4.1.

Lemma 6.3. *Consider two distinct edges $b \dashrightarrow b'$ and $b \dashrightarrow b''$ in $\mathbb{B}(\lambda)^+$. The vertices b' and b'' have a lower bound in this poset.*

Proof. Let $\mu := \mathrm{wt}(b)$, $\nu := \mathrm{wt}(b')$, and $\pi := \mathrm{wt}(b'')$. We have cocovers $\mu \gtrdot \nu$ and $\mu \gtrdot \pi$, so the interval $[\nu \wedge \pi, \mu]$ has one of the structures in Cases A1−A3. Starting from the crystal vertex b, we can apply the f. operators indexed by the labels in the corresponding diagrams, by Lemma 6.2. It suffices to check that these diagrams commute, which follows by using Theorem 5.3 (1)-(2) repeatedly. In fact, we apply this theorem to the corresponding triangles and diamonds, after we verify its hypotheses by inspecting the diagrams. In the case of the pentagons, for instance in Case A2 (a), we let $\mathrm{f}_{ij} := \mathrm{f}_{\alpha_{ij}}$, and calculate:

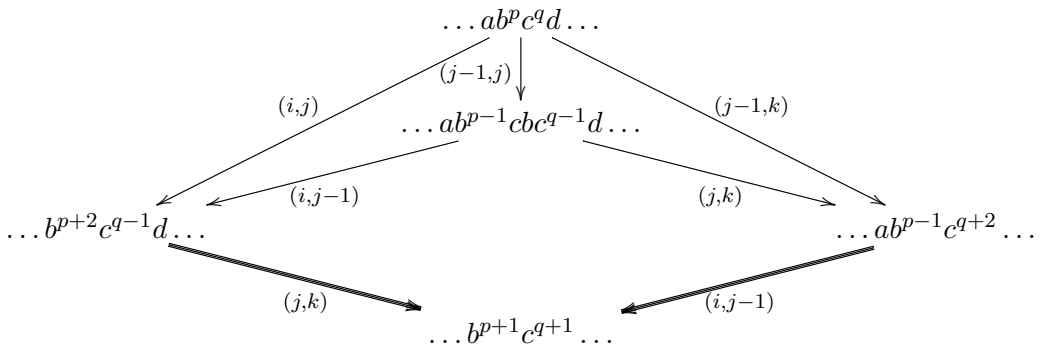
$$\mathrm{f}_{i+2,j}\mathrm{f}_{i+1,i+2}(\mathrm{f}_{i,i+1}(b)) = \mathrm{f}_{i+1,j}\mathrm{f}_{i,i+1}(b) = \mathrm{f}_{i,i+1}\mathrm{f}_{i+1,j}(b)\,;$$

indeed, each of the two equalities follows from Theorem 5.3 (1). □

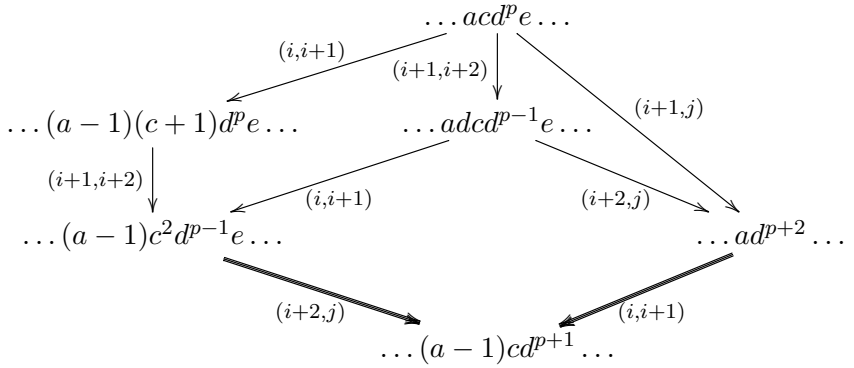
Lemma 6.4. *Consider two distinct edges $b' \dashrightarrow b$ and $b'' \dashrightarrow b$ in $\mathbb{B}(\lambda)^+$. The vertices b' and b'' have an upper bound in this poset.*

Proof. The notation and related conditions are the same as in Section 4.1. Like in the proof of Lemma 6.3, the goal is to lift the diagrams in Cases A1′−A3′ from the dominant weight poset to $\mathbb{B}(\lambda)^+$. In fact, we will perform the lift along the edges of slightly modified diagrams (by starting from the bottom) in the cases indicated below, while we use the same diagrams as in Section 4.1 in Cases A1′ (a) and A1′ (b).

Case A1′ (c).

