AFFINE WEYL GROUPS IN K-THEORY AND REPRESENTATION THEORY

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Abstract. We give an explicit combinatorial Chevalley-type formula for the equivariant K-theory of generalized flag varieties G/P which is a direct generalization of the classical Chevalley formula. Our formula implies a simple combinatorial model for the characters of the irreducible representations of G and, more generally, for the Demazure characters. This model, which we call the alcove path model, can be viewed as a discrete counterpart of the Littelmann path model, and has several advantages. Our construction is given in terms of a certain R-matrix, that is, a collection of operators satisfying the Yang-Baxter equation. It reduces to combinatorics of decompositions in the affine Weyl group and enumeration of saturated chains in the Bruhat order on the (nonaffine) Weyl group. Our model easily implies several symmetries of the coefficients in the Chevalley-type formula. We also derive a simple formula for multiplying an arbitrary Schubert class by a divisor class, as well as a dual Chevalley-type formula. The paper contains other applications and examples.

Contents

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1. Introduction

This paper is concerned with a combinatorial model, which we call the alcove path model, for the T-equivariant K-theory of generalized flag varieties G/B, where G is a complex semisimple Lie group, and B is a Borel subgroup. More precisely, we give a Chevalley-type multiplication formula in $K_T(G/B)$ in terms of our model. In addition, we apply the alcove path model to the representation theory of G.

Another model used in the study of the K-theory of G/B and of the representation theory of G is the celebrated Littelmann path model. Littelmann [Lit1, Lit2] showed that the characters of the irreducible representations V_λ of G (of highest weight λ) can be described by counting certain continuous paths in $\mathfrak{h}^*_\mathbb{R}$. These paths are constructed recursively starting with an initial one, by using certain operators acting on them, which are known as root operators. By making specific choices for the initial path, one can obtain special cases which are described combinatorially. One such class of paths, corresponding to a straight line initial path, is known as the class of Lakshmibai-Seshadri paths (LS-paths). These paths were introduced before Littelmann's work, in the context of standard monomial theory [LaSe]. They have a nonrecursive characterization in terms of the Bruhat order on the quotient W/W_λ of the corresponding Weyl group W modulo the stabilizer W_λ of λ. Recently, Gaussent and Littelmann [GaLi][1], motivated by the study of Mirković-Vilonen cycles, defined another combinatorial model for the irreducible characters of a complex semisimple Lie group. This model is based on LS-galleries, which are certain sequences of faces of alcoves for the corresponding affine Weyl group.

The Chevalley formula [Chev] from Schubert calculus is a multiplication formula in the cohomology ring of G/B. It expresses the product of the class of a Schubert variety with the class of the line bundle $\mathcal{L}_\lambda$ corresponding to the character of B determined by the weight $-\lambda$. This formula implies a rule for the product of a divisor class with an arbitrary Schubert class, known as Monk's rule in type A.

Let us now consider K-theory. Let $K_T(G/B)$ be the Grothendieck ring of T-equivariant coherent sheaves on G/B. According to Kostant and Kumar [KoKu], the ring $K_T(G/B)$ is a free module over the representation ring $R(T)$ of the maximal torus, with basis given by the classes $[\mathcal{O}_{X_w}]$, $w \in W$, of structure sheaves of Schubert varieties. A K-theory Chevalley formula is a formula for the basis expansion of the product of $[\mathcal{O}_{X_w}]$ with the class $[\mathcal{L}_\lambda]$ of the line bundle mentioned above. Fulton and Lascoux [FuLa] gave the first K-theory Chevalley formula; this refers to the equivariant Grothendieck ring $K_T(SL_n/B)$ of the classical flag variety, and is based on the combinatorics of Young tableaux. Other Chevalley-type and Monk-type formulas in $K(SL_n/B)$ were given in [Len1]. The first K-theory Chevalley formula for an arbitrary generalized flag variety G/B was given by Pittie and Ram [PiRa]. They showed that the product of $[\mathcal{O}_{X_w}]$ with $[\mathcal{L}_\lambda]$, where λ is a dominant weight, can be expressed as a nonnegative sum over certain special LS-paths. The fact that the product in the Pittie-Ram formula expands as a nonnegative linear combination was also explained geometrically by Brion [Brion] and Mathieu [Mat]. The coefficients in the Pittie-Ram formula were identified as certain characters by Lakshmibai and Littelmann [LaLi] using geometry. Littelmann and Seshadri [LiSe] showed that the Pittie-Ram formula is a consequence of standard monomial theory [LLM, LaSe, Lit3], and, furthermore, that it is almost equivalent to standard monomial theory. Griffeth and Ram [GrRa] extended the Pittie-Ram formula to antidominant weights λ, and also gave a formula for multiplying an arbitrary Schubert class by a codimension 1 class. Willems

[1]The original version of our paper was posted on `arXiv` on September 12, 2003, around the same time as [GaLi]. Our constructions were developed independently.

[Wil] gave a K-theory Chevalley formula for an arbitrary weight λ; this is not a positive formula when λ is dominant.[2]

In this paper, we present a Chevalley-type formula[3] for the product of $[\mathcal{O}_{X_w}]$ and $[\mathcal{L}_\lambda]$ in the equivariant Grothendieck ring $K_T(G/P)$, where P is a parabolic subgroup of G. Our formula is based on enumerating certain saturated chains in the Bruhat order on the corresponding Weyl group W. This enumeration is determined by an alcove path, which is a sequence of adjacent alcoves for the affine Weyl group W_{aff} of the Langlands dual group $G^\vee$. Thus, we refer to our model as the alcove path model. Alcove paths correspond to decompositions of elements in the affine Weyl group into products of generators. Our Chevalley-type formula is conveniently formulated in terms of a certain R-matrix, that is, in terms of a collection of operators satisfying the Yang-Baxter equation. We express the operator E^λ of multiplication by the class of a line bundle as a composition $R^{[\lambda]}$ of elements of the R-matrix given by a certain alcove path. In order to prove the formula, we simply verify that the operators $R^{[\lambda]}$ satisfy the same commutation relations with the elementary Demazure operators T_i as the operators E^λ.

We now discuss the main features of our Chevalley-type formula. First of all, it is a direct generalization of the classical Chevalley formula. It is nonrecursive and combinatorial, being based on chains in the Bruhat order. It works for any weight λ, and it is equally simple for all such weights (regular and nonregular, dominant and nondominant). It is a positive formula (i.e., cancellation free) when λ is dominant or antidominant. Its proof is completely elementary; indeed, it does not rely on any geometric arguments, and it just uses combinatorics of the affine Weyl group and some algebraic manipulations with R-matrices and Demazure operators. Our formula obviates the Deodhar lifts in the Pittie-Ram formula; these are lifts $W/W_\lambda \to W$ from cosets modulo W_λ, which are defined by a nontrivial recursive procedure [Deo2]. The need for the Deodhar lifts in the Pittie-Ram formula explains why the alcove path model contains additional information and is not equivalent to LS-paths and LS-galleries. Finally, the alcove path model has found several applications, which are mentioned below.

We start by listing some corollaries and related results, all of which appear in this paper. Our formula easily implies a Monk-type formula for products of the classes $[\mathcal{O}_{X_w}]$ with divisor classes. Indeed, the divisor classes are expressed in terms of the classes of line bundles $\mathcal{L}_{-\omega_i}$, where ω_i denotes a fundamental weight. We easily obtain the dual Chevalley-type formula for products of $[\mathcal{L}_\lambda]$ with elements of the dual basis to $\{[\mathcal{O}_{X_w}] \mid w \in W\}$. The alcove path model facilitates the study of certain symmetries of the Chevalley coefficients in equivariant K-theory, which is not easily carried out based on other methods. For instance, one of these symmetries was earlier derived by Brion [Brion] using a nontrivial geometric argument. We also derive a nonnegative combinatorial model for the characters of the irreducible representations of G and for the Demazure characters. Finally, the independence of our formulas from the choice involved in our model (i.e., the choice of an alcove path) follows from the fact that the R-matrices used in the construction satisfy the Yang-Baxter equation.

We also mention some subsequent publications, which present other applications of the alcove path model and of the formulas in this paper. Our Chevalley-type formula implies a Pieri-type formula in $K(SL_n/B)$ [LeSo] for multiplying arbitrary Schubert classes with certain special Schubert classes pulled back from a Grassmannian, that are indexed by cycles. No other such formulas, based on other models, are known. In [LeMa1], we give a model for $K_T(G/B)$ in terms of a certain braided Hopf algebra called the Nichols-Woronowicz algebra. This model has built-in Chevalley-type multiplication operators based on the formula in this paper; therefore, it has potential applications to deriving more general multiplication formulas. In [LeMa2], we define quantum Grothendieck polynomials, which we conjecture to represent Schubert classes in the quantum K-theory [Lee] of the classical flag variety SL_n/B. In support of our conjecture, we prove that the quantum Grothendieck polynomials satisfy (and, in fact, are determined by) the natural quantum generalization of our K-theory Chevalley formula, which is conjectured

[2] [GrRa] and [Wil] were posted on `arXiv` after this paper.

[3] Notational remark: We call a rule for $[\mathcal{L}_\lambda] \cdot [\mathcal{O}_{X_w}]$ a Chevalley-type formula and use the term Monk-type formula for a rule for products $[\mathcal{O}_{X_{w_\circ s_i}}] \cdot [\mathcal{O}_{X_w}]$ of divisor classes $[\mathcal{O}_{X_{w_\circ s_i}}]$ with arbitrary classes $[\mathcal{O}_{X_w}]$. The term Pieri-type formula refers to multiplication with the special Schubert classes pulled back from a Grassmannian.

in Section 17. In [LePo] we develop our model entirely within the representation theory of complex symmetrizable Kac-Moody algebras; in this context, we also derive an explicit Littlewood-Richardson rule for decomposing tensor products of irreducible representations, and describe the corresponding crystal graph structures. Finally, the alcove path model leads to an extensive generalization of the combinatorics of irreducible characters from Lie type A (where the combinatorics is based on Young tableaux) to arbitrary type. For example, in [Len2] we present a combinatorial realization of Lusztig's involution on irreducible crystals, which generalizes Schützenberger's evacuation for Young tableaux. The study of the combinatorics of the alcove path model will be continued in future publications.

As a preview of our main result, let us present here a formula for the product $[\mathcal{L}_\lambda] \cdot [\mathcal{O}_{X_w}]$ of classes in the usual (nonequivariant) Grothendieck ring $K(G/B)$. Let $\mathcal{A}$ be the affine Coxeter arrangement for the Langlands dual group $G^\vee$. The regions of $\mathcal{A}$, called alcoves, correspond to the elements of the affine Weyl group W_{aff}. Fix a weight λ. Let $\pi(t)$ be a continuous path in $\mathfrak{h}^*_{\mathbb{R}}$ that connects a point $\pi(0)$ inside the fundamental alcove with the point $\pi(1) = \pi(0) - \lambda$. Assume that $\pi(t)$ does not pass through pairwise intersections of hyperplanes in $\mathcal{A}$. As t changes from 0 to 1, the path $\pi(t)$ crosses the hyperplanes $H_1, \ldots, H_l \in \mathcal{A}$. Let β_i be the root perpendicular to H_i with the opposite orientation to the path $\pi(t)$. We call a sequence of roots $(\beta_1, \ldots, \beta_l)$ obtained in such a way a λ-chain. In fact, λ-chains are in a bijective correspondence with decompositions of a certain element $v_{-\lambda}$ of the affine Weyl group into products $v_{-\lambda} = s_{i_1} \cdots s_{i_l}$ of the generators of W_{aff}.

For positive roots $\alpha \in \Phi^+$, let us define the Bruhat operators B_α that act on the Grothendieck ring $K(G/B)$ by

$$B_\alpha : [\mathcal{O}_{X_w}] \longmapsto \begin{cases} [\mathcal{O}_{X_{ws_\alpha}}] & \text{if } \ell(ws_\alpha) = \ell(w) - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Also let $B_{-\alpha} = -B_\alpha$. These operators are specializations of the quantum Bruhat operators from [BFP]. The operators $1 + B_\alpha$ satisfy the Yang-Baxter equation.

Theorem 1.1. (K-theory Chevalley formula) *Let λ be any weight (dominant or nondominant, regular or nonregular). Let $(\beta_1, \ldots, \beta_l)$ be a λ-chain. Then, for any $w \in W$, we have*

$$[\mathcal{L}_\lambda] \cdot [\mathcal{O}_{X_w}] = (1 + B_{\beta_l}) \cdots (1 + B_{\beta_1})([\mathcal{O}_{X_w}])$$

in the Grothendieck ring $K(G/B)$.

The number of times a root α appears in the λ-chain $(\beta_1, \ldots, \beta_l)$ minus the number of times $-\alpha$ appears in the λ-chain equals $(\lambda, \alpha^\vee)$. Thus the linear part of the expansion of $(1 + B_{\beta_l}) \cdots (1 + B_{\beta_1})$ is precisely $\sum_{\alpha>0} (\lambda, \alpha^\vee)\, B_\alpha$. This linear part produces the classical Chevalley formula for products of classes in the cohomology ring $H^*(G/B)$.

We say that a λ-chain is reduced if it has minimal possible length. Reduced λ-chains correspond to reduced decompositions in the affine Weyl group. If λ is a dominant weight, then all roots in a reduced λ-chain are positive. In this case, Theorem 1.1 involves only positive terms. If λ is an antidominant weight, then all roots in a reduced λ-chain are negative. In this case, the sign of the coefficient of $[\mathcal{O}_{X_w}]$ in $[\mathcal{L}_\lambda] \cdot [\mathcal{O}_{X_u}]$ equals $(-1)^{\ell(u)-\ell(w)}$, and Theorem 1.1 gives a subtraction-free expression for this coefficient.

Let $s_1, \ldots, s_r$ be the system of simple reflections in the Weyl group (compatible with our choice of Borel subgroup), let $\omega_1, \ldots, \omega_r$ be the corresponding set of fundamental weights, and let $w_\circ$ be the longest element in W. The special classes $[\mathcal{O}_{X_{w_\circ s_i}}] \in K(G/B)$ for codimension 1 Schubert varieties can be expressed as $[\mathcal{O}_{X_{w_\circ s_i}}] = 1 - [\mathcal{L}_{-\omega_i}]$. Note that $(\beta_1, \ldots, \beta_l)$ is a λ-chain if and only if $(-\beta_l, \ldots, -\beta_1)$ is a $(-\lambda)$-chain.

Corollary 1.2. (K-theory Monk formula) *Let us fix a simple reflection s_i. Let $(\beta_1, \ldots, \beta_l)$ be a $(-\omega_i)$-chain. Then, for any $w \in W$, we have*

$$[\mathcal{O}_{X_{w_\circ s_i}}] \cdot [\mathcal{O}_{X_w}] = (1 - (1 - B_{\beta_1}) \cdots (1 - B_{\beta_l}))([\mathcal{O}_{X_w}])$$

in the Grothendieck ring $K(G/B)$.

The special classes $[\mathcal{O}_{X_{w_\circ s_i}}]$ generate the Grothendieck ring $K(G/B)$. Thus Corollary 1.2 gives a complete characterization of the multiplicative structure of the Grothendieck ring.

The general outline of the paper is as follows. In Section 2, we review basic notions related to roots systems and fix our notation. In Section 3, we present some background on the Grothendieck ring $K_T(G/B)$. In Section 4, we discuss the relationship between the Grothendieck ring and the Demazure characters. In Section 5, we remind a few facts about affine Weyl groups. In particular, we show that decompositions of affine Weyl group elements correspond to sequences of adjacent alcoves, which we call alcove paths. In Section 6, we state our combinatorial formula for products in equivariant K-theory, that is, our K_T-Chevalley formula. As a corollary of the K_T-Chevalley formula, we obtain a combinatorial model for the characters of the irreducible representations V_λ and for the Demazure characters. In Section 7, we extend the K_T-Chevalley formula to equivariant K-theory of G/P. In Section 8, we present several applications of our K_T-Chevalley formula. We derive the K_T-Monk formula for the product of an arbitrary class $[\mathcal{O}_{X_w}]$ with a divisor class $[\mathcal{O}_{X_{w_\circ s_i}}]$, as well as the dual K_T-Chevalley formula. Then we study two symmetries of the coefficients in the K_T-Chevalley formula. In the following sections, we develop tools needed to reformulate our rule in a compact operator notation and to prove this rule. In Section 9, we discuss the Yang-Baxter equation. In Section 10, we construct a certain R-matrix and show that it satisfies the Yang-Baxter equation. In Section 11, we derive commutation relations between the elements of the R-matrix and the Demazure operators T_i. These commutation relations are the core of the proof of our formula. In Section 12, we define compositions $R^{[\lambda]}$ of elements of the R-matrix. We use tail-flips of alcove paths to prove that the operators $R^{[\lambda]}$ satisfy the same commutation relations with T_i as the operators E^λ. In Section 13, we reformulate and prove our main result—the K_T-Chevalley formula—using the R-matrix notation. We show that $R^{[\lambda]}$ coincides with the operator E^λ of multiplication by the class of a line bundle in the Grothendieck ring $K_T(G/B)$. In Section 14, we use central points of alcoves to prove the equivalence of the two formulations of our main result. In Sections 15 and 16, we give several examples for types A, B, C, and G_2. In Section 17, we conjecture a natural generalization of our K-theory Monk formula to quantum K-theory. In Appendix 18, we reformulate our model for characters using admissible foldings of galleries and compare our model with LS-galleries and LS-paths.

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2. Notation

Let G be a connected, simply connected, simple complex Lie group. Fix a Borel subgroup B and a maximal torus T such that $G \supset B \supset T$. Let $\mathfrak{h}$ be the corresponding Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G. Let r be the rank of the Cartan subalgebra $\mathfrak{h}$. Let $\Phi \subset \mathfrak{h}^*$ be the corresponding irreducible *root system*. Let $\mathfrak{h}^*_\mathbb{R} \subset \mathfrak{h}^*$ be the real span of the roots. Let $\Phi^+ \subset \Phi$ be the set of positive roots corresponding to our choice of B. Then Φ is the disjoint union of Φ^+ and $\Phi^- = -\Phi^+$. Let $\alpha_1, \dots, \alpha_r \in \Phi^+$ be the corresponding *simple roots*. They form a basis of $\mathfrak{h}^*_\mathbb{R}$. Let (λ, μ) denote the nondegenerate scalar product on $\mathfrak{h}^*_\mathbb{R}$ induced by the Killing form. Given a root α, the corresponding *coroot* is $\alpha^\vee := 2\alpha/(\alpha, \alpha)$. The collection of coroots $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$ forms the *dual root system*.

The *Weyl group* $W \subset \mathrm{Aut}(\mathfrak{h}^*_\mathbb{R})$ of the Lie group G is generated by the reflections $s_\alpha : \mathfrak{h}^*_\mathbb{R} \to \mathfrak{h}^*_\mathbb{R}$, for $\alpha \in \Phi$, given by

$$s_\alpha : \lambda \mapsto \lambda - (\lambda, \alpha^\vee)\, \alpha.$$

In fact, the Weyl group W is generated by the *simple reflections* $s_1, \ldots, s_r$ corresponding to the simple roots $s_i := s_{\alpha_i}$, subject to the *Coxeter relations:*

$$(s_i)^2 = 1 \quad \text{and} \quad (s_i s_j)^{m_{ij}} = 1 \quad \text{for any } i, j \in \{1, \ldots, r\},$$

where m_{ij} is half of the order of the dihedral subgroup generated by s_i and s_j. An expression of a Weyl group element w as a product of generators $w = s_{i_1} \cdots s_{i_l}$ which has minimal length is called a *reduced decomposition* for w; its length $\ell(w) = l$ is called the *length* of w. The Weyl group contains a unique *longest element* $w_\circ$ with maximal length $\ell(w_\circ) = |\Phi^+|$. For $u, w \in W$, we say that u *covers* w, and write $u \gtrdot w$, if $w = us_\beta$, for some $\beta \in \Phi^+$, and $\ell(u) = \ell(w) + 1$. The transitive closure "$>$" of the relation "$\gtrdot$" is called the *Bruhat order* on W.

