HAGLUND-HAIMAN-LOEHR TYPE FORMULAS FOR HALL-LITTLEWOOD POLYNOMIALS OF TYPE B AND C

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Abstract. In previous work we showed that two apparently unrelated formulas for the Hall-Littlewood polynomials of type A are, in fact, closely related. The first is the tableau formula obtained by specializing $q = 0$ in the Haglund-Haiman-Loehr formula for Macdonald polynomials. The second is the type A instance of Schwer's formula (rephrased and rederived by Ram) for Hall-Littlewood polynomials of arbitrary finite type; Schwer's formula is in terms of so-called alcove walks, which originate in the work of Gaussent-Littelmann and of the author with Postnikov on discrete counterparts to the Littelmann path model. We showed that the tableau formula follows by "compressing" Ram's version of Schwer's formula. In this paper, we derive tableau formulas for the Hall-Littlewood polynomials of type B and C by compressing the corresponding instances of Schwer's formula.

1. Introduction

Hall-Littlewood polynomials are at the center of many recent developments in representation theory and algebraic combinatorics. They were originally defined in type A, as a basis for the algebra of symmetric functions depending on a parameter t; this basis interpolates between two fundamental bases: the one of Schur functions, at $t = 0$, and the one of monomial functions, at $t = 1$. Beside the original motivation for defining Hall-Littlewood polynomials which comes from the Hall algebra [18], there are many other applications (see e.g. [12] and the references therein).

Macdonald [20] showed that there is a formula for the spherical functions corresponding to a Chevalley group over a p-adic field which generalizes the formula for the Hall-Littlewood polynomials. Thus, the Macdonald spherical functions generalize the Hall-Littlewood polynomials to all root systems, and the two names are used interchangeably in the literature. There are two families of Hall-Littlewood polynomials of arbitrary type, called P-polynomials and Q-polynomials, which form dual bases for the Weyl group invariants. The P-polynomials specialize to the Weyl characters at $t = 0$. The transition matrix between Weyl characters and P-polynomials is given by Lusztig's t-analog of weight multiplicities (Kostka-Foulkes polynomials of arbitrary type), which are certain affine Kazhdan-Lusztig polynomials [9, 19]. On the combinatorial side, we have the Lascoux-Schützenberger formula for the Kostka-Foulkes polynomials in type A [10], but no generalization of this formula to other types is known. Other applications of the type A Hall-Littlewood polynomials that extend to arbitrary type are those related to fermionic multiplicity formulas

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[2] and affine crystals [11]. We refer to [23, 27] for surveys on Hall-Littlewood polynomials of arbitrary type.

Macdonald [21, 22] defined a remarkable family of orthogonal polynomials depending on parameters q, t, which bear his name. These polynomials generalize the spherical functions for a p-adic group, the Jack polynomials, and the zonal polynomials. At $q = 0$, the Macdonald polynomials specialize to the Hall-Littlewood polynomials, and thus they further specialize to the Weyl characters (upon setting $t = 0$ as well). There has been considerable interest recently in the combinatorics of Macdonald polynomials. This stems in part from a combinatorial formula for the ones corresponding to type A, which is due to Haglund, Haiman, and Loehr [6]. This formula is in terms of fillings of Young diagrams, and uses two statistics, called inv and maj, on such fillings. The Haglund-Haiman-Loehr formula already found important applications, such as new proofs of the positivity theorem for Macdonald polynomials, which states that the two-parameter Kostka-Foulkes polynomials have nonnegative integer coefficients. One of the mentioned proofs, due to Grojnowski and Haiman [5], is based on Hecke algebras, while the other, due to Assaf [3], is purely combinatorial and leads to a positive formula for the two-parameter Kostka-Foulkes polynomials. Moreover, in the one-parameter case (i.e., when $q = 0$), the Haglund-Haiman-Loehr formula was used to give a concise derivation of the Lascoux-Schützenberger formula for the Kostka-Foulkes polynomials of type A [6, Section 7].

An apparently unrelated development, at the level of arbitrary finite root systems, led to Schwer's formula [26], rephrased and rederived by Ram [24], for the Hall-Littlewood polynomials of arbitrary type. The latter formulas are in terms of so-called alcove walks, which originate in the work of Gaussent-Littelmann [4] and of the author with Postnikov [14, 15] on discrete counterparts to the Littelmann path model [16, 17]. Schwer's formula was recently generalized by Ram and Yip to a similar formula for the Macdonald polynomials [25]. The generalization consists in the fact that the latter formula is in terms of alcove walks with both "positive" and "negative" foldings, whereas in the former only "positive" foldings appear.

In [12], we related Schwer's formula to the Haglund-Haiman-Loehr formula. More precisely, we showed that we can group the terms in the type A instance of Schwer's formula (in fact, we used Ram's version of it) for $P_\lambda(x; t)$ into equivalence classes, such that the sum in each equivalence class is a term in the Haglund-Haiman-Loehr formula for $q = 0$. An equivalence class consists of all the terms corresponding to alcove walks that produce the same filling of a Young diagram λ (indexing the Hall-Littlewood polynomial) via a simple construction. In fact, we required that the partition λ has no two parts identical (i.e., it is a regular weight); the general case, which displays additional complexity, will be considered in a future publication. This work was extended in [13], by showing that the type A instance of the Ram-Yip formula for Macdonald polynomials compresses, in a similar way, to a formula which is analogous to the Haglund-Haiman-Loehr one, but has fewer terms.

In this paper we extend the results in [12] to types B and C. More precisely, we derive formulas for the Hall-Littlewood polynomials of type B and C indexed by regular weights in terms of fillings of Young diagrams; we do this by compressing the corresponding instances of Schwer's formula (in fact, we again use Ram's version of it). Note that no tableau formula for the Hall-Littlewood or Macdonald polynomials exists beyond type A so far. Our approach provides a natural way to obtain such formulas, and suggests that this method could be further extended to type D (this case is slightly more complex than types B and C, see below), as well as to Macdonald polynomials; these problems are currently explored, as is the compression in the case of a Hall-Littlewood polynomial indexed by a non-regular weight. Our formula is more complex than the corresponding one in type A (i.e., the Haglund-Haiman-Loehr formula at $q = 0$); however, the statistic we use is, in the case of some special fillings, completely similar to the Haglund-Haiman-Loehr inversion statistic (which is the more intricate of their two statistics). The naturality of our formula is also supported by the fact that the Kashiwara-Nakashima tableaux of type B and C [8] are, essentially, the surviving fillings in this formula when we set $t = 0$. We also note that that the passage from (Ram's version of) Schwer's formula to ours results in a considerably larger reduction in the number of terms in type B and C compared to type A. In terms of applications, it would be very interesting to see whether our formula could be used to derive, in the spirit of [6, Section 7], a positive combinatorial formula for Lusztig's t-analog of weight multiplicities in type B and C, which has been long sought.

2. The tableau formula in type C

Let us start by recalling the Weyl group of type B/C, viewed as the group of signed permutations B_n. Such permutations are bijections w from $[\overline{n}] := \{1 < 2 < \ldots < n < \overline{n} < \overline{n-1} < \ldots < \overline{1}\}$ to $[\overline{n}]$ satisfying $w(\overline{\imath}) = \overline{w(i)}$. Here $\overline{\imath}$ is viewed as $-i$, so $\overline{\overline{\imath}} = i$. We use the window notation $w = w(1)\ldots w(n)$. Given $1 \le i < j \le n$, we denote by (i, j) the reflection which transposes the entries in positions i and j (upon right multiplication). Similarly, we denote by $(i, \overline{\jmath})$, again for $i < j$, the transposition of entries in positions i and j followed by the sign change of those entries. Finally, we denote by $(i, \overline{\imath})$ the sign change in position i. Given w in B_n, we define

$$
\begin{aligned}
\ell_+(w) &:= |\{(k,l) \,:\, 1 \le k < l \le n,\, w(k) > w(l)\}|\,, \\
\ell_-(w) &:= |\{(k,l) \,:\, 1 \le k \le l \le n,\, w(k) > \overline{w(l)}\}|\,.
\end{aligned}
\tag{2.1}
$$

Then the length of w is given by $\ell(w) := \ell_+(w) + \ell_-(w)$.

Let λ be a partition corresponding to a regular weight in type C_n for $n \ge 2$, that is $\lambda = (\lambda_1 > \lambda_2 > \ldots > \lambda_n > 0)$. Consider the shape $\widehat{\lambda}$ obtained from λ by replacing each column of height k with k or $2k-1$ (adjacent) copies of it, depending on the given column being the first one or not. We are representing a filling σ of $\widehat{\lambda}$ as a concatenation of columns C_{ij} and C'_{ik}, where $i = 1, \ldots, \lambda_1$, while for a given i we have $j = 1, \ldots, \lambda'_i$ if $i > 1$, $j = 1$ if $i = 1$, and $k = 2, \ldots, \lambda'_i$; the columns C_{ij} and C'_{ik} have height λ'_i. More precisely, we let

$$
\sigma = \mathcal{C}^{\lambda_1} \ldots \mathcal{C}^1\,,
\tag{2.2}
$$

where

$$
\mathcal{C}^i := \begin{cases} C'_{i2} \ldots C'_{i,\lambda'_i} C_{i1} \ldots C_{i,\lambda'_i} & \text{if } i > 1 \\ C'_{i2} \ldots C'_{i,\lambda'_i} C_{i1} & \text{if } i = 1\,. \end{cases}
$$

Note that the leftmost column is $C_{\lambda_1,1}$, and the rightmost column is C_{11}.

Essentially, the above description says that the column to the right of C_{ij} is $C_{i,j+1}$, whereas the column to the right of C'_{ik} is $C'_{i,k+1}$. Here we are assuming that the mentioned columns exist, up to the following conventions:

$$C_{i,\lambda_i'+1} = \begin{cases} C'_{i-1,2} & \text{if } i>1 \text{ and } \lambda'_{i-1}>1 \\ C_{i-1,1} & \text{if } i>1 \text{ and } \lambda'_{i-1}=1\,, \end{cases} \qquad C'_{i,\lambda_i'+1} = C_{i1}\,. \tag{2.3}$$

We consider the set $\mathcal{F}(\lambda)$ of fillings of $\widehat{\lambda}$ with entries in $[\overline{n}]$ which satisfy the following conditions:

(1) the rows are weakly decreasing from left to right;
(2) no column contains two entries a,b with $a=\pm b$;
(3) each column (with the exception of the leftmost one) is related to its left neighbor as indicated below; essentially, it differs from this neighbor by a "signed cycle", that is, a composition $(r_1,\overline{\jmath})\ldots(r_p,\overline{\jmath})$, where $1\le r_1<\ldots<r_p<j$; furthermore, j varies from 1 to the length of the corresponding column, as we consider the columns from left to right.

Here we let reflections in B_n act on columns C like they do on signed permutations; for instance, $C(a,\overline{b})$ is the column obtained from C by transposing the entries in positions a,b and by changing their signs. Let us first explain the passage from some column C_{ij} to $C_{i,j+1}$. There exist positions $1\le r_1<\ldots<r_p<j$ (possibly $p=0$) such that $C_{i,j+1}$ differs from $D=C_{ij}(r_1,\overline{\jmath})\ldots(r_p,\overline{\jmath})$ only in position j, while $C_{i,j+1}(j)\not\in\{\pm D(r)\;:\;r\in[\lambda_i']\setminus\{j\}\}$ and $C_{i,j+1}(j)\le D(j)$. To include the case $j=\lambda_i'$ in this description, just replace $C_{i,j+1}$ everywhere by $C_{i,j+1}[1,\lambda_i']$ and use the conventions (2.3). Let us now explain the passage from some column C'_{ik} to $C'_{i,k+1}$. There exist positions $1\le r_1<\ldots<r_p<k$ (possibly $p=0$) such that $C'_{i,k+1}=C'_{ik}(r_1,\overline{k})\ldots(r_p,\overline{k})$. This description includes the case $k=\lambda_i'$, based on the conventions (2.3).