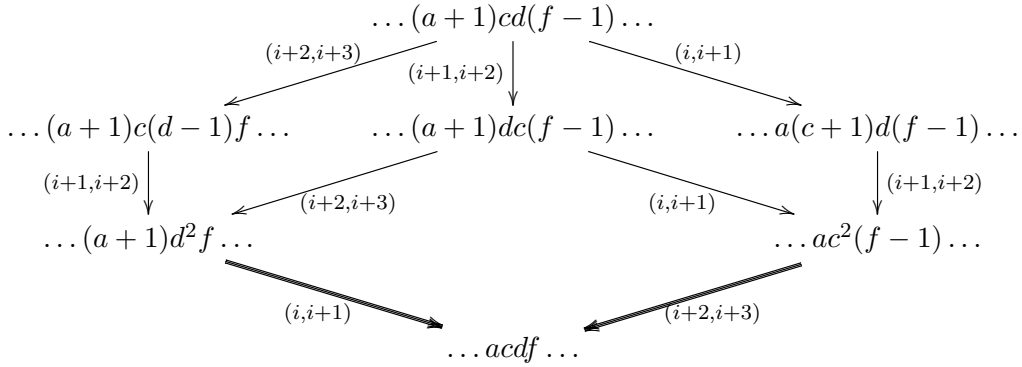


Case A2′ (a).



Case A2′ (b) is similar to Case A2′ (a).

Case A3′.



In Cases A1′ (a) and A1′ (b), the lemma is a direct consequence of Theorem 5.5 (2) and (1), respectively. Note that, in the first case, the extra condition in Theorem 5.5 (2) on the dominant weight $\dots a \dots c \dots e \dots g \dots$ at the bottom of the diagram amounts to $c > e$, where the two covers of the mentioned weight are $\dots a+1 \dots c-1 \dots e \dots g \dots$ and $\dots a \dots c \dots e+1 \dots g-1 \dots$; this condition immediately follows from the fact that the covers are dominant weights themselves.

Thus, it suffices to focus on the three diagrams above. Their distinctive feature is that the weight in the middle is not dominant, but a single pair of consecutive entries is in increasing order. In each of the three cases, we start by applying Theorem 5.5 (2) to the diamond at the bottom (the extra condition in the theorem is part of the assumptions corresponding to the mentioned cases). Then we can apply the corresponding e. operator to the determined vertex of nondominant weight, by Lemma 5.2 (2). Finally, starting from the determined top vertex, we can apply the corresponding f. operators along all the remaining edges of the modified diagrams, by Lemma 6.2. The commutativity of the remaining triangles and diamonds is checked by Theorem 5.3 (1); indeed, in each case we verify the hypothesis by inspecting the corresponding diagram. This concludes the proof. □

Theorem 6.5. *The components of $\mathbb{B}(\lambda)^+$ define a t-atomic decomposition. Moreover, these components are isomorphic to intervals of the form $[\widehat{0}, \mu]$ in the dominant weight poset via the weight projection, and the distinguished vertex $h \in H(\lambda)$ in each of them is chosen to be the corresponding maximum.*

Proof. By Lemma 6.2, the weight projection of a lower order ideal determined by a vertex of weight μ is the interval $[\widehat{0}, \mu]$.

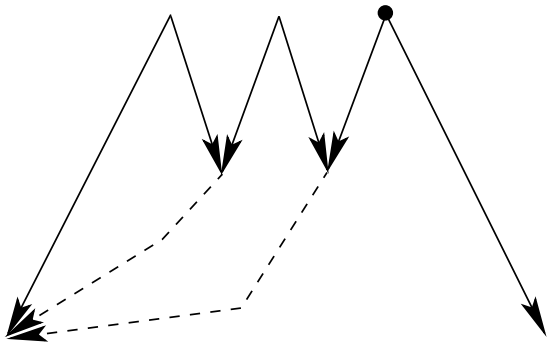


Figure 1: Proof of Theorem 6.5: $\overline{C} \neq \emptyset$.

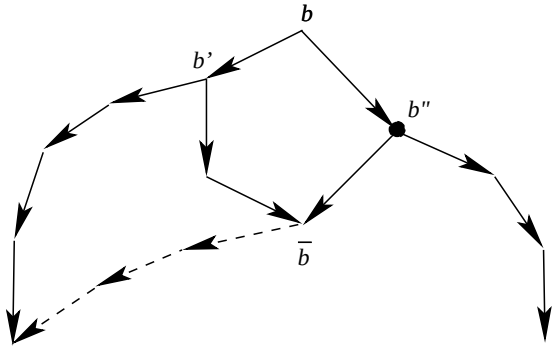


Figure 2: Proof of Theorem 6.5: C has a minimum.

Now fix a weakly connected component C of $\mathbb{B}(\lambda)^+$. As we saw, all of its minimal vertices have weight $\widehat{0}$. We will first prove the uniqueness of a minimal vertex. Assuming the contrary, let $\overline{C}$ be the subposet of C consisting of vertices connected via directed paths to more than one minimal vertex. Find an undirected path in C connecting two distinct minimal vertices, and consider its local maxima and minima which are not endpoints. As noted above, there are directed paths from the local minima to minimal vertices. By considering these paths, we can see that some local maximum must be in $\overline{C}$, so $\overline{C} \neq \emptyset$; see Figure 1. Fixing a minimal vertex b in $\overline{C}$, we must have distinct edges $b \dashrightarrow b'$ and $b \dashrightarrow b''$. By Lemma 6.3, b' and b'' have a lower bound $\overline{b}$ in C. By considering a directed path from $\overline{b}$ to a minimal vertex, we can see that b' or b'' are in $\overline{C}$, which contradicts the minimality of b; see Figure 2. Therefore, C has a minimum $b_{\min}$. The existence of a maximum $b_{\max}$ is proved in a completely similar way, by using Lemma 6.4 instead.

