ON THE COMBINATORICS OF CRYSTAL GRAPHS, I. LUSZTIG'S INVOLUTION

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Abstract. In this paper, we continue the development of a new combinatorial model for the irreducible characters of a complex semisimple Lie group. This model, which will be referred to as the alcove path model, can be viewed as a discrete counterpart to the Littelmann path model. It leads to an extensive generalization of the combinatorics of irreducible characters from Lie type A (where the combinatorics is based on Young tableaux, for instance) to arbitrary type; our approach is type-independent. The main results of this paper are: (1) a combinatorial description of the crystal graphs corresponding to the irreducible representations (this result includes a transparent proof, based on the Yang-Baxter equation, of the fact that the mentioned description does not depend on the choice involved in our model); (2) a combinatorial realization (which is the first direct generalization of *Schützenberger's involution* on tableaux) of Lusztig's involution on the canonical basis exhibiting the crystals as self-dual posets; (3) an analog for arbitrary root systems, based on the Yang-Baxter equation, of Schützenberger's sliding algorithm, which is also known as jeu de taquin (this algorithm has many applications to the representation theory of the Lie algebra of type A).

1. Introduction

We have recently given a simple combinatorial model for the irreducible characters of a complex semisimple Lie group G and, more generally, for the *Demazure characters* [22]. For reasons explained below, we call our model the *alcove path model*. This was extended to complex symmetrizable Kac-Moody algebras in [23] (that is, to infinite root systems). In this context, we also gave a *Littlewood-Richardson rule* for decomposing tensor products of irreducible representations and a branching rule. The exposition in [22] was in the context of the equivariant K-theory of the generalized flag variety G/B; more precisely, we first derived a *Chevalley-type multiplication formula* in $K_T(G/B)$, and then we deduced from it our Demazure character formula. By contrast, the exposition in [23] was purely representation theoretic, being based on Stembridge's combinatorial model for Weyl characters [33].

The alcove path model leads to an extensive generalization of the combinatorics of irreducible characters from Lie type A (where the combinatorics is based on Young tableaux, for instance) to arbitrary type; our approach is type-independent. The present paper continues the study of the combinatorics of the new model, which was started in [22, 23]. A future publication will be concerned with a direct generalization of the notion of the product of Young tableaux in the context of the product of crystals.

The main results of this paper are:

(1) a combinatorial description of the crystal graphs corresponding to the irreducible representations (Corollary 4.9); this result includes a transparent proof, based on the Yang-Baxter equation, of the fact that the mentioned description does not depend on the choice involved in our model (Corollary 4.8);
(2) a combinatorial realization of Lusztig's involution [30] on the canonical basis (Theorem 5.11, see also Example 5.13); this involution exhibits the crystals as self-dual posets, and corresponds to the action of the longest Weyl group element on an irreducible representation; our combinatorial

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realization is the first direct generalization of *Schützenberger's involution* on tableaux (see e.g. [8]);

(3) an analog for arbitrary root systems, based on the Yang-Baxter equation, of Schützenberger's *sliding* algorithm, which is also known as *jeu de taquin* (Section 4); this algorithm has many applications to the representation theory of the Lie algebra of type A (see e.g. [8]).

Our model is based on the choice of an *alcove path*, which is a sequence of adjacent alcoves for the *affine Weyl group* W_{aff} of the Langland's dual group $G^\vee$. An alcove path is best represented as a λ-*chain*, that is, as a sequence of positive roots corresponding to the common walls of successive alcoves in the mentioned sequence of alcoves. These chains extend the notion of a *reflection ordering* [7]. Given a fixed λ-chain, the objects that generalize semistandard Young tableaux are all the subsequences of roots that give rise to saturated increasing chains in Bruhat order (on the Weyl group W) upon multiplying on the right by the corresponding reflections. We call these subsequences *admissible subsets*. In [23] we defined *root operators* on admissible subsets, which are certain partial operators associated with the simple roots; in type A, they correspond to the *coplactic operations* on tableaux [28]. The root operators produce a directed colored graph structure and a poset structure on admissible subsets. We showed in [23] that this graph is isomorphic to the crystal graph of the corresponding irreducible representation if the chosen λ-chain is a special one. All this background information on the alcove path model is explained in more detail in Section 3, following some general background material discussed in Section 2.

In Section 4, we study certain discrete moves which allow us to deform any λ-chain into any other λ-chain (for a fixed dominant weight λ), and to biject the corresponding admissible subsets. We call these moves *Yang-Baxter moves* since they express the fact that certain operators satisfy the *Yang-Baxter equation*. We will explain below the reason for which the Yang-Baxter moves can be considered an analog of jeu de taquin for arbitrary root systems. We show that the Yang-Baxter moves commute with the root operators; this means that the directed colored graph defined by the root operators is invariant under Yang-Baxter moves, and it is thus independent from the choice of a λ-chain. Based on the special case in [23] discussed above, this immediately implies that the mentioned graph is isomorphic to the corresponding crystal graph for any choice of a λ-chain.

In Section 5, we present a combinatorial description of Lusztig's involution η_λ on the *canonical basis*. Such a description was given by Schützenberger in type A in terms of tableaux, and the corresponding procedure is known as *evacuation*. The importance of this involution stems from the fact that it exhibits the crystals as self-dual posets, and it corresponds to the action of the longest Weyl group element on an irreducible representation; it also appears in other contexts, such as the recent realization of the category of crystals as a *coboundary category* [10]. Our description of Lusztig's involution is very similar to that of the evacuation map. The main ingredient in defining the latter map, namely Schützenberger's sliding algorithm (also known as jeu de taquin), is replaced by Yang-Baxter moves. There is another ingredient, which has to do with "reversing" a λ-chain and an associated admissible subset, by analogy with reversing the word of a tableau in the definition of the evacuation map. Our construction also leads to a purely combinatorial proof of the fact that the crystals (as defined by our root operators) are self-dual posets. In Section 6, we present several applications; in particular, we give an intrinsic explanation for the fact that our procedure is an involution.

We will now briefly discuss the relationship between our model and other models for characters. We explained in [23] that our model can be viewed as a discrete counterpart to the *Littelmann path model* [24, 25, 26, 27], which is based on enumerating certain continuous paths in $\mathfrak{h}_{\mathbb{R}}^*$. These paths are constructed recursively, by starting with an initial one, and by applying certain root operators. By making specific choices for the initial path, one can obtain special cases which have more explicit descriptions. For instance, a straight line initial path leads to the *Lakshmibai-Seshadri paths* (LS paths) [16]; these were introduced before Littelmann's work, in the context of *standard monomial theory* [16]. A model closely related to Littelmann paths is the one due to Gaussent and Littelmann [9], which is based on *LS-galleries*. In [22, 23] we discussed in detail the relationship of the alcove path model with

Littelmann paths, LS paths, and LS-galleries. We explained the reasons for which the alcove path model is not simply a translation of the Littelmann path model into a different language. We also showed that our model has certain advantages due to its simplicity and combinatorial nature; it also compares favorably in terms of computational complexities (see also Subsection 3.3).

The results in this paper highlight new advantages of the alcove path model. For instance, we mentioned above our transparent combinatorial explanation, based on the Yang-Baxter moves, for the independence of the directed colored graph defined by the root operators from the choice of a λ-chain (Corollary 4.8). Similarly, it was proved in [25] that the directed colored graph structure on Littelmann paths generated by the corresponding root operators is independent of the initial path. However, this proof, which is based on continuous arguments, is less transparent.

As far as analogs of Schützenberger's jeu de taquin are concerned, let us first mention the extensions to types C_n, B_n, and D_n due to Lecouvey and Sheats [19, 20, 32]. Let us also note that the only such analog known in the Littelmann path model is the one due to van Leeuwen [21]. The goal of the mentioned paper was to use this analog in order to express in a bijective manner the symmetry of the Littlewood-Richardson rule in the Littelmann path model. In a future publication, we will show that van Leeuwen's jeu de taquin realizes precisely the commutator in the category of crystals that was defined in [10].

As far as our combinatorial realization of Lusztig's involution is concerned, let us note that the alcove path model reveals an interesting feature of it, which appears to be new. This feature is related to certain Weyl group elements associated with an admissible subset, which we call *initial* and *final keys* (see Definition 5.2 and Remark 5.3), and which are related to the Demazure character formula in Theorem 6.3. More precisely, Lusztig's involution interchanges the two keys in the sense mentioned in Corollary 6.2. Let us also note that no combinatorial realization of Lusztig's involution is available in the Littelmann path model. However, an explicit description of it is given in [31] in a different model for characters, which is based on *Lusztig's parametrization* and the *string parametrization* of the dual canonical basis [2]. Unlike the combinatorial approach in Schützenberger's evacuation procedure, the involution is now expressed as an affine map whose coefficients are entries of the corresponding Cartan matrix. No intrinsic explanation for the fact that this map is an involution is available.

We believe that the properties of our model that were investigated in [22, 23] as well as in this paper represent just a small fraction of a rich combinatorial structure yet to be explored, which would generalize most of the combinatorics of Young tableaux.

2. Preliminaries

We recall some background information on finite root systems, affine Weyl groups, Demazure characters, and crystal graphs.

2.1. **Root systems.** Let G be a connected, simply connected, simple complex Lie group. Fix a Borel subgroup B and a maximal torus T such that $G \supset B \supset T$. As usual, we denote by B^- be the opposite Borel subgroup, while N and N^- are the unipotent radicals of B and B^-, respectively. Let $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{n}$, and $\mathfrak{n}^-$ be the complex Lie algebras of G, T, N, and N^-, respectively. Let r be the rank of the Cartan subalgebra $\mathfrak{h}$. Let $\Phi \subset \mathfrak{h}^*$ be the corresponding irreducible *root system*, and let $\mathfrak{h}^*_{\mathbb{R}} \subset \mathfrak{h}^*$ be the real span of the roots. Let $\Phi^+ \subset \Phi$ be the set of positive roots corresponding to our choice of B. Then Φ is the disjoint union of Φ^+ and $\Phi^- := -\Phi^+$. We write $\alpha > 0$ (respectively, $\alpha < 0$) for $\alpha \in \Phi^+$ (respectively, $\alpha \in \Phi^-$), and we define $\mathrm{sgn}(\alpha)$ to be 1 (respectively -1). We also use the notation $|\alpha| := \mathrm{sgn}(\alpha)\alpha$. Let $\alpha_1, \ldots, \alpha_r \in \Phi^+$ be the corresponding *simple roots*, which form a basis of $\mathfrak{h}^*_{\mathbb{R}}$. Let $\langle \cdot , \cdot \rangle$ denote the nondegenerate scalar product on $\mathfrak{h}^*_{\mathbb{R}}$ induced by the Killing form. Given a root α, the corresponding *coroot* is $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$. The collection of coroots $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$ forms the *dual root system.*

The *Weyl group* $W \subset \mathrm{Aut}(\mathfrak{h}_{\mathbb{R}}^*)$ of the Lie group G is generated by the reflections $s_\alpha : \mathfrak{h}_{\mathbb{R}}^* \to \mathfrak{h}_{\mathbb{R}}^*$, for $\alpha \in \Phi$, given by

$$s_\alpha : \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \, \alpha.$$

In fact, the Weyl group W is generated by the *simple reflections* $s_1, \ldots, s_r$ corresponding to the simple roots $s_i := s_{\alpha_i}$, subject to the *Coxeter relations:*

$$(s_i)^2 = 1 \quad \text{and} \quad (s_i s_j)^{m_{ij}} = 1 \quad \text{for any } i, j \in \{1, \ldots, r\},$$

where m_{ij} is half of the order of the dihedral subgroup generated by s_i and s_j. An expression of a Weyl group element w as a product of generators $w = s_{i_1} \cdots s_{i_l}$ which has minimal length is called a *reduced decomposition* for w; its length $\ell(w) = l$ is called the *length* of w. The Weyl group contains a unique *longest element* $w_\circ$ with maximal length $\ell(w_\circ) = \#\Phi^+$. For $u, w \in W$, we say that u *covers* w, and write $u \gtrdot w$, if $w = u s_\beta$, for some $\beta \in \Phi^+$, and $\ell(u) = \ell(w) + 1$. The transitive closure "$>$" of the relation "$\gtrdot$" is called the *Bruhat order* on W.

The *weight lattice* Λ is given by

$$\Lambda := \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for any } \alpha \in \Phi\}. \tag{2.1}$$

The weight lattice Λ is generated by the *fundamental weights* $\omega_1, \ldots, \omega_r$, which are defined as the elements of the dual basis to the basis of simple coroots, i.e., $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. The set Λ^+ of *dominant weights* is given by

$$\Lambda^+ := \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for any } \alpha \in \Phi^+\}.$$

Let $\rho := \omega_1 + \cdots + \omega_r = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta$. The *height* of a coroot $\alpha^\vee \in \Phi^\vee$ is $\langle \rho, \alpha^\vee \rangle = c_1 + \cdots + c_r$ if $\alpha^\vee = c_1 \alpha_1^\vee + \cdots + c_r \alpha_r^\vee$. Since we assumed that Φ is irreducible, there is a unique *highest coroot* $\theta^\vee \in \Phi^\vee$ that has maximal height. (In other words, $\theta^\vee$ is the highest root of the dual root system $\Phi^\vee$. It should not be confused with the coroot of the highest root of Φ.) We will also use the *Coxeter number*, that can be defined as $h := \langle \rho, \theta^\vee \rangle + 1$.

2.2. **Affine Weyl groups.** In this subsection, we remind a few basic facts about affine Weyl groups and alcoves, cf. Humphreys [11, Chaper 4] for more details.

Let W_{aff} be the *affine Weyl group* for the Langland's dual group $G^\vee$. The affine Weyl group W_{aff} is generated by the affine reflections $s_{\alpha,k} : \mathfrak{h}_{\mathbb{R}}^* \to \mathfrak{h}_{\mathbb{R}}^*$, for $\alpha \in \Phi$ and $k \in \mathbb{Z}$, that reflect the space $\mathfrak{h}_{\mathbb{R}}^*$ with respect to the affine hyperplanes

$$H_{\alpha,k} := \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \lambda, \alpha^\vee \rangle = k\}. \tag{2.2}$$

Explicitly, the affine reflection $s_{\alpha,k}$ is given by

$$s_{\alpha,k} : \lambda \mapsto s_\alpha(\lambda) + k\,\alpha = \lambda - (\langle \lambda, \alpha^\vee \rangle - k)\,\alpha.$$

The hyperplanes $H_{\alpha,k}$ divide the real vector space $\mathfrak{h}_{\mathbb{R}}^*$ into open regions, called *alcoves.* Each alcove A is given by inequalities of the form

$$A := \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid m_\alpha < \langle \lambda, \alpha^\vee \rangle < m_\alpha + 1 \text{ for all } \alpha \in \Phi^+\},$$

where $m_\alpha = m_\alpha(A)$, $\alpha \in \Phi^+$, are some integers.

A proof of the following important property of the affine Weyl group can be found, e.g., in [11, Chapter 4].

Lemma 2.1. *The affine Weyl group W_{aff} acts simply transitively on the collection of all alcoves.*

The *fundamental alcove* $A_\circ$ is given by

$$A_\circ := \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid 0 < \langle \lambda, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Phi^+\}.$$

Lemma 2.1 implies that, for any alcove A, there exists a unique element v_A of the affine Weyl group W_{aff} such that $v_A(A_\circ) = A$. Hence the map $A \mapsto v_A$ is a one-to-one correspondence between alcoves and elements of the affine Weyl group.

Recall that $\theta^\vee \in \Phi^\vee$ is the highest coroot. Let $\theta \in \Phi^+$ be the corresponding root, and let $\alpha_0 := -\theta$. The fundamental alcove $A_\circ$ is, in fact, the simplex given by

$$A_\circ = \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid 0 < \langle \lambda, \alpha_i^\vee \rangle \text{ for } i = 1, \ldots, r, \text{ and } \langle \lambda, \theta^\vee \rangle < 1\}, \tag{2.3}$$

Lemma 2.1 also implies that the affine Weyl group is generated by the set of reflections $s_0, s_1, \ldots, s_r$ with respect to the walls of the fundamental alcove $A_\circ$, where $s_0 := s_{\alpha_0,-1}$ and $s_1, \ldots, s_r \in W$ are the simple reflections $s_i = s_{\alpha_i,0}$. Like the Weyl group, the affine Weyl group W_{aff} is a Coxeter group. As in the case of the Weyl group, a decomposition $v = s_{i_1} \cdots s_{i_l} \in W_{\text{aff}}$ is called *reduced* if it has minimal length; its length $\ell(v) = l$ is called the length of v.

We say that two alcoves A and B are *adjacent* if B is obtained by an affine reflection of A with respect to one of its walls. In other words, two alcoves are adjacent if they are distinct and have a common wall. For a pair of adjacent alcoves, let us write $A \xrightarrow{\beta} B$ if the common wall of A and B is of the form $H_{\beta,k}$ and the root $\beta \in \Phi$ points in the direction from A to B.

Let Z be the set of the elements of the lattice Λ/h that do not belong to any affine hyperplane $H_{\alpha,k}$ (recall that h is the Coxeter number). Each alcove A contains precisely one element ζ_A of the set Z (cf. [15, 22]); this will be called the *central point* of A. In particular, $\zeta_{A_\circ} = \rho/h$.

Proposition 2.2. [22] *For a pair of adjacent alcoves $A \xrightarrow{\alpha} B$, we have $\zeta_B - \zeta_A = \alpha/h$.*

2.3. **Demazure characters.** The *generalized flag variety* G/B is a smooth projective variety. It decomposes into a disjoint union of *Schubert cells* $X_w^\circ := BwB/B$ indexed by elements $w \in W$ of the Weyl group. The closures of Schubert cells $X_w := \overline{X_w^\circ}$ are called *Schubert varieties.* We have $u > w$ in the Bruhat order (defined above) if and only if $X_u \supset X_w$. Let $\mathcal{O}_{X_w}$ be the structure sheaf of the Schubert variety X_w. Let $\mathcal{L}_\lambda$ be the line bundle over G/B associated with the weight λ, that is, $\mathcal{L}_\lambda := G \times_B \mathbb{C}_{-\lambda}$, where B acts on G by right multiplication, and the B-action on $\mathbb{C}_{-\lambda} = \mathbb{C}$ corresponds to the character determined by $-\lambda$. (This character of T extends to B by defining it to be identically one on the commutator subgroup $[B, B]$.)

For a dominant weight $\lambda \in \Lambda^+$, let V_λ denote the finite dimensional irreducible representation of the Lie group G with highest weight λ. For $\lambda \in \Lambda^+$ and $w \in W$, the *Demazure module* $V_{\lambda,w}$ is the B-module that is dual to the space of global sections of the line bundle $\mathcal{L}_\lambda$ on the Schubert variety X_w:

$$V_{\lambda,w} := H^0(X_w, \mathcal{L}_\lambda)^*. \tag{2.4}$$

For the longest Weyl group element $w = w_\circ$, the space $V_{\lambda,w_\circ} = H^0(G/B, \mathcal{L}_\lambda)^*$ has the structure of a G-module. The classical *Borel-Weil theorem* says that $V_{\lambda,w_\circ}$ is isomorphic to the irreducible G-module V_λ.