Let us now define the content of a filling. For this purpose, we first associate with a filling σ a "compressed" version of it, namely the filling $\overline{\sigma}$ of the partition 2λ. This is defined as follows:

$$\overline{\sigma}=\overline{\mathcal{C}}^{\lambda_1}\ldots\overline{\mathcal{C}}^{1}\,,\quad \text{where } \overline{\mathcal{C}}^{i}:=C'_{i2}C_{i1}\,, \tag{2.4}$$

where the conventions (2.3) are used again. Now define $\mathrm{ct}(\sigma)=(c_1,\ldots,c_n)$, where c_i is half the difference between the number of occurences of the entries i and $\overline{\imath}$ in $\overline{\sigma}$. Sometimes, this vector is written in terms of the coordinate vectors ε_i, as

$$\mathrm{ct}(\sigma)=c_1\varepsilon_1+\ldots+c_n\varepsilon_n=\frac{1}{2}\sum_{b\in\overline{\sigma}}\varepsilon_{\overline{\sigma}(b)}\,; \tag{2.5}$$

here the last sum is over all boxes b of $\overline{\sigma}$, and we set $\varepsilon_{\overline{\imath}}:=-\varepsilon_i$.

We now define two statistics on fillings that will be used in our compressed formula for Hall-Littlewood polynomials. Intervals refer to the discrete set $[\overline{n}]$. Let

$$\sigma_{ab}:=\begin{cases}1 & \text{if } a,b\ge\overline{n}\\ 0 & \text{otherwise}\,.\end{cases} \tag{2.6}$$

Given a sequence of integers w, we write $w[i,j]$ for the subsequence $w(i)w(i+1)\ldots w(j)$. We use the notation $N_{ab}(w)$ for the number of entries $w(i)$ with $a<w(i)<b$.

Given two columns D,C of the same height d such that $D\ge C$ componentwise, we will define two statistics $N(D,C)$ and $\mathrm{des}(D,C)$ in some special cases, as specified below.

Case 0. If $D=C$, then $N(D,C):=0$ and $\mathrm{des}(D,C):=0$.

Case 1. Assume that $C = D(r, \overline{\jmath})$ with $r < j$. Let $a := D(r)$ and $b := D(j)$. In this case, we set

$$N(D, C) := N_{\overline{b}a}(D[r+1, j-1]) + |(\overline{b}, a) \setminus \{\pm D(i) \; : \; i = 1, \ldots, j\}| + \sigma_{ab}\,,$$

and $\mathrm{des}(D, C) := 1$.

Case 2. Assume that $C = D(r_1, \overline{\jmath}) \ldots (r_p, \overline{\jmath})$ where $1 \leq r_1 < \ldots < r_p < j$. Let $D_i := D(r_1, \overline{\jmath}) \ldots (r_i, \overline{\jmath})$ for $i = 0, \ldots, p$, so that $D_0 = D$ and $D_p = C$. We define

$$N(D, C) := \sum_{i=1}^{p} N(D_{i-1}, D_i)\,, \quad \mathrm{des}(D, C) := p\,.$$

Case 3. Assume that C differs from $D' := D(r_1, \overline{\jmath}) \ldots (r_p, \overline{\jmath})$ with $1 \leq r_1 < \ldots < r_p < j$ (possibly $p = 0$) only in position j, while $C(j) \notin \{\pm D'(r) \; : \; r \in [d] \setminus \{j\}\}$ and $C(j) < D'(j)$. We define

$$N(D, C) := N(D, D') + N_{C(j), D'(j)}(D[j+1, d])\,, \quad \mathrm{des}(D, C) := p + 1\,.$$

If the height of C is larger than the height d of D (necessarily by 1), and $N(D, C[1, d])$ can be computed as above, we let $N(D, C) := N(D, C[1, d])$ and $\mathrm{des}(D, C) := \mathrm{des}(D, C[1, d])$. Given a filling σ in $\mathcal{F}(\lambda)$ with columns $C_m, \ldots, C_1$, we set

$$N(\sigma) := \sum_{i=1}^{m-1} N(C_{i+1}, C_i) + \ell_+(C_1)\,;$$

here $\ell_+(C_1)$ is defined like in (2.1). Furthermore, we also set

$$\mathrm{des}(\sigma) := \sum_{i=1}^{m-1} \mathrm{des}(C_{i+1}, C_i)\,.$$

Note that $\mathrm{des}(\sigma)$ essentially counts the descents in the rows of σ.

We can now state our new formula for the Hall-Littlewood polynomials of type C, which follows as a corollary of our main result, i.e., Theorem 4.6. A completely similar formula in type B is discussed in Section 5. We refer to Proposition 2.3 and Remarks 4.7 for more insight into our formula. In particular, note that the Kashiwara-Nakashima tableaux of type C are, essentially, the surviving fillings in this formula when we set $t = 0$, and that, in some special cases, the statistic $N(\sigma)$ is completely similar to the Haglund-Haiman-Loehr inversion statistic (the more intricate of their two statistics).

Theorem 2.1. *Given a regular weight λ, we have*

$$P_\lambda(X; t) = \sum_{\sigma \in \mathcal{F}(\lambda)} t^{N(\sigma)}\,(1-t)^{\mathrm{des}(\sigma)}\,x^{\mathrm{ct}(\sigma)}\,, \tag{2.7}$$

where $x^{(c_1, \ldots, c_n)} := x_1^{c_1} \ldots x_n^{c_n}$.

Example 2.2. Consider the simplest case, namely $n = 2$ and $\lambda = (2, 1)$. This leads to considering fillings of the shape $(3, 2)$ with elements in $[\overline{2}]$, namely

$$\begin{array}{|c|c|c|} \hline e & c & a \\ \hline \end{array}$$
$$\begin{array}{c|c|c|} & d & b \\ \cline{2-3} \end{array}\,.$$

The fillings need to satisfy the following conditions:

(1) $a \leq c \leq e$, $b \leq d$;
(2) $a \neq \pm b$;

(3) either $c = a$ and $d = b$, or $c = \overline{b}$ and $d = \overline{a}$.

For $i \in \{1, 2\}$, let n_i be half the difference between the number of i's and $\overline{\imath}$'s in the multiset $\{a, b, c, d, e, e\}$. Given a proposition A, we let $\chi(A)$ be 1 or 0, depending on the logical value of A being true or false. Then

$$P_{(2,1)}(x_1, x_2; t) = \sum_{(a,b,c,d,e)} t^{\chi(a>b)+\chi(a,b\leq 2,\, a\neq c)}(1-t)^{\chi(a\neq c)+\chi(c\neq e)} x_1^{n_1} x_2^{n_2} \,.$$

It turns out that there are 27 terms in this sum, versus 70 terms in (Ram's version of) Schwer's formula. For instance, the terms contributing to the coefficient of x_2 correspond to the fillings

$$\begin{array}{|c|c|c|}\hline \overline{1} & 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 & 2 \\ \cline{2-3}\end{array}\,, \qquad \begin{array}{|c|c|c|}\hline \overline{1} & 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 1 & 1 \\ \cline{2-3}\end{array}\,, \qquad \begin{array}{|c|c|c|}\hline 2 & 2 & 1 \\ \hline \multicolumn{1}{c|}{} & \overline{1} & \overline{2} \\ \cline{2-3}\end{array}\,;$$

the associated polynomials in t are

$$1-t\,, \qquad t(1-t)\,, \qquad 1-t\,,$$

respectively. Note that these polynomials are obtained by compressing 3, 2, and 2 terms in Schwer's formula, respectively. By symmetry, the coefficients of x_1, x_2, x_1^{-1}, and x_2^{-1} in $P_{(2,1)}(x_1, x_2; t)$ are all $(t+2)(1-t)$. Other fillings have an even larger number of terms in Schwer's formula corresponding to them, such as

$$\begin{array}{|c|c|c|}\hline \overline{1} & \overline{2} & \overline{2} \\ \hline \multicolumn{1}{c|}{} & \overline{1} & \overline{1} \\ \cline{2-3}\end{array}\,,$$

which has 7; in other words, the associated polynomial in t, namely $1 - t$, which contributes to the coefficient of $x_1^{-2}x_2^{-1}$, is the sum of 7 polynomials of the form $t^r(1-t)^s$ in Schwer's formula. In conclusion, we have

$$\begin{aligned} P_{(2,1)}(x_1, x_2; t) = {} & x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_2^{-1} + x_1 x_2^{-2} + x_1^{-1} x_2^2 + x_1^{-2} x_2 + \\ & + x_1^{-1} x_2^{-2} + x_1^{-2} x_2^{-1} + (t+2)(1-t)(x_1 + x_2 + x_1^{-1} + x_2^{-1})\,. \end{aligned}$$

In order to relate our statistic $N(\sigma)$ to the Haglund-Haiman-Loehr *inversion statistic* and to compare our formula to its type A counterpart (see [6][Proposition 8.1] or [12][Theorem 2.10]), let us recall some definitions from [6, 12]. We start by considering fillings τ of the shape λ with entries in $[\overline{n}]$, which are displayed in "Japanese style", as a sequence of columns $\tau = C_{\lambda_1} \dots C_1$; here C_i is a sequence $(C_i(1), \dots, C_i(\lambda'_i))$, so the entry in cell $u = (i, j)$ is $\tau(u) = C_j(i)$. Two cells $u, v \in \lambda$ are said to *attack* each other if they are in one of the following two relative positions:

$$\begin{array}{|c|c|}\hline \bullet & \\ \hline & \\ \hline \bullet & \\ \hline \multicolumn{1}{c|}{} & \\ \cline{2-2}\end{array}\,, \qquad \begin{array}{|c|c|}\hline & \bullet \\ \hline & \\ \hline \bullet & \\ \hline \multicolumn{1}{c|}{} & \\ \cline{2-2}\end{array}\,.$$

An *inversion* of τ is a pair of attacking cells (u, v) which have one of the following two relative positions, where $a := \tau(u) < b := \tau(v)$:

$$\begin{array}{|c|c|}\hline a & \\ \hline & \\ \hline b & \\ \hline \multicolumn{1}{c|}{} & \\ \cline{2-2}\end{array}\,, \qquad \begin{array}{|c|c|}\hline & b \\ \hline & \\ \hline a & \\ \hline \multicolumn{1}{c|}{} & \\ \cline{2-2}\end{array}\,.$$

The Haglund-Haiman-Loehr statistic $\mathrm{inv}(\tau)$ is defined as the number of inversions of τ. The *descent statistic*, denoted $\mathrm{des}(\tau)$ (which is similar to des for fillings of $\widehat{\lambda}$ defined above, see below), is the number of cells $u = (i, j)$ with $j \neq 1$ and

$\tau(u) > \tau(v)$, where $v = (i, j-1)$. Now let, as usual, $n(\lambda) := \sum_i (i-1)\lambda_i$, and assume that τ has the following two properties: (i) $\tau(u) \neq \tau(v)$ whenever u and v attack each other; (ii) τ is weakly decreasing in rows. Then it was shown in [12][Proposition 2.12] that the so-called *complementary inversion statistic* $\operatorname{cinv}(\tau) := n(\lambda) - \operatorname{inv}(\tau)$ counts the triples of cells filled with $a < b < c$ which have the following relative position (here the third cell might be outside the shape λ, in which case we only require $a < b$):

$$\begin{array}{|c|c|}\hline & b \\ \hline & \\ \hline c & a \\ \hline \multicolumn{1}{c|}{} & \\ \cline{2-2}\end{array}\,.$$

Proposition 2.3. *Let σ in $\mathcal{F}(\lambda)$ be a filling satisfying the following properties:* (1) $C'_{i,j+1} = C'_{i,j}$ *for all i and $j = 1, \ldots, \lambda'_i$;* (2) $C_{i,j+1}$ *only differs from C_{ij} in position j. Let $\widetilde{\sigma}$ be the filling of λ given by*

$$\widetilde{\sigma} := C_{\lambda_1,1} C_{\lambda_1-1,1} \ldots C_{11}\,.$$

Then $N(\sigma) = \operatorname{cinv}(\widetilde{\sigma})$ and $\operatorname{des}(\sigma) = \operatorname{des}(\widetilde{\sigma})$.