The *weight lattice* Λ is given by

$$\Lambda := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \text{ for any } \alpha \in \Phi\}. \tag{2.1}$$

The weight lattice Λ is generated by the *fundamental weights* $\omega_1, \ldots, \omega_r$, which are defined as the elements of the dual basis to the basis of simple coroots, i.e., $(\omega_i, \alpha_j^\vee) = \delta_{ij}$. The set Λ^+ of *dominant weights* is given by

$$\Lambda^+ := \{\lambda \in \Lambda \mid (\lambda, \alpha^\vee) \geq 0 \text{ for any } \alpha \in \Phi^+\}.$$

Let $\rho := \omega_1 + \cdots + \omega_r = \frac{1}{2}\sum_{\beta \in \Phi^+} \beta$. The *height* of a coroot $\alpha^\vee \in \Phi^\vee$ is $(\rho, \alpha^\vee) = c_1 + \cdots + c_r$ if $\alpha^\vee = c_1\alpha_1^\vee + \cdots + c_r\alpha_r^\vee$. Since we assumed that Φ is irreducible, there is a unique *highest coroot* $\theta^\vee \in \Phi^\vee$ that has maximal height. (In other words, $\theta^\vee$ is the highest root of the dual root system $\Phi^\vee$. It should not be confused with the coroot of the highest root of Φ.) We will also use the *Coxeter number*, that can be defined as $h := (\rho, \theta^\vee) + 1$.

3. Equivariant K-theory of Generalized Flag Varieties

In this section, we remind a few facts about the Grothendieck ring $K_T(G/B)$. For more details on the Grothendieck ring, we refer to Kostant and Kumar [KoKu], see also Pittie and Ram [PiRa].

The *generalized flag variety* G/B is a smooth projective variety. It decomposes into a disjoint union of *Schubert cells* $X_w^\circ := BwB/B$ indexed by elements $w \in W$ of the Weyl group. The closures of Schubert cells $X_w := \overline{X_w^\circ}$ are called *Schubert varieties.* We have $u > w$ in the Bruhat order (defined as above) if and only if $X_u \supset X_w$. Let $\mathcal{O}_{X_w}$ be the structure sheaf of the Schubert variety X_w.

Let $\mathbb{Z}[\Lambda]$ be the group algebra of the weight lattice Λ. It has a $\mathbb{Z}$-basis of formal exponents $\{e^\lambda \mid \lambda \in \Lambda\}$ with multiplication $e^\lambda \cdot e^\mu := e^{\lambda+\mu}$, i.e., $\mathbb{Z}[\Lambda] = \mathbb{Z}[e^{\pm\omega_1}, \cdots, e^{\pm\omega_r}]$ is the algebra of Laurent polynomials in r variables. The group of characters $X = X(T)$ of the maximal torus T is isomorphic to the weight lattice Λ. Its group algebra $\mathbb{Z}[X] = R(T)$ is the representation ring of T. The rings $\mathbb{Z}[\Lambda]$ and $\mathbb{Z}[X]$ are isomorphic. (However we will distinguish these two rings.) Let us denote by x^λ the element of $\mathbb{Z}[X]$ corresponding to the character determined by λ, as well as to $e^\lambda \in \mathbb{Z}[\Lambda]$. Thus $\mathbb{Z}[X] = \mathbb{Z}[x^{\pm\omega_1}, \cdots, x^{\pm\omega_r}]$. Let $\mathcal{L}_\lambda$ be the line bundle over G/B associated with the weight λ, that is, $\mathcal{L}_\lambda := G \times_B \mathbb{C}_{-\lambda}$, where B acts on G by right multiplication, and the B-action on $\mathbb{C}_{-\lambda} = \mathbb{C}$ corresponds to the character determined by $-\lambda$. (This character of T extends to B by defining it to be identically one on the commutator subgroup $[B, B]$.)

Denote by $K_T(G/B)$ the *Grothendieck ring* of coherent T-equivariant sheaves on G/B. According to Kostant and Kumar [KoKu], the Grothendieck ring $K_T(G/B)$ is a free $\mathbb{Z}[X]$-module, and the classes $[\mathcal{O}_{X_w}] \in K_T(G/B)$ of the structure sheaves of Schubert varieties form its $\mathbb{Z}[X]$-basis. The classes $[\mathcal{L}_\lambda]$ of the line bundles $\mathcal{L}_\lambda$ also span $K_T(G/B)$ as a $\mathbb{Z}[X]$-module.

We now discuss the presentation of the Grothendieck ring $K_T(G/B)$ as a quotient of $\mathbb{Z}[X] \otimes \mathbb{Z}[\Lambda]$. The Weyl group W acts on the group algebra $\mathbb{Z}[\Lambda]$ by $w(e^\lambda) := e^{w(\lambda)}$. Let $\mathbb{Z}[\Lambda]^W$ be the subalgebra of W-invariant elements. The tensor product $\mathbb{Z}[X] \otimes \mathbb{Z}[\Lambda]$ is the algebra of Laurent polynomials in $2r$ variables

$x^{\omega_1}, \ldots, x^{\omega_r}, e^{\omega_1}, \ldots, e^{\omega_r}$ with integer coefficients. Let $i : \mathbb{Z}[\Lambda] \to \mathbb{Z}[X]$ be the natural isomorphism given by $i(e^\lambda) := x^\lambda$. Let $\mathcal{I}$ be the ideal in $\mathbb{Z}[X] \otimes \mathbb{Z}[\Lambda]$ generated by the following elements:

$$\mathcal{I} := \left\langle i(f) \otimes 1 - 1 \otimes f \mid f \in \mathbb{Z}[\Lambda]^W \right\rangle .$$

The Grothendieck ring $K_T(G/B)$ is canonically isomorphic to the quotient ring

$$K_T(G/B) \simeq (\mathbb{Z}[X] \otimes \mathbb{Z}[\Lambda])/\mathcal{I}. \tag{3.1}$$

The isomorphism is given by the $\mathbb{Z}[X]$-linear map $[\mathcal{L}_\lambda] \mapsto e^{-\lambda}$, for $\lambda \in \Lambda$. From now on, we will identify the two rings. Recall from [KoKu] the $\mathbb{Z}$-linear involution $* : K_T(G/B) \to K_T(G/B)$ given by $x^\mu \otimes e^\lambda \mapsto x^{-\mu} \otimes e^{-\lambda}$; in other words, this map takes a vector bundle to its dual. Let $[\mathcal{O}_w] := *[\mathcal{O}_{X_w}]$. Throughout most of this paper, we will work with the classes $[\mathcal{O}_w]$ instead of $[\mathcal{O}_{X_w}]$; it is straightforward to rephrase all results in terms of $[\mathcal{O}_{X_w}]$.

It is possible to express all classes $[\mathcal{O}_w]$ as Laurent polynomials in $\mathbb{Z}[X] \otimes \mathbb{Z}[\Lambda]$ by choosing a representative of the class $[\mathcal{O}_1]$ and by applying Demazure operators, as described below. The action of the Weyl group on $\mathbb{Z}[\Lambda]$ defined above is extended $\mathbb{Z}[X]$-linearly to $\mathbb{Z}[X] \otimes \mathbb{Z}[\Lambda]$. For $i = 1, \ldots, r$, the elementary *Demazure operator* $T_i : \mathbb{Z}[X] \otimes \mathbb{Z}[\Lambda] \to \mathbb{Z}[X] \otimes \mathbb{Z}[\Lambda]$ is the $\mathbb{Z}[X]$-linear operator given by

$$T_i(f) := \frac{f - e^{-\alpha_i} s_i(f)}{1 - e^{-\alpha_i}} . \tag{3.2}$$

Note that the numerator is always divisible by the denominator[4], so the right-hand side is a valid expression in the algebra $\mathbb{Z}[X] \otimes \mathbb{Z}[\Lambda]$. One can verify directly from the definition that the operators T_i satisfy the following relations:

$$T_i^2 = T_i , \tag{3.3}$$

$$(T_i \, T_j)^{m_{ij}} = 1 , \tag{3.4}$$

$$T_i(fg) = f \cdot T_i(g), \quad \text{if } s_i(f) = f . \tag{3.5}$$

Equations (3.3) and (3.4) imply that the operators T_i give an action of the corresponding Hecke algebra $\mathcal{H}_q$ specialized at $q = 0$, e.g., see [Hum]. Equation (3.5) implies that the operators T_i preserve the ideal $\mathcal{I}$. Thus the elementary Demazure operators T_i induce operators acting on the Grothendieck ring $K_T(G/B) \simeq (\mathbb{Z}[X] \otimes \mathbb{Z}[\Lambda])/\mathcal{I}$, which will be denoted by the same symbols.

For a reduced decomposition $w = s_{i_1} \cdots s_{i_l} \in W$, the *Demazure operator* T_w is defined as the following composition of elementary Demazure operators:

$$T_w := T_{i_1} \cdots T_{i_l}. \tag{3.6}$$

The Coxeter relations (3.4) imply that the operator T_w depends only on w, not on the choice of a reduced decomposition. Equation (3.3) implies that an arbitrary product $T_{j_1} \cdots T_{j_m}$ reduces to T_w for some $w \in W$. Kostant and Kumar [KoKu] showed that, for any $w \in W$,

$$[\mathcal{O}_w] = T_{w^{-1}}([\mathcal{O}_1]). \tag{3.7}$$

For type A, the elementary Demazure operators T_i are also called *isobaric divided difference operators.* The polynomial representatives of the classes $[\mathcal{O}_w]$ obtained by applying these operators to a certain polynomial representative of $[\mathcal{O}_1]$ are the *double Grothendieck polynomials* of Lascoux and Schützenberger [LaSc].

The product $e^\lambda \cdot [\mathcal{O}_u]$ in the Grothendieck ring $K_T(G/B)$ can be written as a finite sum

$$e^\lambda \cdot [\mathcal{O}_u] = \sum_{w \in W, \, \mu \in \Lambda} c_{u,w}^{\lambda;\mu} \; x^\mu \, [\mathcal{O}_w] , \tag{3.8}$$

[4]Check this for $f = e^\lambda$.

where $c_{u,w}^{\lambda;\mu}$ are some integer coefficients. Equivalently, we can write

$$[\mathcal{L}_\lambda]\cdot[\mathcal{O}_{X_u}] = \sum_{w\in W,\,\mu\in\Lambda} c_{u,w}^{\lambda;\mu}\, x^{-\mu}\,[\mathcal{O}_{X_w}]\,. \tag{3.9}$$

We will call the coefficients $c_{u,w}^{\lambda;\mu}$ *K_T-Chevalley coefficients,* because they extend the coefficients in the usual Chevalley formula, as shown below in this section. In this paper, we present an explicit combinatorial formula for $c_{u,w}^{\lambda;\mu}$, see Theorems 6.1 and 13.1. We will see that $c_{u,w}^{\lambda;\mu}=0$ unless $w\le u$ in the Bruhat order, and that $c_{u,u}^{\lambda;\mu}=\delta_{\lambda,\mu}$. If λ is a dominant weight, then we will see that all coefficients $c_{u,w}^{\lambda;\mu}$ are nonnegative. In this case, Pittie and Ram [PiRa] showed that $c_{u,w}^{\lambda;\mu}$ count certain LS-paths, cf. also Lakshmibai-Littelmann [LaLi] and Littelmann-Seshadri [LiSe].

For a weight λ, let $E^\lambda : f\mapsto e^\lambda f$ be the operator of multiplication by the exponent e^λ in the ring $\mathbb{Z}[X]\otimes\mathbb{Z}[\Lambda]$. The induced operator on $K_T(G/B)$, which will be denoted by the same symbol E^λ, acts as the operator of multiplication by the class $[\mathcal{L}_{-\lambda}]$ of a line bundle. It follows from the definitions that E^λ and T_i satisfy the following commutation relation:

$$E^\lambda\,T_i = T_i\,E^{s_i(\lambda)} + \frac{E^\lambda - E^{s_i(\lambda)}}{1-E^{-\alpha_i}}. \tag{3.10}$$

The quotient in this expression expands as the Laurent polynomial

$$\frac{E^\lambda - E^{s_i(\lambda)}}{1-E^{-\alpha_i}} = \sum_{0\le k<(\lambda,\alpha_i^\vee)} E^{\lambda-k\alpha_i} - \sum_{(\lambda,\alpha_i^\vee)\le k<0} E^{\lambda-k\alpha_i}.$$

Also, we have

$$E^\lambda([\mathcal{O}_1]) = x^\lambda\,[\mathcal{O}_1]. \tag{3.11}$$

Let $\hat{\mathcal{H}}$ be the ring generated by the operators $T_1,\dots,T_r$ and E^λ, $\lambda\in\Lambda$. Then $\hat{\mathcal{H}}$ is described by relations (3.3), (3.4), and (3.10), i.e., $\hat{\mathcal{H}}$ is a certain degeneration of the affine Hecke algebra. This follows from the fact that the elements $T_{w^{-1}}E^\mu$, $w\in W$, $\mu\in\Lambda$, form a $\mathbb{Z}$-basis of $\hat{\mathcal{H}}$. Indeed, according to the relations, the elements $T_{w^{-1}}E^\mu$ span $\hat{\mathcal{H}}$. On the other hand, these elements are linearly independent, because $T_{w^{-1}}E^\mu([\mathcal{O}_1]) = x^\mu[\mathcal{O}_w]$.

Using the commutation relation in (3.10) repeatedly, we obtain, for any $u\in W$ and $\lambda\in\Lambda$, the following identity in the ring $\hat{\mathcal{H}}$:

$$E^\lambda\,T_{u^{-1}} = \sum_{w\in W,\,\mu\in\Lambda} c_{u,w}^{\lambda;\mu}\;T_{w^{-1}}\,E^\mu, \tag{3.12}$$

for some integer coefficients $c_{u,w}^{\lambda;\mu}$. Applying both sides of this expression to the class $[\mathcal{O}_1]$ and using (3.7) and (3.11), we deduce that the coefficients $c_{u,w}^{\lambda;\mu}$ in (3.12) are equal to the K_T-Chevalley coefficients in (3.8).

The commutation relation (3.10) gives a recursive procedure for calculating the product $e^\lambda\cdot[\mathcal{O}_u]$ in $K_T(G/B)$. In this paper, we present a simple nonrecursive rule for this product. The proof of our rule is based on the following trivial observation, which is implied by the above discussion.

Lemma 3.1. *Let A be an algebra that contains $\mathbb{Z}[X]$, and let $\tilde{K}=K_T(G/B)\otimes_{\mathbb{Z}[X]}A$. The action of the Demazure operators T_i extends A-linearly to $\tilde{K}$. Suppose that R^λ, $\lambda\in\Lambda$, is a family of A-linear operators acting on the space $\tilde{K}$ such that relations* (3.10) *and* (3.11) *hold with E^λ replaced by R^λ. Then the operator R^λ preserves $K_T(G/B)\subset\tilde{K}$ and coincides with E^λ for all λ.*

Proof. The conditions imply that relation (3.12) holds with E^λ replaced by R^λ. Applying this expression to $[\mathcal{O}_1]$, we deduce that $R^\lambda([\mathcal{O}_u]) = E^\lambda([\mathcal{O}_u])$, for any $u\in W$. □

Let us also mention another basis of $K_T(G/B)$ studied by Kostant and Kumar [KoKu], see also the recent paper [GrRa] by Griffeth and Ram. One can easily check that there is an algebra involution ψ of the ring $\hat{\mathcal{H}}$ given by $\psi : T_i \mapsto 1 - T_i$, $i = 1, \dots, r$, and $\psi : E^\lambda \mapsto E^{-\lambda}$. In other words, the operators $\varepsilon_i = 1 - T_i$, for $i = 1, \dots, r$, satisfy relations (3.3), (3.4), and (3.10) with T_i replaced by ε_i and E^λ replaced by $E^{-\lambda}$. Thus one can correctly define the elements $\varepsilon_w := \varepsilon_{i_1} \cdots \varepsilon_{i_l} \in \hat{\mathcal{H}}$, for a reduced decomposition $w = s_{i_1} \cdots s_{i_l} \in W$. For $w \in W$, let $[\mathcal{I}_w]$ be the element of $K_T(G/B)$ given by

$$[\mathcal{I}_w] = \varepsilon_{w^{-1}}([\mathcal{O}_1]). \tag{3.13}$$

It turns out that the elements $[\mathcal{I}_w]$, $w \in W$, form a $\mathbb{Z}[X]$-basis of $K_T(G/B)$, as well. Moreover, the bases $\{[\mathcal{I}_w] \mid w \in W\}$ and $\{[\mathcal{O}_w] \mid w \in W\}$ are related to each other as follows:

$$[\mathcal{I}_w] = \sum_{u \le w} (-1)^{\ell(u)} [\mathcal{O}_u] \qquad \text{and} \qquad [\mathcal{O}_w] = \sum_{u \le w} (-1)^{\ell(u)} [\mathcal{I}_u].$$

These two relations are easy to check by induction on the length of w.

The element $[\mathcal{I}_w]$ can be described geometrically. Up to sign, it is the class $*[\mathcal{I}_{X_w}]$, where $\mathcal{I}_{X_w}$ is the sheaf given by the exact sequence $0 \to \mathcal{I}_{X_w} \to \mathcal{O}_{X_w} \to \mathcal{O}_{\partial X_w} \to 0$, and $\partial X_w = \bigcup_{u<w} X_u$ is the boundary of the Schubert variety X_w (cf. [Mat, Theorem 2.1 (ii)], [LiSe, Equation (4)], and [GrRa, Section 2]). Brion and Lakshmibai [BrLa] showed that the classes $[\mathcal{I}_{X_w}]$ form the dual basis to $\{[\mathcal{O}_{X_w}] \mid w \in W\}$ with respect to the natural intersection pairing in K-theory.

Applying the above involution ψ to both sides of (3.12), we obtain

$$E^{-\lambda}\, \varepsilon_{u^{-1}} = \sum_{w \in W,\, \mu \in \Lambda} c_{u,w}^{\lambda,\mu}\; \varepsilon_{w^{-1}}\, E^{-\mu}.$$

Then applying both sides of this relation to $[\mathcal{O}_1]$, we immediately deduce the following *dual form* of (3.8)

$$e^{-\lambda} \cdot [\mathcal{I}_u] = \sum_{w \in W,\, \mu \in \Lambda} c_{u,w}^{\lambda,\mu}\, x^{-\mu}\, [\mathcal{I}_w], \tag{3.14}$$

where $c_{u,w}^{\lambda,\mu}$ are the same K_T-Chevalley coefficients as those in (3.8) and (3.12).

Note that relations (3.3), (3.4), and (3.10) in the algebra $\hat{\mathcal{H}}$ are equivalent to the relations obtained from them by reversing the order of all terms. This symmetry of the relations implies that the expression

$$T_u\, E^\lambda = \sum_{w \in W,\, \mu \in \Lambda} c_{u,w}^{\lambda,\mu}\; E^\mu\, T_w \tag{3.15}$$

has the same K_T-Chevalley coefficients $c_{u,w}^{\lambda,\mu}$.

The (nonequivariant) Grothendieck ring $K(G/B)$ of coherent sheaves on G/B can be obtained by the specialization $x^\mu \mapsto 1$, for all μ, i.e., by ignoring all exponents x^μ in equivariant K-theory. By a slight abuse of notation, we will use the same symbols $[\mathcal{O}_{X_w}]$ and $[\mathcal{L}_\lambda]$ for the obvious classes in $K(G/B)$ as in equivariant K-theory. The classes $[\mathcal{O}_{X_w}]$, $w \in W$, form a $\mathbb{Z}$-basis of $K(G/B)$.