Let $\mathbb{Z}[\Lambda]$ be the group algebra of the weight lattice Λ, which is isomorphic to the representation ring of T. The algebra $\mathbb{Z}[\Lambda]$ has a $\mathbb{Z}$-basis of formal exponents $\{e^\lambda \mid \lambda \in \Lambda\}$ with multiplication $e^\lambda \cdot e^\mu := e^{\lambda+\mu}$; in other words, $\mathbb{Z}[\Lambda] = \mathbb{Z}[e^{\pm\omega_1}, \cdots, e^{\pm\omega_r}]$ is the algebra of Laurent polynomials in r variables. The formal characters of the modules $V_{\lambda,w}$, called *Demazure characters*, are given by $ch(V_{\lambda,w}) = \sum_{\mu \in \Lambda} m_{\lambda,w}(\mu)\, e^\mu \in \mathbb{Z}[\Lambda]$, where $m_{\lambda,w}(\mu)$ is the multiplicity of the weight μ in $V_{\lambda,w}$. These characters generalize the characters of the irreducible representations $ch(V_\lambda) = ch(V_{\lambda,w_\circ})$. Demazure [5] gave a formula expressing the characters $ch(V_{\lambda,w})$ in terms of certain operators known as *Demazure operators.*

2.4. **Crystal graphs and Lusztig's involution.** Let $U(\mathfrak{g})$ be the universal enveloping algebra of the Lie algebra $\mathfrak{g}$. Let $\mathcal{B}$ be the *canonical basis* of $U(\mathfrak{n}^-)$, and let $\mathcal{B}_\lambda := \mathcal{B} \cap V_\lambda$ be the canonical basis of the irreducible representation V_λ with highest weight λ [12, 29, 30]. Let v_λ and v_λ^{low} be the highest and lowest weight vectors in $\mathcal{B}_\lambda$, respectively. Let $\widetilde{E}_i$, $\widetilde{F}_i$, for $i = 1, \ldots, r$, be Kashiwara's operators [12, 30]; these are also known as raising and lowering operators, respectively. The *crystal graph* of V_λ is the directed colored graph on $\mathcal{B}_\lambda$ defined by arrows $x \to y$ colored i for each $\widetilde{F}_i(x) = y$, or, equivalently, for each $\widetilde{E}_i(y) = x$. (In fact, Kashiwara introduced the notion of a crystal graph of an $U_q(\mathfrak{g})$-representation, where $U_q(\mathfrak{g})$ is the Drinfeld-Jimbo q-deformation of $U(\mathfrak{g})$, also known as a *quantum group*; using the quantum deformation, one can associate a crystal graph to a $\mathfrak{g}$-representation.) One can also define partial orders $\preceq_i$ on $\mathcal{B}_\lambda$ by

$$x \preceq_i y \quad \text{if} \quad x = \widetilde{F}_i^k(y) \text{ for some } k \geq 0\,.$$

We let $\preceq$ denote the partial order generated by all partial orders $\preceq_i$, for $i = 1, \ldots, r$. The poset $(\mathcal{B}_\lambda, \preceq)$ has maximum v_λ and minimum v_λ^{low}.

In order to proceed, we need the following general setup. Let V be a module over an associative algebra U and σ an automorphism of U. The twisted U-module V^σ is the same vector space V but with the new action $u * v := \sigma(u)v$ for $u \in U$ and $v \in V$. Clearly, $V^{\sigma\tau} = (V^\sigma)^\tau$ for every two automorphisms σ and τ of U. Furthermore, if V is a simple U-module, then so is V^σ. In particular, if $U = U(\mathfrak{g})$ and $V = V_\lambda$, then $(V_\lambda)^\sigma$ is isomorphic to $V_{\sigma(\lambda)}$ for some dominant weight $\sigma(\lambda)$. Thus there is an isomorphism of vector spaces $\sigma_\lambda \,:\, V_\lambda \to V_{\sigma(\lambda)}$ such that

$$\sigma_\lambda(uv) = \sigma(u)\sigma_\lambda(v)\,, \quad u \in U(\mathfrak{g})\,,\; v \in V_\lambda\,.$$

By Schur's lemma, σ_λ is unique up to a scalar multiple.

The longest Weyl group element $w_\circ$ defines an involution on the simple roots by $\alpha_i \mapsto \alpha_{i^*} := -w_\circ(\alpha_i)$. Consider the automorphisms of $U(\mathfrak{g})$ defined by

$$\phi(E_i) = F_i\,, \qquad \phi(F_i) = E_i\,, \qquad \phi(H_i) = -H_i\,, \tag{2.5}$$

$$\psi(E_i) = E_{i^*}\,, \qquad \psi(F_i) = F_{i^*}\,, \qquad \psi(H_i) = H_{i^*}\,, \tag{2.6}$$

and $\eta := \phi\psi$. Clearly, these three automorphisms together with the identity automorphism form a group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. It also easily follows from (2.5)-(2.6) that

$$\phi(\lambda) = \psi(\lambda) = -w_\circ(\lambda)\,, \quad \eta(\lambda) = \lambda\,.$$

We can normalize each of the maps ϕ_λ, ψ_λ, and η_λ by the requirement that

$$\phi_\lambda(v_\lambda) = v_{-w_\circ(\lambda)}^{low}\,, \quad \psi_\lambda(v_\lambda) = v_{-w_\circ(\lambda)}\,, \quad \eta_\lambda(v_\lambda) = v_\lambda^{low}\,. \tag{2.7}$$

(Of course, we also set Id_λ to be the identity map on V_λ.) By [30, Proposition 21.1.2], cf. also [1, Proposition 7.1], we have the following result.

Proposition 2.3. [1, 30] (1) *Each of the maps ϕ_λ and ψ_λ sends $\mathcal{B}_\lambda$ to $\mathcal{B}_{-w_\circ(\lambda)}$, while η_λ sends $\mathcal{B}_\lambda$ to itself.*

(2) *For every two (not necessarily distinct) elements σ, τ of the group $\{\mathrm{Id}, \phi, \psi, \eta\}$, we have $(\sigma\tau)_\lambda = \sigma_{\tau(\lambda)}\tau_\lambda$. In particular, the map η_λ is an involution.*

(3) *For every $i = 1, \ldots, r$, we have*

$$\phi_\lambda \widetilde{F}_i = \widetilde{E}_i \phi_\lambda\,, \quad \psi_\lambda \widetilde{F}_i = \widetilde{F}_{i^*} \psi_\lambda\,, \quad \eta_\lambda \widetilde{F}_i = \widetilde{E}_{i^*} \eta_\lambda\,. \tag{2.8}$$

In particular, the poset $(\mathcal{B}_\lambda, \preceq)$ is self-dual, and η_λ is the corresponding antiautomorphism.

Berenstein and Zelevinsky [1] showed that, in type A_{n-1} (that is, in the case of the Lie algebra $\mathfrak{sl}_n$), the operator η_λ is given by Schützenberger's *evacuation* procedure for semistandard Young tableaux (see e.g. [8]). More precisely, it is known that, for each partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq 0)$, the semistandard Young tableaux of shape λ and entries $1, \ldots, n$ parametrize the the canonical basis $\mathcal{B}_\lambda$ of V_λ. Hence, we can transfer the action of η_λ on $\mathcal{B}_\lambda$ to an action on the corresponding tableaux. As mentioned above, the latter action coincides with Schützenberger's evacuation map. One way to realize this map on a tableau T is the following three-step procedure.

(1) Rotate the tableau 180°, such that its row/column words get reversed.
(2) Complement the entries via the map $i \mapsto w_\circ(i) = n + 1 - i$, where $w_\circ$ is the longest element in the symmetric group S_n.

(3) Apply *jeu de taquin* to construct the *rectification* of the skew tableau obtained in the previous step, that is, successively apply Schützenberger's *sliding algorithm* for the inside corners of the mentioned tableau.

For convenience, we will call these steps: REVERSE, COMPLEMENT, SLIDE. They are illustrated in Figure 1 below.

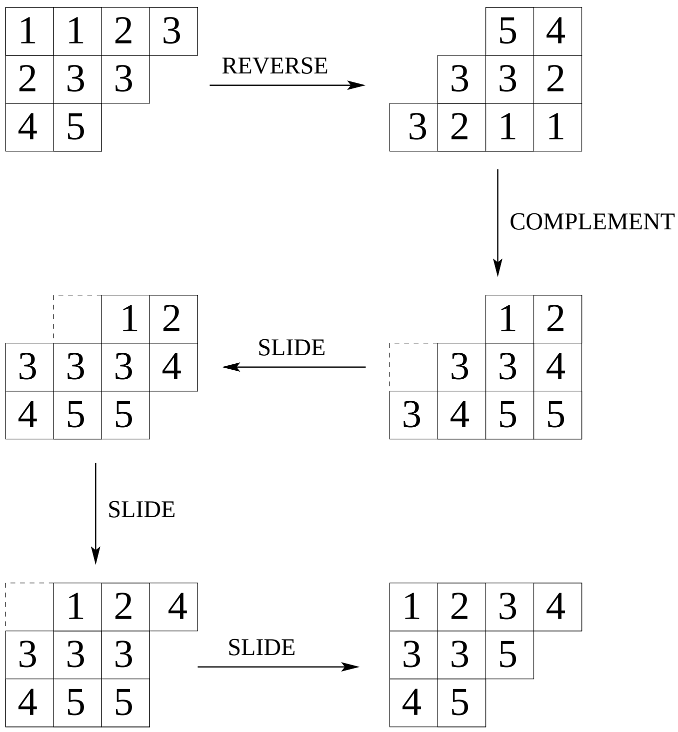

FIGURE 1. The evacuation map.

3. The Alcove Path Model

In this section, we recall the model for the irreducible characters of semisimple Lie algebras that we introduced in [22, 23]. We refer to these papers for more details, including the proofs of the results mentioned below. Although some of these results hold for infinite root systems (cf. [23]), the setup in this paper is that of a finite irreducible root system, as discussed in Section 2.

Our model is conveniently phrased in terms of several sequences, so let us mention some related notation. Given a totally ordered index set $I = \{i_1 < i_2 < \ldots < i_n\}$, a sequence $(a_{i_1}, a_{i_2}, \ldots, a_{i_n})$ is sometimes abbreviated to $\{a_j\}_{j \in I}$. We also let $[n] := \{1, 2, \ldots, n\}$.

3.1. **λ-chains.** The affine translations by weights preserve the set of affine hyperplanes $H_{\alpha,k}$, cf. (2.1) and (2.2). It follows that these affine translations map alcoves to alcoves. Let $A_\lambda := A_\circ + \lambda$ be the alcove obtained by the affine translation of the fundamental alcove $A_\circ$ by a weight $\lambda \in \Lambda$. Let v_λ be the corresponding element of W_{aff}, i.e,. v_λ is defined by $v_\lambda(A_\circ) = A_\lambda$. Note that the element v_λ may not be an affine translation itself.

Let us now fix a dominant weight λ. Let $v \mapsto \bar{v}$ be the homomorphism $W_{\mathrm{aff}} \to W$ defined by ignoring the affine translation. In other words, $\bar{s}_{\alpha,k} = s_\alpha \in W$.

Definition 3.1. A λ*-chain of roots* is a sequence of positive roots $(\beta_1, \ldots, \beta_n)$ which is determined as indicated below by a reduced decomposition $v_{-\lambda} = s_{i_1} \cdots s_{i_n}$ of $v_{-\lambda}$ as a product of generators of W_{aff}:

$$\beta_1 = \alpha_{i_1},\ \beta_2 = \bar{s}_{i_1}(\alpha_{i_2}),\ \beta_3 = \bar{s}_{i_1}\bar{s}_{i_2}(\alpha_{i_3}), \ldots,\ \beta_n = \bar{s}_{i_1} \cdots \bar{s}_{i_{n-1}}(\alpha_{i_n})\,.$$

When the context allows, we will abbreviate "λ-chain of roots" to "λ-chain". The λ*-chain of reflections* associated with the above λ-chain of roots is the sequence $(\widehat{r}_1, \ldots, \widehat{r}_n)$ of affine reflections in W_{aff} given by

$$\widehat{r}_1 = s_{i_1},\ \widehat{r}_2 = s_{i_1}s_{i_2}s_{i_1},\ \widehat{r}_3 = s_{i_1}s_{i_2}s_{i_3}s_{i_2}s_{i_1},\ \ldots,\ \widehat{r}_n = s_{i_1} \cdots s_{i_n} \cdots s_{i_1}.$$

We will present two equivalent definitions of a λ-chain of roots.

Definition 3.2. An *alcove path* is a sequence of alcoves $(A_0, A_1, \ldots, A_n)$ such that A_{i-1} and A_i are adjacent, for $i = 1, \ldots, n$. We say that an alcove path is *reduced* if it has minimal length among all alcove paths from A_0 to A_n.

Given a finite sequence of roots $\Gamma = (\beta_1, \ldots, \beta_n)$, we define the sequence of integers $(l_1^{\emptyset}, \ldots, l_n^{\emptyset})$ by $l_i^{\emptyset} := \#\{j < i \mid \beta_j = \beta_i\}$, for $i = 1, \ldots, n$. We also need the following two conditions on Γ.

- (R1) The number of occurrences of any positive root α in Γ is $\langle \lambda, \alpha^\vee \rangle$.
- (R2) For each triple of positive roots $(\alpha,\, \beta,\, \gamma)$ with $\gamma^\vee = \alpha^\vee + \beta^\vee$, the subsequence of Γ consisting of $\alpha,\, \beta,\, \gamma$ is a concatenation of pairs (α, γ) and (β, γ) (in any order).

Theorem 3.3. [22] *The following statements are equivalent.*

- (a) *The sequence of roots $\Gamma = (\beta_1, \ldots, \beta_n)$ is a λ-chain, and $(\widehat{r}_1, \ldots, \widehat{r}_n)$ is the associated λ-chain of reflections.*
- (b) *We have a reduced alcove path $A_0 \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_n} A_n$ from $A_0 = A_\circ$ to $A_n = A_{-\lambda}$, and $\widehat{r}_i$ is the affine reflection in the common wall of A_{i-1} and A_i, for $i = 1, \ldots, n$.*
- (c) *The sequence Γ satisfies conditions* (R1) *and* (R2) *above, and $\widehat{r}_i = s_{\beta_i, -l_i^{\emptyset}}$, for $i = 1, \ldots, n$.*

We now describe a particular choice of a λ-chain. First note that constructing a λ-chain amounts to defining a total order on the index set

$$I := \{(\alpha, k) \mid \alpha \in \Phi^+,\, 0 \le k < \langle \lambda, \alpha^\vee \rangle\}\,,$$

such that condition (R2) above holds, where the sequence $\Gamma = \{\beta_i\}_{i \in I}$ is defined by $\beta_i = \alpha$ for $i = (\alpha, k)$. Fix a total order on the set of simple roots $\alpha_1 < \alpha_2 < \ldots < \alpha_r$. For each $i = (\alpha, k)$ in I, let $\alpha^\vee = c_1\alpha_1^\vee + \ldots + c_r\alpha_r^\vee$, and define the vector

$$v_i := \frac{1}{\langle \lambda, \alpha^\vee \rangle}(k, c_1, \ldots, c_r)$$

in $\mathbb{Q}^{r+1}$. It turns out that the map $i \mapsto v_i$ is injective. Hence, we can define a total order on I by $i < j$ iff $v_i < v_j$ in the lexicographic order on $\mathbb{Q}^{r+1}$.

Proposition 3.4. [23] *Given the total order on I defined above, the sequence $\{\beta_i\}_{i \in I}$ defined by $\beta_i = \alpha$ for $i = (\alpha, k)$ is a λ-chain.*

3.2. **Admissible subsets.** For the remainder of this section, we fix a λ-chain $\Gamma = (\beta_1, \ldots, \beta_n)$. Let $r_i := s_{\beta_i}$. We now define the centerpiece of our combinatorial model for characters, which is our generalization of semistandard Young tableaux in type A.

Definition 3.5. An *admissible subset* is a subset of $[n]$ (possibly empty), that is, $J = \{j_1 < j_2 < \ldots < j_s\}$, such that we have the following saturated chain in the Bruhat order on W:

$$1 \lessdot r_{j_1} \lessdot r_{j_1}r_{j_2} \lessdot \ldots \lessdot r_{j_1}r_{j_2} \ldots r_{j_s}\,.$$

We denote by $\mathcal{A}(\Gamma)$ the collection of all admissible subsets corresponding to our fixed λ-chain Γ. Given an admissible subset J, we use the notation

$$\mu(J) := -\widehat{r}_{j_1} \ldots \widehat{r}_{j_s}(-\lambda)\,, \qquad w(J) := r_{j_1} \ldots r_{j_s}\,.$$

We call $\mu(J)$ the *weight* of the admissible subset J.

Theorem 3.6. [22, 23] (1) *We have the following character formula:*

$$ch(V_\lambda) = \sum_{J \in \mathcal{A}(\Gamma)} e^{\mu(J)}\,.$$

(2) *More generally, the following Demazure character formula holds for any $u \in W$:*

$$ch(V_{\lambda,u}) = \sum_J e^{-u\,\widehat{r}_{j_1} \cdots \widehat{r}_{j_s}(-\lambda)}\,,$$

where the summation is over all subsets $J = \{j_1 < \cdots < j_s\} \subseteq [n]$ such that

$$u \gtrdot u\,r_{j_1} \gtrdot u\,r_{j_1} r_{j_2} \gtrdot \cdots \gtrdot u\,r_{j_1} r_{j_2} \cdots r_{j_s}$$

is a saturated decreasing chain in the Bruhat order on the Weyl group W.

In addition to the above character formulas, a Littlewood-Richardson rule for decomposing tensor products of irreducible representations is also presented in terms of our model in [23].

Example 3.7. Consider the Lie algebra $\mathfrak{sl}_3$ of type A_2. The corresponding root system Φ can be realized inside the vector space $V := \mathbb{R}^3/\mathbb{R}(1,1,1)$ as $\Phi = \{\alpha_{ij} := \varepsilon_i - \varepsilon_j \mid i \neq j,\ 1 \leq i,j \leq 3\}$, where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in V$ are the images of the coordinate vectors in $\mathbb{R}^3$. The reflection $s_{\alpha_{ij}}$ is denoted by s_{ij}. The simple roots are α_{12} and α_{23}, while $\alpha_{13} = \alpha_{12} + \alpha_{23}$ is the other positive root. Let $\lambda = \omega_1 = \varepsilon_1$ be the first fundamental weight. In this case, there is only one λ-chain $(\beta_1, \beta_2) = (\alpha_{12}, \alpha_{13})$. There are 3 admissible subsets: $\emptyset, \{1\}, \{1,2\}$. The subset $\{2\}$ is not admissible because the reflection s_{13} does not cover the identity element. We have $(l_1^\emptyset, l_2^\emptyset) = (0,0)$. Theorem 3.6 gives the following expression for the character of V_{ω_1}:

$$ch(V_{\omega_1}) = e^{\omega_1} + e^{s_{12}(\omega_1)} + e^{s_{12}s_{13}(\omega_1)}.$$

In Subsections 3.4 and 3.5, we present two alternative ways of viewing admissible subsets, which are closely related to the equivalent definitions of λ-chains in Theorem 3.3 (b) and (c).