Proof. The equality $\operatorname{des}(\sigma) = \operatorname{des}(\widetilde{\sigma})$ is clear, so we concentrate on the first one. Let $m := \lambda_1$ be the number of columns of λ, and let $C_m = C_{m1}, \ldots, C_1 = C_{11}$ be the columns of $\widetilde{\sigma}$, of lengths $c_m := \lambda'_m, \ldots, c_1 := \lambda'_1$; let $C'_k := C_k[1, c_{k+1}]$, for $k = 1, \ldots, m-1$. We refer to a pair (i, j) with $1 \leq i < j \leq c_k$ and $C_k(i) > C_k(j)$ as a (type A) inversion in C_k. It is easy to see that $\widetilde{\sigma}$ satisfies the properties considered above: (i) $\widetilde{\sigma}(u) \neq \widetilde{\sigma}(v)$ whenever u and v attack each other; (ii) $\widetilde{\sigma}$ is weakly decreasing in rows. We start by evaluating $N(\mathcal{C}^k C_{k-1,1})$ (see (2.2)). By definition, we have

$$N(\mathcal{C}^k C_{k-1,1}) = \sum_{i=1}^{c_k - 1} N_{C_{k-1}(i), C_k(i)}(C_k[i+1, c_k])\,.$$

This is the number of inversions (i, j) in C_k for which $C_{k-1}(i) < C_k(j)$. If (i, j) is an inversion in C_k not satisfying the previous condition, then $C_{k-1}(i) > C_k(j)$ (by property (i) of $\widetilde{\sigma}$), and thus (i, j) is an inversion in C'_{k-1} (by property (ii) of $\widetilde{\sigma}$). Moreover, the only inversions of C'_{k-1} which do not arise in this way are those counted by the statistic $\operatorname{cinv}(C_k C'_{k-1})$, so

$$N(\mathcal{C}^k C_{k-1,1}) = \ell_+(C_k) - (\ell_+(C'_{k-1}) - \operatorname{cinv}(C_k C'_{k-1}))\,.$$

We conclude that

$$N(\sigma) - \ell_+(C_1) = \sum_{k=2}^{m} \ell_+(C_k) - \ell_+(C'_{k-1}) + \operatorname{cinv}(C_k C'_{k-1})\,.$$

Now recall that λ has no two parts identical. We clearly have $c_m = 1$ so $\ell_+(C_m) = 0$. Therefore, we have

$$\begin{aligned} N(\sigma) &= \sum_{k=2}^{m} \ell_+(C_{k-1}) - \ell_+(C'_{k-1}) + \operatorname{cinv}(C_k C'_{k-1}) = \\ &= \sum_{k=2}^{m} \operatorname{cinv}(C_k C_{k-1}) = \operatorname{cinv}(\widetilde{\sigma})\,. \end{aligned}$$

□

3. Background on Ram's version of Schwer's formula

We recall some background information on finite root systems and affine Weyl groups.

3.1. **Root systems.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra, whose rank is r. Let $\Phi \subset \mathfrak{h}^*$ be the corresponding irreducible *root system*, $\mathfrak{h}^*_{\mathbb{R}} \subset \mathfrak{h}^*$ the real span of the roots, and $\Phi^+ \subset \Phi$ the set of positive roots. Let $\alpha_1, \ldots, \alpha_r \in \Phi^+$ be the corresponding *simple roots*. We denote by $\langle \cdot\,, \cdot \rangle$ the nondegenerate scalar product on $\mathfrak{h}^*_{\mathbb{R}}$ induced by the Killing form. Given a root α, we consider the corresponding *coroot* $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$ and reflection s_α.

Let W be the corresponding *Weyl group*, whose Coxeter generators are denoted, as usual, by $s_i := s_{\alpha_i}$. The length function on W is denoted by $\ell(\cdot)$. The *Bruhat graph* on W is the directed graph with edges $u \to w$ where $w = us_\beta$ for some $\beta \in \Phi^+$, and $\ell(w) > \ell(u)$; we usually label such an edge by β and write $u \xrightarrow{\beta} w$. The *reverse Bruhat graph* is obtained by reversing the directed edges above. The *Bruhat order* on W is the transitive closure of the relation corresponding to the Bruhat graph.

The *weight lattice* Λ is given by

$$\Lambda := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \;:\; \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for any } \alpha \in \Phi\}. \tag{3.1}$$

The weight lattice Λ is generated by the *fundamental weights* $\omega_1, \ldots, \omega_r$, which form the dual basis to the basis of simple coroots, i.e., $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. The set Λ^+ of *dominant weights* is given by

$$\Lambda^+ := \{\lambda \in \Lambda \;:\; \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for any } \alpha \in \Phi^+\}.$$

The subgroup of W stabilizing a weight λ is denoted by W_λ, and the set of minimum coset representatives in W/W_λ by W^λ. Let $\mathbb{Z}[\Lambda]$ be the group algebra of the weight lattice Λ, which has a $\mathbb{Z}$-basis of formal exponents $\{x^\lambda \;:\; \lambda \in \Lambda\}$ with multiplication $x^\lambda \cdot x^\mu := x^{\lambda+\mu}$.

Given $\alpha \in \Phi$ and $k \in \mathbb{Z}$, we denote by $s_{\alpha,k}$ the reflection in the affine hyperplane

$$H_{\alpha,k} := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \;:\; \langle \lambda, \alpha^\vee \rangle = k\}. \tag{3.2}$$

These reflections generate the *affine Weyl group* W_{aff} for the *dual root system* $\Phi^\vee := \{\alpha^\vee \;:\; \alpha \in \Phi\}$. The hyperplanes $H_{\alpha,k}$ divide the real vector space $\mathfrak{h}^*_{\mathbb{R}}$ into open regions, called *alcoves.* The *fundamental alcove* $A_\circ$ is given by

$$A_\circ := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \;:\; 0 < \langle \lambda, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Phi^+\}.$$

3.2. **Alcove walks.** We say that two alcoves A and B are *adjacent* if they are distinct and have a common wall. Given a pair of adjacent alcoves $A \neq B$ (i.e., having a common wall), we write $A \xrightarrow{\beta} B$ if the common wall is of the form $H_{\beta,k}$ and the root $\beta \in \Phi$ points in the direction from A to B.

Definition 3.1. *An* alcove path *is a sequence of alcoves such that any two consecutive ones are adjacent. We say that an alcove path $(A_0, A_1, \ldots, A_m)$ is reduced if m is the minimal length of all alcove paths from A_0 to A_m.*

We need the following generalization of alcove paths.

Definition 3.2. *An* alcove walk *is a sequence* $\Omega = (A_0, F_1, A_1, F_2, \ldots, F_m, A_m, F_\infty)$ *such that* $A_0, \ldots, A_m$ *are alcoves;* F_i *is a codimension one common face of the alcoves* A_{i-1} *and* A_i*, for* $i = 1, \ldots, m$*; and* F_∞ *is a vertex of the last alcove* A_m*. The weight* F_∞ *is called the* weight *of the alcove walk, and is denoted by* $\mu(\Omega)$*.*

The *folding operator* ϕ_i is the operator which acts on an alcove walk by leaving its initial segment from A_0 to A_{i-1} intact and by reflecting the remaining tail in the affine hyperplane containing the face F_i. In other words, we define

$$\phi_i(\Omega) := (A_0, F_1, A_1, \ldots, A_{i-1}, F'_i = F_i, A'_i, F'_{i+1}, A'_{i+1}, \ldots, A'_m, F'_\infty)\,;$$

here $A'_j := \rho_i(A_j)$ for $j \in \{i, \ldots, m\}$, $F'_j := \rho_i(F_j)$ for $j \in \{i, \ldots, m\} \cup \{\infty\}$, and ρ_i is the affine reflection in the hyperplane containing F_i. Note that any two folding operators commute. An index j such that $A_{j-1} = A_j$ is called a *folding position* of Ω. Let $\mathrm{fp}(\Omega) := \{j_1 < \ldots < j_s\}$ be the set of folding positions of Ω. If this set is empty, Ω is called *unfolded*. Given this data, we define the operator "unfold", producing an unfolded alcove walk, by

$$\mathrm{unfold}(\Omega) = \phi_{j_1} \ldots \phi_{j_s}(\Omega)\,.$$

Definition 3.3. *An alcove walk* $\Omega = (A_0, F_1, A_1, F_2, \ldots, F_m, A_m, F_\infty)$ *is called* positively folded *if, for any folding position* j*, the alcove* $A_{j-1} = A_j$ *lies on the positive side of the affine hyperplane containing the face* F_j*.*

We now fix a dominant weight λ and a reduced alcove path $\Pi := (A_0, A_1, \ldots, A_m)$ from $A_\circ = A_0$ to its translate $A_\circ + \lambda = A_m$. Assume that we have

$$A_0 \xrightarrow{\beta_1} A_1 \xrightarrow{\beta_2} \ldots \xrightarrow{\beta_m} A_m\,,$$

where $\Gamma := (\beta_1, \ldots, \beta_m)$ is a sequence of positive roots. This sequence, which determines the alcove path, is called a λ-*chain* (of roots). Two equivalent definitions of λ-chains (in terms of reduced words in affine Weyl groups, and an interlacing condition) can be found in [14][Definition 5.4] and [15][Definition 4.1 and Proposition 4.4]; note that the λ-chains considered in the mentioned papers are obtained by reversing the ones in the present paper. We also let $r_i := s_{\beta_i}$, and let $\widehat{r}_i$ be the affine reflection in the common wall of A_{i-1} and A_i, for $i = 1, \ldots, m$; in other words, $\widehat{r}_i := s_{\beta_i, l_i}$, where $l_i := |\{j \le i \,:\, \beta_j = \beta_i\}|$ is the cardinality of the corresponding set. Given $J = \{j_1 < \ldots < j_s\} \subseteq [m] := \{1, \ldots, m\}$, we define the Weyl group element $\phi(J)$ and the weight $\mu(J)$ by

$$\phi(J) := r_{j_1} \ldots r_{j_s}\,, \qquad \mu(J) := \widehat{r}_{j_1} \ldots \widehat{r}_{j_s}(\lambda)\,. \tag{3.3}$$

Given $w \in W$, we define the alcove path $w(\Pi) := (w(A_0), w(A_1), \ldots, w(A_m))$. Consider the set of alcove paths

$$\mathcal{P}(\Gamma) := \{w(\Pi) \,:\, w \in W^\lambda\}\,.$$

We identify any $w(\Pi)$ with the obvious unfolded alcove walk of weight $\mu(w(\Pi)) := w(\lambda)$. Let us now consider the set of alcove walks

$$\mathcal{F}_+(\Gamma) := \{\,\text{positively folded alcove walks } \Omega \,:\, \mathrm{unfold}(\Omega) \in \mathcal{P}(\Gamma)\}\,.$$

We can encode an alcove walk Ω in $\mathcal{F}_+(\Gamma)$ by the pair (w, J) in $W^\lambda \times 2^{[m]}$, where

$$\mathrm{fp}(\Omega) = J \quad \text{and} \quad \mathrm{unfold}(\Omega) = w(\Pi)\,.$$

Clearly, we can recover Ω from (w, J) with $J = \{j_1 < \ldots < j_s\}$ by

$$\Omega = \phi_{j_1} \ldots \phi_{j_s}(w(\Pi))\,.$$

Let $\mathcal{A}(\Gamma)$ be the image of $\mathcal{F}_+(\Gamma)$ under the map $\Omega \mapsto (w,J)$. We call a pair (w,J) in $\mathcal{A}(\Gamma)$ an *admissible pair*, and the subset $J \subseteq [m]$ in this pair a *w-admissible subset*. The following straightforward result is taken from [12].