We then need to show that there are no two vertices of the same weight in a component. Assume for contradiction that b and b' are such vertices, with $\mathrm{wt}(b) = \mathrm{wt}(b') = \mu$. Then we can find a saturated decreasing chain from μ to $\widehat{0}$ in the dominant weight poset. According to Lemma 6.2, by applying to b and b' the f. operators corresponding to the labels on the mentioned chain, we obtain directed paths from these vertices to the minimum $b_{\min}$ in the considered component. However, using the reverse sequence of labels and starting from $b_{\min}$, it is clearly impossible to reach two different vertices via the e. operators.

It is now clear that the components of $\mathbb{B}(\lambda)^+$ define an atomic decomposition, where in each component we choose its maximum as the distinguished vertex $h \in H(\lambda)$.

To get the t-atomic decomposition, we first need, according to Definition 2.3, a statistic on $H(\lambda)$. One can use the realization of $B(\lambda)$ in terms of semistandard tableaux. Obviously, the natural candidate for the statistic $\mathrm{c}(\cdot)$ in Definition 2.9 is the Lascoux-Schützenberger charge [16]. For any dominant weight μ (i.e. for any partition with at most n parts), we set

$$A_{\lambda,\mu} = \sum_{\substack{h \in H(\lambda) \\ \mathrm{wt}(h)=\mu}} t^{\mathrm{c}(h)} .$$

We then have for any dominant weight ν

$$K_{\lambda,\nu}(t) = \sum_{T \in B(\lambda)_\nu} t^{\mathrm{c}(T)} = \sum_{\nu \leq \mu \leq \lambda} \sum_{\substack{h \in H(\lambda) \\ \mathrm{wt}(h)=\mu}} \sum_{T \in B(\lambda)_\nu \cap \mathbb{B}(\lambda,h)} t^{\mathrm{c}(T)} .$$

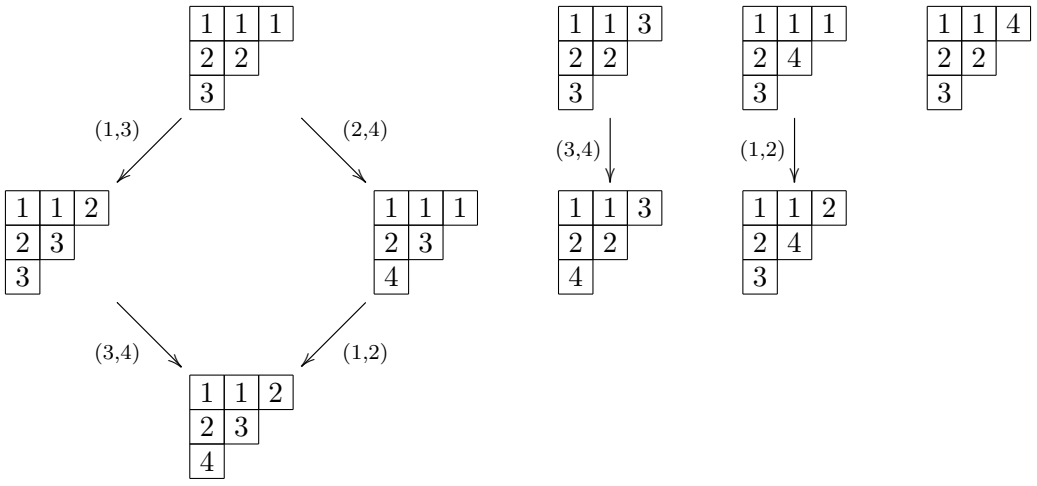


Figure 3: The modified crystal graph $\mathbb{B}(\lambda)^+$ in Example 6.6.

Now by Proposition 5.11, we obtain the equality $\mathrm{c}(T) = \mathrm{c}(h) + \langle \mu - \nu, \rho^\vee \rangle$ for any $T \in B(\lambda)_\nu \cap \mathbb{B}(\lambda, h)$. Indeed we have $\mathrm{wt}(h) - \mathrm{wt}(T) = \mu - \nu$ and T can be obtained from h by applying modified crystal operators f_α, each of them increasing the charge by $\langle \alpha, \rho^\vee \rangle$. Also, the set $B(\lambda)_\nu \cap \mathbb{B}(\lambda, h)$ is reduced to a singleton because the connected component $\mathbb{B}(\lambda, h)$ of $\mathbb{B}(\lambda)^+$ contains exactly one vertex of weight $\nu \leq \lambda$. Thus we can write