3.3. **Computational complexities.** In this subsection, we compare the computational complexity of our model with that of LS-paths constructed via root operators.

Fix a root system of rank r with N positive roots, a dominant weight λ, and a Weyl group element u of length l. We want to determine the character of the Demazure module $V_{\lambda,u}$. Let d be its dimension, and let L be the length of the affine Weyl group element $v_{-\lambda}$ (that is, the number of affine hyperplanes separating the fundamental alcove $A_\circ$ and $A_\circ - \lambda$). Note that $L = 2(\lambda, \rho^\vee)$, where $\rho^\vee = \frac{1}{2}\sum_{\beta \in \Phi^+} \beta^\vee$. We claim that the complexity of the character formula in Theorem 3.6 (2) is $O(d\,l L)$. Indeed, we start by determining an alcove path via the method underlying Proposition 3.4, which involves sorting a sequence of L rational numbers. The complexity is $O(L \log L)$, and note that $\log L$ is, in general, much smaller than d (see below for some examples). Whenever we examine some subword of the word of length L we fixed at the beginning, we have to check at most $L-1$ ways to add an extra reflection at the end. On the other hand, in each case, we have to check whether, upon multiplying by the corresponding nonaffine reflection, the length decreases by precisely 1. The complexity of the latter operation is $O(l)$, based on the Strong Exchange Condition [11, Theorem 5.8]. Then, for each "good" subword, we have to do a calculation, namely applying at most $2l$ affine reflections to $-\lambda$. In fact, it is fairly easy to implement this algorithm.

Now let us examine at the complexity of the algorithm based on root operators for constructing the LS-paths associated with λ. In other words, we are looking at the complexity of constructing the corresponding crystal graph. We have to generate the whole crystal graph first, and then figure out which paths give weights for the Demazure module. For each path, we can apply r root operators. Each path has at most N linear steps, so applying a root operator has complexity $O(N)$. But now we have to check whether the result is a path already determined, so we have to compare the obtained path with the other paths (that were already determined) of the same rank in the crystal graph (viewed as a ranked poset). This has complexity $O(NM)$, where M is the maximum number of elements of the same rank. Since we have at most $N+1$ ranks, M is at least $d/(N+1)$. In conclusion, the complexity is $O(drNM)$, which is at least $O(d^2 r)$.

Let us get a better picture of how the two results compare. Assume we are in a classical type, and let us first take λ to be the i-th fundamental weight, with i fixed, plus $u = w_\circ$. Clearly l is $O(r^2)$, L is $O(r)$, and d is $O(r^i)$, so the complexity of our formula is $O(r^{i+3})$. For LS-paths, we get at least $O(r^{2i+1})$. So the ratio between the complexity in the model based on LS-paths and our model is at least $O(r^{i-2})$.

Let us also take $\lambda = \rho$. In this case $d = 2^N$, and a simple calculation shows that L is $O(r^3)$. Our formula has complexity $O(2^N r^5)$, while the model based on LS-paths has complexity at least $O(2^{2N} r)$. So the ratio between the complexities is at least $O(2^N/r^4)$, where N is $r(r+1)/2$, r^2, and $r^2 - r$ in types A, B/C, and D, respectively.

3.4. **Galleries.**

Definition 3.8. A *gallery* is a sequence $\gamma = (F_0 = \{0\}, A_0 = A_\circ, F_1, A_1, F_2, \ldots, F_n, A_n, F_\infty = \{\mu\})$ such that $A_0, \ldots, A_n$ are alcoves; F_i is a codimension one common face of the alcoves A_{i-1} and A_i, for $i = 1, \ldots, n$; and F_∞ is a vertex of the last alcove A_n. The weight μ is called the *weight* of the gallery and is denoted by $\mu(\gamma)$. The folding operator ϕ_i is the operator which acts on a gallery by leaving its initial segment from A_0 to A_{i-1} intact and by reflecting the remaining tail in the affine hyperplane containing the face F_i. In other words, we define

$$\phi_i(\gamma) := (F_0, A_0, F_1, A_1, \ldots, A_{i-1}, F_i' = F_i, A_i', F_{i+1}', A_{i+1}', \ldots, A_n', F_\infty')\,;$$

here $A_j' := \widehat{t}_i(A_j)$ for $j \in \{i, \ldots, n\}$, $F_j' := \widehat{t}_i(F_j)$ for $j \in \{i, \ldots, n\} \cup \{\infty\}$, and $\widehat{t}_i$ is the affine reflection in the hyperplane containing F_i, as in Theorem 3.3.

The galleries defined above are special cases of the generalized galleries in [9].

Recall that our fixed λ-chain $\Gamma = (\beta_1, \ldots, \beta_n)$ determines a reduced alcove path $A_0 = A_\circ \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_n} A_n = A_{-\lambda}$. This alcove path determines, in turn, an obvious gallery

$$\gamma(\emptyset) = (F_0, A_0, F_1, \ldots, F_n, A_n, F_\infty)$$

of weight $-\lambda$.

Definition 3.9. Given a subset $J = \{j_1 < \cdots < j_s\} \subseteq [n]$, we associate with it the gallery $\gamma(J) := \phi_{j_1} \cdots \phi_{j_s}(\gamma(\emptyset))$. If J is an admissible subset, we call $\gamma(J)$ an *admissible gallery.*

Remarks 3.10. (1) The weight of the gallery $\gamma(J)$, i.e. $\mu(\gamma(J))$, is $-\mu(J)$.

(2) In order to define the gallery $\gamma(J)$, we augmented the index set $[n]$ corresponding to the fixed λ-chain by adding a new minimum 0 and a new maximum ∞. The same procedure is applied when the initial index set is an arbitrary (finite) totally ordered set.

3.5. **Chains of roots.**

Definition 3.11. A *chain of roots* is an object of the form

$$\Gamma = ((\gamma_1, \gamma_1'), \ldots, (\gamma_n, \gamma_n'), \gamma_\infty)\,, \tag{3.1}$$

where (γ_i, γ_i') are pairs of roots with $\gamma_i' = \pm\gamma_i$, for $i = 1, \ldots, n$, and γ_∞ is a weight. Given a chain of roots Γ and i in $[n]$, we let $t_i := s_{\gamma_i}$ and we define

$$\phi_i(\Gamma) := ((\delta_1, \delta_1'), \ldots, (\delta_n, \delta_n'), \delta_\infty))\,,$$

where $\delta_\infty := t_i(\gamma_\infty)$ and

$$(\delta_j, \delta'_j) := \begin{cases} (\gamma_j, \gamma'_j) & \text{if } j < i \\ (\gamma_j, t_i(\gamma'_j)) & \text{if } j = i \\ (t_i(\gamma_j), t_i(\gamma'_j)) & \text{if } j > i\,. \end{cases}$$

Our fixed λ-chain $\Gamma = (\beta_1, \ldots, \beta_n)$ determines the chain of roots

$$\Gamma(\emptyset) := ((\beta_1, \beta_1), \ldots, (\beta_n, \beta_n), \rho)\,;$$

recall that ρ was defined in Subsection 2.1.

Definition 3.12. Given a subset $J = \{j_1 < \cdots < j_s\} \subseteq [n]$, we associate with it the chain of roots $\Gamma(J) := \phi_{j_1} \cdots \phi_{j_s}(\Gamma(\emptyset))$. If J is an admissible subset, we call $\Gamma(J)$ an *admissible folding* (of $\Gamma(\emptyset)$).

Remark 3.13. We can also define folding operators on subsets J of $[n]$ by $\phi_i \,:\, J \mapsto J \triangle \{i\}$, where $\triangle$ denotes the symmetric difference of sets. The folding operators ϕ_i on J, $\gamma(J)$, and $\Gamma(J)$ are commuting involutions (for $J \subseteq [n]$), and their actions are compatible. Throughout this paper, we use J, $\gamma(J)$, and $\Gamma(J)$ interchangeably. We will call the elements of J the *folding positions* in $\gamma(J)$ and $\Gamma(J)$.

Given a fixed subset J of $[n]$, we will now discuss the relationship between the gallery $\gamma(J)$ and the chain of roots $\Gamma(J)$.

Let $\gamma = (F_0, A_0, F_1, \ldots, F_n, A_n, F_\infty)$ be an arbitrary gallery. Let $\widehat{t}_i$ be the affine reflection in the common wall of A_{i-1} and A_i, as usual. We associate with γ a chain of roots $\Gamma(\gamma) = ((\gamma_1, \gamma'_1), \ldots, (\gamma_n, \gamma'_n), \gamma_\infty)$ as follows:

$$\gamma_i := h(\zeta_{A_{i-1}} - \zeta_{\widehat{t}_i(A_{i-1})})\,, \quad \gamma'_i := h(\zeta_{\widehat{t}_i(A_i)} - \zeta_{A_i})\,, \quad \gamma_\infty := h(\zeta_{A_n} - \mu(\gamma))\,; \tag{3.2}$$

here h is the Coxeter number, $i = 1, \ldots, n$, and ζ_A is the central point of the alcove A, as defined in Subsection 2.2. By Proposition 2.2, we have

$$\widehat{t}_i(A_{i-1}) \xrightarrow{\gamma_i} A_{i-1}\,, \qquad A_i \xrightarrow{\gamma'_i} \widehat{t}_i(A_i)\,. \tag{3.3}$$

On the one hand, $\Gamma(\gamma)$ uniquely determines the gallery γ. On the other hand, we have $\Gamma(J) = \Gamma(\gamma(J))$.

Remark 3.14. In [23], we also associated with an admissible subset J a certain piecewise-linear path. This is closely related to $\gamma(J)$ and $\Gamma(J)$; essentially, it is obtained from the path joining the central points of the alcoves in the gallery $\gamma(\emptyset)$ via the folding operators used to construct $\gamma(J)$ from $\gamma(\emptyset)$. However, this path is *not* a Littelmann path in general.

3.6. **Combinatorial properties.** Let J be a fixed admissible subset, and let

$$\gamma(J) = (F_0, A_0, F_1, \ldots, F_n, A_n, F_\infty)\,, \quad \Gamma(J) = ((\gamma_1, \gamma'_1), \ldots, (\gamma_n, \gamma'_n), \gamma_\infty)\,.$$

Let us also fix a simple root α_p. We associate with J the sequence of integers $L(J) = (l_1, \ldots, l_n)$ defined by $F_i \subset H_{-|\gamma_i|, l_i}$ for $i = 1, \ldots, n$. Note that $L(\emptyset) = (l_1^\emptyset, \ldots, l_n^\emptyset)$, as defined in Subsection 3.1. We also define $l_\infty^p := \langle \mu(J), \alpha_p^\vee \rangle$, which means that $F_\infty \subset H_{-\alpha_p, l_\infty^p}$. Finally, we let

$$I(J,p) := \{i \in [n] \mid \gamma_i = \pm\alpha_p\}\,, \quad L(J,p) := (\{l_i\}_{i \in I(J,p)}, l_\infty^p)\,, \quad M(J,p) := \max L(J,p)\,. \tag{3.4}$$

It turns out that $M(J,p) \geq 0$.

Let $I(J,p) = \{i_1 < i_2 < \ldots < i_m\}$. We associate with J and p the sequence $\Sigma(J,p) = (\sigma_1, \ldots, \sigma_{m+1})$, where $\sigma_j := (\mathrm{sgn}(\gamma_{i_j}), \mathrm{sgn}(\gamma'_{i_j}))$ for $j = 1, \ldots, m$, and $\sigma_{m+1} := \mathrm{sgn}(\langle \gamma_\infty, \alpha_p^\vee \rangle)$. We now present some properties of the sequence $\Sigma(J,p)$, which will be used later, and which reflect the combinatorics of admissible subsets, as discussed in [23].

Proposition 3.15. [23] *The sequence $\Sigma(J,p)$ has the following properties:*

(S1) $\sigma_j \in \{(1,1), (-1,-1), (1,-1)\}$ *for* $j = 1, \ldots, m$*;*
(S2) $j = 0$ *or* $\sigma_j = (1,1)$ *implies* $\sigma_{j+1} \in \{(1,1), (1,-1), 1\}$.

The sequence $\Sigma(J,p)$ determines a continuous piecewise-linear function $g_{J,p}\,:\,[0,m+\frac{1}{2}]\rightarrow\mathbb{R}$ as shown below. By a step (h,k) of a function f at $x=a$, we understand that $f(a+h)=f(a)+k$, and that f is linear between a and $a+h$. We set $g_{J,p}(0)=-\frac{1}{2}$ and, by scanning $\Sigma(J,p)$ from left to right while ignoring brackets, we impose the following condition: the ith entry ± 1 corresponds to a step $(\frac{1}{2},\pm\frac{1}{2})$ of $g_{J,p}$ at $x=\frac{i-1}{2}$, respectively.

Proposition 3.16. [23] *The function $g_{J,p}$ encodes the sequence $L(J,p)$ as follows:*

$$l_{i_j}=g_{J,p}\left(j-\frac{1}{2}\right)\,,\;j=1,\ldots,m\,,\quad \textit{and}\quad l_{\infty}^{p}=g_{J,p}\left(m+\frac{1}{2}\right)\,.$$

Example 3.17. Assume that the entries of $\Gamma(J)$ indexed by the elements of $I(J,p)$ are $(\alpha_p,-\alpha_p)$, $(-\alpha_p,-\alpha_p)$, (α_p,α_p), (α_p,α_p), $(\alpha_p,-\alpha_p)$, $(-\alpha_p,-\alpha_p)$, $(\alpha_p,-\alpha_p)$, (α_p,α_p), in this order; also assume that $\operatorname{sgn}(\langle\gamma_\infty,\alpha_p^\vee\rangle)=1$. The graph of $g_{J,p}$ is shown in Figure 2; this graph is separated into segments corresponding to the entries of the sequence $\Sigma(J,p)$.

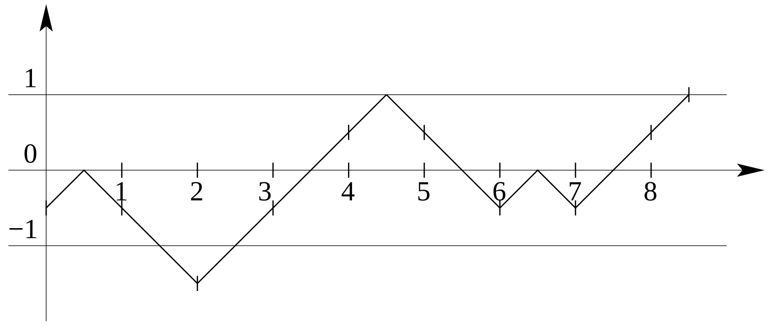

FIGURE 2. The graph of the function $g_{J,p}$ in Example 3.17.

3.7. **Root operators.** We now define partial operators known as *root operators* on the collection $\mathcal{A}(\Gamma)$ of admissible subsets corresponding to our fixed λ-chain. They are associated with a fixed simple root α_p, and are traditionally denoted by F_p (also called a lowering operator) and E_p (also called a raising operator). The notation is the one introduced in the previous subsection.

We first consider F_p on the admissible subset J. This is defined whenever $M(J,p)>0$. Let $m=m_F(J,p)$ be defined by

$$m_F(J,p):=\begin{cases}\min\{i\in I(J,p)\mid l_i=M(J,p)\} & \text{if this set is nonempty}\\ \infty & \text{otherwise}\,.\end{cases}$$

Let $k=k_F(J,p)$ be the predecessor of m in $I(J,p)\cup\{\infty\}$, which always exists. It turns out that $m\in J$ if $m\neq\infty$, but $k\notin J$ (cf. Proposition 3.18 below). Finally, we set

$$F_p(J):=(J\setminus\{m\})\cup\{k\}\,. \tag{3.5}$$

Proposition 3.18. [23] *Given the above setup, the following hold.*

(1) *If $m\neq\infty$, then $\gamma'_m=-\gamma_m=-\alpha_p$. We also have $\gamma_k=\gamma'_k=\alpha_p$ and $l_k=M(J,p)-1$.*
(2) *We have $\mu(F_p(J))=\mu(J)-\alpha_p$.*
(3) *We have $w(F_p(J))=w(J)$ if $m\neq\infty$, and $w(F_p(J))=s_pw(J)$ otherwise.*

Let us now define a partial inverse E_p to F_p. The operator E_p is defined on the admissible subset J whenever $M(J,p)>\langle\mu(J),\alpha_p^\vee\rangle$. Let $k=k_E(J,p)$ be defined by

$$k_E(J,p):=\max\{i\in I(J,p)\mid l_i=M(J,p)\}\,;$$

the above set turns out to be always nonempty. Let $m = m_E(J,p)$ be the successor of k in $I(J,p) \cup \{\infty\}$. It turns out that $k \in J$ but $m \notin J$ (cf. Proposition 3.19 below). Finally, we set

$$E_p(J) := (J \setminus \{k\}) \cup (\{m\} \setminus \{\infty\}) . \tag{3.6}$$

Proposition 3.19. [23] *Given the above setup, the following hold.*

1. *We have $\gamma'_k = -\gamma_k = -\alpha_p$. If $m \neq \infty$, then $\gamma_m = \gamma'_m = -\alpha_p$, and $l_m = M(J,p) - 1$.*
2. *We have $\mu(E_p(J)) = \mu(J) + \alpha_p$.*
3. *We have $w(E_p(J)) = w(J)$ if $m \neq \infty$, and $w(E_p(J)) = s_p w(J)$ otherwise.*

Similarly to Kashiwara's operators (see Subsection 2.4), the root operators above define a directed colored graph structure and a poset structure on the set $\mathcal{A}(\Gamma)$ of admissible subsets corresponding to a fixed λ-chain Γ. According to [23, Proposition 6.9]), the admissible subset $J_{\max} = \emptyset$ is the maximum of the poset $\mathcal{A}(\Gamma)$. The following result related to the special λ-chain in Proposition 3.4, which we denote by Γ^*, was proved in [23].

Theorem 3.20. [23] *The directed colored graph on the set $\mathcal{A}(\Gamma^*)$ defined by the root operators is isomorphic to the crystal graph of the irreducible representation V_λ with highest weight λ. Under this isomorphism, the weight of an admissible subset gives the weight space in which the corresponding element of the canonical basis lies.*

4. Yang-Baxter Moves

In this section, we define the analog of Schützenberger's sliding algorithm in our model, which we call a *Yang-Baxter move*, for reasons explained below. We start with some results on dihedral subgroups of Weyl groups.

4.1. **Dihedral reflection subgroups.** Let $\overline{W}$ be a dihedral Weyl group of order $2q$, that is, a Weyl group of type $A_1 \times A_1$, A_2, B_2, or G_2 (with $q = 2, 3, 4, 6$, respectively). Let $\overline{\Phi}$ be the corresponding root system with simple roots α, β. The sequence

$$\beta_1 := \alpha,\ \ \beta_2 := s_\alpha(\beta),\ \ \beta_3 := s_\alpha s_\beta(\alpha),\ \ \ldots,\ \ \beta_{q-1} := s_\beta(\alpha),\ \ \beta_q := \beta \tag{4.1}$$

is a *reflection ordering* on the positive roots of $\overline{\Phi}$ (cf. [7]). The following Lemma describes the structure of $\overline{W}$ and its action on $\overline{\Phi}$. As an illustration, we present the Bruhat order on the Weyl group of type G_2 in Figure 3. Here, as well as throughout this paper, we label a cover $v \lessdot v s_\gamma$ in Bruhat order by the corresponding root γ.