Proposition 3.4. [12] (1) *We have*

$$\mathcal{A}(\Gamma) = \{\, (w,J) \in W^\lambda \times 2^{[m]} \;:\; J = \{j_1 < \ldots < j_s\}\,, \quad w > wr_{j_1} > \ldots > wr_{j_1}\ldots r_{j_s} = w\phi(J)\}\,; \tag{3.4}$$

here the decreasing chain is in the Bruhat order on the Weyl group, its steps not being covers necessarily.

(2) *If $\Omega \mapsto (w,J)$, then*

$$\mu(\Omega) = w(\mu(J))\,.$$

The formula for the Hall-Littlewood P-polynomials in [26] was rederived in [24] in a slightly different version, based on positively folded alcove walks. Based on Proposition 3.4, we now restate the latter formula in terms of admissible pairs.

Theorem 3.5. [24, 26] *Given a dominant weight λ, we have*

$$P_\lambda(X;t) = \sum_{(w,J)\in\mathcal{A}(\Gamma)} t^{\frac{1}{2}(\ell(w)+\ell(w\phi(J))-|J|)}\,(1-t)^{|J|}\,x^{w(\mu(J))}\,. \tag{3.5}$$

4. Specializing Ram's version of Schwer's formula to type C

We now restrict ourselves to the root system of type C_n. We can identify the space $\mathfrak{h}^*_{\mathbb{R}}$ with $V := \mathbb{R}^n$, the coordinate vectors being $\varepsilon_1,\ldots,\varepsilon_n$. The root system Φ can be represented as $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j \,:\, 1 \le i < j \le n\} \cup \{\pm 2\varepsilon_i \,:\, 1 \le i \le n\}$. The simple roots are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $i = 1,\ldots,n-1$ and $\alpha_n = 2\varepsilon_n$. The fundamental weights are $\omega_i = \varepsilon_1 + \ldots + \varepsilon_i$, for $i = 1,\ldots,n$. The weight lattice is $\Lambda = \mathbb{Z}^n$. A dominant weight $\lambda = \lambda_1\varepsilon_1 + \ldots + \lambda_{n-1}\varepsilon_{n-1} + \lambda_n\varepsilon_n$ is identified with the partition $(\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_{n-1} \ge \lambda_n \ge 0)$ of length at most n. A dominant weight is regular if all the previous inequalities are strict (i.e., the corresponding partition has all parts distinct and nonzero). We fix such a partition λ for the remainder of this paper.

The corresponding Weyl group W is the group of signed permutations B_n. For simplicity, we use the same notation for roots and the corresponding reflections, cf. Section 2. For instance, given $1 \le i < j \le n$, we denote by (i,j) the positive root $\varepsilon_i - \varepsilon_j$, by $(i,\overline{j})$ the positive root $\varepsilon_i + \varepsilon_j$, and by $(i,\overline{i})$ the positive root $2\varepsilon_i$.

Let

$$\Gamma(k) := \Gamma'_2 \ldots \Gamma'_k \Gamma_1(k) \ldots \Gamma_k(k)\,,$$

where

$$\begin{array}{lllll} \Gamma'_j := ((1,\overline{j}), (2,\overline{j}), \ldots, (j-1,\overline{j}))\,, & & & & \\ \Gamma_j(k) := (\,(1,\overline{j}), & (2,\overline{j}), & \ldots, & (j-1,\overline{j}), \\ \qquad\qquad (j,\overline{k+1}), & (j,\overline{k+2}), & \ldots, & (j,\overline{n}), \\ \qquad\qquad (j,\overline{j}), & & & \\ \qquad\qquad (j,n), & (j,n-1), & \ldots, & (j,k+1)\,)\,. \end{array}$$

Lemma 4.1. *$\Gamma(k)$ is an ω_k-chain.*

Proof. We use the criterion for λ-chains in [15][Definition 4.1, Proposition 4.4], cf. [15][Proposition 10.2]. This criterion says that a chain of roots Γ is a λ-chain if and only if it satisfies the following conditions:

(R1) The number of occurrences of any positive root α in Γ is $\langle \lambda, \alpha^\vee \rangle$.

(R2) For each triple of positive roots $(\alpha,\, \beta,\, \gamma)$ with $\gamma^\vee = \alpha^\vee + \beta^\vee$, the subsequence of Γ consisting of α, β, γ is a concatenation of pairs (γ, α) and (γ, β) (in any order).

Letting $\lambda = \omega_k = \varepsilon_1 + \ldots + \varepsilon_k$, condition (R1) is easily checked; for instance, a root $(a, \overline{b})$ appears twice in $\Gamma(k)$ if $a < b \le k$, once if $a \le k < b$, and zero times otherwise. For condition (R2), we use a case by case analysis, as follows, where $a < b < c$:

(1) $\alpha = (a, b)$, $\beta = (b, c)$, $\gamma = (a, c)$;
(2) $\alpha = (a, b)$, $\beta = (b, \overline{c})$, $\gamma = (a, \overline{c})$;
(3) $\alpha = (a, c)$, $\beta = (b, \overline{c})$, $\gamma = (a, \overline{b})$;
(4) $\alpha = (b, c)$, $\beta = (a, \overline{c})$, $\gamma = (a, \overline{b})$;
(5) $\alpha = (a, b)$, $\beta = (b, \overline{b})$, $\gamma = (a, \overline{a})$;
(6) $\alpha = (a, \overline{a})$, $\beta = (b, \overline{b})$, $\gamma = (a, \overline{b})$.

Case (1) is the same as in type A. Each of the cases (2)-(4) has the following three subcases: $k \ge c$, $b \le k < c$, and $a \le k < b$, while each of the cases (5)-(6) has the following two subcases: $k \ge b$, and $a \le k < b$. For instance, if $b \le k < c$ in Case (3), then the subsequence of $\Gamma(k)$ consisting of α, β, γ is $((a, \overline{b}),\, (a, c),\, (a, \overline{b}),\, (b, \overline{c}))$. $\square$

Hence, we can construct a λ-chain as a concatenation $\Gamma := \Gamma^{\lambda_1} \ldots \Gamma^1$, where

$$\Gamma^i = \Gamma(\lambda_i') = \Gamma_{i2}' \ldots \Gamma_{i,\lambda_i'}' \Gamma_{i1} \ldots \Gamma_{i,\lambda_i'}\,, \quad \text{and } \Gamma_{ij} = \Gamma_j(\lambda_i')\,,\ \Gamma_{ij}' = \Gamma_j'\,. \tag{4.1}$$

This λ-chain is fixed for the remainder of this paper. Thus, we can replace the notation $\mathcal{A}(\Gamma)$ with $\mathcal{A}(\lambda)$.

Example 4.2. Consider $n = 3$ and $\lambda = (3, 2, 1)$, for which we have the λ-chain below. The factorization of Γ into subchains is indicated with vertical bars, while the double vertical bars separate the subchains corresponding to different columns. The underlined pairs are only relevant in Example 4.3 below.

$$\begin{aligned} \Gamma &= \Gamma_{31} \,||\, \Gamma_{22}'\Gamma_{21}\Gamma_{22} \,||\, \Gamma_{12}'\Gamma_{13}'\Gamma_{11}\Gamma_{12}\Gamma_{13} = \\ &= ((1,\overline{2}), \underline{(1,\overline{3})}, (1,\overline{1}), (1,3), (1,2) \,|| \\ &\quad \underline{(1,\overline{2})} \,|\, (1,\overline{3}), (1,\overline{1}), (1,3) \,|\, (1,\overline{2}), (2,\overline{3}), \underline{(2,\overline{2})}, \underline{(2,3)} \,|| \\ &\quad (1,\overline{2}) \,|\, (1,\overline{3}), (2,\overline{3}) \,|\, (1,\overline{1}) \,|\, (1,\overline{2}), (2,\overline{2}) \,|\, (1,\overline{3}), (2,\overline{3}), (3,\overline{3}))\,. \end{aligned} \tag{4.2}$$

We represent the Young diagram of λ inside a broken 3×2 rectangle, as below. In this way, a reflection in Γ can be viewed as swapping entries and/or changing signs in the two parts of each column, or only the top part.

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 & 2 \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 \\ \cline{3-3} \end{array}$$

$$\begin{array}{|c|c} \cline{1-1} 2 & \\ \hline 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

Given the λ-chain Γ above, in Section 3.2 we considered subsets $J = \{j_1 < \ldots < j_s\}$ of $[m]$, where m is the length of Γ. Instead of J, it is now convenient to use the subsequence T of the roots in Γ whose positions are in J. This is viewed as a concatenation with distinguished factors T_{ij} and T_{ik}' induced by the factorization (4.1) of Γ.

All the notions defined in terms of J are now redefined in terms of T. As such, from now on we will write $\phi(T)$, $\mu(T)$, and $|T|$, the latter being the size of T, cf. (3.3). If J is a w-admissible subset for some w in B_n, we will also call the corresponding T a *w-admissible sequence*, and (w,T) an admissible pair. We will use the notation $\mathcal{A}(\Gamma)$ and $\mathcal{A}(\lambda)$ accordingly. We denote by $wT_{\lambda_1,1}\ldots T_{ij}$ and $wT_{\lambda_1,1}\ldots T'_{ik}$ the permutations obtained from w via right multiplication by the transpositions in $T_{\lambda_1,1},\ldots,T_{ij}$ and $T_{\lambda_1,1},\ldots,T'_{ik}$, considered from left to right. This agrees with the above convention of using pairs to denote both roots and the corresponding reflections. As such, $\phi(T)$ can now be written simply T.

Example 4.3. We continue Example 4.2, by picking the admissible pair (w,J) with $w=\overline{1}\,\overline{2}\,\overline{3}\in B_3$ and $J=\{2,6,12,13\}$ (see the underlined positions in (4.2)). Thus, we have

$$T=T_{31}||T'_{22}T_{21}T_{22}||T'_{12}T'_{13}T_{11}T_{12}T_{13}=((1,\overline{3})||(1,\overline{2})|\ \ |(2,\overline{2}),(2,3)||\ \ |\ \ |\ \ |\ \ |\ \ ).$$

The corresponding decreasing chain in Bruhat order is the following, where the swapped entries are shown in bold (we represent permutations as broken columns starting with w, as discussed in Example 4.2, and we display the splitting of the chain into subchains induced by the above splitting of T):

$$\begin{array}{ccccccccccccccccccccccccc}
\boxed{\overline{\mathbf{1}}} & & \boxed{3} & & \boxed{\mathbf{3}} & & \boxed{2} & & \boxed{2} & & \boxed{2} & & \boxed{2} & & \boxed{2} & & \boxed{2} & & \boxed{2} & & \boxed{2} & & \boxed{2} & & \boxed{2} \\
 & > & & || & \boxed{\overline{\mathbf{2}}} & > & \boxed{\overline{3}} & | & \boxed{\overline{3}} & | & \boxed{\overline{\mathbf{3}}} & > & \boxed{\mathbf{3}} & > & \boxed{1} & || & \boxed{1} & | & \boxed{1} & | & \boxed{1} & | & \boxed{1} & | & \boxed{1} \\
\boxed{\overline{2}} & & \boxed{\overline{2}} & & & & & & & & & & & & & & \boxed{3} & & \boxed{3} & & \boxed{3} & & \boxed{3} & & \boxed{3} \\
\boxed{\overline{\mathbf{3}}} & & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{\mathbf{1}} & & \boxed{3} & & & & & & & & & &
\end{array}\;.$$

Given a (not necessarily admissible) pair (w,T), with T split into factors T_{ij} and T'_{ik} as above, we consider the permutations

$$\pi_{ij}=\pi_{ij}(w,T):=wT_{\lambda_1,1}\ldots T_{i,j-1}\,,\qquad \pi'_{ik}=\pi'_{ik}(w,T):=wT_{\lambda_1,1}\ldots T'_{i,k-1}\,;$$

when undefined, $T_{i,j-1}$ and $T'_{i,k-1}$ are given by conventions similar to (2.3), based on the corresponding factorization (4.1) of the λ-chain Γ. In particular, $\pi_{\lambda_1,1}=w$.