$$K_{\lambda,\nu}(t) = \sum_{\nu \leq \mu \leq \lambda} t^{\langle \mu - \nu, \rho^\vee \rangle} \sum_{\substack{h \in H(\lambda) \\ \mathrm{wt}(h) = \mu}} t^{\mathrm{c}(h)} = \sum_{\nu \leq \mu \leq \lambda} t^{\langle \mu - \nu, \rho^\vee \rangle} A_{\lambda,\mu}(t)\,,$$

which is equivalent to the t-atomic decomposition by Proposition 2.2. □

Example 6.6. Consider $\lambda = (3, 2, 1)$ in type A_3. The modified crystal graph $\mathbb{B}(\lambda)^+$ is shown in Figure 3. Its vertices are labeled by semistandard Young tableaux whose content is a partition, and its edges are labeled as above. In particular, this graph gives the following atomic decomposition of the character:

$$\chi_\lambda = w_{(3,2,1)} + w_{(2,2,2)} + w_{(3,1,1,1)} + w_{(2,2,1,1)}\,.$$

6.2. **Types B_n, C_n, and D_n.** This section refers to the stable ranges in types B_n, C_n, and D_n, namely to a corresponding graph/poset $\mathbb{B}(\lambda)^+$. We refer freely to Sections 4.2, 4.3, and 4.4, as well as to the results in Section 6.1.

Lemma 6.7. *Lemmas* 6.3 *and* 6.4 *hold in types B_n, C_n, and D_n, in the corresponding stable ranges.*

Proof. We start with types C_n and D_n. In addition to the cases considered in the proofs of Lemmas 6.3 and 6.4, we need to consider the cases involving the new type of cover in the corresponding dominant weight poset, namely Cases C1−C3 (for extending the first lemma, by Proposition 4.2) and Cases C1′−C2′ (for extending the second one, by Proposition 4.3). The goal is the same: to lift the corresponding diagrams from the dominant weight poset to $\mathbb{B}(\lambda)^+$. We use the same reasoning as in the mentioned lemmas, and give more details below.

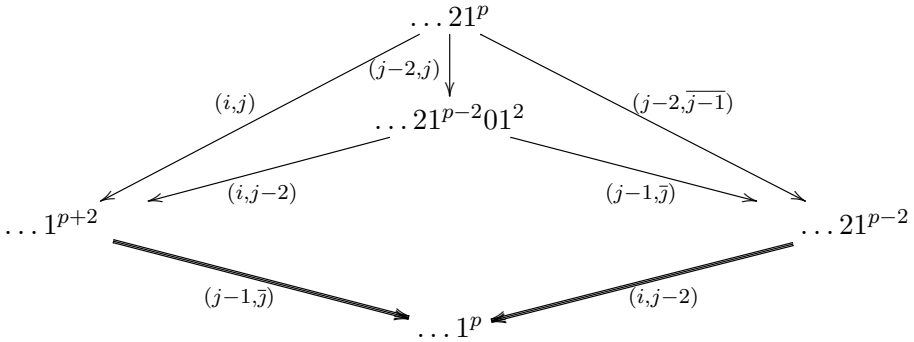
For instance, in order to prove Lemma 6.3 in Case C2, we let $\mathrm{f}_{i\overline{j}} := \mathrm{f}_{\alpha_{i\overline{j}}}$, and calculate:

$$\mathrm{f}_{i+2,\overline{i+3}}\mathrm{f}_{i+1,i+3}(\mathrm{f}_{i,i+1}(b)) = \mathrm{f}_{i+1,\overline{i+2}}\mathrm{f}_{i,i+1}(b) = \mathrm{f}_{i,i+1}\mathrm{f}_{i+1,\overline{i+2}}(b)\,;$$

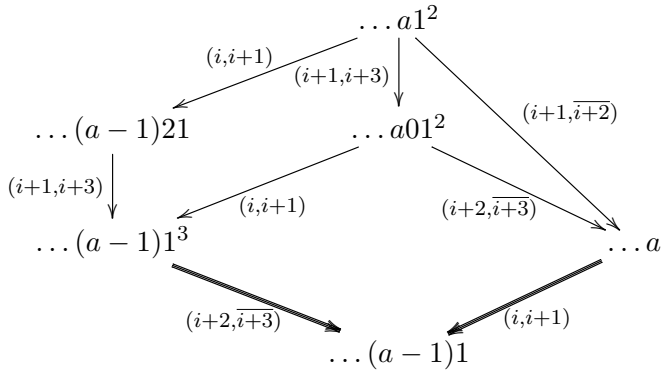
indeed, each of the two equalities follows from Theorem 5.3 (1) after we verify its hypothesis by inspecting the corresponding diagram.