Lemma 4.1. (1) *If $i \leq \frac{q+1}{2}$, then the reflection s_{β_i} sends the roots $\beta_1, \ldots, \beta_{i-1}$ to $-\beta_{2i-1}, \ldots, -\beta_{i+1}$, and the roots $\beta_{2i}, \ldots, \beta_q$ to $\beta_q, \ldots, \beta_{2i}$, respectively. If $i > \frac{q+1}{2}$, then the reflection s_{β_i} sends the roots $\beta_{i+1}, \ldots, \beta_q$ to $-\beta_{i-1}, \ldots, -\beta_{2i-q}$, and the roots $\beta_1, \ldots, \beta_{2i-q-1}$ to $\beta_{2i-q-1}, \ldots, \beta_1$, respectively.*

(2) *Given $\overline{v} \in \overline{W}$ with $a := \ell(\overline{v}) < q$, consider its covers in Bruhat order by defining $\overline{\Phi}(\overline{v}) := \{j \in [q] \mid \ell(\overline{v} s_{\beta_j}) = \ell(\overline{v}) + 1\}$. We have*

$$\overline{\Phi}(\overline{v}) = \begin{cases} \{1, q-a\} & \text{if } \overline{v} \text{ has reduced decomposition } \ldots s_\alpha s_\beta \\ \{a+1, q\} & \text{if } \overline{v} \text{ has reduced decomposition } \ldots s_\beta s_\alpha . \end{cases}$$

With every pair of Weyl group elements $\overline{u} < \overline{w}$ in Bruhat order, we will associate a subset $J(\overline{u}, \overline{w})$ of $[q]$ as follows. Let $a := \ell(\overline{u})$ and $b := \ell(\overline{w})$. Given $\delta \in \{\alpha, \beta\}$, we will use the notation

$$\overline{W}_\delta := \{\overline{v} \in \overline{W} \mid \ell(\overline{v} s_\delta) > \ell(\overline{v})\}, \quad \overline{W}^\delta := \overline{W} \setminus \overline{W}_\delta = \{\overline{v} \in \overline{W} \mid \ell(\overline{v} s_\delta) < \ell(\overline{v})\} .$$

Case 0: $\overline{u} = \overline{w}$. We let $J(\overline{u}, \overline{u}) := \emptyset$.
Case 1: $b - a = 1$. We have the following disjoint subcases.
Case 1.1: $\overline{u} \in \overline{W}_\alpha$, $\overline{w} \in \overline{W}^\alpha$, so $0 \leq a \leq q-1$. We let $J(\overline{u}, \overline{w}) := \{1\}$.

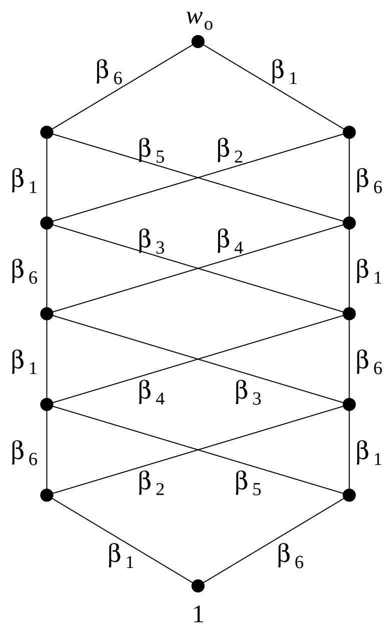

FIGURE 3. The Bruhat order on the Weyl group of type G_2.

Case 1.2: $\overline{u} \in \overline{W}^\beta$, $\overline{w} \in \overline{W}_\alpha$, so $0 < a < q-1$. We let $J(\overline{u}, \overline{w}) := \{q-a\}$.
Case 1.3: $\overline{u} \in \overline{W}_\beta$, $\overline{w} \in \overline{W}^\beta$, so $0 \le a \le q-1$. We let $J(\overline{u}, \overline{w}) := \{q\}$.
Case 1.4: $\overline{u} \in \overline{W}^\alpha$, $\overline{w} \in \overline{W}_\beta$, so $0 < a < q-1$. We let $J(\overline{u}, \overline{w}) := \{a+1\}$.

Case 2: $1 < b-a < q$. We have the following disjoint subcases.
Case 2.1: $\overline{u} \in \overline{W}_\alpha$, $\overline{w} \in \overline{W}_\beta$, so $0 \le a < a+2 \le b < q$.
We let $J(\overline{u}, \overline{w}) := \{1, a+2, a+3, \ldots, b\}$.
Case 2.2: $\overline{u} \in \overline{W}^\beta$, $\overline{w} \in \overline{W}^\beta$, so $0 < a < a+2 \le b \le q$.
We let $J(\overline{u}, \overline{w}) := \{1, a+2, a+3, \ldots, b-1, q\}$.
Case 2.3: $\overline{u} \in \overline{W}_\beta$, $\overline{w} \in \overline{W}_\alpha$, so $0 \le a < a+2 \le b < q$.
We let $J(\overline{u}, \overline{w}) := \{a+1, a+2, \ldots, b-1, q\}$.
Case 2.4: $\overline{u} \in \overline{W}^\alpha$, $\overline{w} \in \overline{W}^\alpha$, so $0 < a < a+2 \le b \le q$.
We let $J(\overline{u}, \overline{w}) := \{a+1, a+2, \ldots, b\}$.

Case 3: $a = 0$ and $b = q$, that is, $\overline{u}$ is the identity and $\overline{w}$ is the longest Weyl group element $\overline{w}_\circ$. In this case, we let $J := [q]$.

In Case 2.2, if $b = a+2$ then the sequence $a+2, a+3, \ldots, b-1$ is considered empty.

Let $J(\overline{u}, \overline{w}) := \{j_1 < j_2 < \ldots < j_{b-a}\}$. We use the notation $r_i := s_{\beta_i}$, as above. In all cases above we have a unique saturated increasing chain in Bruhat order from $\overline{u}$ to $\overline{w}$ whose labels form a subsequence of (4.1); this chain is

$$\overline{u} \lessdot \overline{u} r_{j_1} \lessdot \overline{u} r_{j_1} r_{j_2} \lessdot \ldots \lessdot \overline{u} r_{j_1} \ldots r_{j_{b-a}} = \overline{w}\,.$$

Indeed, this can be easily checked based on Lemma 4.1 (2). More generally, we have the result below for an arbitrary Weyl group W with a dihedral reflection subgroup $\overline{W}$ and corresponding root systems $\Phi \supseteq \overline{\Phi}$. The notation is the same as above. It is known that any element w of W can be written uniquely as $w = \lfloor w \rfloor \overline{w}$, where $\lfloor w \rfloor$ is the minimal representative of the left coset $w\overline{W}$, and $\overline{w} \in \overline{W}$.

Proposition 4.2. *For each pair of elements $u < w$ in the same (left) coset of W modulo $\overline{W}$, we have a unique saturated increasing chain in Bruhat order from u to w whose labels form a subsequence of* (4.1)*;*

this chain is

$$u \lessdot ur_{j_1} \lessdot ur_{j_1}r_{j_2} \lessdot \ldots \lessdot ur_{j_1}\ldots r_{j_{b-a}} = w\,,$$

where $J(\overline{u},\overline{w}) := \{j_1 < j_2 < \ldots < j_{b-a}\}$.

This result can be easily deduced from the corresponding one for $W = \overline{W}$ via the following Lemma about cosets modulo dihedral reflection subgroups, which was discussed in [3].

Lemma 4.3. [3] *The Bruhat order on $\overline{W}$ (viewed as a Coxeter group with generators s_α and s_β) is isomorphic to the partial order on any coset $w\overline{W}$ (induced from the Bruhat order on W). The isomorphism is given by the map $\overline{w} \mapsto \lfloor w \rfloor \overline{w}$. This statement can be rephrased by saying that, for any $\overline{w} \in \overline{W}$ and $\gamma \in \overline{\Phi}$, we have $\overline{w} < \overline{w}s_\gamma$ if and only if $\lfloor w \rfloor \overline{w} < \lfloor w \rfloor \overline{w} s_\gamma$.*

We obtain another reflection ordering by reversing the sequence (4.1). Let us denote the corresponding subset of $[q]$ by $J'(\overline{u},\overline{w})$. We are interested in passing from the chain between u and w compatible with the ordering (4.1) to the chain compatible with the reverse ordering. If we fix $a := \ell(\overline{u})$ and $b := \ell(\overline{w})$, we can realize the passage from $J(\overline{u},\overline{w})$ to $J'(\overline{u},\overline{w})$ via the involution $Y_{q,a,b}$ described below in each of the cases mentioned above.

Case 0: $\emptyset \leftrightarrow \emptyset$ if $a = b$.

Case 1.1: $\{1\} \leftrightarrow \{q\}$ if $0 \le a = b-1 \le q-1$.

Case 1.2: $\{q-a\} \leftrightarrow \{a+1\}$ if $0 < a = b-1 < q-1$.

Case 2.1: $\{1, a+2, a+3, \ldots, b\} \leftrightarrow \{a+1, a+2, \ldots, b-1, q\}$ if $0 \le a < a+2 \le b < q$.

Case 2.2: $\{1, a+2, a+3, \ldots, b-1, q\} \leftrightarrow \{a+1, a+2, \ldots, b\}$ if $0 < a < a+2 \le b \le q$.

Case 3: $[q] \leftrightarrow [q]$ if $a = 0$ and $b = q$.

4.2. **Yang-Baxter moves and their properties.** Let us now consider an index set

$$I := \{\overline{1} < \ldots < \overline{t} < 1 < \ldots < q < \overline{t+1} < \ldots < \overline{n}\}\,, \tag{4.2}$$

and let $\overline{I} := \{\overline{1}, \ldots, \overline{n}\}$. Let $\Gamma = \{\beta_i\}_{i\in I}$ be a λ-chain, denote $r_i := s_{\beta_i}$ as before, and let $\Gamma' = \{\beta'_i\}_{i\in I}$ be the sequence of roots defined by

$$\beta'_i = \begin{cases} \beta_{q+1-i} & \text{if } i \in I \setminus \overline{I} \\ \beta_i & \text{otherwise}\,. \end{cases} \tag{4.3}$$

In other words, the sequence Γ' is obtained from the λ-chain Γ by reversing a certain segment. Now assume that $\{\beta_1, \ldots, \beta_q\}$ are the positive roots of a rank two root system $\overline{\Phi}$ (without repetition). Let $\overline{W}$ be the corresponding dihedral reflection subgroup of the Weyl group W. The following result is easily proved using the correspondence between λ-chains and reduced words for the affine Weyl group element $v_{-\lambda}$ mentioned in Definition 3.1; most importantly, we need to recall from the proof of [22, Lemma 9.3] that the moves $\Gamma \to \Gamma'$ correspond to Coxeter moves (on the mentioned reduced words) in this context.

Proposition 4.4. (1) *The sequence Γ' is also a λ-chain, and the sequence $\beta_1, \ldots, \beta_q$ is a reflection ordering.*

(2) *We can obtain any λ-chain for a fixed dominant weight λ from any other λ-chain by moves of the form $\Gamma \to \Gamma'$.*

Let us now map the admissible subsets in $\mathcal{A}(\Gamma)$ to those in $\mathcal{A}(\Gamma')$. Given $J \in \mathcal{A}(\Gamma)$, let

$$\overline{J} := J \cap \overline{I}\,, \quad u := w(J \cap \{\overline{1}, \ldots, \overline{t}\})\,, \quad \text{and} \quad w := w(J \cap (\{\overline{1}, \ldots, \overline{t}\} \cup [q]))\,. \tag{4.4}$$

Also let

$$u = \lfloor u \rfloor \overline{u}\,, \quad w = \lfloor w \rfloor \overline{w}\,, \quad a := \ell(\overline{u})\,, \quad \text{and} \quad b := \ell(\overline{w})\,, \tag{4.5}$$

as above. It is clear that we have a bijection $Y : \mathcal{A}(\Gamma) \to \mathcal{A}(\Gamma')$ given by

$$Y(J) := \overline{J} \cup Y_{q,a,b}(J \setminus \overline{J})\,. \tag{4.6}$$

We call the moves $J \mapsto Y(J)$ *Yang-Baxter moves* (cf. the discussion following Theorem 4.5). We say that they are of types 0, 1.1, 1.2, 2.1, 2.2, and 3 depending on the cases considered above in relation to the definition of the corresponding map $Y_{q,a,b}$; we also use the term type 1 (respectively 2) for types 1.1 or 1.2 (respectively 2.1 or 2.2). Clearly, a Yang-Baxter move preserves the Weyl group element $w(\,\cdot\,)$ associated to an admissible subset, that is,

$$w(Y(J)) = w(J)\,. \tag{4.7}$$

In addition, Theorem 4.5 below holds.

In order to prove the mentioned result, we need to recall some information from [22]. Consider the ring $K := \mathbb{Z}[\Lambda/h] \otimes \mathbb{Z}[W]$, where $\mathbb{Z}[W]$ is the group algebra of the Weyl group W, and $\mathbb{Z}[\Lambda/h]$ is the group algebra of $\Lambda/h := \{\lambda/h \mid \lambda \in \Lambda\}$ (i.e., of the weight lattice shrunk h times, h being the Coxeter number defined in Subsection 2.1). We define $\mathbb{Z}[\Lambda/h]$-linear operators B_α and X^λ on K, where α is a positive root and λ is a weight:

$$B_\alpha : w \longmapsto \begin{cases} ws_\alpha & \text{if } \ell(ws_\alpha) = \ell(w)+1 \\ 0 & \text{otherwise,} \end{cases} \qquad X^\lambda : w \mapsto e^{w(\lambda/h)}w.$$

The following commutation relation will be needed:

$$B_\alpha\, X^\lambda = X^{s_\alpha(\lambda)}\, B_\alpha\,. \tag{4.8}$$

Theorem 4.5. *The map Y preserves the weight of an admissible subset. In other words, $\mu(Y(J)) = \mu(J)$ for all admissible subsets J.*

Proof. Fix an admissible subset J and, for each $i \in I$, let us set

$$Z_i := \begin{cases} B_{\beta_i} & \text{if } i \in J \\ X^{\beta_i} & \text{otherwise}\,. \end{cases}$$

We can calculate $\mu(J)$ as follows:

$$X^\rho Z_{\overline{n}} \ldots Z_{\overline{t+1}} Z_q \ldots Z_1 Z_{\overline{t}} \ldots Z_{\overline{1}} X^{-\rho}(1) = e^{\mu(J)} w(J)\,. \tag{4.9}$$

Indeed, let us denote the alcoves in the gallery $\gamma(J)$ by A_i for $i \in \{\overline{0}\} \cup I$, and let us also consider the admissible folding $\Gamma(J) = (\{(\gamma_i, \gamma_i')\}_{i\in I}, \gamma_\infty)$. Fix $i \in I$, and let ζ_i and $\zeta_{i'}$ be the central points of A_i and $A_{i'}$, where i' is the predecessor of i in $\{\overline{0}\} \cup I$. Then, based on (3.2), we have

$$Z_i(e^\mu w) = \begin{cases} e^\mu w r_i & \text{if } i \in J \\ e^{\mu+\gamma_i/h} w = e^{\mu+\zeta_{i'}-\zeta_i} w & \text{otherwise}\,. \end{cases}$$

Therefore, $Z_i \ldots Z_{\overline{1}} X^{-\rho}(1) = e^{-\zeta_i} w(J \cap \{j \in I \mid j \le i\})$. Finally, by (3.2), applying the last operator X^ρ amounts to multiplying by $e^{\gamma_\infty/h} = e^{\zeta_{\overline{n}}+\mu(J)}$, where $\zeta_{\overline{n}}$ is the central point of $A_{\overline{n}}$. Denoting the operators Z_i corresponding to $Y(J)$ by Z_i', we will show that the compositions $Z_q \ldots Z_1$ and $Z_q' \ldots Z_1'$ coincide; hence, when plugging them into the left-hand side of (4.9), we obtain the same result.

The cases we now consider correspond to the types of the Yang-Baxter move $J \mapsto Y(J)$. If the set $J \cap [q]$ is empty or equal to $[q]$ (that is, we have a Yang-Baxter move of type 0 or 3), then we clearly have $Z_q \ldots Z_1 = Z_q' \ldots Z_1'$.

Case 1: $J \cap [q] = \{i\}$. We will show that the two compositions coincide, that is,

$$X^{\beta_q} \ldots X^{\beta_{i+1}} B_{\beta_i} X^{\beta_{i-1}} \ldots X^{\beta_1} = X^{\beta_1} \ldots X^{\beta_{i-1}} B_{\beta_i} X^{\beta_{i+1}} \ldots X^{\beta_q}\,,$$

for $i = 1, \ldots, q$. By commuting the two operators B_{β_i} to the right, based on Lemma 4.1 (1) and (4.8), both sides are equal to $X^{\beta_{2i}+\ldots+\beta_q} B_{\beta_i}$ if $i \le \frac{q+1}{2}$, and $X^{\beta_1+\ldots+\beta_{2i-q-1}} B_{\beta_i}$ if $i > \frac{q+1}{2}$.

Case 2: $J\cap[q]=\{1,a+2,a+3,\dots,b\}$ or $J\cap[q]=\{a+1,a+2,\dots,b-1,q\}$ with $0\le a<a+2\le b<q$. Assume that we have

$$Z_q\dots Z_1=X^{\beta_q}\dots X^{\beta_{b+1}}B_{\beta_b}\dots B_{\beta_{a+2}}X^{\beta_{a+1}}\dots X^{\beta_2}B_{\beta_1}\quad\text{and}$$
$$Z'_q\dots Z'_1=B_{\beta_1}X^{\beta_2}\dots X^{\beta_{q+1-b}}B_{\beta_{q+2-b}}\dots B_{\beta_{q-a}}X^{\beta_{q+1-a}}\dots X^{\beta_q}\,.$$

By commuting the two operators B_{β_1} past the operators of the form X^{μ} to their left/right, based on Lemma 4.1 (1) and (4.8), we obtain

$$Z_q\dots Z_1=X^{\beta_q}\dots X^{\beta_{b+1}}B_{\beta_b}\dots B_{\beta_{a+2}}B_{\beta_1}X^{\beta_{q+1-a}}\dots X^{\beta_q}\quad\text{and}$$
$$Z'_q\dots Z'_1=X^{\beta_q}\dots X^{\beta_{b+1}}B_{\beta_1}B_{\beta_{q+2-b}}\dots B_{\beta_{q-a}}X^{\beta_{q+1-a}}\dots X^{\beta_q}\,.$$

The case $J\cap[q]=\{1,a+2,a+3,\dots,b-1,q\}$ or $J\cap[q]=\{a+1,a+2,\dots,b\}$ with $0<a<a+2\le b\le q$ is completely similar. □

We now explain the way in which the Yang-Baxter moves are related to the *Yang-Baxter equation*, which justifies the terminology. In [22], we considered the operators $R_\alpha:=X^\rho(X^\alpha+B_\alpha)X^{-\rho}$ for $\alpha\in\Phi^+$; if $\alpha\in\Phi^-$, we defined R_α by setting $B_\alpha:=-B_{-\alpha}$. It was proved in [22, Theorem 10.1] that the operators $\{R_\alpha\mid\alpha\in\Phi\}$ satisfy the Yang-Baxter equation in the sense of Cherednik [4]. (In fact, the dual of B_α was used in [22], but this does not affect the above result.) The main application of the operators R_α was to show that, given a λ-chain $\Gamma=(\beta_1,\dots,\beta_n)$, we have

$$R_{\beta_n}\dots R_{\beta_1}(1)=\sum_{J\in\mathcal{A}(\Gamma)}e^{\mu(J)}w(J)\,. \tag{4.10}$$

Due to the Yang-Baxter property, the right-hand side of the above formula does not change when we replace the λ-chain Γ by Γ', as defined above. The Yang-Baxter moves described above implement the passage from Γ to Γ' at the level of the individual terms in (4.10).