Definition 4.4. *The* filling map *is the map f from pairs (w,T), not necessarily admissible, to fillings $\sigma=f(w,T)$ of the shape $\widehat{\lambda}$, defined (based on the notation* (2.2)*) by*

$$C_{ij}=\pi_{ij}[1,\lambda'_i]\,,\qquad C'_{ik}=\pi'_{ik}[1,\lambda'_i]\,. \tag{4.3}$$

Example 4.5. Given (w,T) as in Example 4.3, we have

$$f(w,T)=\begin{array}{|c|c|c|c|c|c|c|}\hline \overline{1} & 3 & 2 & 2 & 2 & 2 & 2 \\ \hline \multicolumn{1}{c|}{} & \overline{2} & \overline{3} & \overline{3} & 1 & 1 & 1 \\ \cline{2-7} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & 3 & 3 \\ \cline{5-7}\end{array}\,.$$

The following theorem describes the way in which our tableau formula (2.7) is obtained by compressing Ram's version of Schwer's formula (3.5). Recall that λ is a regular weight, so $B_n^\lambda=B_n$, and thus the pairs (w,J) in $\mathcal{A}(\lambda)$ are only subject to the decreasing chain condition in (3.4); this fact is implicitly used in the proof of the theorem.

Theorem 4.6. (1) *We have $f(\mathcal{A}(\lambda))=\mathcal{F}(\lambda)$.*

(2) *Given any $\sigma\in\mathcal{F}(\lambda)$ and $(w,T)\in f^{-1}(\sigma)$, we have* $\mathrm{ct}(f(w,T))=w(\mu(T))$.

(3) *The following compression formula holds:*

$$\sum_{(w,T)\in f^{-1}(\sigma)} t^{\frac{1}{2}(\ell(w)+\ell(wT)-|T|)}\,(1-t)^{|T|}=t^{N(\sigma)}\,(1-t)^{\mathrm{des}(\sigma)}\,. \tag{4.4}$$

Proof. We start with part (1). The fact that $f(\mathcal{A}(\lambda)) \subseteq \mathcal{F}(\lambda)$ is clear from the definition of the set of fillings $\mathcal{F}(\lambda)$ in Section 2 and the construction (4.1) of the fixed λ-chain Γ. Viceversa, given a filling σ in $\mathcal{F}(\lambda)$, it is not hard to construct an admissible pair (w,T) in $f^{-1}(\sigma)$. We will assign to the columns C_{ij} and C'_{ij} signed permutations ρ_{ij} and ρ'_{ij} in B_n recursively, starting with $\rho_{11} := C_{11}$; in parallel, we construct the reverse $\mathrm{rev}(T)$ of the mentioned chain of roots T, and conclude by letting $w := \rho_{\lambda_1,1}$. Each time we pass to the left neighbor C'_{ik} of a column $C'_{i,k+1} = C'_{ik}(r_1,\overline{k})\ldots(r_p,\overline{k})$, we append to the part of $\mathrm{rev}(T)$ already constructed the roots $(r_p,\overline{k}),\ldots,(r_1,\overline{k})$ and let $\rho'_{ik} := \rho'_{i,k+1}(r_p,\overline{k})\ldots(r_1,\overline{k})$. We proceed similarly when passing to the left neighbor C_{ij} of a column $C_{i,j+1}$, where $C_{i,j+1}$ differs from $D = C_{ij}(r_1,\overline{\jmath})\ldots(r_p,\overline{\jmath})$ only in position j; the only difference is that, in this case, we start by applying to $\rho_{i,j+1}$ and appending to $\mathrm{rev}(T)$ the reflection which exchanges the entry $C_{i,j+1}(j)$ with $D(j)$, and then we proceed as above.

Parts (2) and (3) of the theorem are proved in Section 6 and Section 7, respectively. □

Remarks 4.7. (1) The Kashiwara-Nakashima tableaux [8] of shape λ index the basis elements of the irreducible representation of $\mathfrak{sp}_{2n}$ of highest weight λ. It is shown in Proposition 4.8 below that these tableaux correspond precisely to the surviving fillings in our formula (2.7) when we set $t = 0$.

(2) In (4.4), in general, we cannot replace the filling map f with the map $\overline{f}$, sending (w,T) to the compressed version $\overline{f(w,T)}$ of $f(w,T)$. Indeed, consider $n = 2$, $\lambda = (3,2)$, and the following filling of $2\lambda = (6,4)$, which happens to be the "doubled" version of a Kashiwara-Nakashima tableau:

$$\overline{\sigma} = \begin{array}{|c|c|c|c|c|c|} \hline \overline{2} & \overline{2} & \overline{2} & \overline{2} & 1 & 1 \\ \hline \multicolumn{2}{c|}{} & \overline{1} & \overline{1} & 2 & 2 \\ \cline{3-6} \end{array}\,.$$

If $(w,T) \in \overline{f}^{-1}(\overline{\sigma})$, we must have $w = \overline{21}$ and

$$T \subseteq \Gamma_{21}\Gamma_{22} = ((1,\overline{1}) \mid (1,\overline{2}),(2,\overline{2}))\,,$$

where the full λ-chain factors as follows:

$$\Gamma = \Gamma_{31} \,||\, \Gamma'_{22}\Gamma_{21}\Gamma_{22} \,||\, \Gamma'_{12}\Gamma_{11}\Gamma_{12}\,.$$

There are two elements (w,T^1) and (w,T^2) in $\overline{f}^{-1}(\overline{\sigma})$, namely

$$T^1 = ((1,\overline{2}))\,, \quad \text{and} \quad T^2 = ((1,\overline{1}),(1,\overline{2}),(2,\overline{2}))\,.$$

But we have

$$\sum_{(w,T)\in\overline{f}^{-1}(\overline{\sigma})} t^{\frac{1}{2}(\ell(w)+\ell(wT)-|T|)}\,(1-t)^{|T|} = t(1-t)+(1-t)^3 = (1-t)(1-t+t^2)\,.$$

In general, the above sum has several factors not of the form t or $1-t$, which are hard to control.

(3) In order to measure the compression phenomenon, we define the *compression factor* $c(\lambda)$ like in [12], as the ratio of the number of terms in Ram's version of Schwer's formula for λ and the number of terms in the tableau formula. The compression factor is considerably larger in type C. For instance, for $\lambda = (3,2,1,0)$ in C_4 we have 23,495 terms in the compressed formula, while $c(\lambda) = 44.9$.

Proposition 4.8. *The map $\sigma \mapsto \overline{\sigma}$ (see (2.4)) is a bijection between the fillings σ in $\mathcal{F}(\lambda)$ with $N(\sigma) = 0$ and the "doubled" versions of the type C Kashiwara-Nakashima tableaux of shape λ.*

Proof. It was proved in [1] that for each type C Kashiwara-Nakashima tableau of shape λ there is a unique admissible pair (w, T) whose associated chain in Bruhat order is saturated and ends at the identity, such that the compressed version $\overline{\sigma}$ of $\sigma = f(w, T)$ is the "doubled" version of the given tableau. It follows that the term associated to (w, T) in (3.5) is $t^0(1-t)^{|T|}x^{w(\mu(T))}$, and therefore $N(\sigma) = 0$, by (4.4). On the other hand, since $P_\lambda(x; 0)$ is the irreducible character indexed by λ, which is expressed in terms of Kashiwara-Nakashima tableaux, we conclude that all σ in $\mathcal{F}(\lambda)$ with $N(\sigma) = 0$ arise in this way. □

5. The tableau formula in type B

We now restrict ourselves to the root system of type B_n. This can be represented as $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j \,:\, 1 \le i < j \le n\} \cup \{\pm\varepsilon_i \,:\, 1 \le i \le n\}$. The simple roots are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $i = 1, \ldots, n-1$ and $\alpha_n = \varepsilon_n$. The fundamental weights are $\omega_i = \varepsilon_1 + \ldots + \varepsilon_i$, for $i = 1, \ldots, n-1$ and $\omega_n = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n)$. A dominant weight $\lambda = \alpha_1\omega_1 + \ldots + \alpha_n\omega_n$, where $\alpha_i \in \mathbb{Z}_{\ge 0}$, is identified with the partition $\mu = (n^{\alpha_n}, \ldots, 1^{\alpha_1})$; we let $\ell(\mu) := \alpha_1 + \ldots + \alpha_n$, and write $\mu = (\mu_1, \ldots, \mu_{\ell(\mu)})$. A dominant weight is regular if $\alpha_i > 0$ for all i. Let us now fix such a weight λ.

The corresponding Weyl group W is the same group of signed permutations B_n considered above. For simplicity, we again use the same notation for roots and the corresponding reflections, cf. Section 2. The pairs (i, j) and $(i, \overline{\jmath})$ have the same meaning as in type C, whereas (i) denotes the positive root ε_i. Note that, as a reflection in B_n, (i) is the same as $(i, \overline{\imath})$ in type C.

The canonical ω_k-chains and λ-chains are very similar to those in type C. If $k < n$, let

$$\Gamma(k) := \Gamma'_1 \ldots \Gamma'_k \Gamma_1(k) \ldots \Gamma_k(k)\,,$$

where

$$\begin{array}{rllll}
\Gamma'_j := & ((1,\overline{\jmath}), (2,\overline{\jmath}), \ldots, (j-1,\overline{\jmath}), (j))\,, & & & \\
\Gamma_j(k) := (& (1,\overline{\jmath}), & (2,\overline{\jmath}), & \ldots, & (j-1,\overline{\jmath}), \\
 & (j,\overline{k+1}), & (j,\overline{k+2}), & \ldots, & (j,\overline{n}), \\
 & (j), & & & \\
 & (j,n), & (j,n-1), & \ldots, & (j,k+1)\,)\,.
\end{array}$$

On the other hand, we let

$$\Gamma(n) := \Gamma'_1 \ldots \Gamma'_n = \Gamma_1(n) \ldots \Gamma_n(n)\,.$$

Like in the type C case, we can prove that $\Gamma(k)$ is an ω_k-chain for any k. Hence, we can construct a λ-chain as a concatenation $\Gamma := \Gamma^{\ell(\mu)} \ldots \Gamma^1$, where $\Gamma^i = \Gamma(\mu_i)$.

The filling map is defined like in Definition 4.4. This gives rise to fillings

$$\sigma = \mathcal{C}^{\ell(\mu)} \ldots \mathcal{C}^1\,,$$

where each $\mathcal{C}^i$ is a concatenation of columns of height μ_i, as follows:

$$\mathcal{C}^i := \begin{cases} C'_{i1} \ldots C'_{i,\mu_i} C_{i1} \ldots C_{i,\mu_i} & \text{if } \mu_i < n \\ C_{i1} \ldots C_{i,\mu_i} & \text{if } i \ne 1 \text{ and } \mu_i = n \\ C_{11} & \text{if } i = 1\,. \end{cases}$$

The fillings are subject to the same conditions (1)-(3) as in type C, where condition (3) is made more precise below. In fact, the above λ-chain Γ specifies the way in which each column is related to its left neighbor. Essentially, everything is similar to type C, except for a small difference in the passage from some column C'_{ik} to $C'_{i,k+1}$. Namely, there exist positions $1 \le r_1 < \ldots < r_p < k$ (possibly $p = 0$) such that $C'_{i,k+1} = C'_{ik}(r_1, \overline{k}) \ldots (r_p, \overline{k})$, like in type C, or $C'_{i,k+1} = C'_{ik}(r_1, \overline{k}) \ldots (r_p, \overline{k})(k)$, in which case we also require $C'_{i,k+1}(k) \le n$.

The weight of a filling, and the statistics $N(\sigma)$ and $\operatorname{des}(\sigma)$ are defined completely similarly to type C. The only minor addition is the definition of $N(D, C)$ and $\operatorname{des}(D, C)$ when $C = D(r_1, \overline{k}) \ldots (r_p, \overline{k})(k)$. With the notation in Case 2 of the definition of $N(D, C)$ in Section 2, we set

$$N(D, C) := N(D, D_p) + N(D_p, C)\,, \quad \operatorname{des}(D, C) := p + 1\,.$$

Here $N(D, D_p)$ is defined in Case 2 above, whereas $N(D_p, C)$ is given by the second formula in (7.1); more precisely,

$$N(D_p, C) := \frac{1}{2}|(\overline{a}, a) \setminus \{\pm D_p(i) \,:\, i = 1, \ldots, k\}|\,,$$

where $a := D_p(k)$.