Let us also recall the way in which Schubert calculus in cohomology can be recovered from K-theory. Let $H^*(G/B) := H^*(G/B, \mathbb{Q})$ be the cohomology ring of G/B with rational coefficients. It has a linear basis of classes of Schubert varieties $[X_w]$, $w \in W$, called *Schubert classes.* The cohomology ring is $2\mathbb{Z}$-graded by $\deg([X_w]) = 2(\ell(w_\circ) - \ell(w))$. Let $\mathfrak{h}_\mathbb{Q}^* \subset \mathfrak{h}^*$ be the $\mathbb{Q}$-span of the weight lattice Λ, and let $Sym(\mathfrak{h}_\mathbb{Q}^*)$ be its symmetric algebra, i.e., the ring of polynomials on $\mathfrak{h}_\mathbb{Q}$. The classical *Borel theorem* says that the cohomology ring $H^*(G/B)$ is isomorphic to the following quotient of the symmetric algebra:

$$H^*(G/B) \simeq Sym(\mathfrak{h}_\mathbb{Q}^*)/\mathcal{J},$$

where $\mathcal{J} := \langle f \in Sym(\mathfrak{h}_\mathbb{Q}^*)^W \mid f(0) = 0 \rangle$ is the ideal generated by W-invariant polynomials without constant term. The isomorphism identifies the Chern class $[\lambda] \in H^2(G/B)$ of the line bundle $\mathcal{L}_\lambda$ with the coset of λ modulo $\mathcal{J}$. The product of $[\lambda]$ and a Schubert class $[X_u]$ in the cohomology ring is given by the following classical formula due to Chevalley [Chev]:

$$[\lambda] \cdot [X_u] = \sum_{\alpha\in\Phi^+,\, \ell(us_\alpha)=\ell(u)-1} (\lambda, \alpha^\vee)\, [X_{us_\alpha}]. \tag{3.16}$$

The *Chern character* is the ring isomorphism $ChCh : K(G/B) \otimes \mathbb{Q} \to H^*(G/B)$ that sends the class $[\mathcal{L}_\lambda] \in K(G/B)$ of the line bundle $\mathcal{L}_\lambda$ to $\exp[\lambda] := 1 + [\lambda] + [\lambda]^2/2! + \cdots \in H^*(G/B)$. Then

$$ChCh([\mathcal{O}_w]) = [X_w] + \text{higher degree terms}.$$

This shows that the Chevalley formula (3.16) for the product $[\lambda]\cdot[X_u]$ in $H^*(G/B)$ is obtained from the expression $[\mathcal{L}_\lambda]\cdot[\mathcal{O}_{X_u}] - [\mathcal{O}_{X_u}]$ in $K_T(G/B)$ by expanding it using (3.9), ignoring the exponents $x^{-\mu}$, applying the Chern character map, and then extracting terms of degree $\deg([X_u]) + 2$. In other words, for $\lambda \in \Lambda$, $u \in W$, $\alpha \in \Phi^+$ such that $\ell(us_\alpha) = \ell(u) - 1$, the coefficient in the Chevalley formula equals

$$(\lambda, \alpha^\vee) = \sum_{\mu\in\Lambda} c_{u,us_\alpha}^{\lambda,\mu}. \tag{3.17}$$

A rule for computing the coefficients $c_{u,w}^{\lambda,\mu}$ can be thought of as a generalization of the Chevalley formula to T-equivariant K-theory.

Remark 3.2. In fact, Pittie and Ram [PiRa] worked in a more general setup than the Grothendieck ring $K_T(G/B)$. Their construction implies that the same K_T-Chevalley coefficients $c_{u,w}^{\lambda,\mu}$ as in (3.9) give the product of the classes of $\mathcal{L}_\lambda$ and $\mathcal{O}_{X_u}$ in the K-theory of a G/B-bundle over a smooth base. Thus, the results of the present paper apply to this more general case as well.

4. Demazure Characters

Lakshmibai-Littelmann [LaLi] and Littelmann-Seshadri [LiSe] indicated that the product $[\mathcal{L}_\lambda]\cdot[\mathcal{O}_{X_u}]$ in the Grothendieck ring $K_T(G/B)$ is related to representation theory. This relation is also implicit in the Pittie-Ram formula [PiRa]. Kumar [Kum] pointed out that the Demazure characters can be expressed in terms of the K_T-Chevalley coefficients, as shown below.

For a dominant weight $\lambda \in \Lambda^+$, let V_λ denote the finite dimensional irreducible representation of the Lie group G with highest weight λ. For $\lambda \in \Lambda^+$ and $w \in W$, the *Demazure module* $V_{\lambda,w}$ is the B-module that is dual to the space of global sections of the line bundle $\mathcal{L}_\lambda$ on the Schubert variety X_w:

$$V_{\lambda,w} := H^0(X_w, \mathcal{L}_\lambda)^*. \tag{4.1}$$

For the longest Weyl group element $w = w_\circ$, the space $V_{\lambda,w_\circ} = H^0(G/B, \mathcal{L}_\lambda)^*$ has the structure of a G-module. The classical *Borel-Weil theorem* says that $V_{\lambda,w_\circ}$ is isomorphic to the irreducible G-module V_λ. The formal characters of these modules, called *Demazure characters*, are given by $ch(V_{\lambda,w}) = \sum_{\mu\in\Lambda} m_{\lambda,w}(\mu)\, e^\mu \in \mathbb{Z}[\Lambda]$, where $m_{\lambda,w}(\mu)$ is the multiplicity of the weight μ in $V_{\lambda,w}$. They generalize the characters of the irreducible representations $ch(V_\lambda) = ch(V_{\lambda,w_\circ})$. The *Demazure character formula* [Dem] says that the character $ch(V_{\lambda,w})$ is given by

$$ch(V_{\lambda,w}) = T_w(e^\lambda), \tag{4.2}$$

where T_w is the Demazure operator (3.6).

Lemma 4.1. *For any $\lambda \in \Lambda^+$ and $u \in W$, the Demazure character $ch(V_{\lambda,u})$ can be expressed in terms of the K_T-Chevalley coefficients $c_{u,w}^{\lambda,\mu}$ in* (3.8) *as follows:*

$$ch(V_{\lambda,u}) = \sum_{w\in W,\, \mu\in\Lambda} c_{u,w}^{\lambda,\mu}\, e^\mu.$$

In particular, the character of the irreducible representation V_λ of G is equal to

$$ch(V_\lambda) = \sum_{w\in W,\, \mu\in\Lambda} c_{w_\circ,w}^{\lambda,\mu}\, e^\mu.$$

Proof. Applying both sides of identity (3.15) to $[\mathcal{O}_{w_\circ}] = 1$ and using $T_w(1) = 1$, we obtain

$$T_u(e^\lambda) = \sum_{w\in W,\, \mu\in\Lambda} c_{u,w}^{\lambda;\mu}\ e^\mu,$$

which, together with the Demazure character formula (4.2), proves the lemma. □

Let us also give a geometric argument that proves Lemma 4.1. It is implicit in [LaLi] and [LiSe] and was reported to us by Kumar [Kum]. Let $\chi : K_T(G/B) \to \mathbb{Z}[\Lambda]$ be the *Euler characteristic map* given by

$$\chi : [\mathcal{V}] \longmapsto \sum_{i\geq 0} (-1)^i\, ch(H^i(G/B, \mathcal{V})^*),$$

for a coherent sheaf $\mathcal{V}$ on G/B. For a dominant weight λ, the Euler characteristic $\chi([\mathcal{L}_\lambda]\cdot[\mathcal{O}_{X_u}])$ is equal to the Demazure character $ch(V_{\lambda,u})$. Indeed, this follows from (4.1), the fact that

$$H^i(G/B, \mathcal{L}_\lambda \otimes \mathcal{O}_{X_u}) = H^i(X_u, \mathcal{L}_\lambda)\,,$$

and the vanishing of the cohomologies $H^i(X_u, \mathcal{L}_\lambda)$, for $i \geq 1$. In particular, we have $\chi([\mathcal{O}_{X_w}]) = 1$, for any $w \in W$. Thus $\chi(x^{-\mu}[\mathcal{O}_{X_w}]) = e^\mu$. Applying the Euler characteristic map χ to both sides of (3.9), we obtain Lemma 4.1.

5. Affine Weyl Groups

In this section, we remind a few basic facts about affine Weyl groups and alcoves, cf. Humphreys [Hum, Chaper 4] for more details. Then we define λ-chains that will be used in the rest of the paper.

Let W_{aff} be the *affine Weyl group* for the Langlands dual group $G^\vee$. The affine Weyl group W_{aff} is generated by the affine reflections $s_{\alpha,k} : \mathfrak{h}^*_\mathbb{R} \to \mathfrak{h}^*_\mathbb{R}$, for $\alpha \in \Phi$ and $k \in \mathbb{Z}$, that reflect the space $\mathfrak{h}^*_\mathbb{R}$ with respect to the affine hyperplanes

$$H_{\alpha,k} := \{\lambda \in \mathfrak{h}^*_\mathbb{R} \mid (\lambda, \alpha^\vee) = k\}. \tag{5.1}$$

Explicitly, the affine reflection $s_{\alpha,k}$ is given by

$$s_{\alpha,k} : \lambda \mapsto s_\alpha(\lambda) + k\,\alpha = \lambda - ((\lambda,\alpha^\vee) - k)\,\alpha.$$

The hyperplanes $H_{\alpha,k}$ divide the real vector space $\mathfrak{h}^*_\mathbb{R}$ into open regions, called *alcoves.* Each alcove A is given by inequalities of the form

$$A := \{\lambda \in \mathfrak{h}^*_\mathbb{R} \mid m_\alpha < (\lambda, \alpha^\vee) < m_\alpha + 1 \text{ for all } \alpha \in \Phi^+\},$$

where $m_\alpha = m_\alpha(A)$, $\alpha \in \Phi^+$, are some integers.

A proof of the following important property of the affine Weyl group can be found, e.g., in [Hum, Chapter 4].

Lemma 5.1. *The affine Weyl group W_{aff} acts simply transitively on the collection of all alcoves.*

The *fundamental alcove* $A_\circ$ is given by

$$A_\circ := \{\lambda \in \mathfrak{h}^*_\mathbb{R} \mid 0 < (\lambda, \alpha^\vee) < 1 \text{ for all } \alpha \in \Phi^+\}.$$

Lemma 5.1 implies that, for any alcove A, there exists a unique element v_A of the affine Weyl group W_{aff} such that $v_A(A_\circ) = A$. Hence the map $A \mapsto v_A$ is a one-to-one correspondence between alcoves and elements of the affine Weyl group.

Recall that $\theta^\vee \in \Phi^\vee$ is the highest coroot. Let $\theta \in \Phi^+$ be the corresponding root, and let $\alpha_0 := -\theta$. The fundamental alcove $A_\circ$ is, in fact, the simplex given by

$$A_\circ = \{\lambda \in \mathfrak{h}^*_\mathbb{R} \mid 0 < (\lambda, \alpha_i^\vee) \text{ for } i = 1, \ldots, r, \text{ and } (\lambda, \theta^\vee) < 1\}, \tag{5.2}$$

Lemma 5.1 also implies that the affine Weyl group is generated by the set of reflections $s_0, s_1, \ldots, s_r$ with respect to the walls of the fundamental alcove $A_\circ$, where $s_0 := s_{\alpha_0,-1}$ and $s_1, \ldots, s_r \in W$ are the simple reflections $s_i = s_{\alpha_i,0}$. As before, a decomposition $v = s_{i_1} \cdots s_{i_l} \in W_{\mathrm{aff}}$ is called *reduced* if it has minimal length; its length $\ell(v) = l$ is called the length of v.

Like the Weyl group, the affine Weyl group W_{aff} is a Coxeter group, i.e., it is described by the relations

$$(s_i)^2 = 1 \quad \text{and} \quad (s_i s_j)^{m_{ij}} = 1, \quad \text{for any } i, j \in \{0, \ldots, r\}, \tag{5.3}$$

where m_{ij} is half of the order of the dihedral subgroup generated by s_i and s_j.

We say that two alcoves A and B are *adjacent* if B is obtained by an affine reflection of A with respect to one of its walls. In other words, two alcoves are adjacent if they are distinct and have a common wall. For a pair of adjacent alcoves, let us write $A \xrightarrow{\beta} B$ if the common wall of A and B is of the form $H_{\beta,k}$ and the root $\beta \in \Phi$ points in the direction from A to B. By the definition, all alcoves that are adjacent to the fundamental alcove $A_\circ$ are obtained from $A_\circ$ by the reflections $s_0, \cdots, s_r$, and $A_\circ \xrightarrow{-\alpha_i} s_i(A_\circ)$.

Definition 5.2. An *alcove path* is a sequence of alcoves $(A_0, A_1, \ldots, A_l)$ such that A_{j-1} and A_j are adjacent, for $j = 1, \ldots, l$. Let us say that an alcove path is *reduced* if it has minimal length among all alcove paths from A_0 to A_l.

Let $v \mapsto \bar{v}$ be the homomorphism $W_{\mathrm{aff}} \to W$ defined by ignoring the affine translation. In other words, $\bar{s}_{\alpha,k} = s_\alpha \in W$.

The following lemma, which is essentially well-known, summarizes some properties of decompositions in affine Weyl groups, cf. [Hum].

Lemma 5.3. *Let v be any element of W_{aff}, and let $A = v(A_\circ)$ be the corresponding alcove. Then the decompositions $v = s_{i_1} \cdots s_{i_l}$ of v (reduced or not) as a product of generators in W_{aff} are in one-to-one correspondence with alcove paths $A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_l} A_l$ from the fundamental alcove $A_0 = A_\circ$ to $A_l = A$. This correspondence is explicitly given by $A_j = s_{i_1} \cdots s_{i_j}(A_\circ)$, for $j = 0, \ldots, l$; and the roots $\beta_1, \ldots, \beta_l$ are given by*

$$\beta_1 = \alpha_{i_1},\ \beta_2 = \bar{s}_{i_1}(\alpha_{i_2}),\ \beta_3 = \bar{s}_{i_1}\bar{s}_{i_2}(\alpha_{i_3}), \ldots,\ \beta_l = \bar{s}_{i_1} \cdots \bar{s}_{i_{l-1}}(\alpha_{i_l}).$$

Let $r_j \in W_{\mathrm{aff}}$ denote the affine reflection with respect to the common wall of the alcoves A_{j-1} and A_j, for $j = 1, \ldots, l$. Then the affine reflections $r_1, \ldots, r_l$ are given by

$$r_1 = s_{i_1},\ r_2 = s_{i_1}s_{i_2}s_{i_1},\ r_3 = s_{i_1}s_{i_2}s_{i_3}s_{i_2}s_{i_1},\ \ldots,\ r_l = s_{i_1} \cdots s_{i_l} \cdots s_{i_1}.$$

We have $\bar{r}_i = s_{\beta_i}$ and $v = s_{i_1} \cdots s_{i_l} = r_l \cdots r_1$. Moreover, the following claims are equivalent:

(a) *$v = s_{i_1} \cdots s_{i_l}$ is a reduced decomposition;*
(b) *$(A_0, A_1, \ldots, A_l)$ is a reduced alcove path;*
(c) *all affine reflections $r_1, \ldots, r_l$ are distinct;*
(d) *$\beta_i \neq -\beta_j$, for any i and j.*

Finally, for any $\alpha \in \Phi^+$, we have $m_\alpha(A) = \#\{j \mid \beta_j = -\alpha\} - \#\{j \mid \beta_j = \alpha\}$.

Proof. Let $v = s_{i_1} \cdots s_{i_l}$ be a decomposition and $A_j = s_{i_1} \cdots s_{i_j}(A_\circ)$, for $j = 0, \ldots, l$. Then $A_0 = A_\circ$ and $A_l = v(A_\circ) = A$. Applying $s_{i_1} \cdots s_{i_{j-1}}$ to the adjacent pair $A_\circ \xrightarrow{-\alpha_{i_j}} s_{i_j}(A_\circ)$, we deduce that the pair $A_{j-1} \xrightarrow{-\beta_j} A_j$ is adjacent as well, where $\beta_j = \bar{s}_{i_1} \cdots \bar{s}_{i_{j-1}}(\alpha_{i_j})$. Thus $(A_0, \ldots, A_l)$ is an alcove path from $A_\circ$ to A. The reflection s_{i_j} switches the alcoves $A_\circ$ and $s_{i_j}(A_\circ)$. Thus the reflection $r_j = s_{i_1} \cdots s_{i_j} \cdots s_{i_1}$ is the reflection with respect to the common wall of A_{j-1} and A_j.

On the other hand, let $(A_0, \ldots, A_l)$ be any alcove path from $A_\circ$ to A, and let r_j be the reflection with respect to the common wall of A_{j-1} and A_j, for $j = 1, \ldots, l$. Then $A_j = r_j \cdots r_1(A_\circ)$. Applying $(r_{j-1} \cdots r_1)^{-1} = r_1 \cdots r_{j-1}$ to the adjacent pair (A_{j-1}, A_j), we obtain the adjacent pair $(A_\circ, s(A_\circ))$, where $s = r_1 \cdots r_{j-1} r_j r_{j-1} \cdots r_1$. Thus s should be a reflection with respect to one of the walls of $A_\circ$. Thus there are $i_1, \ldots, i_l \in \{0, \ldots, r\}$ such that $r_1 \cdots r_{j-1} r_j r_{j-1} \cdots r_1 = s_{i_j}$, for $j = 1, \ldots, l$. The affine Weyl group element $s_{i_1} \cdots s_{i_l} = r_l \cdots r_1$ maps $A_\circ$ to A, and is equal to v.

(a) $\Leftrightarrow$ (b). This is clear, because a decomposition and the corresponding alcove path have the same length.

(b) $\Leftrightarrow$ (c). The fact that all affine reflections $r_1, \ldots, r_l$ are distinct for a reduced decomposition is given in [Hum, Lemma 4.5]. On the other hand, the length l of any alcove path should be at least the number of hyperplanes of the form $H_{\alpha,k}$ that separate A_0 and A_l. If all affine reflections $r_1, \ldots, r_l$ are distinct, then the path never crosses the same hyperplane twice, and, thus, its length equals the number of hyperplanes that separate A_0 and A_l.

(c) $\Leftrightarrow$ (d). If $\beta_i = -\beta_j = \alpha$, then the alcove path crosses two parallel hyperplanes $H_{\alpha,k}$ and $H_{\alpha,l}$ in opposite directions. It follows that the path crosses one of these hyperplanes twice, and, thus, the affine reflections $r_1, \ldots, r_l$ are not distinct. On the other hand, if $r_1, \ldots, r_l$ are not distinct, then the path crosses the same hyperplane more than once. It follows that the path should cross this hyperplane in opposite directions. Thus $\beta_i = -\beta_j$ for some i and j.

The last claim follows from the fact that, each time the alcove path crosses a hyperplane of the form $H_{\alpha,k}$, $\alpha \in \Phi^+$, in positive (respectively negative) direction, the number m_α increases (respectively decreases) by 1, and all other m_β's do not change. □

The affine translations by weights preserve the set of affine hyperplanes $H_{\alpha,k}$, cf. (2.1) and (5.1). It follows that these affine translations map alcoves to alcoves. Let $A_\lambda = A_\circ + \lambda$ be the alcove obtained by the affine translation of the fundamental alcove $A_\circ$ by a weight $\lambda \in \Lambda$. Let $v_\lambda = v_{A_\lambda}$ be the corresponding element of W_{aff}, i.e,. v_λ is defined by $v_\lambda(A_\circ) = A_\lambda$. Note that the element v_λ may not be an affine translation itself.

Definition 5.4. Let λ be a weight, and let $v_{-\lambda} = s_{i_1} \cdots s_{i_l}$ be any decomposition, reduced or not, of $v_{-\lambda}$ as a product of generators of W_{aff}. Let us say that the λ-*chain of roots* associated with this decomposition is the sequence $(\beta_1, \ldots, \beta_l)$ of the roots in Φ given by

$$\beta_1 = \alpha_{i_1},\ \beta_2 = \bar{s}_{i_1}(\alpha_{i_2}),\ \beta_3 = \bar{s}_{i_1}\bar{s}_{i_2}(\alpha_{i_3}), \ldots,\ \beta_l = \bar{s}_{i_1} \cdots \bar{s}_{i_{l-1}}(\alpha_{i_l})\,.$$

Sometimes we will abbreviate "λ-chain of roots" as, simply, "λ-chain." Let us also say that the λ-*chain of reflections* associated with the above decomposition for $v_{-\lambda}$ is the sequence $(r_1, \ldots, r_l)$ of the affine reflections in W_{aff} given by

$$r_1 = s_{i_1},\ r_2 = s_{i_1}s_{i_2}s_{i_1},\ r_3 = s_{i_1}s_{i_2}s_{i_3}s_{i_2}s_{i_1},\ \ldots,\ r_l = s_{i_1} \cdots s_{i_l} \cdots s_{i_1}.$$

In particular, $\bar{r}_i = s_{\beta_i}$.