We now turn to Lemma 6.4. Like in the proof of its type A version, we will perform the lift along the edges of slightly modified diagrams (by starting from the bottom) in the cases indicated below, while we use the same diagram as in Section 4.2 in Case C1′ (a). The reasoning is completely similar, based on repeatedly applying Theorems 5.5 (2) and 5.3 (1), after carefully verifying their hypotheses each time. Note that the special condition in Theorem 5.5 (2) requires $n > 4$ in type D_n, but this is clearly true in the stable range.

Case C1′ (b).

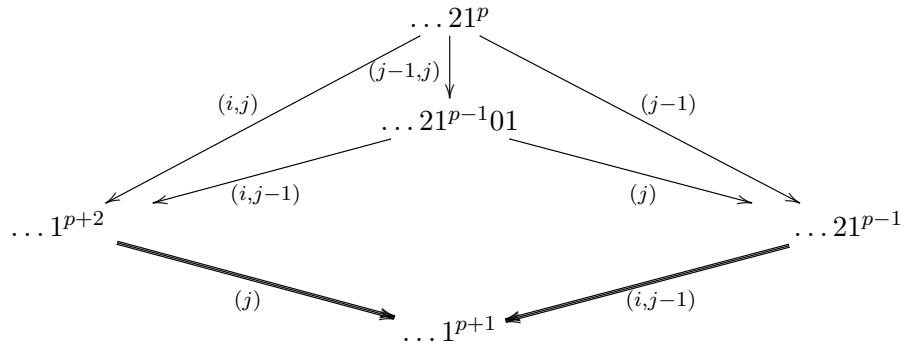


Case C2′.

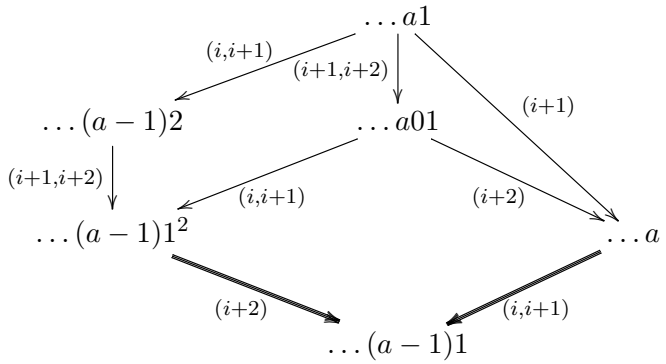


As for type B_n, the reasoning is completely similar, based on Theorem 5.21. To be more precise, for the f. operators, we use Cases B1−B3 in Section 4.4; for the e. operators, the two diagrams above (for Cases C1′ (b) and C2′) are replaced with the following ones below (for Cases B1′ (b) and B2′), respectively.

Case B1′ (b).



Case B2′.



□

Theorem 6.8. *The components of* $\mathbb{B}(\lambda)^+$ *define an atomic decomposition. Moreover, these components are isomorphic to intervals of the form* $[\widehat{0}, \mu]$ *in the dominant weight poset via the weight projection.*

Proof. We use the same reasoning as in the first part of the proof of Theorem 6.5 (the one referring to the atomic decomposition, as opposed to the t-atomic decomposition). The proof is based on Lemma 6.7 instead. □

Example 6.9. Consider $\lambda = (2, 1, 1)$ in type C_3. The modified crystal graph $\mathbb{B}(\lambda)^+$ is shown in Figure 4. Its vertices are labeled by Kashiwara-Nakashima tableaux of dominant weight, and its edges are labeled as above. In particular, this graph gives the following atomic decomposition of the character:

$$\chi_\lambda = w_{(2,1,1)} + 2w_{(1,1,0)} + w_{(0,0,0)}\,.$$

7. Additional facts and perspectives

7.1. **The t-atomic decomposition for the adjoint representation.** Let $\tilde{\alpha}$ be the highest root of the Lie algebra $\mathfrak{g}$. When the root system of $\mathfrak{g}$ is simply laced, $\tilde{\alpha}$ is the unique positive root which is also a dominant weight. Otherwise, there is another positive root $\hat{\alpha}$ which is dominant, and we have $0 \leq \hat{\alpha} \leq \tilde{\alpha}$. More precisely, both $\hat{\alpha}$ and $\tilde{\alpha} - \hat{\alpha}$ are short roots of R_+. In the crystal $B(\tilde{\alpha})$, there are vertices b_α of weight α, one for each root α of $\mathfrak{g}$ and r vertices of weight 0.

Lemma 7.1. *For any simple* α_i *and any index* $j \in \{1, \ldots, r\}$*, we have* $\tilde{f}_j(b_{\alpha_i}) \neq 0$ *if and only if* $i = j$*.*

Proof. Since $\alpha_i \neq -\tilde{\alpha}$ the lowest weight in $B(\tilde{\alpha})$, there is at least an index j such that $f_j(b_{\alpha_i}) \neq 0$. If $j \neq i$, the vertex $f_j(b_{\alpha_i})$ has weight $\alpha_i - \alpha_j = \alpha \in R$. When $\alpha \in R_+$ (resp. when $-\alpha \in R_+$), we get a contradiction because $\alpha_i = \alpha_j + \alpha$ is not simple (resp. $\alpha_j = \alpha_i + (-\alpha)$ is not simple). □

It follows from the lemma that the vertices $b_i = \tilde{f}_{\alpha_i}(b_{\alpha_i})$, $i = 1, \ldots, r$, are the zero weight vertices in $B(\tilde{\alpha})$. Recall also that

$$K_{\tilde{\alpha},0}(t) = \sum_{i=1}^{r} t^{m_i}\,,$$

where $m_1, \ldots, m_r$ are the classical exponents of $\mathfrak{g}$. We can choose $m_1 = |\tilde{\alpha}|$ since $|\tilde{\alpha}|$ is the greatest exponent.