Given the above constructions, the proof of the following theorem is completely similar to its counterparts in type C, since no new situations arise.

Theorem 5.1. *Theorems* 2.1 *and* 4.6 *hold in type B, with the appropriate constructions explained above.*

Remark 5.2. The situation in type D is slightly more complex. In this case, applying the above ideas leads to an analog of the compression formula (4.4) which contains factors of the form $1 - t^k$ with $k > 1$ in the right-hand side. However, these factors are not hard to control, while no extra factors appear.

6. Proof of Theorem 4.6 (2)

Recall the λ-chain Γ in Section 4. Let us write $\Gamma = (\beta_1, \ldots, \beta_m)$, as in Section 3.2. As such, we recall the hyperplanes H_{β_k, l_k} and the corresponding affine reflections $\widehat{r}_k = s_{\beta_k, l_k} = s_{\beta_k} + l_k \beta_k$.

Now fix a signed permutation w in B_n and a subset $J = \{j_1 < \ldots < j_s\}$ of $[m]$ (not necessarily w-admissible). Let Π be the alcove path corresponding to Γ, and define the alcove walk Ω as in Section 3.2, by

$$\Omega := \phi_{j_1} \ldots \phi_{j_s}(w(\Pi))\,.$$

Given k in $[m]$, let $i = i(k)$ be the largest index in $[s]$ for which $j_i < k$, and let $\gamma_k := w r_{j_1} \ldots r_{j_i}(\beta_k)$. Then the hyperplane containing the face F_k of Ω (cf. Definition 3.2) is of the form H_{γ_k, m_k}; in other words

$$H_{\gamma_k, m_k} = w \widehat{r}_{j_1} \ldots \widehat{r}_{j_i}(H_{\beta_k, l_k})\,.$$

Our first goal is to describe m_k purely in terms of the filling associated to (w, J).

Let $\widehat{t}_k$ be the affine reflection in the hyperplane H_{γ_k, m_k}. Note that

$$\widehat{t}_k = w \widehat{r}_{j_1} \ldots \widehat{r}_{j_i} \widehat{r}_k \widehat{r}_{j_i} \ldots \widehat{r}_{j_1} w^{-1}\,.$$

Thus, we can see that

$$w \widehat{r}_{j_1} \ldots \widehat{r}_{j_i} = \widehat{t}_{j_i} \ldots \widehat{t}_{j_1} w\,.$$

Let $T=((a_1,b_1),\ldots,(a_s,b_s))$ be the subsequence of Γ indexed by the positions in J, cf. Section 4. Let T^i be the initial segment of T with length i, let $w_i:=wT^i$, and $\sigma_i:=\overline{f(w,T^i)}$, see (2.4). In particular, σ_0 is the filling with all entries in row i equal to $w(i)$, and $\sigma:=\sigma_s=\overline{f(w,T)}$. The columns of a filling of 2λ are numbered left to right by $2\lambda_1$ to 1. We split each segment Γ^k of Γ into two parts: the head which is a concatenation of $\Gamma'_{k\cdot}$, and the tail which is a concatenation of $\Gamma_{k\cdot}$, see (4.1). We say that the head corresponds to column $2k$ of the Young diagram 2λ, whereas the tail corresponds to column $2k-1$ (cf. the construction of $f(w,T)$ in Section 4 and (2.4)). If $\beta_{j_{i+1}}=(a_{i+1},b_{i+1})=(a,b)$ falls in the segment of Γ corresponding to column p of 2λ, then σ_{i+1} is obtained from σ_i by replacing the entry $w_i(a)$ with $w_i(b)$ in the columns $p-1,\ldots,1$ of σ_i, as well as, possibly, the entry $w_i(\overline{b})$ with $w_i(\overline{a})$ in the same columns.

Now fix a position k, and consider $i=i(k)$ and the roots β_k, $\gamma:=\gamma_k$, as above, where γ_k might be negative. Assume that β_k falls in the segment of Γ corresponding to column q of 2λ. Given a filling ϕ, we denote by $\phi[p]$ and $\phi(p,q]$ the parts of ϕ consisting of the columns $2\lambda_1,2\lambda_1-1,\ldots,p$ and $p-1,p-2,\ldots,q$, respectively. We also recall the definition (2.5) and conventions related to the content of a filling; this definition now applies to any filling of some Young diagram.

Proposition 6.1. *With the above notation, we have*

$$m_k=\langle \mathrm{ct}(\sigma[q]),\gamma^\vee\rangle\,.$$

Proof. We apply induction on i, which starts at $i=0$, when the verification is straightforward. We will now proceed from $j_1<\ldots<j_i<k$, where $i=s$ or $k\le j_{i+1}$, to $j_1<\ldots<j_{i+1}<k$, and we will freely use the notation above. Assume that $\beta_{j_{i+1}}$ falls in the segment of Γ corresponding to column p of 2λ, where $p\ge q$.

We need to compute

$$w\widehat{r}_{j_1}\ldots\widehat{r}_{j_{i+1}}(H_{\beta_k,l_k})=\widehat{t}_{j_{i+1}}\ldots\widehat{t}_{j_1}w(H_{\beta_k,l_k})=\widehat{t}_{j_{i+1}}(H_{\gamma,m})\,,$$

where $m=\langle \mathrm{ct}(\sigma_i[q]),\gamma^\vee\rangle$, by induction. Let $\gamma':=\gamma_{j_{i+1}}$, and $\widehat{t}_{j_{i+1}}=s_{\gamma',m'}$, where $m'=\langle \mathrm{ct}(\sigma_i[p]),(\gamma')^\vee\rangle$, by induction. We will use the following formula:

$$s_{\gamma',m'}(H_{\gamma,m})=H_{s_{\gamma'}(\gamma),m-m'\langle\gamma',\gamma^\vee\rangle}\,.$$

Thus, the proof is reduced to showing that

$$m-m'\langle\gamma',\gamma^\vee\rangle=\langle \mathrm{ct}(\sigma_{i+1}[q]),s_{\gamma'}(\gamma^\vee)\rangle\,.$$

An easy calculation, based on the above information, shows that the latter equality is non-trivial only if $p>q$, in which case it is equivalent to

$$\langle \mathrm{ct}(\sigma_{i+1}(p,q])-\mathrm{ct}(\sigma_i(p,q]),\gamma^\vee\rangle=\langle\gamma',\gamma^\vee\rangle\,\langle \mathrm{ct}(\sigma_{i+1}(p,q]),(\gamma')^\vee\rangle\,.$$

This equality is a consequence of the fact that

$$\mathrm{ct}(\sigma_{i+1}(p,q])=s_{\gamma'}(\mathrm{ct}(\sigma_i(p,q]))\,,$$

which follows from the construction of σ_{i+1} from σ_i explained above. □

Proof of Theorem 4.6 (2). We apply induction on the size of T, using freely the notation above. We prove the statement for $T=(\beta_{j_1},\ldots,\beta_{j_{s+1}})$, assuming it holds for $T^s=(\beta_{j_1},\ldots,\beta_{j_s})$. We have

$$w(\mu(T)=w\widehat{r}_{j_1}\ldots\widehat{r}_{j_{s+1}}(\lambda)=\widehat{t}_{j_{s+1}}\ldots\widehat{t}_{j_1}w(\lambda)=\widehat{t}_{j_{s+1}}(\mathrm{ct}(\sigma_s))\,,$$

by induction. We need to show that

$$\widehat{t}_{j_{s+1}}(\mathrm{ct}(\sigma_s)) = \mathrm{ct}(\sigma_{s+1})\,. \tag{6.1}$$

Let $\gamma := \gamma_{j_{s+1}}$ and assume that $\beta_{j_{s+1}}$ falls in the segment of Γ corresponding to column p of 2λ. Based on Proposition 6.1, (6.1) is rewritten as

$$s_\gamma(\mathrm{ct}(\sigma_s)) + \langle \mathrm{ct}(\sigma_s[p]), \gamma^\vee\rangle\gamma = \mathrm{ct}(\sigma_{s+1})\,. \tag{6.2}$$

Decomposing $\mathrm{ct}(\sigma_s)$ as $\mathrm{ct}(\sigma_s[p])+\mathrm{ct}(\sigma_s(p,1])$ (cf. the notation above), and $\mathrm{ct}(\sigma_{s+1})$ in a similar way, (6.2) would follow from

$$\begin{aligned}
&s_\gamma(\mathrm{ct}(\sigma_s[p])) + \langle \mathrm{ct}(\sigma_s[p]), \gamma^\vee\rangle\gamma = \mathrm{ct}(\sigma_{s+1}[p])\,,\\
&s_\gamma(\mathrm{ct}(\sigma_s(p,1])) = \mathrm{ct}(\sigma_{s+1}(p,1])\,.
\end{aligned}$$

The first equality is clear since $\sigma_s[p] = \sigma_{s+1}[p]$, while the second one follows from the construction of σ_{s+1} from σ_s explained above. □

7. Proof of Theorem 4.6 (3)

We start by recalling some basic facts about the group B_n and some notation from Section 2. We will use the following notation related to a word $w = w_1 \dots w_l$ with integer letters, which is sometimes identified with its set of letters:

$$w[i,j] := w_i \dots w_j\,,\quad N_{ab}(w) := |(a,b)\cap w|\,,\quad N_{ab}(\pm w) := N_{ab}(w) + N_{ab}(-w)\,,$$

where $-w := \overline{w_1}\dots\overline{w_l}$. Given words $w^1,\dots,w^p$, we let

$$N_{ab}(w^1,\dots,w^p) := N_{ab}(w^1) + \dots + N_{ab}(w^p)\,.$$

We also let

$$\tau_{ab} := \begin{cases} 1 & \text{if } a,b \le n\\ 0 & \text{otherwise}\,. \end{cases}$$

With this notation, given a signed permutation w in B_n and $1 \le i < j \le n$, $a := w(i)$, $b := w(j)$, we have the following facts:

$$\begin{aligned}
\frac{\ell(w(i,j)) - \ell(w) - 1}{2} &= N_{ab}(w[i,j])\,,\\
\frac{\ell(w(i,\overline{\imath})) - \ell(w) - 1}{2} &= N_{a\overline{a}}(w[i,n])\,,\\
\frac{\ell(w(i,\overline{\jmath})) - \ell(w) - 1}{2} &= N_{a\overline{b}}(w[i,j-1], \pm w[j+1,n]) + \tau_{ab}\,,
\end{aligned} \tag{7.1}$$

assuming that the left-hand side is nonnegative (i.e., that we go up in Bruhat order upon applying the indicated reflection); these facts are used implicitly throughout.

Given a chain of roots Δ, we define $\mathcal{A}^r(\Delta)$ like in (3.4) except that we impose an increasing chain condition and $w \in W$. The following simple lemma will be useful throughout, for splitting chains into subchains.

Lemma 7.1. *Consider (w,T) with T written as a concatenation $S_1 \dots S_p$. Let $w_i := wS_1\dots S_i$, so $w_0 = w$. Then*

$$\frac{1}{2}(\ell(w)+\ell(wT)-|T|) = \frac{1}{2}(\ell(w_{p-1})+\ell(w_p)-|S_p|) + \sum_{i=1}^{p-1}\frac{1}{2}(\ell(w_{i-1})-\ell(w_i)-|S_i|)\,.$$

Let Δ be the chain

$$\begin{array}{llll}\Delta := (\,(1,p+1), & (1,p+2), & \ldots, & (1,n),\\ (1,\overline{1}), & & & \\ (1,\overline{n}), & (1,\overline{n-1}), & \ldots, & (1,\overline{p+1})\,)\,.\end{array}$$

Proposition 7.2. *Consider a signed permutation w in B_n with $a := w(1)$, a position $1 \le p \le n$, and a value $b \in \{\pm a\} \cup (\pm w[p+1,n])$ such that $b \ge a$. Then we have*

$$\sum_{\substack{T\,:\,(w,T)\in\mathcal{A}^r(\Delta)\\ wT(1)=b}} t^{\frac{1}{2}(\ell(wT)-\ell(w)-|T|)}(1-t)^{|T|} = t^{N_{ab}(w[2,p])}(1-t)^{1-\delta_{ab}}\,; \tag{7.2}$$

here δ_{ab} is the Kronecker delta.