According to Lemma 5.3, we can equivalently define a λ-chain as a sequence of roots $(\beta_1, \ldots, \beta_l)$ such that there exists an alcove path $A_0 \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_l} A_l$ from $A_0 = A_\circ$ to $A_l = A_{-\lambda}$ with edges labeled by the roots $-\beta_1, \ldots, -\beta_l$. The j-th element of the corresponding λ-chain of reflections $(r_1, \ldots, r_l)$ is the affine reflection r_j with respect to the common walls of the alcoves A_{j-1} and A_j, for $j = 1, \ldots, l$.

Finally, we say that a λ-chain is *reduced* if it is associated with a reduced decomposition for $v_{-\lambda}$.

Remark 5.5. If $A \xrightarrow{\beta} B$ is a pair of adjacent alcoves, then $(A+\lambda) \xrightarrow{\beta} (B+\lambda)$, for any affine translation of the alcoves by the weight λ. Thus, a translation of an alcove path by a weight λ is an alcove path labeled by the same sequence of roots. For a λ-chain of roots $(\beta_1, \ldots, \beta_l)$, let us translate the corresponding alcove path $A_\circ \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_l} A_{-\lambda}$ by the weight λ, and then reverse its direction. We obtain the alcove path $A_\circ \xrightarrow{\beta_l} \cdots \xrightarrow{\beta_1} A_\lambda$ associated with the $(-\lambda)$-chain $(-\beta_l, \ldots, -\beta_1)$.

6. The K_T-Chevalley Formula

In this section, we formulate our main result and give its several specializations and applications to characters.

Theorem 6.1. (K_T-Chevalley formula) *Fix any weight λ. Let $(r_1,\ldots,r_l)$ and $(\beta_1,\ldots,\beta_l)$ be the λ-chain of reflections and the λ-chain of roots associated with a decomposition $v_{-\lambda}=s_{i_1}\cdots s_{i_l}\in W_{\mathrm{aff}}$, which may or may not be reduced. Let $u,w\in W$, and $\mu\in\Lambda$. Then the K_T-Chevalley coefficient $c_{u,w}^{\lambda,\mu}$, i.e., the coefficient of $x^\mu\,[\mathcal{O}_w]$ in the expansion of the product $e^\lambda\cdot[\mathcal{O}_u]$, can be expressed as follows:*

$$c_{u,w}^{\lambda,\mu}=\sum_J(-1)^{n(J)}\,, \tag{6.1}$$

where the summation is over all subsets $J=\{j_1<\cdots<j_s\}$ of $\{1,\ldots,l\}$ satisfying the following conditions:

(a) *$u\gtrdot u\,\bar r_{j_1}\gtrdot u\,\bar r_{j_1}\bar r_{j_2}\gtrdot\cdots\gtrdot u\,\bar r_{j_1}\bar r_{j_2}\cdots\bar r_{j_s}=w$ is a saturated decreasing chain from u to w in the Bruhat order on the Weyl group W;*
(b) *$-\mu=u\,r_{j_1}\cdots r_{j_s}(-\lambda)$,*

and $n(J)$ is the number of negative roots in $\{\beta_{j_1},\ldots,\beta_{j_s}\}$.

In Section 13, we reformulate this theorem in a compact form and then prove it, using a certain R-matrix. In Sections 15 and 16, we give several examples that illustrate this theorem.

Lemma 5.3 implies the following statement.

Lemma 6.2. *Let $(\beta_1,\ldots,\beta_l)$ be a reduced λ-chain of roots. Let $\alpha\in\Phi$ be a root such that $(\lambda,\alpha^\vee)\geq 0$. Then $\#\{i\mid\beta_i=\alpha\}=(\lambda,\alpha^\vee)$ and $\#\{i\mid\beta_i=-\alpha\}=0$.*

In particular, if λ is a dominant weight, then all roots $\beta_1,\ldots,\beta_l$ are positive. Also, if λ is an antidominant weight, that is, $-\lambda\in\Lambda^+$, then all roots $\beta_1,\ldots,\beta_l$ are negative.

In the special cases corresponding to dominant and antidominant weights λ, Theorem 6.1 can be reformulated in a more explicit way. In these cases, for reduced λ-chains, Theorem 6.1 gives a manifestly positive formula, which is not the case in general.

Corollary 6.3. *Consider the setup in Theorem 6.1. Assume that $v_{-\lambda}=s_{i_1}\cdots s_{i_l}$ is a reduced decomposition in W_{aff}.*

If λ is a dominant weight, then $c_{u,w}^{\lambda,\mu}$ equals the number of subsets $J\subseteq\{1,\ldots,l\}$ that satisfy conditions (a) *and* (b) *in Theorem* 6.1.

If λ is an antidominant weight, then $(-1)^{\ell(u)-\ell(w)}\,c_{u,w}^{\lambda,\mu}$ equals the number of subsets $J\subseteq\{1,\ldots,l\}$ that satisfy conditions (a) *and* (b) *in Theorem* 6.1.

Proof. For a dominant weight λ, all roots $\beta_1,\ldots,\beta_l$ are positive; thus $n(J)=0$. For an antidominant weight λ, all roots $\beta_1,\ldots,\beta_l$ are negative; thus $n(J)=|J|=\ell(u)-\ell(w)$. □

Theorem 6.1 specializes to following rule for products in the (nonequivariant) Grothendieck ring $K(G/B)$.

Corollary 6.4. *The coefficient $c_{u,w}^{\lambda}$ of $[\mathcal{O}_w]$ in the product $e^\lambda\cdot[\mathcal{O}_u]$ of classes in $K(G/B)$ has the same combinatorial description as in Theorem* 6.1, *except that condition* (b) *on the weights involved is dropped.*

Proof. We have $c_{u,w}^{\lambda}=\sum_{\mu\in\Lambda}c_{u,w}^{\lambda,\mu}$. □

Theorem 6.1 implies the following combinatorial model for the Demazure characters $ch(V_{\lambda,u})$ and, in particular, for the characters $ch(V_\lambda)$ of the irreducible representations V_λ of the Lie group G.

Corollary 6.5. *Let λ be a dominant weight, let $u \in W$, and let $(r_1, \ldots, r_l)$ be a reduced λ-chain of reflections. Then the Demazure character $ch(V_{\lambda,u})$ is equal to the sum*

$$ch(V_{\lambda,u}) = \sum_J e^{-u\, r_{j_1} \cdots r_{j_s}(-\lambda)}$$

over all subsets $J = \{j_1 < \cdots < j_s\} \subset \{1, \ldots, l\}$ such that

$$u \gtrdot u\,\bar{r}_{j_1} \gtrdot u\,\bar{r}_{j_1}\bar{r}_{j_2} \gtrdot \cdots \gtrdot u\,\bar{r}_{j_1}\bar{r}_{j_2}\cdots\bar{r}_{j_s}$$

is a saturated decreasing chain in the Bruhat order on the Weyl group W.

Proof. Apply Corollary 6.3 and Lemma 4.1. □

We can slightly simplify the formula for the characters $ch(V_\lambda) = ch(V_{\lambda,w_\circ})$ of the irreducible representations of G, as follows.

Corollary 6.6. *Consider the setup in Corollary 6.5. We have*

$$ch(V_\lambda) = \sum_J e^{-r_{j_1}\cdots r_{j_s}(-\lambda)}\,,$$

where the summation is over all subsets $J = \{j_1 < \cdots < j_s\} \subset \{1, \ldots, l\}$ such that

$$1 \lessdot \bar{r}_{j_1} \lessdot \bar{r}_{j_1}\bar{r}_{j_2} \lessdot \cdots \lessdot \bar{r}_{j_1}\bar{r}_{j_2}\cdots\bar{r}_{j_s}$$

is a saturated increasing chain in the Bruhat order on the Weyl group W.

Proof. Multiplying elements in a decreasing chain by $w_\circ$ on the left results in an increasing chain in Bruhat order. On the other hand, we can remove $w_\circ$ from the exponent because the character $ch(V_\lambda)$ is W-invariant. □

In the rest of this section, we show how to construct λ-chains of reflections $(r_1, \ldots, r_l)$ and λ-chains of roots $(\beta_1, \ldots, \beta_l)$. Clearly, there are many possible choices.

Let us fix an arbitrary weight λ. Let $\pi : [0,1] \to \mathfrak{h}^*_\mathbb{R}$ be a sufficiently generic continuous path such that $\pi(0) \in A_\circ$ and $\pi(1) \in A_{-\lambda}$. Here "sufficiently generic" means that the path π does not cross any face of an alcove of codimension 2 or higher. For example, the path $\pi : t \mapsto -t\,\lambda + \gamma$, where γ is a generic point in $A_\circ$, will suffice. Suppose that the path π passes through the sequence of alcoves $A_\circ, \ldots, A_{-\lambda}$ as t varies from 0 to 1. This sequence is an alcove path. Let $H_1, \ldots, H_l$ be the affine hyperplanes of the form $H_{\alpha,k}$ that the path π crosses as t varies from 0 to 1. According to Lemma 5.3, the sequence $(r_1, \ldots, r_l)$ of affine reflections with respect to $H_1, \ldots, H_l$ is a λ-chain of reflections.

In order to make our formula completely combinatorial, we present one particular choice for a λ-chain of reflections and the corresponding λ-chain of roots. The construction depends on the choice of a total order $\alpha_1 < \cdots < \alpha_r$ on the simple roots in Φ. Suppose that $\pi = \pi_\varepsilon : [0,1] \to \mathfrak{h}^*_\mathbb{R}$ is the path given by

$$\pi_\varepsilon : t \mapsto -t\,\lambda + \varepsilon\,\omega_1 + \varepsilon^2\omega_2 + \cdots + \varepsilon^r\omega_r,$$

where ε is a sufficiently small positive real number. Let $\mathcal{R} = \mathcal{R}_\lambda \subset W_{\mathrm{aff}}$ be the set of affine reflections with respect to affine hyperplanes $H_{\alpha,k}$ that separate the alcoves $A_\circ$ and $A_{-\lambda}$. This set is given by

$$\mathcal{R} = \mathcal{R}_\lambda = \bigcup_{\alpha\in\Phi^+} \begin{cases} \{s_{\alpha,k} \mid 0 \geq k > -(\lambda,\alpha^\vee)\} & \text{if } (\lambda,\alpha^\vee) > 0\,, \\ \{s_{\alpha,k} \mid 0 < k \leq -(\lambda,\alpha^\vee)\} & \text{if } (\lambda,\alpha^\vee) < 0\,, \\ \emptyset & \text{if } (\lambda,\alpha^\vee) = 0\,. \end{cases}$$

For any $s_{\alpha,k} \in \mathcal{R}$, $\alpha \in \Phi^+$, the path π_ε crosses the affine hyperplane $H_{\alpha,k}$ at the point $t = t_{\alpha,k} = (\lambda,\alpha^\vee)^{-1}(-k + \sum_{i=1}^r (\omega_i,\alpha^\vee)\,\varepsilon^i)$. Note that $(\lambda,\alpha^\vee) \neq 0$, for $s_{\alpha,k} \in \mathcal{R}$. Let $h : \mathcal{R} \to \mathbb{R}^{r+1}$ be the map given by

$$h : s_{\alpha,k} \mapsto (\lambda,\alpha^\vee)^{-1}\,(-k, (\omega_1,\alpha^\vee), \ldots, (\omega_r,\alpha^\vee)), \tag{6.2}$$

for any $s_{\alpha,k} \in \mathcal{R}$ with $\alpha \in \Phi^+$. Then, for sufficiently small $\varepsilon > 0$, we have $t_{\alpha,k} < t_{\alpha',k'}$ if and only if $h(s_{\alpha,k})$ is less than $h(s_{\alpha',k'})$ in the lexicographic order on $\mathbb{R}^{r+1}$. We claim that the map h is injective. Indeed, if $h(s_{\alpha,k}) = h(s_{\alpha',k'})$, then $\alpha = \alpha'$. Otherwise, the root system $\Phi^\vee$ would contain two proportional positive coroots $\alpha^\vee \neq (\alpha')^\vee$, which is not possible. Also, the fact that $\alpha = \alpha'$ implies that $k = k'$.

Let $b : \{\text{affine reflections}\} \to \Phi$ be the map given by

$$b : s_{\alpha,k} \longmapsto \begin{cases} \alpha & \text{if } k \leq 0 \text{ and } \alpha \in \Phi^+, \\ -\alpha & \text{if } k > 0 \text{ and } \alpha \in \Phi^+. \end{cases}$$

We obtain the following result by using Lemma 5.3.

Proposition 6.7. *Let $\mathcal{R} = \{r_1 < r_2 < \cdots < r_l\}$ be the total order on the set $\mathcal{R}$ such that $h(r_1) < h(r_2) < \cdots < h(r_l)$ in the lexicographic order on $\mathbb{R}^{r+1}$. Then $(r_1, \ldots, r_l)$ is the λ-chain of reflections and $(\beta_1, \ldots, \beta_l) = (b(r_1), \ldots, b(r_l))$ is the λ-chain of roots associated with a certain reduced decomposition of $v_{-\lambda}$.*

Example 16.1 illustrates this proposition.

7. Generalization to G/P

Let P be a parabolic subgroup in G such that $P \supset B$. In this section, we show that the K_T-Chevalley formula can be easily extended to equivariant K-theory of the *generalized partial flag variety* G/P.

Let Δ_P be the subset of the simple roots associated with the parabolic subgroup P. Let $\Phi_P \subset \Phi$ be the set of roots that can be written as sums of roots in Δ_P, and let $\Phi_P^+ = \Phi_P \cap \Phi^+$. Then Φ_P is a root system itself, with the Weyl group $W_P \subset W$ generated by the simple reflections s_i, for $\alpha_i \in \Delta_P$. Each coset $\bar{w} = wW_P$ in W/W_P has a unique representative of maximal length. Let us denote the set of maximal coset representatives by $W^P \subset W$, and let us identify it with W/W_P. The Bruhat order on W induces the Bruhat order on $W^P \simeq W/W_P$. According to Deodhar [Deo1], the covering relations in W^P are of the form $u \gtrdot w$, where $w = us_\beta$, for some $\beta \in \Phi^+ \setminus \Phi_P^+$, and $\ell(u) = \ell(w) + 1$. In particular, every covering relation in W^P is a covering relation in the Bruhat order on W.

The generalized partial flag variety G/P decomposes into Schubert cells $X_{\bar{w}}^\circ = B\bar{w}P/P$ indexed by $\bar{w} \in W/W_P$. Their closures $X_{\bar{w}} := \overline{X_{\bar{w}}^\circ}$ are called Schubert varieties. Let $\mathcal{O}_{X_{\bar{w}}}^P$, $\bar{w} \in W/W_P$, be the structure sheaf of the Schubert variety $X_{\bar{w}}$. If λ is a weight satisfying $(\lambda, \beta) = 0$, for all β in Δ_P (or, equivalently, $W_P \subseteq W_\lambda$, where W_λ is the stabilizer of λ), then $-\lambda$ determines a character of P, and thus a line bundle $\mathcal{L}_\lambda^P := G \times_P \mathbb{C}_{-\lambda}$ on G/P. Let $[\mathcal{O}_{X_{\bar{w}}}^P]$ and $[\mathcal{L}_\lambda^P]$ be the corresponding classes in $K_T(G/P)$. The classes $[\mathcal{O}_{X_{\bar{w}}}^P]$ form a $\mathbb{Z}[X]$-basis of $K_T(G/P)$, and the classes $[\mathcal{L}_\lambda^P]$ span $K_T(G/P)$ over $\mathbb{Z}[X]$. Let $[\mathcal{O}_{\bar{w}}^P] := *[\mathcal{O}_{X_{\bar{w}}}^P]$, where the involution $*$ on $K_T(G/P)$ is defined like the one on $K_T(G/B)$.

The equivariant K-theory of G/P can be recovered from $K_T(G/B)$, as stated in [KoKu]. We have the canonical projection $\pi_P : G/B \to G/P$. This determines an injective $\mathbb{Z}[X]$-linear homomorphism $\pi_P^* : K_T(G/P) \to K_T(G/B)$. Moreover, the image of this map, with which $K_T(G/P)$ can be identified, consists precisely of the W_P-invariants in $K_T(G/B)$. It is straightforward to show that

$$\pi_P^*([\mathcal{O}_{\bar{w}}^P]) = [\mathcal{O}_w], \quad \text{and} \quad \pi_P^*([\mathcal{L}_\lambda^P]) = [\mathcal{L}_\lambda], \tag{7.1}$$

where $w \in W^P$ is the maximal coset representative of $\bar{w} \in W/W_P$, and the weight λ is such that $W_P \subseteq W_\lambda$. By abuse of notation, we will denote the class $[\mathcal{L}_{-\lambda}^P]$ in $K_T(G/P)$ by e^λ, as well.

Let us define the integer coefficients $c_{\bar{u},\bar{w}}^{\lambda,\mu}$, for $\bar{u}, \bar{w} \in W/W_P$ and $\lambda, \mu \in \Lambda$, with $W_P \subseteq W_\lambda$, by the following expansion of the product in $K_T(G/P)$:

$$e^\lambda \cdot [\mathcal{O}_{\bar{u}}^P] = \sum_{\bar{w} \in W/W_P,\, \mu \in \Lambda} c_{\bar{u},\bar{w}}^{\lambda,\mu}\; x^\mu\, [\mathcal{O}_{\bar{w}}^P]\,. \tag{7.2}$$

Our combinatorial Chevalley-type formula for $K_T(G/B)$ can be generalized to $K_T(G/P)$, as follows.

Corollary 7.1. *Let $u, w \in W^P$ be the maximal coset representatives of $\bar{u}, \bar{w} \in W/W_P$, and let $\lambda, \mu \in \Lambda$ such that $W_P \subseteq W_\lambda$. Then we have $c_{\bar{u},\bar{w}}^{\lambda,\mu} = c_{u,w}^{\lambda,\mu}$, where $c_{u,w}^{\lambda,\mu}$ is the K_T-Chevalley coefficient for $K_T(G/B)$, which have the combinatorial description given in Theorem* 6.1*. Moreover, if we work with reduced λ-chains, then all the elements of the corresponding saturated chains in the Bruhat order lie in W^P.*

Proof. The first part of the proof is immediate by applying the map π_P^* to both sides of (7.2), and by using (7.1). The second statement follows from the fact that, given the choice of λ, we have $(\lambda, \beta^\vee) = 0$, for all β in Φ_P. Indeed, by Lemma 5.3, a reduced λ-chain of roots does not contain any roots in Φ_P. Therefore, the conclusion follows from the above description of the Bruhat order on W^P. □

8. Applications: K_T-Monk Formula and Duality Formulas

In this section, we present several applications of our K_T-Chevalley formula. First, we give a rule for products $[\mathcal{O}_{w_\circ s_i}] \cdot [\mathcal{O}_u]$, which we call the *$K_T$-Monk formula.* We also give the *dual K_T-Chevalley formula* for products $e^\lambda \cdot [\mathcal{I}_u]$. Then we derive two *duality formulas* for the K_T-Chevalley coefficients. The first one has been already stated for $K(G/B)$, in a slightly imprecise way, by Brion in [Brion, Theorem 4], and proved using some fairly involved geometric arguments. We present a concise combinatorial proof, based on our K_T-Chevalley formula. The two dualities came from the two involutions $w \mapsto w w_\circ$ and $w \mapsto w_\circ w$ on W. Our K_T-Chevalley formula is symmetric with respect to these involutions, because they map increasing chains in the Bruhat order to decreasing chains.

Let us call the classes $[\mathcal{O}_{w_\circ s_i}] \in K_T(G/B)$ the *special classes*; they correspond to the structure sheaves of codimension 1 Schubert varieties $X_{w_\circ s_i}$,

Lemma 8.1. (a) [Brion] *For a simple reflection s_i, we have*

$$[\mathcal{O}_{w_\circ s_i}] = 1 - x^{w_\circ(\omega_i)} e^{-\omega_i}$$

in the Grothendieck ring $K_T(G/B)$.