Furthermore, let us note that Theorem 4.5 also follows by combining Proposition 4.2 with [22, Theorem 10.1], that was mentioned above. However, the proof of the latter theorem is based on an involved case by case check in [3], while even the part of the proof in [22] is not transparent. By contrast, the proof of Theorem 4.5 presented here, based on making the map Y explicit, is a direct and simple one.

4.3. Yang-Baxter moves and root operators.

In this subsection, we present the main result related to Yang-Baxter moves.

We start with a Lemma regarding the action of a root operator, which will be used several times below, and which is based on the combinatorics of admissible subsets discussed in Subsection 3.6. As mentioned above, this combinatorics is best understood by graphing the piecewise-linear function $g_{J,p}$ associated to a simple root α_p and an admissible subset J. Let us also recall the definition of the set $I(J,p)$, of the sequence $L(J,p)$, and of the integer $M(J,p)$ in (3.4), as well as of the sequence $\Sigma(J,p)$. Finally, recall the definition of the positions $k_F(J,p)$ and $m_F(J,p)$ at the beginning of Subsection 3.7, as well as Proposition 3.18, which are all related to the root operator F_p.

Lemma 4.6. *Let $I(J,p)=\{i_1<i_2<\ldots<i_m\}$ and $\Gamma(J)=(\{(\gamma_i,\gamma'_i)\}_{i\in I},\gamma_\infty)$.*

(1) *If we have*

$$(\gamma_j,\gamma'_j)=\begin{cases}(-\alpha_p,-\alpha_p) & \text{if } j=i_c\\ (\alpha_p,-\alpha_p) & \text{if } j=i_{c+1},i_{c+2},\dots,i_{d-1}\\ (\alpha_p,\alpha_p) & \text{if } j=i_d\,,\end{cases}$$

for some $1\le c<d\le m$, then $k_F(J,p)\ne i_d$.

(2) *If we have*

$$(\gamma_j,\gamma'_j)=\begin{cases}(\alpha_p,-\alpha_p) & \text{if } j=i_c,i_{c+1},\dots,i_{d-1}\\ (\alpha_p,\alpha_p) & \text{if } j=i_d\,,\end{cases}$$

for some $1 \leq c < d \leq m$*, then* $m_F(J,p) \neq i_c$.

Proof. Let $\Sigma(J,p) = (\sigma_1, \ldots, \sigma_{m+1})$.

(1) Assume that $k_F(J,p) = i_d$. Then, by Proposition 3.18 (1), we have $M(J,p) = l_{i_d} + 1$. We clearly have $l_{i_c} = l_{i_d}$. By Proposition 3.15 (S2), we have $c > 1$ and $\sigma_{c-1} \in \{(1,-1),\,(-1,-1)\}$. Therefore, by Proposition 3.16, we have $l_{i_{c-1}} = l_{i_c} + 1 = M(J,p)$, which contradicts the definition of $k_F(J,p)$.

(2) Assume that $m_F(J,p) = i_c$. Then $M(J,p) = l_{i_c}$. We clearly have $l_{i_c} = l_{i_d}$. By Proposition 3.15 (S2), we have $\sigma_{d+1} \in \{(1,1),\,(1,-1),\,1\}$. Therefore, by Proposition 3.16, we have $l_{i_{d+1}} = l_{i_d} + 1 = M(J,p) + 1$ if $d < m$, or $l_\infty^p = l_{i_d} + 1 = M(J,p) + 1$ if $d = m$. Both contradict the definition of $M(J,p)$. □

Theorem 4.7. *The root operators commute with the Yang-Baxter moves, that is, a root operator* F_p *is defined on an admissible subset* J *if and only if it is defined on* $Y(J)$ *and we have*

$$Y(F_p(J)) = F_p(Y(J))\,.$$

Proof. The setup is the one described above, particularly in (4.4)-(4.6). Fix an admissible subset J in $\mathcal{A}(\Gamma)$, and consider the corresponding admissible folding $\Gamma(J) = (\{(\gamma_i, \gamma_i')\}_{i\in I}, \gamma_\infty)$. Let

$$\overline{\Gamma}(J) = ((\overline{\gamma}_1, \overline{\gamma}_1'), \ldots, (\overline{\gamma}_q, \overline{\gamma}_q')) := ((\lfloor u\rfloor^{-1}(\gamma_1), \lfloor u\rfloor^{-1}(\gamma_1')), \ldots, (\lfloor u\rfloor^{-1}(\gamma_q), \lfloor u\rfloor^{-1}(\gamma_q')))\,.$$

Clearly, this sequence consists only of roots in $\overline{\Phi}$. We also consider restrictions of $\overline{\Gamma}(J)$ to subsets of consecutive elements $\{i,\, i+1,\, \ldots,\, j\}$ of the set $[q]$, which we denote by

$$\overline{\Gamma}(J)_i^j = ((\overline{\gamma}_i, \overline{\gamma}_i'), (\overline{\gamma}_{i+1}, \overline{\gamma}_{i+1}'), \ldots, (\overline{\gamma}_j, \overline{\gamma}_j'))\,.$$

Similar notation is used for any admissible subset, in particular for $Y(J)$.

Let $\alpha := \beta_1$ and $\beta := \beta_q$, as in (4.1). Note that the only indices $i \in [q]$ for which γ_i or $-\gamma_i$ is a simple root are the ones for which $\overline{\gamma}_i$ belongs to $\{\pm\alpha,\, \pm\beta\}$. Indeed, if $\overline{\gamma}_i = c\alpha + d\beta$, then $\gamma_i = c\lfloor u\rfloor(\alpha) + d\lfloor u\rfloor(\beta)$, where $\lfloor u\rfloor(\alpha)$ and $\lfloor u\rfloor(\beta)$ are positive roots in Φ since $\ell(\lfloor u\rfloor s_\alpha) > \ell(\lfloor u\rfloor)$ and $\ell(\lfloor u\rfloor s_\beta) > \ell(\lfloor u\rfloor)$ (cf. [11, Proposition 5.7]). Hence, in order to compare the action of a root operator F_p on J and $Y(J)$, it is enough to consider the positions in $\overline{\Gamma}(J)$ and $\overline{\Gamma}(Y(J))$ in which the roots $\pm\alpha$ and $\pm\beta$ appear.

For simplicity, we denote the pairs of roots (γ,γ) and $(\gamma,-\gamma)$ by γ and $\pm\gamma$, respectively. It is also convenient to define

$$\delta(i) = \begin{cases} \alpha & \text{if } i \text{ odd} \\ \beta & \text{if } i \text{ even}\,. \end{cases}$$

The cases we now consider, which depend on $\overline{u}$ and $\overline{w}$, are precisely the ones considered above in relation to the definition of the set $J(\overline{u},\overline{w})$; as discussed above, they give the type of the Yang-Baxter move $J \mapsto Y(J)$. The analysis below makes it clear that $F_p(J)$ and $F_p(Y(J))$ are both defined or undefined, so we assume that they are both defined whenever we mention them. If a root operator F_p does not modify $J\cap[q]$ and $Y(J)\cap[q]$, then $F_p(J)$ and $F_p(Y(J))$ are clearly matched by a Yang-Baxter move of the same type as the one matching J and $Y(J)$. Hence it suffices to assume that the root operator F_p modifies $J\cap[q]$ or $Y(J)\cap[q]$.

Case 0: $J\cap[q] = \emptyset$. It is easy to see that $F_p(J)$ and $F_p(Y(J))$ are matched by a Yang-Baxter move of type 1.

Case 1.1: $b - a = 1$, $\overline{u} \in \overline{W}_\alpha$, $\overline{w} \in \overline{W}^\alpha$, so $0 \leq a \leq q-1$ and $J\cap[q] = \{1\}$. (Case 1.3 above is also treated here, since $Y(J)$ satisfies its conditions.) It is not hard to show that we have

$$\overline{\Gamma}(J)_{a+1}^{a+2} = (-\delta(a+1), \delta(a+2))\,, \quad \overline{\Gamma}(Y(J))_a^{a+1} = (-\delta(a+1), \delta(a+2))\,, \quad \text{if } 0 < a < q-1\,, \tag{4.11}$$

$$\overline{\Gamma}(J)_1^2 = (\pm\delta(1), \delta(2))\,, \quad \overline{\Gamma}(Y(J)) = (\delta(2), \ldots, \pm\delta(1))\,, \quad \text{if } a = 0\,, \tag{4.12}$$

$$\overline{\Gamma}(J) = (\pm\delta(q-1), \ldots, -\delta(q))\,, \quad \overline{\Gamma}(Y(J))_{q-1}^q = (-\delta(q), \pm\delta(q-1))\,, \quad \text{if } a = q-1\,. \tag{4.13}$$

We present the proof of the first part of (4.11), while the other facts can be proved similarly. Let w_i be the element of $\overline{W}$ having length i and reduced decomposition of the form $s_\alpha s_\beta \ldots$. We have

$$\beta_i = w_{i-1}(\delta(i))\,,\quad w_{i+1} = w_i s_{\delta(i+1)}\,,\quad \text{and}\quad w(J \cap \{\overline{1}, \ldots, \overline{t}, 1\}) = u s_\alpha = \lfloor u \rfloor (\overline{u} s_\alpha) = \lfloor u \rfloor w_{a+1}^{-1}\,.$$

Hence

$$\begin{aligned} \overline{\gamma}_{a+1} &= w_{a+1}^{-1}(\beta_{a+1}) = s_{\delta(a+1)} w_a^{-1} w_a(\delta(a+1)) = -\delta(a+1)\,, \\ \overline{\gamma}_{a+2} &= w_{a+1}^{-1}(\beta_{a+2}) = w_{a+1}^{-1} w_{a+1}(\delta(a+2)) = \delta(a+2)\,. \end{aligned} \tag{4.14}$$

Note that the roots $\pm\alpha$ and $\pm\beta$ do not appear in other positions in $\overline{\Gamma}(J)$ and $\overline{\Gamma}(Y(J))$ beside the ones indicated in (4.11)-(4.13). For instance, one can show this for the first part of (4.11) by an argument completely similar to the one used in Case 2.1 below relative to the first part of (4.16).

In (4.11), the root operator F_p must insert $a+2$ into J and $a+1$ into $Y(J)$; hence $F_p(J)$ and $F_p(Y(J))$ are matched by a Yang-Baxter move of type 2.1 if $a < q-2$ (more precisely, $\{1, a+2\} \leftrightarrow \{a+1, q\}$), and by a move of type 2.2 if $a = q-2$ (more precisely, $\{1, q\} \leftrightarrow \{q-1, q\}$). In (4.12), the root operator F_p must either remove 1 from J and q from $Y(J)$, or insert 2 into J and 1 into $Y(J)$; hence $F_p(J)$ and $F_p(Y(J))$ are matched by a Yang-Baxter move of type 0, 2.1 (more precisely, $\{1, 2\} \leftrightarrow \{1, q\}$), or 3 (this case is the analog of the previous one for $\overline{\Phi}$ of type $A_1 \times A_1$). In (4.13), the root operator F_p must remove 1 from J and q from $Y(J)$; hence $F_p(J)$ and $F_p(Y(J))$ are matched by a Yang-Baxter move of type 0.

Case 1.2: $b - a = 1$, $\overline{u} \in \overline{W}^\beta$, $\overline{w} \in \overline{W}_\alpha$, so $0 < a < q-1$ and $J \cap [q] = \{q-a\}$. (Case 1.4 above is also treated here, since $Y(J)$ satisfies its conditions.) In a similar way to (4.11)-(4.13), we can prove that we have

$$\overline{\Gamma}(J)_{q-a}^{q-a} = (\pm\delta(a))\,,\quad \overline{\Gamma}(Y(J))_a^{a+2} = (-\delta(a-1), \pm\delta(a), \delta(a+1))\,. \tag{4.15}$$

As in the previous case, one can easily show that the roots $\pm\alpha$ and $\pm\beta$ do not appear in other positions in $\overline{\Gamma}(J)$ and $\overline{\Gamma}(Y(J))$ beside the ones indicated in (4.15).

In (4.15), the root operator F_p must remove $q-a$ from J and $a+1$ from $Y(J)$. Hence $F_p(J)$ and $F_p(Y(J))$ are matched by a Yang-Baxter move of type 0. Note that F_p cannot insert $a+2$ into $Y(J)$, by Lemma 4.6 (1).

Case 2.1: $1 < b-a < q$, $\overline{u} \in \overline{W}_\alpha$, $\overline{w} \in \overline{W}_\beta$, so $0 \le a < a+2 \le b < q$ and $J \cap [q] = \{1, a+2, a+3, \ldots, b\}$. (Case 2.3 above is also treated here, since $Y(J)$ satisfies its conditions.) We start by showing that we have

$$\left.\begin{aligned} \overline{\Gamma}(J)_{a+1}^{b+1} &= (-\delta(a+1), \pm\delta(a+2), \pm\delta(a+3), \ldots, \pm\delta(b), \delta(b+1)) \\ \overline{\Gamma}(Y(J))_a^b &= (-\delta(a+1), \pm\delta(a+2), \pm\delta(a+3), \ldots, \pm\delta(b), \delta(b+1)) \end{aligned}\right. \quad \text{if } a > 0\,, \tag{4.16}$$

as well as

$$\left.\begin{aligned} \overline{\Gamma}(J)_1^{b+1} &= (\pm\delta(1), \pm\delta(2), \ldots, \pm\delta(b), \delta(b+1)) \\ \overline{\Gamma}(Y(J))_1^b &= (\pm\delta(2), \pm\delta(3), \ldots, \pm\delta(b), \delta(b+1)) \end{aligned}\right. \quad \text{if } a = 0\,. \tag{4.17}$$

We present the proof of the first part of (4.16), while the other facts can be proved similarly. The roots $\overline{\gamma}_{a+1}$ and $\overline{\gamma}_{a+2}$ can be computed as in (4.14). For $i = a+3, \ldots, b+1$, we calculate based on Lemma 4.1 (1) and (4.14):

$$\begin{aligned} \overline{\gamma}_i &= w_{a+1}^{-1} s_{\beta_{a+2}} \ldots s_{\beta_{i-1}}(\beta_i) = w_{a+1}^{-1} s_{\beta_{a+2}} \ldots s_{\beta_{i-2}}(-\beta_{i-2}) = w_{a+1}^{-1} s_{\beta_{a+2}} \ldots s_{\beta_{i-3}}(\beta_{i-2}) \\ &= \ldots = \begin{cases} w_{a+1}^{-1}(-\beta_{a+1}) = \delta(a+1) = \delta(i) & \text{if } i-a \text{ odd} \\ w_{a+1}^{-1}(\beta_{a+2}) = \delta(a+2) = \delta(i) & \text{if } i-a \text{ even}\,. \end{cases} \end{aligned} \tag{4.18}$$

Let us also note that the roots $\pm\alpha$ and $\pm\beta$ do not appear in other positions in $\overline{\Gamma}(J)$ and $\overline{\Gamma}(Y(J))$ beside the ones indicated in (4.16)-(4.17). For instance, in the first part of (4.16), we have $\overline{\gamma}_i =$

$\pm w_{a+1}^{-1}(\beta_i) \not\in \{\pm\alpha, \pm\beta\}$ for $i = 1, \ldots, a$, due to (4.14). Similarly, in the same case, we have $\overline{\gamma}_j = w_{a+1}^{-1} s_{\beta_{a+2}} \ldots s_{\beta_b}(\beta_j) \not\in \{\pm\alpha, \pm\beta\}$ for $j = b+2, \ldots, q$, based on (4.18) for $i = b, b+1$.

One way in which the operator F_p can act on $J \cap [q]$ and $Y(J) \cap [q]$ is to insert $b+1$ into J and b into $Y(J)$. This can happen both in (4.16) and in (4.17), but in the former case only if $b-a$ is odd, by Lemma 4.6 (1). Hence $F_p(J)$ and $F_p(Y(J))$ are matched by a move of the form $\{1, a+2, a+3, \ldots, b, b+1\} \leftrightarrow \{a+1, a+2, \ldots, b-1, b, q\}$. This is a Yang-Baxter move of type 2.1 if $b < q-1$, of type 2.2 if $b = q-1,\ a > 0$, and of type 3 if $b = q-1,\ a = 0$.

Finally, we consider the case when F_p removes certain elements from $J \cap [q]$ and $Y(J) \cap [q]$. Let us first concentrate on the case $a > 0$. Then F_p must remove $a+2$ from J and $a+1$ from $Y(J)$, but this can only happen if $b-a$ is even, by Lemma 4.6 (2). Thus $F_p(J)$ and $F_p(Y(J))$ are matched by a move of the form $\{1, a+3, \ldots, b\} \leftrightarrow \{a+2, \ldots, b-1, q\}$. This is a Yang-Baxter move of type 2.1 if $a+3 \le b$, and of type 1.1 if $b = a+2$. Now let us turn to the case $a = 0$. If b is odd, then F_p must remove 1 from J and 2 from $Y(J)$, by Lemma 4.6 (2). Thus $F_p(J)$ and $F_p(Y(J))$ are matched by a move of the form $\{2, 3, \ldots, b\} \leftrightarrow \{1, 3, 4, \ldots, b-1, q\}$, which is a Yang-Baxter move of type 2.2. If b is even, then F_p must remove 2 from J and 1 from $Y(J)$, by Lemma 4.6 (2). Thus $F_p(J)$ and $F_p(Y(J))$ are matched by a move of the form $\{1, 3, 4, \ldots, b\} \leftrightarrow \{2, 3, \ldots, b-1, q\}$, which is a Yang-Baxter move of type 2.1 if $b > 2$ and of type 1.1 if $b = 2$.