Proof. Given $s \in \{\overline{1}, \pm(p+1), \ldots, \pm n\}$, we let Δ_s be the subchain of Δ starting with $(1,s)$. We also let

$$S(w,s) := \sum_{\substack{T\,:\,(w,T)\in\mathcal{A}^r(\Delta_s)\\ wT(1)=b}} t^{\frac{1}{2}(\ell(wT)-\ell(w)-|T|)}(1-t)^{|T|}\,.$$

We consider three cases, depending on $b = w(q)$, $b = \overline{w(q)}$, and $b = \overline{a}$. The proof in the first case is identical to that of the analogous result for type A, namely [12][Proposition 5.3].

Case 2. Let $c := w(q) = \overline{b}$, and $p < q \le s$. We start by showing

$$S(w,\overline{s}) = t^{N_{a\overline{c}}(w[2,q-1],w[q+1,s],\pm w[s+1,n])+\tau_{ac}}(1-t)\,. \tag{7.3}$$

We use induction on s, which starts at $s = q$. For $s > q$, let $w^1 := w[2,q-1]$, $w^2 := w[q+1,s-1]$, $w^3 := w[s+1,n]$, and $d := w(s)$. The sum $S(w,\overline{s})$ splits into two sums, depending on $(1,\overline{s}) \notin T$, and $(1,\overline{s}) \in T$. By induction, the first sum is

$$S(w,\overline{s-1}) = t^{N_{a\overline{c}}(w^1,w^2,\pm dw^3)+\tau_{ac}}(1-t)\,.$$

Again by induction, if $a < \overline{d} < \overline{c}$, then the second sum is

$$\begin{aligned}&t^{N_{a\overline{d}}(w^1cw^2,\pm w^3)+\tau_{ad}}(1-t)S(w(1,\overline{s}),\overline{s-1})\\ =&t^{N_{a\overline{d}}(w^1cw^2,\pm w^3)+N_{\overline{d}\overline{c}}(w^1,w^2,\pm\overline{a}w^3)+\tau_{ad}+\tau_{\overline{d}c}}(1-t)^2\,;\end{aligned}$$

otherwise, it is empty. Adding up the two sums into which $S(w,\overline{s})$ splits, we obtain

$$t^{N_{a\overline{c}}(w^1,w^2d,\pm w^3)+\tau_{ac}}(1-t)\,.$$

The last claim rests on the easily verified facts that, if $a < \overline{d} < \overline{c}$, then

$$\tau_{ad} + \tau_{\overline{d}c} = \tau_{ac}\,, \qquad N_{a\overline{d}}(c) + N_{\overline{d}\overline{c}}(\overline{a}) = N_{a\overline{c}}(d)\,.$$

Still assuming that $c = w(q) = \overline{b}$ and $p < q$, we now show that

$$S(w,\overline{1}) = t^{N_{a\overline{c}}(w[2,q-1],w[q+1,n])+\tau'_{ac}}(1-t)\,, \tag{7.4}$$

where

$$\tau'_{ac} := \begin{cases} 1 & \text{if } a < c \le n\\ 0 & \text{otherwise}\,.\end{cases}$$

Let $w^1 := w[2,q-1]$, like before, and $w^2 := w[q+1,n]$. The sum $S(w,\overline{1})$ splits into two sums, depending on $(1,\overline{1}) \notin T$, and $(1,\overline{1}) \in T$. By (7.3), the first sum is

$$S(w,\overline{n}) = t^{N_{a\overline{c}}(w^1,w^2)+\tau_{ac}}(1-t)\,.$$

Again by (7.3), if $c < a \le n$, then the second sum is

$$t^{N_{a\overline{a}}(w^1cw^2)}(1-t)S(w(1,\overline{1}),\overline{n}) = t^{N_{a\overline{a}}(w^1cw^2)+N_{\overline{a}\overline{c}}(w^1,w^2)+\tau_{\overline{a}c}}(1-t)^2\,;$$

otherwise, it is empty. Adding up the two sums into which $S(w,\overline{s})$ splits, we obtain

$$t^{N_{a\overline{c}}(w^1,w^2)+\tau'_{ac}}(1-t)\,.$$

Assuming that $c = w(q) = \overline{b}$ and $p < q < s$, we now show that

$$S(w,s) = t^{N_{a\overline{c}}(w[2,q-1],w[q+1,s-1])+\tau'_{ac}}(1-t)\,. \tag{7.5}$$

We use decreasing induction on s. Like before, we let $w^1 := w[2,q-1]$, $w^2 := w[q+1,s-1]$, and $d := w(s)$. The sum $S(w,s)$ splits into two sums, depending on $(1,s) \notin T$, and $(1,s) \in T$. By induction, the first sum is

$$S(w,s+1) = t^{N_{a\overline{c}}(w^1,w^2d)+\tau'_{ac}}(1-t)\,.$$

Again by induction, if $a < d < \overline{c}$, then the second sum is

$$t^{N_{ad}(w^1cw^2)}(1-t)S(w(1,s),s+1) = t^{N_{ad}(w^1cw^2)+N_{d\overline{c}}(w^1,w^2a)+\tau'_{dc}}(1-t)^2\,;$$

otherwise, it is empty. (In both calculations, induction works by substituting $\overline{1}$ for $n+1$ when $s = n$, and by using (7.4) in this case.) Adding up the two sums into which $S(w,s)$ splits, we obtain

$$t^{N_{a\overline{c}}(w^1,w^2)+\tau'_{ac}}(1-t)\,.$$

The last claim rests on the easily verified fact that, if $a < d < \overline{c}$, then

$$N_{ad}(c) + \tau'_{dc} = \tau'_{ac}\,.$$

Case 3. Let us now assume that $b = \overline{a}$. We need to show that

$$S(w,p+1) = t^{N_{a\overline{a}}(w[2,p])}(1-t)\,. \tag{7.6}$$

We use decreasing induction on p, which starts at $p = n$; in this case Δ only contains the pair $(1,\overline{1})$, so the above convention of substituting $\overline{1}$ for $n+1$ works well here too. For $p < n$, we let $d := w(p+1)$. The sum $S(w,p+1)$ splits into two sums, depending on $(1,p+1) \notin T$, and $(1,p+1) \in T$. By induction, the first sum is

$$S(w,p+2) = t^{N_{a\overline{a}}(w[2,p]d)}(1-t)\,.$$

If $a < d < \overline{a}$, then by (7.5) of Case 2, the second sum is

$$t^{N_{ad}(w[2,p])}(1-t)S(w(1,p+1)),p+2) = t^{N_{ad}(w[2,p])+N_{d\overline{a}}(w[2,p])+\tau'_{da}}(1-t)^2\,;$$

otherwise, it is empty. Adding up the two sums into which $S(w,p+1)$ splits, we obtain the desired result.

Case 2 *(continued).* Assuming that $c = w(q) = \overline{b}$ and $p < q$, we now show that

$$S(w,q) = t^{N_{a\overline{c}}(w[2,q-1])}(1-t)\,. \tag{7.7}$$

The sum $S(w,q)$ splits into two sums, depending on $(1,q) \notin T$, and $(1,q) \in T$. By (7.5) of Case 2, the first sum is

$$S(w,q+1) = t^{N_{a\overline{c}}(w[2,q-1])+\tau'_{ac}}(1-t)\,.$$

If $a < c \le n$, then by (7.6) of Case 3, the second sum is

$$t^{N_{ac}(w[2,q-1])}(1-t)S(w(1,q)),q+1) = t^{N_{ac}(w[2,q-1])+N_{c\overline{c}}(w[2,q])}(1-t)^2\,;$$

otherwise, it is empty. Adding up the two sums into which $S(w,q)$ splits, we obtain the desired result.

The final step in Case 2 is to prove that

$$S(w,p+1) = t^{N_{a\overline{c}}(w[2,p])}(1-t)\,. \tag{7.8}$$

This can be done by decreasing induction on p, starting with $p=q-1$, which is the case proved in (7.7). The procedure is completely similar to the ones above, and, in fact, to the one for type A in [12][Proposition 5.3]. □

Let us consider the chain

$$\begin{array}{llllll} \Phi := \Gamma_1(n)\dots\Gamma_n(n) = (& (1,\overline{1}), & & & & \\ & (1,\overline{2}), & (2,\overline{2}), & & & \\ & & \dots & & & \\ & (1,\overline{n}), & (2,\overline{n}), & \dots, & (n-1,\overline{n})\,)\,. \end{array} \tag{7.9}$$

We denote by Φ_{ij} the subchain of Φ starting with $(i,\overline{\jmath})$. Given a signed permutation w, recall the definition (2.1) of $\ell_+(w)$ and $\ell_-(w)$. Given (i,j) with $1\le i\le j\le n$, we also define

$$\begin{aligned} \ell_-^{ij}(w) &:= |\{(k,l)\,:\,(k,\overline{l})\in\Phi\setminus\Phi_{ij}\,,\; w(k)>\overline{w(l)}\}|\,,\\ \overline{\ell}_-^{ij} &:= \ell_-(w)-\ell_-^{ij}(w)\,. \end{aligned} \tag{7.10}$$

Proposition 7.3. *Fix (i,j) with $1\le i\le j\le n$ and a signed permutation w in B_n. We have*

$$\sum_{T\,:\,(w,T)\in\mathcal{A}(\Phi_{ij})} t^{\frac{1}{2}(\ell(w)+\ell(wT)-|T|)}\,(1-t)^{|T|} = t^{\ell_+(w)+\ell_-^{ij}(w)}\,. \tag{7.11}$$

In particular, if the above sum is over $(w,T)\in\mathcal{A}(\Phi)$, then the right-hand side is $t^{\ell_+(w)}$.

Proof. Let us denote the sum in the left-hand side of (7.11) by $S(w,i,j)$, and the corresponding sum over $(w,T)\in\mathcal{A}(\Phi_{ij}\setminus\{(i,\overline{\jmath})\})$ by $S'(w,i,j)$. We view the chain Φ as a total order on the pairs $(i,\overline{\jmath})$, with $(1,\overline{1})$ being the smallest pair. With this in mind, we use decreasing induction on pairs $(i,\overline{\jmath})$. Given such a pair, if $w(i)<\overline{w(j)}$ then the induction step is clear, so assume the contrary. We can now split $S(w,i,j)$ into two sums, depending on $(i,\overline{\jmath})\notin T$ and $(i,\overline{\jmath})\in T$. By induction, the first sum is

$$S'(w,i,j) = t^{1+\ell_+(w)+\ell_-^{ij}(w)}\,.$$

By induction and Lemma 7.1, the second sum is

$$\begin{aligned} &(1-t)t^{\frac{1}{2}(\ell(w)-\ell(w(i,\overline{\jmath}))-1)}S'(w(i,\overline{\jmath}),i,j)\\ =&(1-t)t^{\frac{1}{2}(\ell(w)-\ell(w(i,\overline{\jmath}))-1)+\ell_+(w(i,\overline{\jmath}))+\ell_-^{ij}(w(i,\overline{\jmath}))}\,. \end{aligned}$$

The induction step is completed once we show that

$$\ell_+(w)+\ell_-^{ij}(w) = \frac{1}{2}\left(\ell(w)-\ell(w(i,\overline{\jmath}))-1\right)+\ell_+(w(i,\overline{\jmath}))+\ell_-^{ij}(w(i,\overline{\jmath}))\,.$$

The latter equality can be rewritten as

$$\Delta\ell_+(w)+\Delta\ell_-^{ij}(w)-1 = \Delta\overline{\ell}_-^{ij}(w)\,,$$

where $\Delta\ell_+(w) := \ell_+(w)-\ell_+(w(i,\overline{\jmath}))$, and similarly for the other two variations. In order to prove this, let us first fix k between i and j, and analyze the contribution to the three variations of the pairs (i,k) and (k,j), cf. (2.1) and (7.10). For simplicity, let $a := w(i)$, $b := w(k)$, and $c := w(j)$, where $a > \overline{c}$. The mentioned nonzero contributions are as follows:

(1) the pair (i,k) contributes 1 to $\Delta\ell_+(w)$ if $a > b > \overline{c}$;
(2) the pair (k,j) contributes -1 to $\Delta\ell_+(w)$ if $\overline{a} < b < c$, which is equivalent to $a > \overline{b} > \overline{c}$;
(3) the pair (i,k) contributes 1 to $\Delta\ell_-^{ij}(w)$ if $a > \overline{b} > \overline{c}$;
(4) the pair (k,j) contributes 1 to $\Delta\overline{\ell}_-^{ij}(w)$ if $a > b > \overline{c}$.