(b) *The special classes $[\mathcal{O}_{w_\circ s_i}]$, $i = 1, \ldots, r$, generate the Grothendieck ring $K_T(G/B)$ as an algebra over $\mathbb{Z}[X]$.*

Brion proved that $[\mathcal{O}_{X_{w_\circ s_i}}] = 1 - [\mathcal{L}_{-\omega_i}]$ in $K(G/B)$ using a simple geometric argument based on the exact sheaf sequence $0 \to \mathcal{L}_{-\omega_i} \to \mathcal{O}_{G/B} \to \mathcal{O}_{X_{w_\circ s_i}} \to 0$. Brion also mentioned that this argument extends to T-equivariant K-theory.

Proof. (a) Let us apply Theorem 6.1, for $u = w_\circ$ and $\lambda = -\omega_i$. Every saturated chain in the Bruhat order decreasing from $w_\circ$ should start with a simple reflection. For a reduced $(-\omega_i)$-chain of reflections $(r_1, \ldots, r_l)$, exactly one of the reflections $\bar{r}_1, \ldots, \bar{r}_l$ is simple. Namely, $\bar{r}_l = s_i$ and, moreover, $r_l = s_{\alpha_i,1}$. Thus the expansion of the product $e^{-\omega_i} \cdot [\mathcal{O}_{w_\circ}]$ consists of the two terms corresponding to the subsets $J = \emptyset$ and $J = \{l\}$. This expansion is $e^{-\omega_i} \cdot [\mathcal{O}_{w_\circ}] = x^{-w_\circ(\omega_i)}[\mathcal{O}_{w_\circ}] - x^{-w_\circ(\omega_i)}[\mathcal{O}_{w_\circ s_i}]$. Since $[\mathcal{O}_{w_\circ}] = 1$, we obtain the required identity.

(b) Let us identify $K_T(G/B)$ with the quotient in (3.1). There is a *finite* set D of exponents e^μ that spans $K_T(G/B)$ as a $\mathbb{Z}[X]$-module. Indeed, we can take all exponents in some representatives for the classes $[\mathcal{O}_w]$ in $\mathbb{Z}[X] \otimes \mathbb{Z}[\Lambda]$. For a weight $\lambda \in \Lambda$, the exponent e^λ is an invertible element in $K_T(G/B)$; and, thus, the set $e^\lambda D = \{e^{\lambda+\mu} \mid e^\mu \in D\}$ also spans $K_T(G/B)$. For a sufficiently large antidominant weight λ, all exponents in the set $e^\lambda D$ correspond to antidominant weights. On the other hand, according to (a), we have $e^{-\omega_i} = x^{-w_\circ(\omega_i)}(1 - [\mathcal{O}_{w_\circ s_i}])$; thus, all classes $e^\mu = [\mathcal{L}_{-\mu}]$, for antidominant weights μ, can be expressed in terms of the special classes $[\mathcal{O}_{w_\circ s_i}]$. This implies the statement. □

The second part of Corollary 6.3, for $\lambda = -\omega_i$, and Lemma 8.1(a) imply the combinatorial rule below for products of the special classes with the basis elements in $K_T(G/B)$. Note that, if ω_i is a minuscule weight (i.e., $(\omega_i, \alpha^\vee) = 0$ or 1 for any $\alpha \in \Phi^+$), then all reflections r_j in a reduced $(-\omega_i)$-chain of reflections have the form $r_j = s_{\beta_j,1}$, and therefore they all fix ω_i.

Corollary 8.2. (K_T-Monk formula) *Fix a simple reflection s_i, and let $(r_1, \ldots, r_l)$ be a reduced $(-\omega_i)$-chain of reflections. Then, for any $u \in W$, we have*

$$[\mathcal{O}_{w_\circ s_i}] \cdot [\mathcal{O}_u] = (1 - x^{w_\circ(\omega_i) - u(\omega_i)})\,[\mathcal{O}_u] + \sum_J (-1)^{|J|-1}\, x^{\nu(J)}\,[\mathcal{O}_{w(J)}],$$

where the sum is over nonempty subsets $J = \{j_1, \ldots, j_s\}$ in $\{1, \ldots, l\}$ such that $u \gtrdot u\,\bar{r}_{j_1} \gtrdot u\,\bar{r}_{j_1}\bar{r}_{j_2} \gtrdot \cdots \gtrdot u\,\bar{r}_{j_1}\bar{r}_{j_2}\cdots\bar{r}_{j_s} = w$ is a saturated decreasing chain in the Bruhat order from u to $w = w(J)$, and $\nu(J) := w_\circ(\omega_i) - u\,r_{j_1}\cdots r_{j_s}(\omega_i)$. If ω_i is minuscule, then the above formula has the simpler form:

$$[\mathcal{O}_{w_\circ s_i}] \cdot [\mathcal{O}_u] = [\mathcal{O}_u] + x^{w_\circ(\omega_i) - u(\omega_i)} \left(\sum_J (-1)^{|J|-1}\,[\mathcal{O}_{w(J)}] \right),$$

where the notation is as above, but we drop the condition $J \neq \emptyset$.

Since the special classes $[\mathcal{O}_{w_\circ s_i}]$ generate the Grothendieck ring $K_T(G/B)$, Corollary 8.2 completely characterizes the multiplicative structure of this ring.

Remark 8.3. (1) It is not hard to see that $\nu(J) := w_\circ(\omega_i) - u\,r_{j_1}\cdots r_{j_s}(\omega_i)$ is a sum of negative roots. Hence, the coefficient of a Schubert class $[\mathcal{O}_w]$ in the expansion of $[\mathcal{O}_{w_\circ s_i}] \cdot [\mathcal{O}_u]$ can be expressed as $(-1)^{\ell(w)-\ell(u)-1}$ times a sum of products of factors $x^{-\theta}$ and $e^{-\theta} - 1$, where θ is a positive root. This supports the conjecture made by Griffeth and Ram in [GrRa], related to the structure constants of $K_T(G/B)$.

(2) In the equivariant case, the expansion of $[\mathcal{O}_{w_\circ s_i}] \cdot [\mathcal{O}_u]$ contains the term $[\mathcal{O}_u]$ with a nonzero coefficient. This term vanishes in the nonequivariant case of $K(G/B)$. A similar phenomenon happens in the Monk-type formula for equivariant cohomology, which can be derived from Corollary 8.2.

Recall that the classes $[\mathcal{I}_w]$, $w \in W$, given by (3.13) form the dual basis to $\{[\mathcal{O}_w] \mid w \in W\}$ with respect to the natural pairing in K-theory. Define the *dual K_T-Chevalley coefficients* $d_{u,w}^{\lambda,\mu}$, for $u, w \in W$, $\lambda, \mu \in \Lambda$, by the expansion

$$e^\lambda \cdot [\mathcal{I}_u] = \sum_{w \in W,\, \mu \in \Lambda} d_{u,w}^{\lambda,\mu}\, x^\mu\, [\mathcal{I}_w].$$

Corollary 8.4. (dual K_T-Chevalley formula) *The dual K_T-Chevalley coefficients are related to the K_T-Chevalley coefficients as $d_{u,w}^{\lambda,\mu} = c_{u,w}^{-\lambda,-\mu}$. Thus Theorem* 6.1 *provides a combinatorial description for the coefficients $d_{u,w}^{\lambda,\mu}$.*

Proof. Follows from (3.14). □

Remark 8.5. Based on LS-paths, Griffeth and Ram [GrRa] extended the Pittie-Ram formula by giving four different formulas for the products $e^\lambda \cdot [\mathcal{O}_w]$, $e^{-\lambda} \cdot [\mathcal{O}_w]$, $e^{w_\circ(\lambda)} \cdot [\mathcal{O}_w]$, and $[\mathcal{O}_{w_\circ s_i}] \cdot [\mathcal{O}_w]$, as well as a dual Chevalley formula; all these formulas refer to a dominant weight λ. They also derived Lemma 8.1(a) above and Theorem 8.6 below, for dominant λ. Since the Pittie-Ram formula does not work for nondominant weights, Griffeth and Ram had to derive the new formulas separately. From our point of view, all the formulas mentioned above are given by various specializations of the K_T-Chevalley formula, for arbitrary λ.

Let us now discuss symmetries of the K_T-Chevalley coefficients. In order to make our notation compatible with that in [Brion], we define the coefficients $c_u^w(\lambda)$ in $\mathbb{Z}[X]$ by

$$e^\lambda \cdot [\mathcal{O}_u] = \sum_{w \in W} c_u^w(\lambda)\,[\mathcal{O}_w]\,.$$

In other words, the $c_u^w(\lambda)$ are expressed in terms of the K_T-Chevalley coefficients, as follows: $c_u^w(\lambda) = \sum_{\mu\in\Lambda} c_{u,w}^{\lambda,\mu}\, x^\mu$, see (3.8).

Theorem 8.6. [Brion, Theorem 4] *We have the following duality formula for an arbitrary weight λ:*

$$c_u^w(\lambda) = (-1)^{\ell(u)-\ell(w)} c_{ww_\circ}^{uw_\circ}(w_\circ\lambda)\,.$$

Proof. Let $(\beta_1,\ldots,\beta_l)$ and $(r_1,\ldots,r_l)$ be the λ-chain of roots and the λ-chain of reflections associated with some alcove path. Let us translate this alcove path by λ, reverse its direction (cf. Remark 5.5), and then apply the map $A\mapsto -w_\circ(A)$ to the corresponding alcoves. Note that $-w_\circ(A_\circ)=A_\circ$. The resulting alcove path corresponds to the $(w_\circ\lambda)$-chain of roots $(w_\circ\beta_l,\ldots,w_\circ\beta_1)$ and a certain $w_\circ(\lambda)$-chain of reflections $(r_l',\ldots,r_1')$. We can express the affine reflections r_j', as follows. Let γ and t_λ be the operators on $\mathfrak{h}_\mathbb{R}^*$ given by $\gamma:\mu\mapsto-\mu$ and $t_\lambda:\mu\mapsto\mu+\lambda$. Then $r_j' = w_\circ\gamma\, t_\lambda r_j t_{-\lambda}\gamma\, w_\circ$. Thus $\bar r_j' = w_\circ \bar r_j w_\circ$.

Clearly, to each sequence $J=(j_1,j_2,\ldots,j_s)$ with

$$u \gtrdot u\bar r_{j_1} \gtrdot u\bar r_{j_1}\bar r_{j_2} \gtrdot\cdots\gtrdot u\bar r_{j_1}\bar r_{j_2}\cdots\bar r_{j_s} = w\,,$$

corresponds the sequence $J'=(j_s,j_{s-1},\ldots,j_1)$ with

$$ww_\circ \gtrdot ww_\circ\bar r_{j_s}' \gtrdot ww_\circ\bar r_{j_s}'\bar r_{j_{s-1}}' \gtrdot\cdots\gtrdot ww_\circ\bar r_{j_s}'\bar r_{j_{s-1}}'\cdots\bar r_{j_1}' = uw_\circ\,.$$

This correspondence is a bijection. Since $w_\circ$ maps positive roots to negative roots, we have $n(J') = s-n(J) = \ell(u)-\ell(w)-n(J)$, so $(-1)^{n(J)} = (-1)^{\ell(u)-\ell(w)}(-1)^{n(J')}$. This takes care of the sign in the duality formula.

It remains to check that the sequences J and J' produce the same weight, see condition (b) in Theorem 6.1. It suffices to show that

$$r_{j_1}r_{j_2}\ldots r_{j_s}(-\lambda) = \bar r_{j_1}\bar r_{j_2}\ldots\bar r_{j_s} w_\circ r_{j_s}' r_{j_{s-1}}'\ldots r_{j_1}' w_\circ(-\lambda)\,.$$

Let us denote $v = r_{j_1}\cdots r_{j_s}\in W_{\rm aff}$. Then the left-hand side of this expression is $v(-\lambda)$. We can write the right-hand side of this expression as

$$\bar r_{j_1}\cdots\bar r_{j_s}\,\gamma\, t_\lambda r_{j_s}\cdots r_{j_1} t_{-\lambda}\gamma\,(-\lambda) = -\bar v\, t_\lambda v^{-1}(0).$$

We claim that

$$v(-\lambda) = -\bar v\, t_\lambda\, v^{-1}(0), \tag{8.1}$$

for any $v\in W_{\rm aff}$ and $\lambda\in\Lambda$. Indeed, if $v(-\lambda)=\bar v(-\lambda)+\mu$, then $v^{-1}(0)=\bar v^{-1}(0-\mu) = -\bar v^{-1}(\mu)$. Thus $\bar v\, t_\lambda v^{-1}(0) = \bar v(\lambda)-\mu$, as needed. □

Let us also present a new duality formula. We denote by ι the involutory automorphism of $\mathbb{Z}[X]$ given by $\iota: x^\mu\mapsto x^{-w_\circ\mu}$.

Theorem 8.7. *We have the following duality formula for an arbitrary weight λ:*

$$c_u^w(\lambda) = (-1)^{\ell(u)-\ell(w)}\iota(c_{w_\circ w}^{w_\circ u}(-\lambda))\,.$$

Proof. Let $(\beta_1,\ldots,\beta_l)$ and $(r_1,\ldots,r_l)$ be the λ-chain of roots and the λ-chain of reflections associated with some alcove path. Let us translate the alcove path and reverse its direction, as discussed in Remark 5.5. We obtain the $(-\lambda)$-chain of roots $(-\beta_l,\ldots,-\beta_1)$ and the corresponding $(-\lambda)$-chain of roots $(r_l',\ldots,r_1')$. Let t_λ be the operator of translation by λ, as before. Then $r_j' = t_\lambda r_j t_{-\lambda}$. Thus $\bar r_j' = \bar r_j$. In an almost identical way to the proof of Theorem 8.6, we can now construct a bijection between the appropriate decreasing saturated chains from u to w, and those from $w_\circ w$ to $w_\circ u$. The discussion about the signs is also similar. It remains to verify the weight condition:

$$r_{j_1} r_{j_2} \cdots r_{j_s}(-\lambda) = -\bar{r}_{j_1} \bar{r}_{j_2} \cdots \bar{r}_{j_s} r'_{j_s} r'_{j_{s-1}} \cdots r'_{j_1}(\lambda)\,.$$

This identity can be written as $v(-\lambda) = -\bar{v}\, t_\lambda v^{-1} t_{-\lambda}(\lambda)$, for $v = r_{j_1} \cdots r_{j_s}$, which is equivalent to (8.1). □

The two duality formulas above imply the following formula.

Corollary 8.8. *Given an arbitrary weight λ, we have*

$$c_u^w(\lambda) = \iota(c_{w_\circ u w_\circ}^{w_\circ w w_\circ}(-w_\circ \lambda))\,.$$

Note each of the two duality formulas in Theorems 8.6 and 8.7 can be obtained from the other one combined with Corollary 8.8.

Kumar provided us with the following geometric explanation of Corollary 8.8. This duality in equivariant K-theory is induced by the standard involution on G/B, which interchanges the Schubert varieties X_w and $X_{w_\circ w w_\circ}$. Let us denote by θ the canonical isomorphism (3.1) from $(\mathbb{Z}[X] \otimes \mathbb{Z}[\Lambda])/\mathcal{I}$ to $K_T(G/B)$.

Proposition 8.9. *There is an involutive automorphism ω on $K_T(G/B)$ such that*

(a) *the involution ω maps each class $[\mathcal{O}_w]$ to $[\mathcal{O}_{w_\circ w w_\circ}]$;*
(b) *under the isomorphism θ, the involution ω maps $x^\mu \otimes e^\lambda$ to $x^{-w_\circ \mu} \otimes e^{-w_\circ \lambda}$, for $\lambda, \mu \in \Lambda$.*

Algebraic proof. The involutive automorphism of $\mathbb{Z}[X] \otimes \mathbb{Z}[\Lambda]$ given by $x^\mu \otimes e^\lambda \mapsto x^{-w_\circ(\mu)} \otimes e^{-w_\circ(\lambda)}$ preserves the ideal $\mathcal{I}$ and, thus, induces an involutive automorphism ω on $K_T(G/B) \simeq (\mathbb{Z}[X] \otimes \mathbb{Z}[\Lambda])/\mathcal{I}$. Applying this involution to the definition of the elementary Demazure operators T_i in (3.2), we deduce that $\omega\, T_i\, \omega = T_j$, where j is given by $\alpha_j = -w_\circ(\alpha_i)$, or equivalently, $s_j = w_\circ s_i w_\circ$. Thus $\omega\, T_w\, \omega = T_{w_\circ w w_\circ}$, for any $w \in W$. Kostant-Kumar's formula (3.7) implies that $\omega : [\mathcal{O}_w] \mapsto [\mathcal{O}_{w_\circ w w_\circ}]$. □

Geometric proof (due to Kumar [Kum]). Let $c : G \to G$ be the Chevalley isomorphism. This is an algebraic group isomorphism mapping $t \mapsto t^{-1}$ for t in T, and $B \mapsto B^-$, where B^- is the *opposite Borel* subgroup. Also let $c_{w_\circ} : G \to G$ be the automorphism given by $g \mapsto \overline{w}_\circ g \overline{w}_\circ^{-1}$, where $\overline{w}_\circ$ in $N(T)$ is a representative of $w_\circ$. Let $\phi : G \to G$ be the composite $c \circ c_{w_\circ}$. Then $\phi(B) = B$. Thus ϕ induces a variety isomorphism $\overline{\phi} : G/B \to G/B$. Moreover, since c induces the identity map on the Weyl group, we see that $\overline{\phi}(X_w) = X_{w_\circ w w_\circ}$. Thus $\overline{\phi}$ induces the involution ω on $K_T(G/B)$ such that $\omega : [\mathcal{O}_w] \mapsto [\mathcal{O}_{w_\circ w w_\circ}]$.

To show that, under the isomorphism θ, we have $\omega : e^\lambda \mapsto e^{-w_\circ \lambda}$, we identify G/B with K/T, where K is a maximal compact subgroup of G. Let us consider the following bundle morphism.

$$\begin{array}{ccc} K \times_T \mathbb{C}_{-w_\circ \lambda} & \xrightarrow{\;\widehat{\phi}\;} & K \times_T \mathbb{C}_\lambda \\ \downarrow & & \downarrow \\ K/T & \xrightarrow[\;\overline{\phi}\;]{} & K/T \end{array}$$

Here we let $\widehat{\phi}(k, v_\circ) := (\phi(k), \overline{v}_\circ)$, where $v_\circ$ is a generator of $\mathbb{C}_{-w_\circ \lambda}$, and $\overline{v}_\circ$ is a generator of $\mathbb{C}_\lambda$. It is easy to see that $\widehat{\phi}$ is well defined. Thus, we have $\omega \circ \theta(1 \otimes e^\lambda) = \theta(1 \otimes e^{-w_\circ \lambda})$. The proof of $\omega : x^\mu \mapsto x^{-w_\circ \mu}$ is similar. □

Note that the map $\overline{\phi}$ in the above proof is not T-equivariant, whence the involution ω is not a $\mathbb{Z}[X]$-linear map.

Let $c_{u,v}^w \in \mathbb{Z}[X]$ be the structure constants of $K_T(G/B)$ with respect to the basis of classes of structure sheaves of Schubert varieties:

$$[\mathcal{O}_u] \cdot [\mathcal{O}_v] = \sum_w c_{u,v}^w \, [\mathcal{O}_w] \, .$$

The coefficients $c_u^w(\pm\omega_i)$ are related to certain structure constants $c_{u,v}^w$, as follows.

Corollary 8.10. cf. [Brion] *For $v \neq w$, we have*

(a) $c_u^w(-\omega_i) = -x^{-w_\circ(\omega_i)} \, c_{w_\circ s_i, u}^w$;

(b) $c_u^w(\omega_i) = (-1)^{\ell(u)-\ell(w)-1} x^{\omega_i} \, c_{s_i w_\circ, w w_\circ}^{u w_\circ}$;

(c) $c_u^w(\omega_i) = (-1)^{\ell(u)-\ell(w)-1} x^{\omega_i} \, \iota(c_{w_\circ s_i, w_\circ w}^{w_\circ u})$.

Also, we have $c_{w_\circ s_i, u}^u = 1 - x^{w_\circ(\omega_i)-u(\omega_i)}$.

The first two formulas (a) and (b) were given by Brion [Brion] for $K(G/B)$ in a slightly imprecise form.