Case 2.2: $1 < b - a < q$, $\overline{u} \in \overline{W}^{\beta}$, $\overline{w} \in \overline{W}^{\beta}$, so $0 < a < a+2 \le b \le q$ and $J \cap [q] = \{1, a+2, a+3, \ldots, b-1, q\}$. (Case 2.4 above is also treated here, since $Y(J)$ satisfies its conditions.) In a similar way to (4.16)-(4.17), we can prove that we have

$$\begin{aligned} \overline{\Gamma}(J)_{a+1}^{b} &= (-\delta(a+1), \pm\delta(a+2), \pm\delta(a+3), \ldots, \pm\delta(b-1), \delta(b)) \\ \overline{\Gamma}(Y(J))_{a}^{b+1} &= (-\delta(a+1), \pm\delta(a+2), \pm\delta(a+3), \ldots, \pm\delta(b), \pm\delta(b+1), \delta(b+2)) \end{aligned} \quad \text{if } b < q\,, \tag{4.19}$$

as well as

$$\begin{aligned} \overline{\Gamma}(J)_{a+1}^{q} &= (-\delta(a+1), \pm\delta(a+2), \pm\delta(a+3), \ldots, \pm\delta(q-1), \pm\delta(q)) \\ \overline{\Gamma}(Y(J))_{a}^{q} &= (-\delta(a+1), \pm\delta(a+2), \pm\delta(a+3), \ldots, \pm\delta(q), \pm\delta(q+1)) \end{aligned} \quad \text{if } b = q\,. \tag{4.20}$$

As in the previous cases, one can easily show that the roots $\pm\alpha$ and $\pm\beta$ do not appear in other positions in $\overline{\Gamma}(J)$ and $\overline{\Gamma}(Y(J))$ beside the ones indicated in (4.19)-(4.20).

One way in which the operator F_p can act on $J \cap [q]$ and $Y(J) \cap [q]$ is to insert b into J and $b+1$ into $Y(J)$. This can happen in (4.19), but only if $b-a$ is even, by Lemma 4.6 (1). Hence $F_p(J)$ and $F_p(Y(J))$ are matched by a move of the form $\{1, a+2, a+3, \ldots, b-1, b, q\} \leftrightarrow \{a+1, a+2, \ldots, b, b+1\}$. This is always a Yang-Baxter move of type 2.2.

Finally, we consider the case when F_p removes certain elements from $J \cap [q]$ and $Y(J) \cap [q]$. Then it must remove $a+2$ from J and $a+1$ from $Y(J)$. This can happen both in (4.19) and in (4.20), but in the former case only if $b-a$ is odd, by Lemma 4.6 (2). Hence $F_p(J)$ and $F_p(Y(J))$ are matched by a move of the form $\{1, a+3, a+4, \ldots, b-1, q\} \leftrightarrow \{a+2, a+3, \ldots, b\}$. This is a Yang-Baxter move of type 2.2 with the exception of the case $b = q$, $a = q-2$, when it is of type 1.1.

Case 3: $a = 0$ and $b = q$, so $[q] \subseteq J$. In this case we have

$$\overline{\Gamma}(J) = (\pm\delta(1), \pm\delta(2), \ldots)\,, \quad \overline{\Gamma}(Y(J)) = (\pm\delta(2), \pm\delta(3), \ldots)\,.$$

Our root operator F_p must either remove 1 from J and 2 from $Y(J)$, or 2 from J and 1 from $Y(J)$. Hence $F_p(J)$ and $F_p(Y(J))$ are matched by a Yang-Baxter move of type 2.2 (more precisely, $\{1, 3, 4, \ldots, q-1, q\} \leftrightarrow \{2, 3, \ldots, q\}$) or 1.1 (this case is the analog of the previous one for $\overline{\Phi}$ of type $A_1 \times A_1$). $\square$

Theorem 4.7 asserts that the map Y above is an isomorphism between $\mathcal{A}(\Gamma)$ and $\mathcal{A}(\Gamma')$ as directed colored graphs. Given two arbitrary λ-chains Γ and Γ', we know from Proposition 4.4 (2) that they can be related by a sequence of λ-chains $\Gamma = \Gamma_0, \Gamma_1, \ldots, \Gamma_m = \Gamma'$ to which correspond Yang-Baxter moves $Y_1, \ldots, Y_m$. Hence the composition $Y_m \ldots Y_1$ is an isomorphism between $\mathcal{A}(\Gamma)$ and $\mathcal{A}(\Gamma')$ as directed colored graphs. Since every directed graph $\mathcal{A}(\Gamma)$ has a unique source (cf. [23, Proposition 6.9]), its automorphism group as a directed colored graph consists only of the identity. Thus, we have the following corollary of Theorem 4.7.

Corollary 4.8. *Given two arbitrary λ-chains Γ and Γ', the directed colored graph structures on $\mathcal{A}(\Gamma)$ and $\mathcal{A}(\Gamma')$ are isomorphic. This isomorphism is unique and, therefore, is given by the composition of Yang-Baxter moves corresponding to* any *sequence of λ-chains relating Γ and Γ'.*

We have given a transparent combinatorial explanation for the independence of the directed colored graph defined by our root operators from the chosen λ-chain. Similarly, it was proved in [25] that the directed colored graph structure on Littelmann paths generated by the corresponding root operators is independent of the initial path. However, this proof, which is based on continuous arguments, is less transparent.

Based on Corollary 4.8, Theorem 3.20 immediately leads to its generalization below.

Corollary 4.9. *Given* any *λ-chain Γ, the directed colored graph on the set $\mathcal{A}(\Gamma)$ defined by the root operators is isomorphic to the crystal graph of the irreducible representation V_λ with highest weight λ. Under this isomorphism, the weight of an admissible subset gives the weight space in which the corresponding element of the canonical basis lies.*

Based on Corollary 4.9, we will now identify the elements of the canonical basis with the corresponding admissible subsets.

Remark 4.10. We suggest that root operators and Yang-Baxter moves would be able to explain the whole combinatorics of our model. Note the analogy with type A, where we have left strings and right strings, defined via root operators and jeu de taquin, respectively (cf. [17]).

Define an action of a simple reflection s_p on an admissible subset J by

$$s_p(J) := F_p^{\langle \mu(J), \alpha_p^\vee \rangle}(J)\,. \tag{4.21}$$

Up to the isomorphism in Corollary 4.9, this action coincides with the one on crystals defined by Kashiwara in [13] and [14, Theorem 11.1]; hence it leads to an action of the full Weyl group W.

Corollary 4.11. *Equation* (4.21) *defines a W-action on admissible subsets. We have $\mu(w(J)) = w(\mu(J))$ for all w in W and all admissible subsets J.*

5. Lusztig's Involution

In this section, we present an explicit description of the involution η_λ in Subsection 2.4 in the spirit of Schützenberger's evacuation. We will show that the role of jeu de taquin in the definition of the evacuation map is played by the Yang-Baxter moves.

5.1. **Reversing λ-chains and admissible subsets.** Throughout the remainder of this paper, we fix an index set $I := \{\overline{1} < \ldots < \overline{q} < 1 < \ldots < n\}$ and a λ-chain $\Gamma = \{\beta_i\}_{i \in I}$ such that $l_i^{\emptyset} = 0$ if and only if $i \in \overline{I} := \{\overline{1} < \ldots < \overline{q}\}$. In other words, the second occurence of a root can never be before the first occurence of another root. We will also write $\Gamma := (\beta_{\overline{1}}, \ldots, \beta_{\overline{q}}, \beta_1, \ldots, \beta_n)$. Let us recall the notation $r_i := s_{\beta_i}$ for $i \in I$.

Given a Weyl group element w, we denote by $\lfloor w \rfloor$ and $\lceil w \rceil$ the minimal and the maximal representatives of the coset wW_λ, respectively (where W_λ is the stabilizer of the weight λ). Let $w_\circ^\lambda$ be the longest element of W_λ. Based on the discussion in Subsection 3.1, it is easy to see that we have the saturated increasing chain in Bruhat order

$$1 \lessdot r_{\overline{1}} \lessdot r_{\overline{1}} r_{\overline{2}} \lessdot \ldots \lessdot r_{\overline{1}} \ldots r_{\overline{q}}$$

from 1 to $\lfloor w_\circ \rfloor = w_\circ w_\circ^\lambda$. Hence the set $J_{\min} := \overline{I}$ is an admissible subset.

Proposition 5.1. *The admissible subset $J_{\min}$ is the minimum of the poset $\mathcal{A}(\Gamma)$.*

Proof. It suffices to show that, for any admissible subset $J \neq J_{\min}$, there exists $p \in [r]$ such that $M(J, p) > 0$; in other words, the root operator F_p is defined on J. Indeed, given such J, let $j = \min \overline{I} \setminus J$, which exists. Let $\Gamma(J) = (\{(\gamma_i, \gamma_i')\}_{i\in I}, \gamma_\infty)$. It follows from definitions that $\gamma_j = \gamma_j'$ is a simple root α_p. Proposition 3.15 (S2) then implies $M(J, p) > 0$. □

Definition 5.2. Let J be an admissible subset. Let $J \cap \overline{I} = \{\overline{j}_1 < \ldots < \overline{j}_a\}$ and $J \setminus \overline{I} = \{j_1 < \ldots < j_s\}$. The *initial key* $\kappa_0(J)$ and the *final key* $\kappa_1(J)$ of J are the Weyl group elements defined by

$$\kappa_0(J) := r_{\overline{j}_1} \ldots r_{\overline{j}_a}\,, \qquad \kappa_1(J) := w(J) = \kappa_0(J) r_{j_1} \ldots r_{j_s}\,.$$

Remark 5.3. The keys $\kappa_0(J)$ and $\kappa_1(J)$ are the generalizations of the left and right keys of a semistandard Young tableau [18], respectively. They are interchanged by Lusztig's involution (cf. Corollary 6.2) and are related to the Demazure character formula in Theorem 6.3. Now recall the bijection in [23, Section 9] between LS chains (in the orbit of $-\lambda$) and admissible subsets for the special λ-chain. It is not hard to show that $\kappa_0(J)(-\lambda)$ and $\kappa_1(J)(-\lambda)$ are the initial and the final directions of the LS chain associated to J, respectively. If, instead, we use LS chains in the orbit of λ (as we usually do), then $\kappa_0(J)(\lambda)$ and $\kappa_1(J)(\lambda)$ are the final and the initial directions of the corresponding LS chain, respectively.

We associate with our fixed λ-chain Γ another sequence $\Gamma^{\text{rev}} := \{\beta_i'\}_{i\in I}$ by

$$\beta_i' := \begin{cases} \beta_i & \text{if } i \in \overline{I} \\ w_\circ^\lambda(\beta_{n+1-i}) & \text{otherwise}\,. \end{cases}$$

In other words, we have

$$\Gamma^{\text{rev}} = (\beta_{\overline{1}}, \ldots, \beta_{\overline{q}}, w_\circ^\lambda(\beta_n), w_\circ^\lambda(\beta_{n-1}), \ldots, w_\circ^\lambda(\beta_1))\,. \tag{5.1}$$

Proposition 5.4. *Γ^{rev} is a λ-chain.*

Proof. Note first that $w_\circ^\lambda$ permutes the roots in $\Phi^+ \setminus \Phi_\lambda$, because so does any simple reflection in W_λ; here Φ_λ is the parabolic subroot system corresponding to W_λ. Therefore, since the λ-chain Γ consists only of roots in $\Phi^+ \setminus \Phi_\lambda$, so does Γ^{rev}.

We use the characterization of λ-chains in Theorem 3.3 (c). We observe first that the number of occurences of any positive root α in Γ^{rev} is $\langle \lambda, \alpha^\vee\rangle$. Indeed, if $\alpha \in \Phi^+ \setminus \Phi_\lambda$, we have $\langle \lambda, w_\circ^\lambda(\alpha)^\vee\rangle = \langle \lambda, \alpha^\vee\rangle$.

Let us now fix three positive roots α, β, γ such that $\gamma^\vee = \alpha^\vee + \beta^\vee$. Assume first that $\langle \lambda, \alpha^\vee\rangle$ and $\langle \lambda, \beta^\vee\rangle$ are both nonzero. Consider the subsequence of $\{\beta_i\}_{i\in I\setminus\overline{I}}$ consisting of $w_\circ^\lambda(\alpha)$, $w_\circ^\lambda(\beta)$, and $w_\circ^\lambda(\gamma)$. This starts with $w_\circ^\lambda(\gamma)$, and continues with a concatenation of pairs $(w_\circ^\lambda(\alpha), w_\circ^\lambda(\gamma))$ and $(w_\circ^\lambda(\beta), w_\circ^\lambda(\gamma))$. Hence, the subsequence of $\{\beta_i'\}_{i\in I\setminus\overline{I}}$ consisting of α, β, and γ starts with γ and continues with a concatenation of pairs (α, γ) and (β, γ). Also, the subsequence of $\{\beta_i'\}_{i\in\overline{I}}$ consisting of α, β, and γ is either (α, γ, β) or (β, γ, α).

Now assume that $\langle \lambda, \alpha^\vee\rangle = 0$ and $\langle \lambda, \beta^\vee\rangle > 0$. The subsequence of $\{\beta_i\}_{i\in I\setminus\overline{I}}$ consisting of $w_\circ^\lambda(\alpha)$, $w_\circ^\lambda(\beta)$, and $w_\circ^\lambda(\gamma)$ is a concatenation of pairs $(w_\circ^\lambda(\beta), w_\circ^\lambda(\gamma))$. Hence, the subsequence of $\{\beta_i'\}_{i\in I\setminus\overline{I}}$ consisting of α, β, and γ is a concatenation of pairs (β, γ). Also, the subsequence of $\{\beta_i'\}_{i\in\overline{I}}$ consisting of α, β, and γ is (β, γ). □

Let $r_i' := s_{\beta_i'}$ for $i \in I$. Fix an admissible subset

$$J = \{\overline{j}_1 < \ldots < \overline{j}_a < j_1 < \ldots < j_s\} \tag{5.2}$$

in $\mathcal{A}(\Gamma)$, where $\{\overline{j}_1 < \ldots < \overline{j}_a\} \subseteq \overline{I}$ and $\{j_1 < \ldots < j_s\} \subseteq I \setminus \overline{I}$. Let $u := \kappa_0(J)$ and $w := \kappa_1(J)$. We have the increasing saturated chain

$$1 \lessdot r_{\overline{j}_1} \lessdot r_{\overline{j}_1} r_{\overline{j}_2} \lessdot \ldots \lessdot r_{\overline{j}_1} \ldots r_{\overline{j}_a} = u \lessdot u r_{j_1} \lessdot u r_{j_1} r_{j_2} \lessdot \ldots \lessdot u r_{j_1} \ldots r_{j_s} = w\,. \tag{5.3}$$

According to [7], there is a unique saturated increasing chain in Bruhat order of the form

$$1 \lessdot r'_{\overline{k}_1} \lessdot r'_{\overline{k}_1} r'_{\overline{k}_2} \lessdot \ldots \lessdot r'_{\overline{k}_1} \ldots r'_{\overline{k}_b} = \lfloor w_\circ w \rfloor = w_\circ w w_\circ^\lambda \,,$$

where $\{\overline{k}_1 < \overline{k}_2 < \ldots < \overline{k}_b\} \subseteq \overline{I}$. Define

$$J^{\mathrm{rev}} := \{\overline{k}_1 < \ldots < \overline{k}_b < k_1 < \ldots < k_s\} \,, \tag{5.4}$$

where $k_i := n+1-j_{s+1-i}$ for $i=1,\ldots,s$. Note that $\beta'_{k_i} = w_\circ^\lambda(\beta_{j_{s+1-i}})$ for $i=1,\ldots,s$.

Proposition 5.5. *J^{rev} is an admissible subset in $\mathcal{A}(\Gamma^{\mathrm{rev}})$. We have*

$$\kappa_0(J^{\mathrm{rev}}) = \lfloor w_\circ \kappa_1(J) \rfloor \,, \quad \kappa_1(J^{\mathrm{rev}}) = \lfloor w_\circ \kappa_0(J) \rfloor \,, \tag{5.5}$$

as well as $(J^{\mathrm{rev}})^{\mathrm{rev}} = J$.

Proof. We have $r'_{k_i} = w_\circ^\lambda r_{j_{s+1-i}} w_\circ^\lambda$. Therefore, according to (5.3), we have the saturated increasing chain

$$\begin{aligned}\lfloor w_\circ w \rfloor &= w_\circ w w_\circ^\lambda \lessdot w_\circ w w_\circ^\lambda r'_{k_1} = w_\circ w r_{j_s} w_\circ^\lambda \lessdot w_\circ w w_\circ^\lambda r'_{k_1} r'_{k_2} = w_\circ w r_{j_s} r_{j_{s-1}} w_\circ^\lambda \lessdot \\ &\ldots \lessdot w_\circ w w_\circ^\lambda r'_{k_1} \ldots r'_{k_s} = w_\circ w r_{j_s} \ldots r_{j_1} w_\circ^\lambda = w_\circ u w_\circ^\lambda = \lfloor w_\circ u \rfloor \,.\end{aligned}$$

This completes the proof of (5.5), which then easily implies the last statement. □

We now present a direct way to obtain the gallery $\gamma(J^{\mathrm{rev}})$ from $\gamma(J)$. Let us write

$$\gamma(J) = (F_{\overline{0}}, A_{\overline{0}}, F_{\overline{1}}, \ldots, F_{\overline{q}}, A_0, F_1, A_1, \ldots, A_n, F_\infty) \,;$$

the corresponding augmented index set is $\{\overline{0} < \overline{1} < \ldots < \overline{q} = 0 < 1 < \ldots < n < \infty\}$. Let $\mu := -\mu(J)$, that is, $F_\infty = \{\mu\}$. Now define another gallery in the following way:

$$\gamma^\omega := (F'_{\overline{0}}, A'_{\overline{0}}, F'_{\overline{1}}, \ldots, F'_{\overline{q}}, A'_0, F'_1, A'_1, \ldots, A'_n, F'_\infty) \,.$$

The notation is as follows:

- ω is the map on $\mathfrak{h}^*_{\mathbb{R}}$ defined by $x \mapsto -w_\circ(x-\mu)$;
- $A'_i := \omega(A_{n-i})$ for $i=0,\ldots,n$, $F'_i := \omega(F_{n+1-i})$ for $i=1,\ldots,n$, and $F'_\infty = \{w_\circ(\mu)\}$;
- $(F'_{\overline{0}}, A'_{\overline{0}}, F'_{\overline{1}}, \ldots, F'_{\overline{q}})$ is the initial segment of the gallery $\gamma(J^{\mathrm{rev}})$.

Let us justify this construction. First of all, note that $\omega(A_n) = -w_\circ w(A_\circ)$. Secondly, it is easy to show that the alcove indexed by $\overline{q}=0$ in the gallery $\gamma(K)$ associated to some admissible subset K in $\mathcal{A}(\Gamma^{\mathrm{rev}})$ is $-\lceil \kappa_0(K) \rceil(A_\circ)$; indeed, this is true for $K=\emptyset$, so, for an arbitrary K, one only needs to apply $\kappa_0(K)$ to the alcove indexed by $\overline{q}=0$ in $\gamma(\emptyset)$. We conclude that the alcove indexed by $\overline{q}=0$ in J^{rev} is $\omega(A_n)$ since $\lceil \lfloor w_\circ w \rfloor \rceil = w_\circ w$. This means that γ^ω is a gallery.