Note that the second and third contributions cancel out, whereas the first one is equal to the fourth one. The analysis is completely similar if $k < i$ or $k > j$. The pair (i,j) only contributes 1 to $\Delta\ell_-^{ij}(w)$. As far as the pairs (i,i) and (j,j) are concerned, the contribution of the first one to $\Delta\ell_-^{ij}(w)$ and of the second one to $\Delta\overline{\ell}_-^{ij}(w)$ are both equal to σ_{ac}, see (2.6). $\square$

Proof of Theorem 4.6 (3). Fix a filling σ in $\mathcal{F}(\lambda)$ with columns C_{ij} and C'_{ij}, as explained in Section 2. Recall the chain $\Phi := \Gamma_1(n)\dots\Gamma_n(n) = \Gamma_{11}\dots\Gamma_{1n}$ in (7.9). By splitting the λ-chain Γ into the tail Φ and its complement, and by using Lemma 7.1, the sum in the left-hand side of (4.4) can be rewritten as

$$\sum_{(w,T)\in f^{-1}(\sigma)} t^{\frac{1}{2}(\ell(w)+\ell(wT)-|T|)}\,(1-t)^{|T|} = \tag{7.12}$$

$$= \left(\sum_{\substack{(w,T)\in f^{-1}(\sigma)\\ T_{11}=\ldots=T_{1n}=\emptyset}} t^{\frac{1}{2}(\ell(w)-\ell(wT)-|T|)}\,(1-t)^{|T|}\right)\times$$

$$\times\left(\sum_{T\,:\,(C_{11},T)\in\mathcal{A}(\Phi)} t^{\frac{1}{2}(\ell(C_{11})+\ell(C_{11}T)-|T|)}\,(1-t)^{|T|}\right);$$

here the column C_{11}, which has height n, is viewed as a signed permutation in B_n. By Proposition 7.3, the second bracket is $t^{\ell_+(C_{11})}$.

In order to evaluate the first bracket, we will reverse all chains. Let us start by recalling the construction (4.1) of the λ-chain Γ, and in particular the order in which the subchains Γ_{ij} and Γ'_{ij} are concatenated (including the conventions in Section 2 related to $\Gamma_{i,j+1}$ and $\Gamma'_{i,j+1}$). We denote by Γ_{ij}^r and $(\Gamma'_{ij})^r$ the corresponding reverse chains. Also recall that we defined $\mathcal{A}^r(\,\cdot\,)$ like in (3.4) except that we imposed an increasing chain condition and $w\in W$. We consider pairs (w_{ij},S_{ij}) in $\mathcal{A}^r(\Gamma_{ij}^r)$ and (w'_{ij},S'_{ij}) in $\mathcal{A}^r((\Gamma'_{ij})^r)$, where w_{ij} and w'_{ij} are defined by

$$w_{ij} := C_{11}S'_{1,\lambda'_1}\dots S_{i,j+1}\,,\qquad w'_{ij} := C_{11}S'_{1,\lambda'_1}\dots S_{i,j+1}\,,$$

where the concatenation order for S_{ij} and S'_{ij} comes from that for Γ_{ij} and Γ'_{ij}; in particular, $w'_{1,\lambda'_1} = C_{11}$. Given this notation, we define the sum

$$\Sigma_{ij} := \sum_{\substack{S_{ij}\,:\,(w_{ij},S_{ij})\in\mathcal{A}^r(\Gamma_{ij}^r)\\ w_{ij}S_{ij}[1,\lambda'_i]=C_{ij}}} t^{\frac{1}{2}(\ell(w_{ij}S_{ij})-\ell(w_{ij})-|S_{ij}|)}\,(1-t)^{|S_{ij}|}\,,$$

and the similar sum Σ'_{ij}. We can now evaluate the first bracket in the right-hand side of (7.12):

$$\sum_{\substack{(w,T)\in f^{-1}(\sigma)\\ T_{11}=\ldots=T_{1n}=\emptyset}} t^{\frac{1}{2}(\ell(w)-\ell(wT)-|T|)}\,(1-t)^{|T|} = \Sigma_{\lambda_1,1}\ldots\Sigma'_{ij}\ldots\Sigma_{ij}\ldots\Sigma'_{1,\lambda'_1}\,.$$

In fact, we first write the sum in the left-hand side as an iterated sum, which factors in the way shown above because Σ_{ij} only depends on $w_{ij}[1,\lambda'_i] = C_{i,j+1}[1,\lambda'_i]$ (rather than the whole w_{ij}), by Proposition 7.2.

We conclude the proof by calculating the sum Σ_{ij}, the calculation for Σ'_{ij} being similar but simpler. For simplicity, let $d := \lambda'_i$, $w = w_{ij}$, $C := C_{i,j+1}[1,\lambda'_i]$, and $D := C_{ij}$. Assume that C differs from $D' := D(r_1,\overline{\jmath})\ldots(r_p,\overline{\jmath})$ with $1 \le r_1 < \ldots < r_p < j$ (possibly $p = 0$) only in position j. Let $\Gamma^r_{ij} = \Delta\Delta'$, where

$$\begin{aligned}\Delta := (\,&(j,d+1),\quad (j,d+2),\quad \ldots,\quad (j,n),\\ &(j,\overline{\jmath}),\\ &(j,\overline{n}),\qquad (j,\overline{n-1}),\quad \ldots,\quad (j,\overline{d+1})\,)\,,\\ \Delta' := (&(j-1,\overline{\jmath}),\ldots,(2,\overline{\jmath}),(1,\overline{\jmath}))\,.\end{aligned}$$

Correspondingly, the chains S_{ij} split into a head S, which can vary, and a fixed tail

$$S' := ((r_p,\overline{\jmath}),\ldots,(r_1,\overline{\jmath}))\,.$$

We have

$$\Sigma_{ij} = t^e(1-t)^p \sum_{\substack{S\,:\,(w,S)\in\mathcal{A}^r(\Delta)\\ wS(j)=D'(j)}} t^{\frac{1}{2}(\ell(wS)-\ell(w)-|S|)}\,(1-t)^{|S|}\,,$$

where $e := \frac{1}{2}(\ell(wSS') - \ell(wS) - p)$. By Proposition 7.2, the sum in the right-hand side is

$$t^{N_{C(j),D'(j)}(D[j+1,d])}(1-t)\,;$$

note that this sum is missing when $D' = C$, which is another possibility. The exponent e is calculated based on (7.1). □

References

[1] W. Adamczak and C. Lenart. The alcove path model and Young tableaux, 2009. `math.albany.edu/math/pers/lenart`.

[2] E. Ardonne and R. Kedem. Fusion products of Kirillov-Reshetikhin modules and fermionic multiplicity formulas. *J. Algebra*, 308:270–294, 2007.

[3] S. Assaf. A combinatorial proof of LLT and Macdonald positivity, 2008. `http://www-math.mit.edu/~sassaf`.

[4] S. Gaussent and P. Littelmann. LS-galleries, the path model and MV-cycles. *Duke Math. J.*, 127:35–88, 2005.

[5] I. Grojnowski and M. Haiman. Affine Hecke algebras and positivity of LLT and Macdonald polynomials, 2007. `http://math.berkeley.edu/~mhaiman`.

[6] J. Haglund, M. Haiman, and N. Loehr. A combinatorial formula for Macdonald polynomials. *J. Amer. Math. Soc.*, 18:735–761, 2005.

[7] J. E. Humphreys. *Reflection Groups and Coxeter Groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[8] M. Kashiwara and T. Nakashima. Crystal graphs for representations of the q-analogue of classical Lie algebras. *J. Algebra*, 165:295–345, 1994.

[9] S. Kato. Spherical functions and a q-analog of Kostant's weight multiplicity formula. *Invent. Math.*, 66:461–468, 1982.

[10] A. Lascoux and M.-P. Schützenberger. Sur une conjecture de H. O. Foulkes. *C. R. Acad. Sci. Paris Sér. I Math.*, 288:95–98, 1979.

[11] C. Lecouvey and M. Shimozono. Lusztig's q-analogue of weight multiplicity and one-dimensional sums for affine root systems. *Adv. Math.*, 208:438–466, 2007.

[12] C. Lenart. Hall-Littlewood polynomials, alcove walks, and fillings of Young diagrams, I. `arXiv:math.CO/0804.4715`.

[13] C. Lenart. On combinatorial formulas for Macdonald polynomials. *Adv. Math.*, 220:324–340, 2009.

[14] C. Lenart and A. Postnikov. Affine Weyl groups in K-theory and representation theory. *Int. Math. Res. Not.*, pages 1–65, 2007. Art. ID rnm038.

[15] C. Lenart and A. Postnikov. A combinatorial model for crystals of Kac-Moody algebras. *Trans. Amer. Math. Soc.*, 360:4349–4381, 2008.

[16] P. Littelmann. A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras. *Invent. Math.*, 116:329–346, 1994.

[17] P. Littelmann. Paths and root operators in representation theory. *Ann. of Math.* (2), 142:499–525, 1995.

[18] D. Littlewood. On certain symmetric functions. *Proc. London Math. Soc. (3)*, 11:485–498, 1961.

[19] G. Lusztig. Singularities, character formulas, and a q-analog of weight multiplicities. In *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, volume 101 of *Astérisque*, pages 208–229. Soc. Math. France, Paris, 1983.

[20] I. Macdonald. *Spherical functions on a group of p-adic type*. Ramanujan Institute, Centre for Advanced Study in Mathematics,University of Madras, Madras, 1971. Publications of the Ramanujan Institute, No. 2.

[21] I. Macdonald. Schur functions: theme and variations. In *Séminaire Lotharingien de Combinatoire (Saint-Nabor, 1992)*, volume 498 of *Publ. Inst. Rech. Math. Av.*, pages 5–39. Univ. Louis Pasteur, Strasbourg, 1992.

[22] I. Macdonald. Orthogonal polynomials associated with root systems. *Sém. Lothar. Combin.*, 45:Art. B45a, 40 pp. (electronic), 2000/01.

[23] K. Nelsen and A. Ram. Kostka-Foulkes polynomials and Macdonald spherical functions. In *Surveys in combinatorics, 2003 (Bangor)*, volume 307 of *London Math. Soc. Lecture Note Ser.*, pages 325–370. Cambridge Univ. Press, Cambridge, 2003.

[24] A. Ram. Alcove walks, Hecke algebras, spherical functions, crystals and column strict tableaux. *Pure Appl. Math. Q.*, 2:963–1013, 2006.

[25] A. Ram and M. Yip. A combinatorial formula for Macdonald polynomials. `arXiv:math/0803.1146`.

[26] C. Schwer. Galleries, Hall-Littlewood polynomials, and structure constants of the spherical Hecke algebra. *Int. Math. Res. Not.*, pages Art. ID 75395, 31, 2006.

[27] J. Stembridge. Kostka-Foulkes polynomials of general type. `www.math.lsa.umich.edu/~jrs`. Lecture notes for the *Generalized Kostka Polynomials Workshop*, American Institute of Mathematics, July 2005.

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