Proof. Identity (a) is obtained from the formula in Lemma 8.1(a) by multiplying both sides by $[\mathcal{O}_u]$. Identity (b) is obtained from (a) and the duality formula in Theorem 8.6, as follows:

$$\begin{aligned} c_u^w(\omega_i) &= (-1)^{\ell(u)-\ell(w)} c_{w w_\circ}^{u w_\circ}(w_\circ(\omega_i)) = (-1)^{\ell(u)-\ell(w)} c_{w w_\circ}^{u w_\circ}(-\omega_j) \\ &= (-1)^{\ell(u)-\ell(w)-1} x^{-w_\circ(\omega_j)} \, c_{w_\circ s_j, w w_\circ}^{u w_\circ} = (-1)^{\ell(u)-\ell(w)-1} x^{\omega_i} \, c_{s_i w_\circ, w w_\circ}^{u w_\circ} \, . \end{aligned}$$

Here we used the fact that $-w_\circ \alpha_i$ is the simple root α_j such that $s_j = w_\circ s_i w_\circ$. Similarly, we obtain identity (c) using the duality formula in Theorem 8.7. □

Remark 8.11. We can easily expand the product $[\mathcal{O}_{w_\circ s_i}] \cdot [\mathcal{O}_u]$ using our K_T-Chevalley formula, as shown in Corollary 8.2. However, it is hard to apply the Pittie-Ram formula directly to the calculation of this expansion, because the latter formula works for dominant weights only. In order to use this formula, one needs to invert the operator of multiplication by e^{ω_i} acting on the $|W|$-dimensional space $K_T(G/B)$. Alternatively, one can use Brion's geometric argument to derive the second formula in Corollary 8.10. But then, one needs to apply the Pittie-Ram formula for computing *all* products $e^{\omega_j} \cdot [\mathcal{O}_{w w_\circ}]$, for $w \in W$, and extract the coefficient of $[\mathcal{O}_{u w_\circ}]$ in each result, where j is given by $s_j = w_\circ s_i w_\circ$. Indeed, we have no way of knowing in advance to which Weyl group element an LS-path leads, via Deodhar's lift operator. In other words, it is hard to "invert" the Pittie-Ram construction based on LS-paths and Deodhar's lifts.

9. The Yang-Baxter Equation

Our construction is based on a certain R-matrix, that is, a collection of operators satisfying the Yang-Baxter equation. In this section, we discuss the Yang-Baxter equation, following the approach of Cherednik [Cher].

For a pair of roots $\alpha, \beta \in \Phi$ such that $(\alpha, \beta) \le 0$, the subset of roots $\Delta \subset \Phi$ obtained from α and β by a sequence of reflections s_α and s_β is a rank 2 root system of type $A_1 \times A_1$, A_2, B_2, or G_2. The reflections s_α and s_β generate a dihedral subgroup in W of order $2m$, where $m = 2, 3, 4, 6$, for types $A_1 \times A_1$, A_2, B_2, G_2, respectively. The condition $(\alpha, \beta) \le 0$ implies that α, β form a system of simple roots for Δ. The m roots in Δ expressible as nonnegative linear combinations of α and β can be normally ordered as follows: $\alpha, s_\alpha(\beta), s_\alpha s_\beta(\alpha), \ldots, s_\beta(\alpha), \beta$.

The following definition was given by Cherednik [Cher, Definition 2.1a] in a slightly different form.

Definition 9.1. We say that a collection of invertible operators $\{R_\alpha \mid \alpha \in \Phi\}$ labeled by roots satisfies the *Yang-Baxter equation* if $R_{-\alpha} = (R_\alpha)^{-1}$ and, for any pair of roots $\alpha, \beta \in \Phi$ such that $(\alpha, \beta) \le 0$, we have

$$R_\alpha R_{s_\alpha(\beta)} R_{s_\alpha s_\beta(\alpha)} \cdots R_{s_\beta(\alpha)} R_\beta = R_\beta R_{s_\beta(\alpha)} \cdots R_{s_\alpha s_\beta(\alpha)} R_{s_\alpha(\beta)} R_\alpha. \tag{9.1}$$

A collection of operators $\{R_\alpha \mid \alpha \in \Phi\}$ satisfying the Yang-Baxter equation is also called an *R-matrix.*

For example, the operators R_α and R_β commute whenever $(\alpha, \beta) = 0$. If Δ is of type A_2, then the Yang-Baxter equation (9.1) says that

$$R_\alpha R_{\alpha+\beta} R_\beta = R_\beta R_{\alpha+\beta} R_\alpha.$$

The following two lemmas are implicit in [Cher].

Lemma 9.2. *Consider a collection $\{R_\alpha \mid \alpha \in \Phi^+\}$ of invertible operators labeled by positive roots which satisfies the Yang-Baxter equation (9.1), for any pair of positive roots $\alpha, \beta \in \Phi^+$ such that $(\alpha, \beta) \le 0$. Let us extend this collection to all roots $\alpha \in \Phi$ by $R_{-\alpha} := (R_\alpha)^{-1}$. Then the collection $\{R_\alpha \mid \alpha \in \Phi\}$ is an R-matrix.*

Proof. Let us multiply the Yang-Baxter equation (9.1) by $R_{-\beta}$ on the left and on the right. We get

$$R_{-\beta} R_\alpha R_{s_\alpha(\beta)} R_{s_\alpha s_\beta(\alpha)} \cdots R_{s_\beta(\alpha)} = R_{s_\beta(\alpha)} \cdots R_{s_\alpha s_\beta(\alpha)} R_{s_\alpha(\beta)} R_\alpha R_{-\beta}.$$

This is the same equation with (α, β) replaced by the pair $(s_\beta(\beta), s_\beta(\alpha))$. Applying this procedure repeatedly, we can always transform the pair (α, β) into a pair of positive roots. □

For a decomposition $v = s_{i_1} \cdots s_{i_l} \in W_{\mathrm{aff}}$, reduced or not, of an affine Weyl group element v, let $(\beta_1, \dots, \beta_l)$ be the corresponding λ-chain of roots. For an R-matrix $\{R_\alpha \mid \alpha \in \Phi\}$, let us define $R^{(s_{i_1} \cdots s_{i_l})} = R_{\beta_l} R_{\beta_{l-1}} \cdots R_{\beta_2} R_{\beta_1}$.

Lemma 9.3. *Let $\{R_\alpha \mid \alpha \in \Phi\}$ be an R-matrix. Then the operator $R^{(s_{i_1} \cdots s_{i_l})}$ depends only on the affine Weyl group element $v = s_{i_1} \cdots s_{i_l}$, not on the choice of the decomposition.*

Proof. The Coxeter relations (5.3) imply that any two decompositions of v can be related by a sequence of local moves of the following two types: (1) adding or removing segments $s_i s_i$; (2) the Coxeter moves

$$s_{i_1} \cdots s_{i_a} \overbrace{(s_i s_j s_i \cdots)}^{m_{ij} \text{ terms}} s_{i_b} \cdots s_{i_l} \quad \longrightarrow \quad s_{i_1} \cdots s_{i_a} \overbrace{(s_j s_i s_j \cdots)}^{m_{ij} \text{ terms}} s_{i_b} \cdots s_{i_l}. \tag{9.2}$$

Adding or removing a segment $s_i s_i$ in a decomposition for v results in adding or removing a segment $\beta, -\beta$ in the sequence of roots $(\beta_1, \dots, \beta_l)$. This does not change the operator $R_{\beta_l} \cdots R_{\beta_1}$, because $R_\beta R_{-\beta} = 1$. A Coxeter move (9.2) results in applying the Yang-Baxter transformation

$$\alpha, s_\alpha(\beta), \dots, s_\beta(\alpha), \beta \quad \longrightarrow \quad \beta, s_\beta(\alpha), \dots, s_\alpha(\beta), \alpha$$

to the segment $(\beta_{a+1}, \dots, \beta_{b-1}) = (\alpha, s_\alpha(\beta), \cdots, \beta)$ in the sequence $(\beta_1, \dots, \beta_l)$. Here we have $\alpha = \bar{s}_{i_1} \cdots \bar{s}_{i_a}(\alpha_i)$ and $\beta = \bar{s}_{i_1} \cdots \bar{s}_{i_a}(\alpha_j)$. Note that $(\alpha, \beta) = (\alpha_i, \alpha_j) \le 0$. The Yang-Baxter equation (9.1) guarantees that this transformation of the sequence $(\beta_1, \dots, \beta_l)$ does not change the operator $R_{\beta_l} \cdots R_{\beta_1}$. □

10. Bruhat Operators

In this section, we present a class of solutions of the Yang-Baxter equation.

It will be convenient to extend the ring of coefficients $\mathbb{Z}[X] = R(T)$ in $K_T(G/B)$ as follows. Let us shrink the weight lattice h times by defining $\Lambda/h := \{\lambda/h \mid \lambda \in \Lambda\}$, where $h := (\rho, \theta^\vee) + 1$ is the Coxeter number. Let $\mathbb{Z}[\tilde{X}]$ be the group algebra of Λ/h, which has formal exponents $x^{\lambda/h}$, for $\lambda \in \Lambda$. This is the algebra of Laurent polynomials $\mathbb{Z}[\tilde{X}] = \mathbb{Z}[x^{\pm\omega_1/h}, \dots, x^{\pm\omega_r/h}]$. Let

$$\tilde{K}_T(G/B) := K_T(G/B) \otimes_{\mathbb{Z}[X]} \mathbb{Z}[\tilde{X}].$$

The space $\tilde{K}_T(G/B)$ has the $\mathbb{Z}[\tilde{X}]$-linear basis given by the classes $[\mathcal{O}_w]$, for $w \in W$.

For a positive root $\alpha \in \Phi^+$, let us define the *Bruhat operator* B_α acting $\mathbb{Z}[\tilde{X}]$-linearly on $\tilde{K}_T(G/B)$ by

$$B_\alpha : [\mathcal{O}_w] \longmapsto \begin{cases} [\mathcal{O}_{ws_\alpha}] & \text{if } \ell(ws_\alpha) = \ell(w) - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{10.1}$$

Also define $B_\alpha := -B_{-\alpha}$, if α is a negative root. The operators B_α move Weyl group elements one step down in Bruhat order.

For a weight λ, define the $\mathbb{Z}[\tilde{X}]$-linear operators X^λ acting on $\tilde{K}_T(G/B)$ by

$$X^\lambda : [\mathcal{O}_w] \mapsto x^{w(\lambda/h)}[\mathcal{O}_w]. \tag{10.2}$$

For $\alpha \in \Phi$ and $\lambda, \mu \in \Lambda$, these operators satisfy the following relations:

$$(B_\alpha)^2 = 0\,, \tag{10.3}$$

$$X^\lambda\, X^\mu = X^{\lambda+\mu}\,, \tag{10.4}$$

$$B_\alpha\, X^\lambda = X^{s_\alpha(\lambda)}\, B_\alpha\,. \tag{10.5}$$

For a fixed weight λ and $k \in \mathbb{Z}$, we define a family of operators $\{R_\alpha \mid \alpha \in \Phi\}$ labeled by roots $\alpha \in \Phi$ acting on $\tilde{K}_T(G/B)$ as follows:

$$R_\alpha = X^{k\alpha} + X^{(\lambda,\alpha^\vee)\,\alpha}\, B_\alpha = X^\lambda\,(X^{k\alpha} + B_\alpha)\, X^{-\lambda}. \tag{10.6}$$

Using relations (10.3) and (10.5), we obtain

$$R_{-\alpha} = X^{-k\alpha} - X^{(\lambda,\alpha^\vee)\,\alpha}\, B_\alpha = (R_\alpha)^{-1}.$$

Theorem 10.1. *Fix a weight λ and $k \in \mathbb{Z}$. The family of operators $\{R_\alpha \mid \alpha \in \Phi\}$ given by* (10.6) *satisfies the Yang-Baxter equation* (9.1)*.*

Proof. Let us first assume that $\lambda = 0$ and $k = 0$. In this case $R_\alpha = 1 + B_\alpha$. In [BFP], we proved the Yang-Baxter equation for a general class of operators by checking it for all the rank 2 root systems (that is, for types $A_1 \times A_1$, A_2, B_2, and G_2). In particular, the results of [BFP] imply that the family of operators $\{1 + B_\alpha \mid \alpha \in \Phi^+\}$ satisfies the Yang-Baxter equation (9.1). Also $R_{-\alpha} = 1 - B_\alpha = (1 + B_\alpha)^{-1} = (R_\alpha)^{-1}$. According to Lemma 9.2, the collection $\{1 + B_\alpha \mid \alpha \in \Phi\}$ is an R-matrix.

Let us now consider the general case. For $\alpha \in \Phi$ and $n \in \mathbb{Z}$, let us define

$$\hat{R}^n_\alpha := 1 + X^{n\alpha}\, B_\alpha.$$

Then $R_\alpha = X^{k\alpha}\, \hat{R}_\alpha^{(\lambda,\alpha^\vee)-k}$. For $\mu \in \Lambda$, we get, using (10.5),

$$\hat{R}^n_\alpha\, X^\mu = X^\mu\, \hat{R}_\alpha^{n-(\mu,\alpha^\vee)}. \tag{10.7}$$

Let us write the left-hand side of the Yang-Baxter equation (9.1) as follows:

$$R_{\gamma_1} \cdots R_{\gamma_m} = X^{k\gamma_1}\, \hat{R}^{n_1}_{\gamma_1}\, X^{k\gamma_2}\, \hat{R}^{n_2}_{\gamma_2} \cdots X^{k\gamma_m}\, \hat{R}^{n_l}_{\gamma_m},$$

where $(\gamma_1, \ldots, \gamma_m) = (\alpha, s_\alpha(\beta), \cdots, s_\beta(\alpha), \beta)$ and $n_i = (\lambda, \gamma_i^\vee) - k$. Using (10.7) to commute all $X^{k\gamma_i}$ to the left, we obtain the expression

$$R_{\gamma_1} \cdots R_{\gamma_m} = X^{k(\gamma_1 + \cdots + \gamma_m)}\, \hat{R}^{n'_1}_{\gamma_1}\, \hat{R}^{n'_2}_{\gamma_2} \cdots \hat{R}^{n'_l}_{\gamma_m},$$

where

$$n'_i = n_i - \sum_{j=i+1}^{m} k(\gamma_j, \gamma_i^\vee) = (\lambda - k(\gamma_{i+1} - \cdots - \gamma_m), \gamma_i^\vee) - k.$$

Let us show that

$$(\gamma_1 + \cdots + \gamma_{i-1}, \gamma_i^\vee) = (\gamma_{i+1} + \cdots + \gamma_m, \gamma_i^\vee)\,,$$

for all $i = 1, \dots, m$. Suppose that $i \le (m+1)/2$. The reflection s_{γ_i} sends the roots $\gamma_1, \dots, \gamma_{i-1}$ to $-\gamma_{2i-1}, \dots, -\gamma_{i+1}$, and the roots $\gamma_{2i}, \dots, \gamma_m$ to $\gamma_m, \dots, \gamma_{2i}$, respectively. Thus

$$(\gamma_1 + \dots + \gamma_{i-1}, \gamma_i^\vee) = (\gamma_{i+1} + \dots + \gamma_{2i-1}, \gamma_i^\vee) \quad \text{and} \quad (\gamma_{2i} + \dots + \gamma_m, \gamma_i^\vee) = 0,$$

as needed. Since $(\gamma_i, \gamma_i^\vee) = 2$, we get

$$n_i' = (\lambda - k(\gamma_{i+1} + \dots + \gamma_m), \gamma_i^\vee) - k = (\lambda - k\varrho, \gamma_i^\vee),$$

where $\varrho = \frac{1}{2}(\gamma_1 + \dots + \gamma_m)$ is the "rho" for the rank 2 root system Δ generated by α and β.

This shows that

$$R_{\gamma_1} \cdots R_{\gamma_m} = X^{2k\varrho} \hat{R}_{\gamma_1}^{(\mu,\gamma_1^\vee)} \cdots R_{\gamma_l}^{(\mu,\gamma_m^\vee)} = X^{\mu+2k\varrho} \hat{R}_{\gamma_1}^0 \cdots \hat{R}_{\gamma_m}^0 X^{-\mu},$$

where $\mu = \lambda - k\varrho$. Analogously, the right-hand side of the Yang-Baxter equation (9.1) can be written as

$$R_{\gamma_m} \cdots R_{\gamma_1} = X^{\mu+2k\varrho} \hat{R}_{\gamma_m}^0 \cdots \hat{R}_{\gamma_1}^0 X^{-\mu}.$$

The fact that the operators $\hat{R}_\alpha^0 = 1 + B_\alpha$ satisfy the Yang-Baxter equation implies that the family $\{R_\alpha \mid \alpha \in \Phi\}$ satisfies the Yang-Baxter equation as well. This concludes the proof. □

In the rest of the paper, we only use a special case of the operators R_α defined in (10.6), namely we set $\lambda := \rho$ and $k := 1$, which leads to

$$R_\alpha = X^\alpha + X^{(\rho,\alpha^\vee)\,\alpha} B_\alpha = X^\rho \,(X^\alpha + B_\alpha)\, X^{-\rho}, \quad \text{for } \alpha \in \Phi. \tag{10.8}$$

11. Commutation Relations

Let T_i be the operator on $\tilde{K}_T(G/B)$ induced by the elementary Demazure operator (3.2), for $i = 1, \dots, r$. In view of (3.3) and (3.7), this operator acts $\mathbb{Z}[\tilde{X}]$-linearly on $\tilde{K}_T(G/B)$ as

$$T_i : [\mathcal{O}_w] \longmapsto \begin{cases} [\mathcal{O}_{ws_i}] & \text{if } \ell(ws_i) = \ell(w) + 1, \\ [\mathcal{O}_w] & \text{if } \ell(ws_i) = \ell(w) - 1. \end{cases}$$

Let $B_i := B_{\alpha_i}$ be the Bruhat operator for a simple reflection, which is the $\mathbb{Z}[\tilde{X}]$-linear operator on $\tilde{K}_T(G/B)$ defined by

$$B_i : [\mathcal{O}_w] \mapsto \begin{cases} [\mathcal{O}_{ws_i}] & \text{if } \ell(ws_i) = \ell(w) - 1, \\ 0 & \text{if } \ell(ws_i) = \ell(w) + 1. \end{cases}$$

Let us define a similar $\mathbb{Z}[\tilde{X}]$-linear operator B_i^* by

$$B_i^* : [\mathcal{O}_w] \mapsto \begin{cases} [\mathcal{O}_{ws_i}] & \text{if } \ell(ws_i) = \ell(w) + 1, \\ 0 & \text{if } \ell(ws_i) = \ell(w) - 1. \end{cases}$$

Since both operators B_i^* and B_i map $[\mathcal{O}_w]$ to $[\mathcal{O}_{ws_i}]$ or to zero, we have

$$X^\mu B_i^* = B_i^* X^{s_i(\mu)}, \quad \text{and} \quad X^\mu B_i = B_i X^{s_i(\mu)}, \tag{11.1}$$

for any weight $\mu \in \Lambda$.

The operator B_i^* can be expressed in terms of T_i and B_i as follows.

Lemma 11.1. *We have $B_i^* = T_i\,(1 - B_i) = (1 + B_i)(T_i - 1)$, for $i = 1, \dots, r$.*

Proof. It is enough to check this claim for restrictions of the operators on the 2-dimensional invariant subspace spanned by $[\mathcal{O}_w]$ and $[\mathcal{O}_{ws_i}]$, for any $w \in W$ such that $\ell(ws_i) = \ell(w) + 1$. The required identity is

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix},$$

which we leave to the reader as an exercise. □

Recall that B_β are the Bruhat operators given by (10.1).

Lemma 11.2. cf. Deodhar [Deo1, Lemma 2.1] *We have* $B_\beta\, B_i^* = B_i^*\, B_{s_i(\beta)}$, *for* $i = 1, \dots, r$ *and* $\beta \in \Phi$ *such that* $\beta \neq \pm\alpha_i$.