Proposition 5.6. *The gallery γ^ω coincides with $\gamma(J^{\mathrm{rev}})$. In particular, we have $\mu(J^{\mathrm{rev}}) = w_\circ(\mu(J))$.*

Proof. We will show that the admissible foldings corresponding to the two galleries coincide. In other words, we will prove that $\Gamma(\gamma^\omega) = \Gamma(J^{\mathrm{rev}})$, cf. the notation in Subsection 3.5. Let

$$\Gamma(J) = (\{(\gamma_i, \gamma'_i)\}_{i\in I}, \gamma_\infty) \,, \quad \Gamma(J^{\mathrm{rev}}) = (\{(\delta_i, \delta'_i)\}_{i\in I}, \delta_\infty) \,, \quad \text{and} \quad \Gamma(\gamma^\omega) = (\{(\varepsilon_i, \varepsilon'_i)\}_{i\in I}, \varepsilon_\infty) \,.$$

By definition, the initial segments in $\Gamma(J^{\mathrm{rev}})$ and $\Gamma(\gamma^\omega)$ corresponding to $i \in \overline{I}$ coincide. We will now show that $\delta_i = \varepsilon_i$, for all $i \in [n]$; similarly, it can be shown that $\delta'_i = \varepsilon'_i$ and $\delta_\infty = \varepsilon_\infty$. Assume that $k_t < i \le k_{t+1}$ for some t in $\{0,1,\ldots,s\}$ (if $t=0$ or $t=s$, one of the two inequalities is missing). Based on definitions and the fact that $r'_{k_p} = w_\circ^\lambda r_{j_{s+1-p}} w_\circ^\lambda$, we have

$$\begin{aligned}\delta_i &= \lfloor w_\circ w \rfloor r'_{k_1} \ldots r'_{k_t}(\beta'_i) = w_\circ w r_{j_s} r_{j_{s-1}} \ldots r_{j_{s+1-t}} w_\circ^\lambda (w_\circ^\lambda(\beta_{n+1-i})) \\ &= w_\circ w r_{j_s} r_{j_{s-1}} \ldots r_{j_{s+1-t}}(\beta_{n+1-i}) = w_\circ u r_{j_1} \ldots r_{j_{s-t}}(\beta_{n+1-i}) \,.\end{aligned} \tag{5.6}$$

On the other hand, note that ε_i is determined by $A'_{i-1} = \omega(A_{n+1-i})$ and $F'_i = \omega(F_{n+1-i})$. More precisely, we have $\varepsilon_i = -w_\circ(-\gamma'_{n+1-i})$. The proof is completed by observing that $j_{s-t} \le n+1-i < j_{s+1-t}$, which implies that

$$\varepsilon_i = w_\circ(\gamma'_{n+1-i}) = w_\circ u r_{j_1} \dots r_{j_{s-t}}(\beta_{n+1-i})\,. \tag{5.7}$$

Indeed, the expressions for δ_i and ε_i in (5.6) and (5.7) coincide. □

5.2. **The map $J \mapsto J^{\mathrm{rev}}$ and root operators.** We will now present the main result related to the map $J \mapsto J^{\mathrm{rev}}$, which involves its commutation with the root operators. In order to do this, we need two lemmas. We will use once again the notation from Subsection 3.6. In particular, given J in $\mathcal{A}(\Gamma)$ as above and a simple root α_p, we consider the set $I(J,p)$ and the sequence $\Sigma(J,p)$. We let $\Gamma(J) = (\{(\gamma_i, \gamma'_i)\}_{i \in I}, \gamma_\infty)$ and

$$\begin{aligned} &I(J,p) \cap \overline{I} = \{\overline{i}_1 < \ldots < \overline{i}_c\}\,, \quad I(J,p) \setminus \overline{I} = \{i_1 < \ldots < i_d\}\,,\\ &\Sigma(J,p) = (\overline{\sigma}_1, \ldots, \overline{\sigma}_c, \sigma_1, \ldots, \sigma_d, \sigma_{d+1})\,, \end{aligned} \tag{5.8}$$

where $c, d \ge 0$. Also recall that we set

$$L(J) = \{l_i\}_{i \in I}\,, \quad L(\emptyset) = \{l_i^\emptyset\}_{i \in I}\,, \quad l_\infty^p := \langle \mu(J), \alpha_p^\vee \rangle\,, \quad u := \kappa_0(J)\,, \quad w := \kappa_1(J)\,.$$

Lemma 5.7. *If $c > 0$, we have $\overline{\sigma}_1 = \ldots = \overline{\sigma}_{c-1} = (1,-1)$, and either $\overline{\sigma}_c = (1,-1)$ or $\overline{\sigma}_c = (1,1)$; in the first case we have $w_\circ^\lambda u^{-1}(\alpha_p) < 0$, while in the second one we have $w_\circ^\lambda u^{-1}(\alpha_p) > 0$. If $c = 0$, then $w_\circ^\lambda u^{-1}(\alpha_p) < 0$.*

Proof. We start by noting that, for $i \in \overline{I}$, the hyperplane $H_{-|\gamma_i|, l_i}$ is obtained from the hyperplane $H_{-\beta_i, l_i^\emptyset} = H_{\beta_i, 0}$ by applying a nonaffine reflection; therefore, $l_i = 0$ for $i \in \overline{I}$. By Propositions 3.16 and 3.15, we can have $l_{\overline{i}_1} = \ldots = l_{\overline{i}_c} = 0$ only if $\overline{\sigma}_i$ is as above, for $i = 1, \ldots, c$.

Let $\overline{J} := J \cap \overline{I}$, and

$$\begin{aligned} &\Gamma(\overline{J}) = (\{(\overline{\gamma}_i, \overline{\gamma}'_i)\}_{i \in I}, \overline{\gamma}_\infty)\,, \quad L(\overline{J}) = \{\overline{l}_i\}_{i \in I}\,, \quad \overline{l}_\infty^p := \langle \mu(\overline{J}), \alpha_p^\vee \rangle\,,\\ &I(\overline{J}, p) = \{\overline{i}_1 < \ldots < \overline{i}_c < h_1 < \ldots < h_e\}\,, \quad \Sigma(\overline{J}, p) = (\overline{\sigma}_1, \ldots, \overline{\sigma}_c, \pi_1, \ldots, \pi_e, \pi_{e+1})\,, \end{aligned}$$

where $e \ge 0$. Let $\beta := |u^{-1}(\alpha_p)|$, and $\sigma := \mathrm{sgn}(u^{-1}(\alpha_p)) = \mathrm{sgn}(u(\beta))$.

Assume first that $\langle \lambda, \beta^\vee \rangle = |\langle u(\lambda), \alpha_p^\vee \rangle| = |\langle \mu(\overline{J}), \alpha_p^\vee \rangle| = |\overline{l}_\infty^p|$ is nonzero. For $i = 1, \ldots, e$, we have $\beta_{h_i} = \beta$ and $l_{h_i}^\emptyset = i$, which implies $\overline{l}_{h_i} = \sigma i$ and $\overline{l}_\infty^p = \sigma(e+1)$. If $\sigma = 1$, we must have $c > 0$ and $\overline{\sigma}_c = (1,1)$ (by Proposition 3.16). Similarly, if $\sigma = -1$, we must have $c > 0$ (for this we also need Proposition 3.15 (2)) and $\overline{\sigma}_c = (1,-1)$. Finally, the root $u^{-1}(\alpha_p) = \sigma\beta$ does not belong to the parabolic subroot system Φ_λ corresponding to W_λ, so $\sigma = \mathrm{sgn}(u^{-1}(\alpha_p)) = \mathrm{sgn}(w_\circ^\lambda u^{-1}(\alpha_p))$. Indeed, if δ is a simple root in Φ_λ, then s_δ sends a root in $\Phi^+ \setminus \Phi_\lambda$ to another such root.

Now assume that $\langle \lambda, \beta^\vee \rangle = \overline{l}_\infty^p = 0$, in which case we necessarily have $e = 0$. If $c > 0$, we must have $\overline{\sigma}_c = (1,-1)$ and $\pi_1 = 1$ (by Propositions 3.16 and 3.15). But $\pi_1 = \mathrm{sgn}(\langle \overline{\gamma}_\infty, \alpha_p^\vee \rangle) = \mathrm{sgn}(\langle u(\rho), \alpha_p^\vee \rangle) = \mathrm{sgn}(u^{-1}(\alpha_p))$. On the other hand, $u^{-1}(\alpha_p) = \beta$ lies in Φ_λ, so $w_\circ^\lambda u^{-1}(\alpha_p) < 0$. If $c = 0$, we must have $\pi_1 = 1$. The case $c = 0$ is completely similar. □

In addition to the notation in (5.8) related to the admissible subset J, we need the following one related to J^{rev}:

$$\begin{aligned} &I(J^{\mathrm{rev}}, p^*) \cap \overline{I} = \{\overline{h}_1 < \ldots < \overline{h}_e\}\,, \quad I(J^{\mathrm{rev}}, p^*) \setminus \overline{I} = \{h_1 < \ldots < h_f\}\,,\\ &\Sigma(J^{\mathrm{rev}}, p^*) = (\overline{\pi}_1, \ldots, \overline{\pi}_e, \pi_1, \ldots, \pi_f, \pi_{f+1})\,. \end{aligned} \tag{5.9}$$

Let us define $\sigma_0 \in \{-1, 1\}$ by $\overline{\sigma}_c = (1, \sigma_0)$ if $c > 0$, and by $\sigma_0 := -1$ if $c = 0$. We define π_0 similarly, based on e and $\overline{\pi}_e$. Given a pair of integers (a,b), we also set $-(a,b) := (-a,-b)$.

Lemma 5.8. *We have $d = f$, as well as $h_j = n+1-i_{d+1-j}$ for $j = 1, \ldots, d$ and $\pi_j = -\sigma_{d+1-j}$ for $j = 0, 1, \ldots, d+1$.*

Proof. Let $\Gamma(J^{\mathrm{rev}}) = (\{(\delta_i, \delta_i')\}_{i\in I}, \delta_\infty)$. We will show that $\gamma_i = \pm\alpha_p$ implies $\delta'_{n+1-i} = \mp\alpha_{p^*}$, where $i \in [n]$; this, in turn, immediately implies $d = f$, as well as $h_j = n+1-i_{d+1-j}$ and $\pi_j = -\sigma_{d+1-j}$ for $j = 1, \ldots, d$. Indeed, recall the setup related to the definition of J^{rev} in (5.2)-(5.4), and assume that $k_t \le n+1-i < k_{t+1}$ for some t in $\{0, 1, \ldots, s\}$ (if $t = 0$ or $t = s$, one of the two inequalities is missing); by (5.6), we have

$$\delta'_{n+1-i} = \lfloor w_\circ w \rfloor r'_{k_1} \ldots r'_{k_t}(\beta'_{n+1-i}) = w_\circ u r_{j_1} \ldots r_{j_{s-t}}(\beta_i) = w_\circ(\gamma_i)\,,$$

where the last equality follows from the fact that $j_{s-t} < i \le j_{s+1-t}$.

At this point, it suffices to show that $\pi_{d+1} = -\sigma_0$. By Proposition 5.5 and an easy computation, we have

$$\pi_{d+1} = \mathrm{sgn}(\langle w(J^{\mathrm{rev}})(\rho), \alpha_{p^*}\rangle) = \mathrm{sgn}(\langle w_\circ u w_\circ^\lambda(\rho), \alpha_{p^*}\rangle) = -\mathrm{sgn}(w_\circ^\lambda u^{-1}(\alpha_p)) = -\sigma_0\,;$$

the last equality is the content of Lemma 5.7. □

Theorem 5.9. *A root operator F_p is defined on the admissible subset J if and only if E_{p^*} is defined on J^{rev}, and we have*

$$(F_p(J))^{\mathrm{rev}} = E_{p^*}(J^{\mathrm{rev}})\,.$$

Proof. We use the setup above, particularly (5.2)-(5.4) and (5.8)-(5.9). We will compare $F_p(J)$ and $E_{p^*}(J^{\mathrm{rev}})$ in several cases. Let us assume first that $c, e > 0$. Consider the functions $f : [-c+\frac{1}{2}, d+1] \to \mathbb{R}$ and $g : [-e+\frac{1}{2}, d+1] \to \mathbb{R}$ defined by

$$f(x) = g_{J,p}\left(x + c - \frac{1}{2}\right)\,, \quad g(x) = g_{J^{\mathrm{rev}},p^*}\left(x + e - \frac{1}{2}\right)\,.$$

Based on Proposition 3.16, these functions and the following observations related to them will be used below (sometimes implicitly) in order to construct $F_p(J)$ and $E_{p^*}(J^{\mathrm{rev}})$. By Lemmas 5.7 and 5.8, we have $f(0) = g(0) = 0$ and $g'(x) = -f'(d+1-x)$, for all $x \in [0, d+1] \setminus \frac{1}{2}\mathbb{Z}$. This means that x_0 is the first global maximum of f on $[0, d+1]$ if and only if $d+1-x_0$ is the last global maximum of g on $[0, d+1]$. By Proposition 3.15, the local maxima of f and g can only be attained at integer points.

Case 0: F_p is not defined on J, so $M(J,p) = 0$. We have $f(x) \le 0 = f(0)$ for all x in its domain, and therefore $g(x) \le g(d+1)$. This means that E_{p^*} is not defined on J^{rev}.

In fact, the above reasoning allows us to prove that F_p is defined on J if and only if E_{p^*} is defined on J^{rev}. The remaining cases deal with this situation.

Case 1: $m_F(J,p) \ne \infty$ and $k_F(J,p) \in I \setminus \overline{I}$. This case is illustrated by the example in Figure 4 below, where the graph on the left is of the function f, while the one on the right is of the function g; the dashed lines show the effect of applying the root operators F_p to J and E_{p^*} to J^{rev}. Let $m_F(J,p) = i_j \in J$ and $k_F(J,p) = i_{j-1} \notin J$, where $1 < j \le d$. Using Lemma 5.8 and the above observations, we have

$$k_E(J^{\mathrm{rev}}, p^*) = n+1-i_j = h_{d+1-j} \in J^{\mathrm{rev}} \quad \text{and} \quad m_E(J^{\mathrm{rev}}, p^*) = h_{d+2-j} = n+1-i_{j-1} \notin J^{\mathrm{rev}}\,.$$

Hence, we have

$$(F_p(J))^{\mathrm{rev}} = ((J \setminus \{i_j\}) \cup \{i_{j-1}\})^{\mathrm{rev}} \quad \text{and} \quad E_{p^*}(J^{\mathrm{rev}}) = (J^{\mathrm{rev}} \setminus \{n+1-i_j\}) \cup \{n+1-i_{j-1}\}\,.$$

In order to prove that the two admissible subsets above coincide, it suffices to show that their intersections with $\overline{I}$ coincide. The second intersection is $J^{\mathrm{rev}} \cap \overline{I}$, while $(F_p(J))^{\mathrm{rev}} \cap \overline{I}$ is computed based on $w(F_p(J))$. But this computation is the same as the one leading to $J^{\mathrm{rev}} \cap \overline{I}$, because we have $w(F_p(J)) = w(J)$ by Proposition 3.18 (3).

Case 2: $m_F(J,p) = \infty$ and $k_F(J,p) \in I \setminus \overline{I}$. This case is illustrated by the example in Figure 5 below. In this case, the function f has a unique global maximum (on its domain) at $d+1$, while g has a unique global maximum on $[0, d+1]$ at 0, and $\overline{\pi}_e = (1, -1)$. Hence

$$k_F(J,p) = i_d \notin J\,, \quad k_E(J^{\mathrm{rev}}, p^*) = \overline{h}_e \in J^{\mathrm{rev}}\,, \quad m_E(J^{\mathrm{rev}}, p^*) = h_1 = n+1-i_d \notin J^{\mathrm{rev}}\,.$$

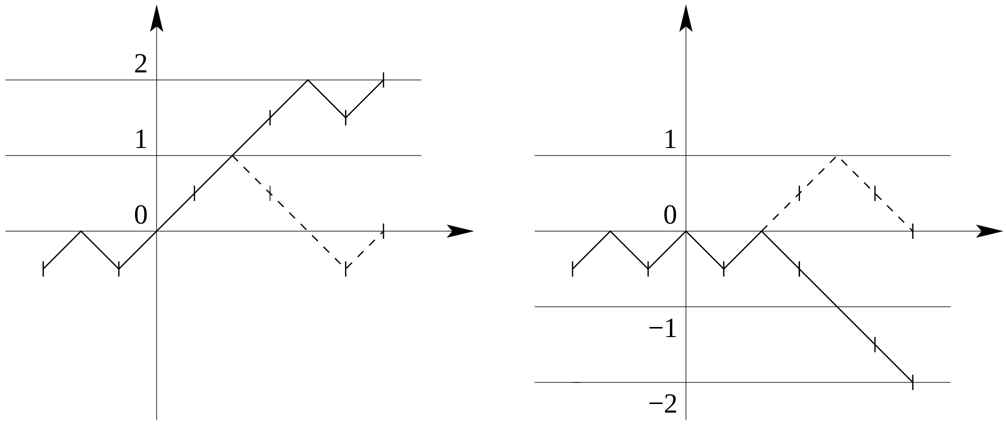

FIGURE 4. Case 1.

Thus, we have

$$F_p(J) = J \cup \{i_d\} \quad \text{and} \quad (E_{p^*}(J^{\text{rev}}))^{\text{rev}} = ((J^{\text{rev}} \setminus \{\overline{h}_e\}) \cup \{n+1-i_d\})^{\text{rev}}\,.$$

These two admissible subsets coincide by a similar argument to the one used in Case 1. Note that we now need to use Proposition 3.19 (3) and Proposition 5.5, namely the fact that $w(E_{p^*}(J^{\text{rev}})) = w(J^{\text{rev}}) = w_\circ \kappa_0(J) w_\circ^\lambda$. Indeed, this implies that the two admissible subsets above have the same initial key.

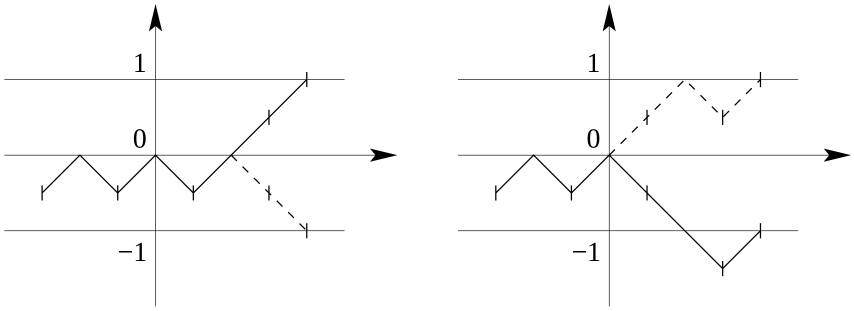

FIGURE 5. Case 2.