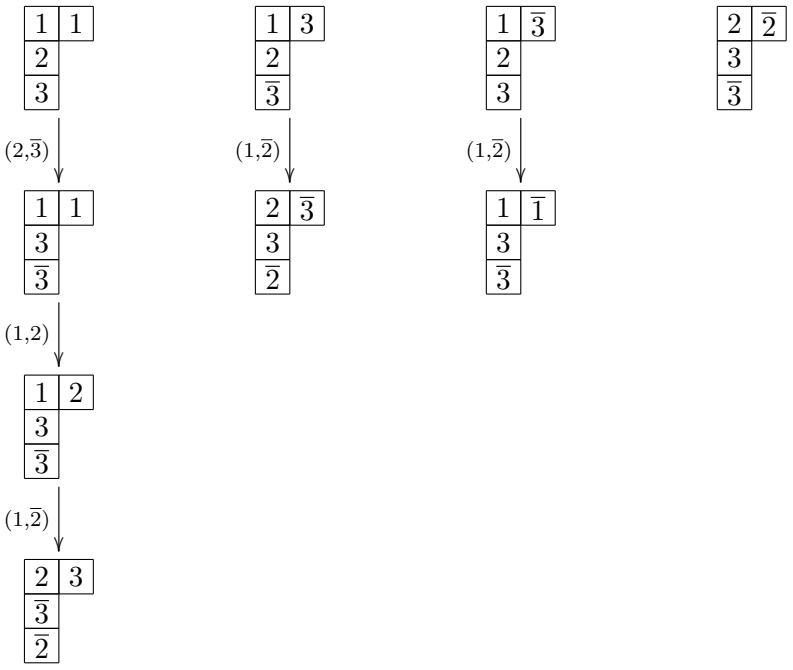


Figure 4: The modified crystal graph $\mathbb{B}(\lambda)^+$ in Example 6.9.

Assume first that α_1 is short, that is, the root system is not of type B_r or F_4. In the simply laced case, the highest root $\tilde{\alpha}$ is in the orbit of α_1 and, since $\langle\tilde{\alpha},\tilde{\alpha}\rangle > 0$, we derive by Lemma 5.2 that $\mathrm{f}_{\tilde{\alpha}}(\tilde{\alpha}) \neq 0$ is a vertex of zero weight in $B(\tilde{\alpha})$. In fact, the previous lemma also implies that $\mathrm{f}_{\tilde{\alpha}}(\tilde{\alpha}) = b_1$, because $\mathrm{f}_\alpha(b_i) = \mathrm{e}_\alpha(b_i) = 0$ for any $i \neq 1$ and any positive root α. Indeed, each vertex b_i has zero weight and thus is fixed under the action of the Weyl group, while $\tilde{f}_1(b_i) \neq 0$ if and only if $i = 1$. In the non-simply laced case (that is, in types C_r and G_2 under our assumption), we get similarly $\mathrm{f}_{\tilde{\alpha}-\hat{\alpha}}(\tilde{\alpha}) = b_{\hat{\alpha}}$ (because $\langle\tilde{\alpha},\tilde{\alpha}-\hat{\alpha}\rangle > 0$) and $\mathrm{f}_{\hat{\alpha}}(\hat{\alpha}) = b_1$. In all cases, the previous actions of the modified operators correspond to coverings in the dominant weight poset. Therefore, with the notation of Section 2.3, we have $H(\tilde{\alpha}) = \{b_{\tilde{\alpha}}, b_2, \ldots, b_r\}$, with $\mathbb{B}(\tilde{\alpha}, b_i) = \{b_i\}$ for any $i = 2, \ldots, r$, and

$$\mathbb{B}(\tilde{\alpha},\tilde{\alpha}) : \begin{cases} b_{\tilde{\alpha}} \overset{\tilde{\alpha}}{\dashrightarrow} b_1 \text{ in the simply laced cases} \\ b_{\tilde{\alpha}} \overset{\tilde{\alpha}-\hat{\alpha}}{\dashrightarrow} b_{\tilde{\alpha}} \overset{\hat{\alpha}}{\dashrightarrow} b_1 \text{ in types } C_r \text{ and } G_2\,. \end{cases}$$