Proof. We may assume that $\beta \in \Phi^+$. Let $\beta' = s_i(\beta)$. Then $\beta' \in \Phi^+$ and $\beta' \neq \alpha_i$. Both operators $B_\beta\, B_i^*$ and $B_i^*\, B_{\beta'}$ map $[\mathcal{O}_w]$ to $[\mathcal{O}_{ws_is_\beta}] = [\mathcal{O}_{ws_{\beta'}s_i}]$ or to zero. Thus, we need to show that $B_\beta\, B_i^*([\mathcal{O}_w])$ is nonzero if and only if $B_i^*\, B_{\beta'}([\mathcal{O}_w])$ is nonzero.

Suppose that this is not true. One possibility is that we have $B_\beta\, B_i^*([\mathcal{O}_w]) = 0$ and $B_i^*\, B_{\beta'}([\mathcal{O}_w]) \neq 0$. Then $\ell(w) = \ell(ws_{\beta'}) + 1 = \ell(ws_i) + 1 = \ell(ws_{\beta'}s_i)$. Indeed, $B_i^*\, B_{\beta'}([\mathcal{O}_w]) \neq 0$ implies that $\ell(ws_{\beta'}) = \ell(w) - 1$ and $\ell(ws_{\beta'}s_i) = \ell(ws_{\beta'}) + 1$, while $B_\beta\, B_i^*([\mathcal{O}_w]) = 0$ implies that $\ell(ws_i) \neq \ell(w) + 1$, and, thus, $\ell(ws_i) = \ell(w) - 1$.

Let us choose a reduced decomposition for $w = s_{i_1} \cdots s_{i_l}$ such that $i_l = i$. By the Strong Exchange Condition [Hum, Theorem 5.8], the fact that $\ell(w) = \ell(ws_{\beta'}) + 1$ implies that there exists $k \in \{1, \dots, l\}$ such that $s_{i_1} \cdots \widehat{s_{i_k}} \cdots s_{i_l}$ is a reduced decomposition for $ws_{\beta'}$. Furthermore, we have $\beta' = s_{i_l} \cdots s_{i_{k+1}}(\alpha_{i_k})$. Since $\beta' \neq \alpha_i$, we have $k \neq l$. We obtain a reduced decomposition for $ws_{\beta'}$ that ends with s_i. Thus $\ell(ws_{\beta'}s_i) = \ell(ws_{\beta'}) - 1$, which is a contradiction.

Now suppose that we have $B_\beta\, B_i^*([\mathcal{O}_w]) \neq 0$ and $B_i^*\, B_{\beta'}([\mathcal{O}_w]) = 0$. Then $\ell(w) = \ell(ws_i) - 1 = \ell(ws_{\beta'}) - 1 = \ell(ws_{\beta'}s_i)$ or, equivalently, $\ell(w') = \ell(w's_i) + 1 = \ell(w's_\beta) + 1 = \ell(w's_\beta s_i)$, for $w' = ws_i$. The above argument shows that this is impossible. □

Remark 11.3. The contradictions derived in the above proof are essentially the content of Lemma 2.1 in [Deo1], which is proved in a similar way.

Let $\{R_\alpha \mid \alpha \in \Phi\}$ be the R-matrix given by (10.8). The main technical result of this section is the following statement that gives a commutation relation between this R-matrix and the Demazure operators T_i.

Proposition 11.4. *For any* $\beta \in \Phi$ *and* $i = 1, \dots, r$, *we have*

(a) $R_{\alpha_i}\, T_i = T_i\, R_{-\alpha_i} + R_{\alpha_i}$,
(b) $R_{-\alpha_i}\, T_i = T_i\, R_{\alpha_i} - R_{\alpha_i}$,
(c) $R_\beta\, T_i = T_i\, R_{-\alpha_i}\, R_{s_i(\beta)}\, R_{\alpha_i}$ *if* $\beta \neq \pm\alpha_i$.

Proof. We have $R_{\alpha_i} = X^{\alpha_i}\,(1 + B_i)$ and $R_{-\alpha_i} = (1 - B_i)\, X^{-\alpha_i}$.

(a) By Lemma 11.1, $(1 + B_i)\,(T_i - 1) = T_i\,(1 - B_i)$. Thus

$$X^{\alpha_i}\,(1 + B_i)\, T_i = X^{\alpha_i}\, T_i\,(1 - B_i) + X^{\alpha_i}\,(1 + B_i).$$

Then use (11.1) to commute X^{α_i} with $T_i\,(1 - B_i) = B_i^*$ in the first term in the right-hand side. This produces (a).

(b) Multiply (a) by $R_{-\alpha_i}$ on the left and by R_{α_i} on the right.

(c) Let $\beta' = s_i(\beta)$. Identity (c) can be written as

$$(X^\beta + X^{k\beta}\, B_\beta)\, T_i = T_i\,(1 - B_i)\, X^{-\alpha_i}\,(X^{\beta'} + X^{k'\beta'}\, B_{\beta'})\, X^{\alpha_i}\,(1 + B_i),$$

where $k = (\rho, \beta^\vee)$ and $k' = (\rho, (\beta')^\vee) = (s_i(\rho), \beta^\vee) = (\rho - \alpha_i, \beta^\vee)$. The right-hand side of this identity can be written as

$$T_i\,(1 - B_i)\,(X^{\beta'} + X^{k\beta'}\, B_{\beta'})\,(1 + B_i).$$

Indeed, $X^{k'\beta' - \alpha_i}\, B_{\beta'} X^{\alpha_i} = X^{k\beta'} B_{\beta'}$, because $k'\beta' - \alpha_i + s_{\beta'}(\alpha_i) = (\rho - \alpha_i, \beta^\vee)\,\beta' - (\alpha_i, (\beta')^\vee)\,\beta' = (\rho, \beta^\vee)\,\beta' = k\beta'$. Commuting $X^{\beta'}$ and $X^{k\beta'}\, B_{\beta'}$ with $T_i\,(1 - B_i) = B_i^*$ using (11.1) and Lemma 11.2, we can rewrite this as

$$(X^\beta + X^{k\beta}\, B_\beta)\, B_i^*(1 + B_i) = (X^\beta + X^{k\beta}\, B_\beta)\, T_i,$$

which is equal to the left-hand side of required identity. ∎

12. Path Operators

Recall that $v_{-\lambda} \in W_{\mathrm{aff}}$, $\lambda \in \Lambda$, is the unique element of the affine Weyl group such that $v_{-\lambda}(A_\circ) = A_{-\lambda} = A_\circ - \lambda$. Each decomposition $v_{-\lambda} = s_{i_1} \cdots s_{i_l}$ in W_{aff} corresponds to an alcove path $A_\circ \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_l} A_{-\lambda}$; and the sequence of roots $(\beta_1, \dots, \beta_l)$ is called a λ-chain, see Definition 5.4. Also recall that there is an associated alcove path $A_\circ \xrightarrow{\beta_l} \cdots \xrightarrow{\beta_1} A_\lambda$, as discussed in Remark 5.5.

For $\lambda \in \Lambda$, let us define the operator $R^{[\lambda]}$ acting on $\tilde{K}_T(G/B)$ by

$$R^{[\lambda]} := R_{\beta_l} R_{\beta_{l-1}} \cdots R_{\beta_2} R_{\beta_1}, \tag{12.1}$$

where $(\beta_1, \dots, \beta_l)$ is a λ-chain, and the R-matrix $\{R_\alpha \mid \alpha \in \Phi\}$ is given by (10.8).

Remark 12.1. Theorem 10.1 and Lemma 9.3 imply that the operator $R^{[\lambda]}$ depends only on the weight λ and does not depend on the choice of a λ-chain.

The following result is not used in subsequent proofs. We state it because it exhibits the commutativity of the operators E^λ and E^μ in our combinatorial model, based on Remark 12.1.

Proposition 12.2. *For any $\lambda, \mu \in \Lambda$, we have $R^{[\lambda]} \cdot R^{[\mu]} = R^{[\lambda+\mu]}$.*

Proof. Let us choose a λ-chain $(\beta_1, \dots, \beta_l)$ and a μ-chain $(\beta'_1, \dots, \beta'_m)$. They correspond to alcove paths $A_\circ \xrightarrow{\beta_l} \cdots \xrightarrow{\beta_1} A_\lambda$ and $A_\circ \xrightarrow{\beta'_m} \cdots \xrightarrow{\beta'_1} A_\mu$. If we translate all alcoves in the second path λ, we obtain the alcove path $A_\lambda \xrightarrow{\beta'_m} \cdots \xrightarrow{\beta'_1} A_{\lambda+\mu}$. Let us concatenate the first path from $A_\circ$ to A_λ with the translated path from A_λ to $A_{\lambda+\mu}$. We obtain the alcove path

$$A_\circ \xrightarrow{\beta_l} \cdots \xrightarrow{\beta_1} A_\lambda \xrightarrow{\beta'_m} \cdots \xrightarrow{\beta'_1} A_{\lambda+\mu}.$$

This shows that the sequence $(\beta'_1, \dots, \beta'_m, \beta_1, \dots, \beta_l)$ is a $(\lambda+\mu)$-chain. Thus

$$R^{[\lambda]} \cdot R^{[\mu]} = R_{\beta_l} \cdots R_{\beta_1} R_{\beta'_m} \cdots R_{\beta'_1} = R^{[\lambda+\mu]},$$

as needed. ∎

Lemma 12.3. *Let $(\beta_1, \dots, \beta_l)$ be a λ-chain. Then, for any $i = 1, \dots, r$, the sequence of roots $(\alpha_i, s_i(\beta_1), \dots, s_i(\beta_l), -\alpha_i)$ is an $s_i(\lambda)$-chain.*

Proof. Applying the reflection s_i to the alcove path $A_\circ \xrightarrow{\beta_l} \cdots \xrightarrow{\beta_1} A_\lambda$, we obtain the alcove path $s_i(A_\circ) \xrightarrow{s_i(\beta_l)} \cdots \xrightarrow{s_i(\beta_1)} s_i(A_\lambda)$. We have $A_\circ \xrightarrow{-\alpha_i} s_i(A_\circ)$. Translating this relation by $s_i(\lambda)$, we obtain $(s_i(A_\circ) + s_i(\lambda)) \xrightarrow{\alpha_i} (A_\circ + s_i(\lambda))$, or, equivalently, $s_i(A_\lambda) \xrightarrow{\alpha_i} A_{s_i(\lambda)}$. Thus

$$A_\circ \xrightarrow{-\alpha_i} s_i(A_\circ) \xrightarrow{s_i(\beta_l)} \cdots \xrightarrow{s_i(\beta_1)} s_i(A_\lambda) \xrightarrow{\alpha_i} A_{s_i(\lambda)}$$

is an alcove path, and $(\alpha_i, s_i(\beta_1), \dots, s_i(\beta_l), -\alpha_i)$ is an $s_i(\lambda)$-chain. ∎

Lemma 12.4. *Let $(\beta_1, \dots, \beta_l)$ be a λ-chain, and let $A_0 \xrightarrow{\beta_l} \cdots \xrightarrow{\beta_1} A_l$ be the corresponding alcove path from $A_0 = A_\circ$ to $A_l = A_\lambda$. Assume that $\pm\beta_j = \alpha_i$ is a simple root, for some $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, l\}$. Then*

$$(\alpha_i, s_i(\beta_1), \dots, s_i(\beta_{j-1}), \beta_{j+1}, \dots, \beta_l)$$

is an $s(\lambda)$-chain, where $s = s_{\alpha_i,k}$ denotes the affine reflection with respect to the common wall of the alcoves $A_{l-j} \xrightarrow{\beta_j} A_{l-j+1}$.

Proof. Let us apply the following tail-flip to the alcove path $A_0 \xrightarrow{\beta_l} \cdots \xrightarrow{\beta_1} A_l$. We leave the initial segment $A_0 \xrightarrow{\beta_l} \cdots \xrightarrow{\beta_{j+1}} A_{l-j}$ unmodified and apply the affine reflection s to the remaining tail: $s(A_{l-j+1}) \xrightarrow{\bar s(\beta_{j-1})} s(A_{l-j+2}) \xrightarrow{\bar s(\beta_{j-2})} \cdots \xrightarrow{\bar s(\beta_1)} s(A_l)$. Note that $A_{l-j} = s(A_{l-j+1})$ and $\bar s = s_i$. Also note that $s(A_l) = s(A_\circ + \lambda) = s_i(A_\circ) + s(\lambda)$, and, thus, $s(A_l) \xrightarrow{\alpha_i} A_{s_i(\lambda)}$. Let us add the step $s(A_l) \xrightarrow{\alpha_i} A_{s_i(\lambda)}$ at the end of the alcove path with flipped tail. We obtain the alcove path

$$A_0 \xrightarrow{\beta_l} \cdots \xrightarrow{\beta_{j+1}} A_{l-j} \xrightarrow{s_i(\beta_{j-1})} s(A_{l-j+2}) \xrightarrow{s_i(\beta_{j-2})} \cdots \xrightarrow{s_i(\beta_1)} s(A_l) \xrightarrow{\alpha_i} A_{s_i(\lambda)}.$$

from $A_\circ$ to $A_{s_i(\lambda)}$. Thus $(\alpha_i, s_i(\beta_1), \ldots, s_i(\beta_{j-1}), \beta_{j+1}, \ldots, \beta_l)$ is an $s(\lambda)$-chain. □

Proposition 12.5. *For any $\lambda \in \Lambda$ and $i \in \{1, \ldots, r\}$, we have*

$$R^{[\lambda]} \cdot T_i = T_i \cdot R^{[s_i(\lambda)]} + \sum_{0 \le k < (\lambda, \alpha_i^\vee)} R^{[\lambda - k\alpha_i]} - \sum_{(\lambda, \alpha_i^\vee) \le k < 0} R^{[\lambda - k\alpha_i]}.$$

Proof. Let us choose a λ-chain $(\beta_1, \ldots, \beta_l)$. Let $A_0 \xrightarrow{\beta_l} \cdots \xrightarrow{\beta_1} A_l$ be the corresponding alcove path from $A_0 = A_\circ$ to $A_l = A_\lambda$. And let r_j be the affine reflection with respect to the common wall of the alcoves $A_{l-j} \xrightarrow{\beta_j} A_{l-j+1}$.

Then $R^{[\lambda]} = R_{\beta_l} \cdots R_{\beta_1}$. Using the relations in Proposition 11.4 repeatedly to commute T_i with $R_{\beta_l} \cdots R_{\beta_1}$, we obtain

$$\begin{aligned} R_{\beta_l} \cdots R_{\beta_1} T_i &= T_i R_{-\alpha_i} R_{s_i(\beta_l)} \cdots R_{s_i(\beta_1)} R_{\alpha_i} \\ &+ \sum_{j:\, \beta_j = \alpha_i} R_{\beta_l} \cdots R_{\beta_{j+1}} R_{s_i(\beta_{j-1})} \cdots R_{s_i(\beta_1)} R_{\alpha_i} \\ &- \sum_{j:\, \beta_j = -\alpha_i} R_{\beta_l} \cdots R_{\beta_{j+1}} R_{s_i(\beta_{j-1})} \cdots R_{s_i(\beta_1)} R_{\alpha_i}. \end{aligned}$$

According to Lemmas 12.3 and 12.4, the right-hand side of this expression can be written as

$$R^{[\lambda]} \cdot T_i = T_i \cdot R^{[s_i(\lambda)]} + \sum_{j:\, \beta_j = \alpha_i} R^{[r_j(\lambda)]} - \sum_{j:\, \beta_j = -\alpha_i} R^{[r_j(\lambda)]}.$$

For a hyperplane H of the form $H_{\alpha_i, k}$, $k \in \mathbb{Z}$, let p_k be the number of times the alcove path $A_\circ \xrightarrow{\beta_l} \cdots \xrightarrow{\beta_1} A_\lambda$ crosses H in the positive direction, and n_k be the number of times the path crosses H in the negative direction. In other words, $p_k = \#\{j \mid \beta_j = \alpha_i,\ r_j = s_{\alpha_i,k}\}$ and $n_k = \#\{j \mid \beta_j = -\alpha_i,\ r_j = s_{\alpha_i,k}\}$. Then $p_k - n_k$ is nonzero if and only if H separates the alcoves $A_\circ$ and A_λ. More specifically,

$$p_k - n_k = \begin{cases} 1 & \text{if } 0 < k \le (\lambda, \alpha_i^\vee), \\ -1 & \text{if } 0 \ge k > (\lambda, \alpha_i^\vee), \\ 0 & \text{otherwise.} \end{cases}$$

This shows that

$$R^{[\lambda]} \cdot T_i = T_i \cdot R^{[s_i(\lambda)]} + \sum_{0 < k \le (\lambda, \alpha_i^\vee)} R^{[s_{\alpha_i,k}(\lambda)]} - \sum_{(\lambda, \alpha_i^\vee) < k \le 0} R^{[s_{\alpha_i,k}(\lambda)]},$$

which is equivalent to the claim of the proposition. □

13. The K_T-Chevalley Formula: Operator Notation

We can formulate and prove our main result—the equivariant K-theory Chevalley formula—using the operator notation, as follows. Recall that

$$R^{[\lambda]} = R_{\beta_l} \cdots R_{\beta_1} = X^{\rho}\,(X^{\beta_l} + B_{\beta_l}) \cdots (X^{\beta_2} + B_{\beta_2})\,(X^{\beta_1} + B_{\beta_1})\,X^{-\rho},$$

where $(\beta_1, \ldots, \beta_l)$ is a λ-chain.

Theorem 13.1. *For any weight λ, the operator $R^{[\lambda]}$ preserves the space $K_T(G/B)$. For any $u \in W$, we have*

$$e^{\lambda} \cdot [\mathcal{O}_u] = R^{[\lambda]}([\mathcal{O}_u]),$$

i.e., the operator $R^{[\lambda]}$ acts on the space $K_T(G/B)$ as the operator of multiplication by the class e^{λ} of the corresponding line bundle.

Proof. Proposition 12.5 says that the operators $R^{[\lambda]}$ satisfy the same commutation relations with the elementary Demazure operators T_i as the operators E^{λ}, see (3.10). Also $R^{[\lambda]}([\mathcal{O}_1]) = x^{\lambda}\,[\mathcal{O}_1]$, by Proposition 14.5. Now Lemma 3.1 implies that the operator $R^{[\lambda]}$ preserves $K_T(G/B) \subset \tilde{K}_T(G/B)$ and acts as the operator E^{λ} of multiplication by the class e^{λ} of a line bundle. □

In Section 14, we show that Theorem 13.1 is equivalent to Theorem 6.1. In Sections 15 and 16, we illustrate Theorems 6.1 and 13.1 by several examples.

Remark 13.2. If λ is a dominant weight, then, according to Lemma 6.2, the operator $R^{[\lambda]}$ expands as a positive expression in the Bruhat operators B_{α}, $\alpha \in \Phi^+$, and the operators X^{μ}. Indeed, a reduced λ-chain involves only positive roots. In this case, Theorem 13.1 gives a positive formula for $e^{\lambda} \cdot [\mathcal{O}_u]$.

Specializing $x^{\mu} \mapsto 1$, we obtain the nonequivariant K-theory Chevalley formula. By a slight abuse of notation, we will use the same symbols e^{λ} and $[\mathcal{O}_w]$ for the obvious classes in $K(G/B)$ as in $K_T(G/B)$.

Corollary 13.3. *Let $\lambda \in \Lambda$ and $(\beta_1, \ldots, \beta_l)$ be a λ-chain. Then the operator*

$$R^{[\lambda]}_{x=1} = (1 + B_{\beta_l}) \cdots (1 + B_{\beta_1})$$

acts on the Grothendieck ring $K(G/B)$ as the operator of multiplication by the class e^{λ} of the corresponding line bundle.

Remark 13.4. We claim that Corollary 13.3 implies the classical Chevalley formula (3.16). In order to derive this formula, we need to collect linear terms in the expansion of the product $(1+B_{\beta_l}) \cdots (1+B_{\beta_1})$. Indeed, the coefficient $c^{\lambda}_{u,us_\alpha}$, for $\ell(us_\alpha) = \ell(u) - 1$, equals to the number of times the term B_{α} appears in the expansion minus the number of times $B_{-\alpha}$ appears in the expansion. According to Lemma 5.3, for any $\alpha \in \Phi^+$, this coefficient is

$$\#\{j \mid \beta_j = \alpha\} - \#\{j \mid \beta_j = -\alpha\} = -m_{\alpha}(A_{-\lambda}) = (\lambda, \alpha^{\vee}),$$

which is exactly the coefficient in the Chevalley formula. Thus, (3.17) and (3.16) follow.