Case 3: $m_F(J,p) \neq \infty$ and $k_F(J,p) \in \overline{I}$. This case is illustrated by the example in Figure 6 below. In this case, the function f has its first global maximum (on its domain) at 1, while g has its last global maximum at d, and $\overline{\sigma}_c = (1,1)$. Hence

$$m_F(J,p) = i_1 \in J\,, \quad k_F(J,p) = \overline{i}_c \notin J\,, \quad k_E(J^{\text{rev}},p^*) = n+1-i_1 = h_d \in J^{\text{rev}}\,, \quad m_E(J^{\text{rev}},p^*) = \infty.$$

Thus, we have

$$(F_p(J))^{\text{rev}} = ((J \setminus \{i_1\}) \cup \{\overline{i}_c\})^{\text{rev}} \quad \text{and} \quad E_{p^*}(J^{\text{rev}}) = J^{\text{rev}} \setminus \{n+1-i_1\}\,.$$

These two admissible subsets coincide by a similar argument to the one used in Case 1.

Case 4: $m_F(J,p) = \infty$ and $k_F(J,p) \in \overline{I}$. This case is illustrated by the example in Figure 7 below. In this case we have $d = 0$, $\overline{\sigma}_c = (1,1)$, $\sigma_1 = 1$, $\overline{\pi}_e = (1,-1)$, and $\pi_1 = -1$. Hence

$$k_F(J,p) = \overline{i}_c \notin J\,, \quad k_E(J^{\text{rev}},p^*) = \overline{h}_e \in J^{\text{rev}}\,.$$

Thus, we have

$$(F_p(J))^{\text{rev}} = (J \cup \{\overline{i}_c\})^{\text{rev}} \quad \text{and} \quad E_{p^*}(J^{\text{rev}}) = J^{\text{rev}} \setminus \{\overline{h}_e\}\,. \tag{5.10}$$

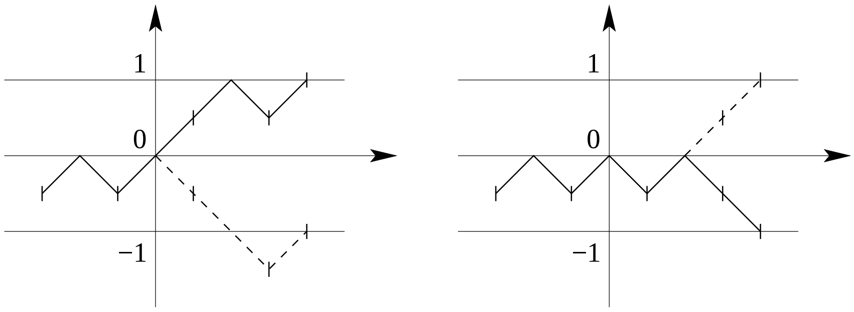

FIGURE 6. Case 3.

By Proposition 3.18 (3), we have $w(F_p(J)) = s_p w(J)$, so

$$\kappa_0((F_p(J))^{\mathrm{rev}}) = w_\circ s_p w(J) w_\circ^\lambda = s_{p^*} w_\circ w(J) w_\circ^\lambda = s_{p^*} \kappa_0(J^{\mathrm{rev}})\,.$$

By a completely similar proof to the one of Proposition 3.19 (3) in [23], we have $\kappa_0(J^{\mathrm{rev}} \setminus \{\overline{h}_e\}) = s_{p^*}\kappa_0(J^{\mathrm{rev}})$. Therefore, we have $\kappa_0(E_{p^*}(J^{\mathrm{rev}})) = \kappa_0((F_p(J))^{\mathrm{rev}})$. This implies that the two admissible subsets in (5.10) coincide.

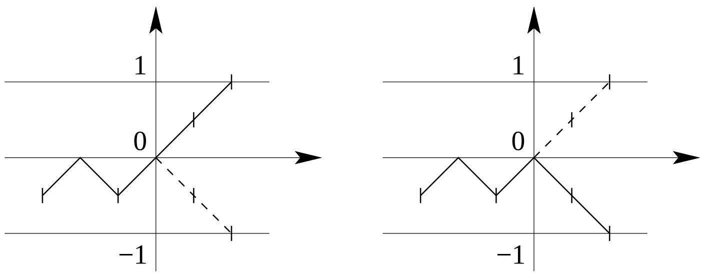

FIGURE 7. Case 4.

We conclude the proof by discussing the case when $c = 0$ or $e = 0$. This is reduced to the simple observations below.

- If $c = 0$ and $M(J, p) = 0$ then $g_{J,p}$ attains its global maximum at $\frac{1}{2}$ (by Proposition 3.15 (S2)), and therefore g_{J^{rev},p^*} attains its global maximum at $e + d - \frac{1}{2}$ and at the endpoint $e + d + \frac{1}{2}$; indeed, $\pi_{d+1} = 1$ by Lemma 5.8. Hence, E_{p^*} is not defined on J^{rev}.
- Case 1 is treated in the same way if $c = 0$ or $e = 0$.
- In Case 2 we cannot have $e = 0$ because then $\pi_1 = (-1, -1)$, and this is impossible by Proposition 3.15 (S2). If $c = 0$, then Case 2 is treated in the same way.
- Case 3 does not make sense for $c = 0$, and is treated in the same way if $e = 0$.
- Case 4 does not exist.

□

We can summarize the construction in this section (based on Propositions 5.4 and 5.5) as follows: given the λ-chain Γ (for a fixed dominant weight λ), we defined the λ-chain Γ^{rev}, and given $J \in \mathcal{A}(\Gamma)$, we defined $J^{\mathrm{rev}} \in \mathcal{A}(\Gamma^{\mathrm{rev}})$. Hence we can map J^{rev} to an admissible subset $J^* \in \mathcal{A}(\Gamma)$ using Yang-Baxter moves, as it is described in Section 4 and it is recalled below. To be more precise, let $R : \mathcal{A}(\Gamma) \to \mathcal{A}(\Gamma^{\mathrm{rev}})$ denote the bijection $J \mapsto J^{\mathrm{rev}}$. On the other hand, we know from Proposition 4.4 (2) that the λ-chains Γ^{rev} and Γ can be related by a sequence of λ-chains $\Gamma^{\mathrm{rev}} = \Gamma_0, \Gamma_1, \ldots, \Gamma_m = \Gamma$ to which correspond Yang-Baxter moves $Y_1, \ldots, Y_m$. By Corollary 4.8, the composition $Y := Y_m \ldots Y_1$ does not depend on the sequence of intermediate λ-chains, and it defines a bijection from $\mathcal{A}(\Gamma^{\text{rev}})$ to $\mathcal{A}(\Gamma)$. We let $J^* := YR(J)$ and conclude that it is a bijection on $\mathcal{A}(\Gamma)$. The main result of this section, namely Theorem 5.11 below, now follows directly from Theorems 4.7 and 5.9.

Remark 5.10. We claim that we can choose the λ-chains $\Gamma^{\text{rev}} = \Gamma_0, \Gamma_1, \ldots, \Gamma_m = \Gamma$ such that their initial segments indexed by $\overline{1}, \ldots, \overline{q}$ are identical. Indeed, this is true for Γ^{rev} and Γ by definition. On the other hand, let us recall the correspondence between λ-chains and reduced words for the affine Weyl group element $v_{-\lambda}$ mentioned in Definition 3.1; most importantly, we recall from the proof of [22, Lemma 9.3] that the moves $\Gamma_{i-1} \to \Gamma_i$ (for $i = 1, \ldots, m$) correspond to Coxeter moves (on the mentioned reduced words) in this context. The claim is now justified by noting that two reduced words for $v_{-\lambda}$ with identical initial segments can be related by Coxeter moves which do not involve the mentioned initial segment.

Theorem 5.11. *The bijection $J \mapsto J^*$ constructed above coincides with Lusztig's involution η_λ on the canonical basis. In other words, a root operator F_p is defined on the admissible subset J if and only if E_{p^*} is defined on J^*, and we have*

$$(J_{\min})^* = J_{\max}\,, \quad (J_{\max})^* = J_{\min}\,, \quad \textit{and} \quad (F_p(J))^* = E_{p^*}(J^*)\,, \ \textit{for } p = 1, \ldots, r\,. \tag{5.11}$$

In particular, the map $J \mapsto J^*$ expresses combinatorially the self-duality of the poset $\mathcal{A}(\Gamma)$.

Proof. The first equality in (5.11) is obvious. For the second one, note that $(J_{\max})^{\text{rev}} = J_{\min}$ and use Remark 5.10 to show that the map Y fixes $J_{\min}$. The last equality follows directly from Theorems 4.7 and 5.9. Now recall that, based on the (directed colored graph) isomorphism in Corollary 4.9, we identified the vertex sets $\mathcal{B}_\lambda$ and $\mathcal{A}(\Gamma)$ of the corresponding directed colored graphs. By comparing (2.7)-(2.8) with (5.11), and by noting that the bijection specified by these conditions is unique, we conclude that the bijection $J \mapsto J^*$ coincides with η_λ (via the isomorphism mentioned above). □

Remark 5.12. The above construction is analogous to the definition of Schützenberger's evacuation map (see, for instance, [8]). Below, we recall from Subsection 2.4 the three-step procedure defining this map and we discuss the analogy with our construction in the case of each step.

(1) REVERSE: We rotate a given semistandard Young tableau by 180°. This corresponds to reversing its word, in the same way as we reversed the direction of our gallery, cf. Proposition 5.6.
(2) COMPLEMENT: We complement each entry via the map $i \mapsto w_\circ(i)$, where $w_\circ$ is the longest element in the corresponding symmetric group. This corresponds to using $w_\circ$ for the arbitrary Weyl group in the definition (5.4) of J^{rev}.
(3) SLIDE: We apply jeu de taquin on the obtained skew tableau. This corresponds to the Yang-Baxter moves $Y_1, \ldots, Y_m$ discussed above.

Example 5.13. Consider the Lie algebra $\mathfrak{sl}_3$ of type A_2, cf. Example 3.7. Consider the dominant weight $\lambda = 4\varepsilon_1 + 2\varepsilon_2$ and the following λ-chain:

$$\begin{array}{rcccccccc} & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 \\ \Gamma = (& \alpha_{12}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & \underline{\alpha_{12}}, & \alpha_{13}, & \underline{\alpha_{23}}, & \alpha_{13})\,. \end{array}$$

Here we indicated the index corresponding to each root, using the notation in Subsection 5.1; more precisely, we have $I = \{\overline{1} < \overline{2} < \overline{3} < 1 < 2 < 3 < 4 < 5\}$ and $\overline{I} = \{\overline{1} < \overline{2} < \overline{3}\}$. By the defining relation (5.1), we have

$$\begin{array}{rcccccccc} & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 \\ \Gamma^{\text{rev}} = (& \underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & \underline{\alpha_{23}}, & \alpha_{13}, & \underline{\alpha_{12}}, & \alpha_{13})\,. \end{array}$$

Consider the admissible subset $J = \{2, 4\}$. This is indicated above by the underlined roots in Γ. In order to define J^{rev}, cf. (5.4), we need to compute

$$\kappa_0(J^{\text{rev}}) = w_\circ w(J) = (s_{12}s_{23}s_{12})(s_{12}s_{23}) = s_{12}\,.$$

Hence we have $J^{\text{rev}} = \{\overline{1}, 2, 4\}$. This is indicated above by the underlined positions in Γ^{rev}.

In order to transform the λ-chain Γ^{rev} into Γ, we need to perform a single Yang-Baxter move; this consists of reversing the order of the bracketed roots below:

$$\begin{array}{rlllllllll} & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 & \\ \Gamma^{\text{rev}} = (& \underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & (\underline{\alpha_{23}}, & \alpha_{13}, & \underline{\alpha_{12}}), & \alpha_{13}) & \longrightarrow \\ & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 & \\ \Gamma \quad = (& \underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & (\alpha_{12}, & \underline{\alpha_{13}}, & \underline{\alpha_{23}}), & \alpha_{13}) . & \end{array}$$

The underlined roots indicate the way in which the Yang-Baxter move $J^{\text{rev}} \mapsto Y(J^{\text{rev}}) = J^*$ works. All we need to know is that there are two saturated chains in Bruhat order between the permutations u and w, cf. the notation in (4.4):

$$u = s_{12} \lessdot s_{12}s_{23} \lessdot s_{12}s_{23}s_{12} = w\,, \quad u = s_{12} \lessdot s_{12}s_{13} \lessdot s_{12}s_{13}s_{23} = w\,.$$

The first chain is retrieved as a subchain of Γ^{rev} and corresponds to J^{rev}, while the second one is retrieved as a subchain of Γ and corresponds to J^*. Hence we have $J^* = \{\overline{1}, 3, 4\}$.

6. Other Applications

We can give an intrinsic explanation for the fact that the map $J \mapsto J^*$ is an involution on $\mathcal{A}(\Gamma)$; this explanation is only based on the results in Sections 4 and 5, so it does not rely on Proposition 2.3 (2). Let us first recall the bijections $R : \mathcal{A}(\Gamma) \to \mathcal{A}(\Gamma^{\text{rev}})$ and $Y : \mathcal{A}(\Gamma^{\text{rev}}) \to \mathcal{A}(\Gamma)$ defined above. We claim that $YR = R^{-1}Y^{-1}$, which would prove that the composition YR is an involution. In the same way as we proved Theorem 5.11 (that is, as a direct consequence of Theorems 4.7 and 5.9), we can verify that the composition $R^{-1}Y^{-1}$ satisfies the conditions in (5.11). Since these conditions uniquely determine the corresponding map from $\mathcal{A}(\Gamma)$ to itself, our claim follows.

Remark 6.1. According to the above discussion, we have a second way of realizing Lusztig's involution η_λ on the canonical basis, namely as $R^{-1}Y^{-1}$. In some sense, this is the analog of the construction of the evacuation map based on the *promotion* operation (see, for instance, [8, p. 184]). To be more precise, the mentioned procedure has the following three steps.

(1) Perform a sequence of sliding operations into the upper left corner of the given semistandard Young tableau, from which entries are removed successively.
(2) Place the removed entries into the corresponding outside corners that are vacated as a result of the sliding operations.
(3) Complement the entries of the newly obtained filling of the corresponding Young diagram.

In one word, the sliding operations precede the complementation.

We have the following corollary of Propositions 5.5 and 5.6. According to this corollary, the alcove path model reveals an interesting feature of Lusztig's involution, which appears to be new. More precisely, it easily follows from our previous results that the involution $J \mapsto J^*$ interchanges the initial and the final keys in the sense mentioned below.

Corollary 6.2. *For any $J \in \mathcal{A}(\Gamma)$, we have*

$$\mu(J^*) = w_\circ(\mu(J))\,, \quad \kappa_0(J^*) = \lfloor w_\circ \kappa_1(J) \rfloor\,, \quad \kappa_1(J^*) = \lfloor w_\circ \kappa_0(J) \rfloor\,. \tag{6.1}$$

Proof. The first equality follows directly from Proposition 5.6 and the fact that a Yang-Baxter move preserves the weight of an admissible subset (cf. Theorem 4.5). The second equality follows from the definition of J^{rev} in (5.4) combined with the fact that $\kappa_0(J^*) = \kappa_0(J^{\text{rev}})$; the latter claim is a direct consequence of Remark 5.10. The third equality follows from (5.5) and the fact that a Yang-Baxter move preserves the Weyl group element $w(\,\cdot\,)$ associated to an admissible subset (cf. (4.7)). $\square$

Recall the Demazure module $V_{\lambda,u}$ and its character $ch(V_{\lambda,u})$. Theorem 3.6 (2) provides a formula for this character. We now give a new formula, which we prove by setting up a bijection between the combinatorial objects indexing its terms and the combinatorial objects corresponding to the formula in Theorem 3.6 (2).

Theorem 6.3. *For any $u \in W$ and any λ-chain Γ, we have*

$$ch(V_{\lambda,u}) = \sum_{\substack{J \in \mathcal{A}(\Gamma) \\ w(J) \leq u}} e^{\mu(J)}\,.$$

Proof. By (4.7) and Theorem 4.5, it suffices to consider a λ-chain Γ (and the corresponding index set I) having the special form discussed at the beginning of Section 5. Let us assume first that u is a maximal (left) coset representative modulo W_λ. We know from Theorem 3.6 (2) that

$$ch(V_{\lambda,u}) = \sum e^{-ur_{\overline{j}_1}\ldots r_{\overline{j}_a}\widehat{r}_{j_1}\ldots \widehat{r}_{j_s}(-\lambda)}\,, \tag{6.2}$$

where the summation is over all subsets $J = \{\overline{j}_1 < \ldots < \overline{j}_a < j_1 < \ldots < j_s\}$ of I such that we have a saturated decreasing chain in Bruhat order

$$u \gtrdot ur_{\overline{j}_1} \gtrdot \ldots \gtrdot ur_{\overline{j}_1}\ldots r_{\overline{j}_a} \gtrdot ur_{\overline{j}_1}\ldots r_{\overline{j}_a}r_{j_1} \gtrdot \ldots \gtrdot ur_{\overline{j}_1}\ldots r_{\overline{j}_a}r_{j_1}\ldots r_{j_s}\,;$$

here it is assumed that $J \cap \overline{I} = \{\overline{j}_1 < \ldots < \overline{j}_a\}$. Let $u' := w_\circ ur_{\overline{j}_1}\ldots r_{\overline{j}_a}$, which is a minimal coset representative modulo W_λ. There is a unique subset $\{\overline{k}_1 < \ldots < \overline{k}_b\}$ of $\overline{I}$ such that

$$1 \lessdot r_{\overline{k}_1} \lessdot r_{\overline{k}_1}r_{\overline{k}_2} \lessdot \ldots \lessdot r_{\overline{k}_1}\ldots r_{\overline{k}_b} = u'$$

is a saturated increasing chain in Bruhat order from 1 to u' (cf. Dyer [7]). Thus, $K := \{\overline{k}_1 < \ldots < \overline{k}_b < j_1 < \ldots < j_s\}$ is an admissible subset. In fact, the map $J \mapsto K$ is a bijection between the subsets J in (6.2) and the admissible subsets K with $\kappa_0(K) \geq w_\circ u$. Hence we have

$$ch(V_{\lambda,u}) = \sum e^{w_\circ(\mu(K))} = \sum e^{\mu(K^*)}\,,$$

where the summations are over all admissible subsets K with $\kappa_0(K) \geq w_\circ u$. But, by Corollary 6.2 and the properties of the Bruhat order summarized in [6, Lemma 2.1], the latter condition is equivalent to $\kappa_1(K^*) = w(K^*) \leq \lfloor u \rfloor$. The theorem now follows by using the fact that $ch(V_{\lambda,u}) = ch(V_{\lambda,\lceil u \rceil})$, for any u in W, as well as the equivalence of $w(J) \leq u$ and $w(J) \leq \lfloor u \rfloor$, where J is an admissible subset (cf. [6, Lemma 2.1]). $\square$

Remark 6.4. Theorem 6.3 is the analog of the Demazure character formula due to Littelmann [24], [26, Theorem 9.1]. Compared to the Demazure character formula in Theorem 3.6 (2), the one above has the advantage of realizing *all* Demazure characters $ch(V_{\lambda,u})$ (for a fixed λ) in terms of the *same* combinatorial objects, i.e., in terms of certain subsets of $\mathcal{A}(\Gamma)$.

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