Now define a statistic c on $H(\tilde{\alpha})$ such that $\mathrm{c}(b_{\tilde{\alpha}}) = 0$ and $\{\mathrm{c}(b_2), \ldots, \mathrm{c}(b_r)\} = \{m_2, \ldots, m_r\}$. We can then extend it to $B(\tilde{\alpha})^+$ by setting

$$\mathrm{c}(b_1) = \mathrm{c}(b_{\tilde{\alpha}}) + \langle\tilde{\alpha},\rho^\vee\rangle = |\tilde{\alpha}|\,, \quad \text{and} \quad \mathrm{c}(b_{\hat{\alpha}}) = \mathrm{c}(b_{\tilde{\alpha}}) + \langle\tilde{\alpha}-\hat{\alpha},\rho^\vee\rangle = |\tilde{\alpha}| - |\hat{\alpha}|\,.$$

We then get

$$A_{\tilde{\alpha},\tilde{\alpha}}(t) = \sum_{\substack{h\in H(\tilde{\alpha}) \\ \mathrm{wt}(h)=\tilde{\alpha}}} t^{\mathrm{c}(h)} = 1 \quad \text{and} \quad A_{\tilde{\alpha},0}(t) = \sum_{\substack{h\in H(\tilde{\alpha}) \\ \mathrm{wt}(h)=0}} t^{\mathrm{c}(h)} = t^{m_2} + \cdots + t^{m_r}\,.$$

For types C_r and G_2, we also have

$$A_{\tilde{\alpha},\hat{\alpha}}(t) = \sum_{\substack{h\in H(\tilde{\alpha}) \\ \mathrm{wt}(h)=\hat{\alpha}}} t^{\mathrm{c}(h)} = 0\,.$$

Finally, by Proposition 2.2, we get the desired t-atomic decomposition

$$K_{\tilde{\alpha},0}(t) = \begin{cases} t^0 A_{\tilde{\alpha},0}(t) + t^{|\tilde{\alpha}|} A_{\tilde{\alpha},\tilde{\alpha}}(t) \text{ in the simply laced case} \\ t^0 A_{\tilde{\alpha},0}(t) + t^{|\tilde{\alpha}|-|\hat{\alpha}|} A_{\tilde{\alpha},\hat{\alpha}}(t) + t^{|\tilde{\alpha}|} A_{\tilde{\alpha},\tilde{\alpha}}(t) \text{ for types } C_r \text{ and } G_2\,. \end{cases}$$

Now it remains to consider types B_r and F_4 where α_1 is a long root. Recall Stembridge's result [37] stating that any simple root gives a cover in dominant weight poset. As we have already seen for type B_r, we cannot define only the modified crystal operators based on $\tilde{f}_1$, because this would not permit to get the covering relations corresponding to short roots. We shall also need the modified crystal operators obtained by Weyl group conjugation of the ordinary crystal operator $\tilde{f}_r$ with α_r the short simple root. In fact, for the adjoint representation in type B_r and F_4, the coverings we need only make the short roots appear. So we only need, for any short root $\alpha \in R_+$, the modified operators $\mathrm{f}_\alpha = u_\alpha \tilde{f}_r u_\alpha^{-1}$, where $u_\alpha \in W$ is of minimal length such that $u(\alpha_r) = \alpha$. By using the same arguments as above, we then also get a t-atomic decomposition for $K_{\tilde{\alpha},0}(t)$, this time with $\mathbb{B}(\tilde{\alpha}, b_i) = \{b_i\}$ for any $i \neq n$ and

$$\mathbb{B}(\tilde{\alpha},\tilde{\alpha}) : b_{\tilde{\alpha}} \overset{\tilde{\alpha}-\hat{\alpha}}{\dashrightarrow} b_{\hat{\alpha}} \overset{\hat{\alpha}}{\dashrightarrow} b_n\,.$$

Remark 7.2. In simply laced cases, we expect that the modified crystal operators defined from $\tilde{f}_1$ suffice to derive an atomic decomposition of crystals with mild assumptions on the highest weight considered (as in Example 2.6). With similar restrictions and by considering also the modified crystal operators defined from $\tilde{f}_n$, this should also hold in the non simply laced cases even if new commutation relations will be probably needed in types F_4 and G_2.

7.2. **More about the charge.** In Section 2.3, we defined the t-atomic decomposition of $B(\lambda)$ from a charge statistic on $H(\lambda)$ which propagates on the vertices b of each component $\mathbb{B}(\lambda, h)$, $h \in H(\lambda)$, by the formula (21). When such an atomic decomposition exists, it yields a combinatorial description of $K_{\lambda,\mu}(t)$ by (24). In type A, Theorem 6.5 gives a t-atomic decomposition of crystals, where c is the Lascoux-Schützenberger charge statistic.

Conversely, if we fix $\nu \in P_+$ and assume that we have both a combinatorial description of the Kostka-Foulkes polynomials $K_{\lambda,\mu}(t)$ with $\lambda \leq \nu$ due to a statistic c defined on the crystals $B(\lambda)$, $\lambda \leq \nu$, and an atomic decomposition of each $B(\lambda)$ in which (21) holds, we will obtain a t-atomic decomposition of $B(\lambda)$ based on c, exactly as in the proof of Theorem 6.5.

Now assume that ν is a partition and $\mathfrak{g}$ is of type C_r, with r sufficiently large. In [19], a statistic was defined on Kashiwara-Nakashima tableaux which, conjecturally, gives a combinatorial description of the Kostka-Foulkes polynomials. The definition of this statistic is involved since it is based on the cyclage operation in the symplectic plactic monoid, which is much more complex than in type A. It is even challenging to decide if it (or one of its versions) could satisfy (21) when the atomic decomposition of $B(\lambda)$ based on the modified crystal operators is used. In [22], we also defined a statistic on King tableaux of zero weight, and proved that it yields a combinatorial description of the generalized exponents $K_{\lambda,0}(t)$. It is then tempting to try to connect this description with the charge defined in [19] by using the *Sheats bijection* [35] between King and Kashiwara-Nakashima tableaux, as the crystal structure is only known on the latter (an implementation of this bijection is available, see [7]). Unfortunately, it was established in [10] that this bijection does not intertwine the two statistics (a counterexample was found in type C_3 for $\lambda = (2,1,1)$). One can also imagine combining the Sheats bijection, the atomic decomposition of the crystal $B(\lambda)$ for λ a partition of even size, and (21), in order to define another statistic on the whole $B(\lambda)^+$. An interesting question would then arise: does this new statistic give a combinatorial description of the Lusztig t-analogue? If the answer is affirmative, we would get, in type C: (1) a generalization of the Lascoux-Schützenberger charge; (2) an efficient algorithm, based on crystal combinatorics, for calculating $K_{\lambda,\mu}(t)$ starting from $K_{\lambda,0}(t)$.

7.3. **Atomic decompositions for stable one-dimensional sums.** One-dimensional sums are q-analogues of tensor product multiplicities defined using the energy function on tensor products of *Kirillov-Reshetikhin (KR) crystals*. In affine type A, they are known to coincide with the (finite type A) Kostka polynomials when the affine KR crystals considered have row or column shapes. For the other classical affine types, one-dimensional sums corresponding to tensor products of row and column KR crystals admit stable versions which are special Kostka polynomials. Nevertheless, the two families do not coincide in general. We refer the interested reader to [23], where we establish, in any classical affine type, the atomic decomposition for the stable one-dimensional sums associated to tensor products of row and column KR crystals. As mentioned previously, we do expect that the atomic decomposition holds in full generality, for all the Kostka-Foulkes polynomials (up to mild assumptions on the rank of the root system considered), and not only when these polynomials coincide with one-dimensional sums.

8. Geometric interpretation

We give an interpretation of the combinatorial atomic decomposition in terms of the *geometric Satake correspondence*. For a reductive group G, this important theory exhibits a geometric realization of the irreducible representation $V(\lambda)$ of highest weight λ of the Langlands dual group, as the *intersection cohomology* $IH^*(\overline{Gr_\lambda})$ of the *Schubert variety* denoted $\overline{Gr_\lambda}$ in the *affine Grassmannian* Gr_G for G; there is also a geometric basis of *MV cycles* [28]. However, it is hard to give concrete formulas for the MV cycles and the action. We will show how one can understand the combinatorics of the geometric Satake correspondence via our combinatorial atomic decomposition.

The Schubert variety $\overline{Gr^\lambda}$ in Gr_G has a *Bott-Samelson desingularization* $\widehat{\Sigma} \to \overline{Gr^\lambda} \hookrightarrow Gr_G$. Thus, we have cohomology maps

$$H^*(Gr_G) \to H^*(\overline{Gr^\lambda}) \hookrightarrow H^*(\widehat{\Sigma}) \simeq IH^*(\overline{Gr^\lambda}) \oplus \text{other summands}\,. \tag{34}$$

The direct sum decomposition, as $H^*(Gr_G)$-modules, is given by the *Decomposition Theorem*, see e.g. [6].

$IH^*(\overline{Gr^\lambda})$ has the *truncation filtration* (or *standard Grothendieck filtration*), which gives the Kostka-Foulkes polynomials when restricted to the weight spaces [9]. The degree 0 piece in this filtration is the cohomology of the constant sheaf, i.e., $H^*(\overline{Gr^\lambda})$. This has the basis of classes of Schubert varieties inside $\overline{Gr^\lambda}$, which are indexed by the weights of $V(\lambda)$ considered without multiplicity, as recorded by the layer sum polynomials. In this language, the atomic decomposition decomposition in Definition 2.3 is expressing the fact that there is a refinement of the truncation filtration (with the $H^*(Gr_G)$-action), whose successive quotients are isomorphic to $H^*(\overline{Gr^\mu})$ for $\mu \in P^+(\lambda)$. These quotients correspond precisely to the blocks of the partition in the combinatorial atomic decomposition, see Definition 2.8.

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