14. Central Points of Alcoves

In this section, we show that Theorem 6.1 is equivalent to Theorem 13.1. In order to do this, we show explicitly the way in which the operator $R^{[\lambda]}$ acts on basis elements $[\mathcal{O}_u]$. It is convenient to do this using central points of alcoves.

Let us define the set $Z \subset \mathfrak{h}^*_{\mathbb{R}}$ as

$$Z := \{\zeta \in \Lambda/h \mid (\zeta, \alpha^{\vee}) \notin \mathbb{Z} \text{ for any } \alpha \in \Phi\},$$

i.e., Z is the set of the elements of the lattice Λ/h that do not belong to any hyperplane $H_{\alpha,k}$, where h is the Coxeter number. Then every element of Z belongs to some alcove. The affine Weyl group W_{aff} preserves the set Z. This set was considered by Kostant [Kost].

Lemma 14.1. [Kost] *Each alcove contains precisely one element of the set Z. The only element of Z in the fundamental alcove $A_\circ$ is ρ/h.*

Proof. It is enough to prove the statement only for the fundamental alcove, because W_{aff} acts transitively on the alcoves. Let us express the highest coroot as a linear combination of simple coroots: $\theta^\vee = c_1\,\alpha_1^\vee + \cdots + c_r\,\alpha_r^\vee$. Then c_i are strictly positive integers and $h = c_1 + \cdots + c_r + 1$. Every element ζ of Z can be written as $\zeta = (a_1\,\omega_1 + \cdots + a_r\,\omega_r)/h$, where $a_1, \dots, a_r \in \mathbb{Z}$. The condition that $\zeta \in Z \cap A_\circ$ can be written as $a_1, \dots, a_r > 0$ and $(a_1\,c_1 + \cdots + a_r\,c_r)/(c_1 + \cdots + c_r + 1) < 1$, see (5.2). The only sequence of integers $(a_1, \dots, a_r)$ that satisfies these conditions is $(1, \dots, 1)$. Thus $Z \cap A_\circ$ consists of the single element $(\omega_1 + \cdots + \omega_r)/h = \rho/h$. □

For an alcove A, the only element ζ_A of $Z \cap A$ is called the *central point* of the alcove A. In particular, $\zeta_{A_\circ} = \rho/h$. The map $A \mapsto \zeta_A$ is a one-to-one correspondence between the set of all alcoves and Z.

Lemma 14.2. *For a pair of adjacent alcoves $A \xrightarrow{\alpha} B$, we have $\zeta_B - \zeta_A = \alpha/h$.*

Proof. It is enough to prove this lemma for the fundamental alcove $A = A_\circ$. All alcoves adjacent to $A_\circ$ are obtained from $A_\circ$ by the reflections $s_0, s_1, \dots, s_r$; and $A_\circ \xrightarrow{-\alpha_i} s_i(A_\circ)$. Applying these reflections to the central point $\zeta_{A_\circ} = \rho/h$, we obtain $s_i(\zeta_{A_\circ}) - \zeta_{A_\circ} = -\alpha_i/h$, for $i = 0, \dots, r$. □

In fact, in the simply-laced case, the converse statement is true as well.

Lemma 14.3. *Suppose that Φ is a root system of type A-D-E. Then $A \xrightarrow{\alpha} B$ if and only if $\zeta_B - \zeta_A = \alpha/h$.*

Proof. Again, we can assume that $A = A_\circ$ is the fundamental alcove. In view of Lemma 14.2, it remains to show that $\mu = \rho/h + \alpha/h \notin Z$, for any root $\alpha \in \Phi \setminus \{-\alpha_1, \dots, -\alpha_r, \theta\}$. For any such α, there is a simple root α_i such that $\alpha + \alpha_i$ is a root. Thus $(\alpha, \alpha_i^\vee) = -1$ and $(\mu, \alpha_i^\vee) = 0$. This implies that μ belongs to the hyperplane $H_{\alpha_i,0}$ and, thus, $\mu \notin Z$. □

Remark 14.4. In the case of a nonsimply-laced root system, the statement converse to Lemma 14.2 is not true. In other words, there are nonadjacent alcoves A and B such that $\zeta_B - \zeta_A = \alpha/h$ for some root α.

Let us now fix an alcove path $A_\circ \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_l} A_{-\lambda}$ and the associated λ-chain $(\beta_1, \dots, \beta_l)$. By the definition, the operator $R^{[\lambda]}$ can be expressed as

$$R^{[\lambda]} = X^\rho\,(X^{\beta_l} + B_{\beta_l}) \cdots (X^{\beta_2} + B_{\beta_2})\,(X^{\beta_1} + B_{\beta_1})\,X^{-\rho}. \tag{14.1}$$

We can expand $R^{[\lambda]}$ as a sum of 2^l terms. For a subset $J \subset \{1, \dots, l\}$, let $R_J^{[\lambda]}$ be the term that contains B_{β_j}, if $j \in J$, and X^{β_j}, otherwise. It is convenient to give the following interpretation for the term $R_J^{[\lambda]}$ using tail-flips.

Let $\pi = (0, \pi_0, \pi_1, \dots, \pi_l, \mu)$ be a collection of points in $\mathfrak{h}_\mathbb{R}^*$. We can think of this collection as a continuous piecewise-linear path in $\mathfrak{h}_\mathbb{R}^*$ from 0 to μ. Let j be an index such that $\pi_{j-1} \neq \pi_j$, and let r_j be the affine reflection with respect to the perpendicular bisector of the segment $[\pi_{j-1}, \pi_j]$. In other words, the affine reflection r_j is given by the condition $r_j(\pi_{j-1}) = \pi_j$. For such an index j, we define the j-th *tail-flip* of π as

$$f_j(\pi) = (0, \pi_0, \dots, \pi_{j-1}, r_j(\pi_{j+1}), \dots, r_j(\pi_l), r_j(\mu)).$$

Then $f_j(\pi)$ corresponds to a path from 0 to $r_j(\mu)$. Let us associate with π the following composition of operators

$$X_\pi := X^{h(\pi_l - \mu)} X^{h(\pi_{l-1} - \pi_l)} \cdots X^{h(\pi_0 - \pi_1)} X^{h(0 - \pi_0)} = X^{-h\mu}.$$

Then $X_{f_j(\pi)} = X^{-h r_j(\mu)}$.

Let us now assume that $\pi = (0, \zeta_{A_0}, \ldots, \zeta_{A_l}, -\lambda)$, i.e., π_i's are the central points of the alcoves A_i. Then

$$X_\pi = X^\rho\, X^{\beta_l} \cdots X^{\beta_1}\, X^{-\rho} = X^{h\lambda}.$$

Indeed, $h(0-\zeta_{A_\circ}) = -\rho$, $h(\zeta_{A_{j-1}} - \zeta_{A_j}) = \beta_j$, and $h(\zeta_{A_{-\lambda}} - (-\lambda)) = \rho$, see Lemmas 14.1 and 14.2. The expression X_π is precisely the term $R^{[\lambda]}_{\emptyset}$ in the expansion of (14.1).

In this case, r_j is the affine reflection with respect to the common face of A_{j-1} and A_j and $\bar r_j = s_{\beta_j}$, for $j = 1, \ldots, l$. Suppose that the subset J consists of a single element j. The corresponding term $R^{[\lambda]}_{\{j\}}$ in the expansion of (14.1) is obtained from the above expression X_π by replacing the term X^{β_j} with B_{β_j}. Let us commute B_{β_j} all the way to the left using relation (10.5). We obtain

$$\begin{aligned} R^{[\lambda]}_{\{j\}} &= X^\rho\, X^{\beta_l} \cdots X^{\beta_{j+1}} B_{\beta_j} X^{\beta_{j-1}} \cdots X^{\beta_1}\, X^{-\rho} \\ &= B_{\beta_j}\, X^{\bar r_j(\rho)}\, X^{\bar r_j(\beta_l)} \cdots X^{\bar r_j(\beta_{j+1})} X^{\beta_{j-1}} \cdots X^{\beta_1}\, X^{-\rho}. \end{aligned}$$

The product of X's in the last expression is precisely the operator $X_{f_j(\pi)}$ for the j-th tail-flip π. In other words, $R^{[\lambda]}_{\{j\}} = B_{\beta_j}\, X_{f_j(\pi)}$.

In general, for a subset $J = \{j_1 < \cdots < j_s\} \subset \{1, \ldots, l\}$, we have

$$R^{[\lambda]}_J = B_{\beta_{j_s}} \cdots B_{\beta_{j_1}} X_{f_{j_1} \cdots f_{j_s}(\pi)}.$$

Indeed, let us start with the expression X_π. Replace the term $X^{\beta_{j_s}}$ in it with $B_{\beta_{j_s}}$, and commute it all the way to the left. This leads to the expression $B_{\beta_{j_s}}\, X_{f_{j_s}(\pi)}$. Then replace the term $X^{\beta_{j_{s-1}}}$ with $B_{\beta_{j_{s-1}}}$ and commute it to the left. This leads to the expression $B_{\beta_{j_s}}\, B_{\beta_{j_{s-1}}}\, X_{f_{j_{s-1}} f_{j_s}(\pi)}$, etc.

We have

$$X_{f_{j_1} \cdots f_{j_s}(\pi)} = X^{-h\, r_{j_1} \cdots r_{j_s}(-\lambda)}.$$

According to (10.2), this operator is explicitly given by

$$X_{f_{j_1} \cdots f_{j_s}(\pi)} : [\mathcal{O}_u] \longmapsto x^{-u\, r_{j_1} \cdots r_{j_s}(-\lambda)}\, [\mathcal{O}_u].$$

Let us summarize our calculations.

Proposition 14.5. *Let $\lambda \in \Lambda$ be a weight. Let $(r_1, \ldots, r_l)$ and $(\beta_1, \ldots, \beta_l)$ be the λ-chain of reflections and the λ-chain of roots associated with a decomposition $v_{-\lambda} = s_{i_1} \cdots s_{i_l}$. Then the operator $R^{[\lambda]}$ is given by*

$$R^{[\lambda]} : [\mathcal{O}_u] \longmapsto \sum_J x^{-u\, r_{j_1} \cdots r_{j_s}(-\lambda)}\, B_{\beta_{j_s}} \cdots B_{\beta_{j_1}}([\mathcal{O}_u]),$$

over all subsets $J = \{j_1 < \cdots < j_s\} \subset \{1, \ldots, l\}$.

We can now finish the proof Theorem 6.1.

Proof of Theorem 6.1. This follows from Theorem 13.1 and Proposition 14.5. □

15. Examples for Type A

In this and the next sections we illustrate our results by presenting several examples.

Suppose that $G = SL_n$. Then the root system Φ is of type A_{n-1} and the Weyl group W is the symmetric group S_n. We can identify the space $\mathfrak{h}^*_{\mathbb{R}}$ with the quotient space $V := \mathbb{R}^n/\mathbb{R}(1, \ldots, 1)$, where $\mathbb{R}(1, \ldots, 1)$ denotes the subspace in $\mathbb{R}^n$ spanned by the vector $(1, \ldots, 1)$. The action of the symmetric group S_n on V is obtained from the (left) S_n-action on $\mathbb{R}^n$ by permutation of coordinates. Let $\varepsilon_1, \ldots, \varepsilon_n \in V$ be the images of the coordinate vectors in $\mathbb{R}^n$. The root system Φ can be represented as $\Phi = \{\alpha_{ij} := \varepsilon_i - \varepsilon_j \mid i \neq j,\ 1 \le i, j \le n\}$. The simple roots are $\alpha_i = \alpha_{i\,i+1}$, for $i = 1, \ldots, n-1$. The longest coroot is $\theta^\vee = \alpha^\vee_{1n}$. The fundamental weights are $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$, for $i = 1, \ldots, n-1$. We have $\rho = n\varepsilon_1 + (n-1)\varepsilon_2 + \cdots + 2\varepsilon_{n-1} + \varepsilon_n$, and the Coxeter number is $h = (\rho, \theta^\vee) + 1 = n$. The weight lattice is $\Lambda = \mathbb{Z}^n/\mathbb{Z}(1,\dots,1)$. We use the notation $[\lambda_1,\dots,\lambda_n]$ for a weight, as the coset of $(\lambda_1,\dots,\lambda_n)$ in $\mathbb{Z}^n$.

Let $nZ \subset \Lambda$ be the set Z of central points of alcoves scaled by the factor $h = n$. The fundamental alcove corresponds to the point ρ in nZ. According Lemma 14.3, two alcoves are adjacent $A \xrightarrow{\alpha} B$, $\alpha \in \Phi$, if and only if the corresponding elements of nZ are related by $n\zeta_B - n\zeta_A = \alpha$. In this case, we write $n\zeta_A \xrightarrow{\alpha} n\zeta_B$. Thus, we have the structure of a directed graph with labeled edges on the set nZ. Alcove paths correspond to paths in this graph. The set nZ can be explicitly described as

$$nZ = \{[\mu_1,\dots,\mu_n] \in \Lambda \mid \mu_1,\dots,\mu_n \text{ have distinct residues modulo } n\}.$$

For an element $\mu = [\mu_1,\dots,\mu_n] \in nZ$, there exists an edge $\mu \xrightarrow{\alpha_{ij}} (\mu + \alpha_{ij})$ if and only if $\mu_i + 1 \equiv \mu_j \bmod n$. Given a weight λ, the corresponding λ-chains are in one-to-one correspondence with directed paths in the graph nZ from ρ to $\rho - n\lambda$.

Example 15.1. Suppose that $n = 4$ and $\lambda = \omega_2 = [1,1,0,0]$. The directed path

$$[4,3,2,1] \xrightarrow{-\alpha_{23}} [4,2,3,1] \xrightarrow{-\alpha_{13}} [3,2,4,1] \xrightarrow{-\alpha_{24}} [3,1,4,2] \xrightarrow{-\alpha_{14}} [2,1,4,3]$$

from $\rho = [4,3,2,1]$ to $\rho - n\,\omega_2 = [0,-1,2,1] = [2,1,4,3]$ produces the ω_2-chain $(\alpha_{23},\alpha_{13},\alpha_{24},\alpha_{14})$.

Example 15.2. For an arbitrary n, we have $\omega_1 = \varepsilon_1 = [1,0,\dots,0]$. The path

$$[n,n-1,\dots,1] \xrightarrow{-\alpha_{12}} [n-1,n,n-2,\dots,1] \xrightarrow{-\alpha_{13}} [n-2,n,n-1,n-3,\dots,1]$$
$$\xrightarrow{-\alpha_{14}} [n-3,n,n-1,n-2,n-4,\dots,1] \xrightarrow{-\alpha_{15}} \cdots \xrightarrow{-\alpha_{1n}} [1,n,n-1,\dots,2].$$

from ρ to $\rho - n\,\omega_1$ gives the ω_1-chain $(\alpha_{12},\alpha_{13},\alpha_{14},\dots,\alpha_{1n})$. In general, for any $k = 1,\dots,n$, we have the ε_k-chain

$$(\alpha_{k\,k+1},\alpha_{k\,k+2},\dots,\alpha_{k\,n},\alpha_{k\,1},\alpha_{k\,2},\dots,\alpha_{k\,k-1}) \tag{15.1}$$

given by the corresponding path from ρ to $\rho - n\varepsilon_k$.

Recall that $v_{-\lambda}$ is the unique element of W_{aff} such that $v_{-\lambda}(A_\circ) = A_{-\lambda}$. Equivalently, we can define $v_{-\lambda}$ in terms of central points of alcoves by the condition $v_{-\lambda}(\rho/h) = \rho/h - \lambda$.

Lemma 15.3. *Suppose that Φ is of type A_{n-1}. Then, for $k = 1,\dots,n-1$, the affine Weyl group element $v_{-\omega_k}$ belongs, in fact, to $S_n \subset W_{\mathrm{aff}}$. This permutation is given by*

$$v_{-\omega_k} = \begin{pmatrix} 1 & 2 & \cdots & n-k & n-k+1 & \cdots & n \\ k+1 & k+2 & \cdots & n & 1 & \cdots & k \end{pmatrix} \in S_n \subset W_{\mathrm{aff}}.$$

Proof. This permutation maps $\rho = [n,\dots,1]$ to $[k,k-1,\dots,1,n,n-1,\dots,k+1] = [0,-1,\dots,-k+1,n-k,n-k-1,\dots,1] = \rho - n\,\omega_k$, as needed. □

Let $R_{ij} := R_{\alpha_{ij}}$. Theorem 13.1 implies the following statement.

Corollary 15.4. *For $k = 1,\dots,n$, the operator of multiplication by e^{ε_k} in the Grothendieck ring $K_T(SL_n/B)$ is given by*

$$R^{[\varepsilon_k]} = R_{k\,k-1}R_{k\,k-2}\cdots R_{k\,1}R_{k\,n}R_{k\,n-1}\cdots R_{k\,k+1}.$$

For $k = 1,\dots,n-1$, the operator of multiplication by the class e^{ω_k} corresponding to the k-th fundamental weight ω_k is given by

$$R^{[\omega_k]} = R^{[\varepsilon_1]}\cdots R^{[\varepsilon_k]} = \overrightarrow{\prod_{i=1,\dots,k}}\ \overleftarrow{\prod_{j=k+1,\dots,n}} R_{ij}. \tag{15.2}$$

The combinatorial formula for multiplication by e^{ω_k} in the Grothendieck ring $K(SL_n/B)$ that follows from formula (15.2) was originally found in [Len1].

Proof. The expression for $R^{[\varepsilon_k]}$ is given by the ε_k-chain (15.1). The expression for $R^{[\omega_k]}$ can be obtained by simplifying $R^{[\varepsilon_1]}\cdots R^{[\varepsilon_k]}$, as shown in [Len1]. Alternatively, the reduced decomposition $v_{-\omega_k} = (s_k\cdots s_{n-1})(s_{k-1}\cdots s_{n-2})\cdots(s_1\cdots s_{n-k})$ for the permutation $v_{-\omega_k}$ given by Lemma 9.3 corresponds to an ω_k-chain, see Definition 5.4. This ω_k-chain produces the needed expression for $R^{[\omega_k]}$. □

Example 15.5. For $n = 3$, Corollary 15.4 says that

$$R^{[\omega_1]} = R_{13}\,R_{12} \quad \text{and} \quad R^{[\omega_2]} = R_{13}\,R_{23}.$$

For a weight $\lambda = a_1\omega_1 + \cdots + a_r\omega_r$, we can obtain an expression for $R^{[\lambda]}$ by concatenation of a_1 copies of $R^{[\omega_1]}$, a_2 copies of $R^{[\omega_2]}$, etc.

Theorem 6.1 says that that the coefficient of $[\mathcal{O}_w]$ in the product $e^\lambda\cdot[\mathcal{O}_u]$ in $K_T(G/B)$ is given by the sum over subsequences in the λ-chain $(\beta_1,\dots,\beta_l)$ that give saturated decreasing chains $u \gtrdot \cdots \gtrdot w$ in the Bruhat order on W. Let us illustrate this theorem by the following two examples.

Example 15.6. Suppose that $n = 3$, $\lambda = \omega_1$, and $u = w_\circ = s_1s_2s_1 \in W$. Let us calculate the product $e^\lambda\cdot[\mathcal{O}_u]$ in $K_T(SL_n/B)$ using Theorem 6.1. The ω_1-chain $(\beta_1,\beta_2) = (\alpha_{12},\alpha_{13})$ is associated with the reduced decomposition $s_1s_2 = v_{-\omega_1}$. The corresponding ω_1-chain of reflections is $(r_1,r_2) = (s_1, s_1s_2s_1) = (s_{\alpha_{12},0}, s_{\alpha_{13},0})$. Three out of four subsequences in (β_1,β_2) correspond to decreasing chains in Bruhat order starting at $w_\circ$: (empty subsequence), (α_{12}), and $(\alpha_{12},\alpha_{13})$. Thus we have

$$e^{\omega_1}\cdot[\mathcal{O}_{w_\circ}] = x^{-w_\circ(-\omega_1)}[\mathcal{O}_{w_\circ}] + x^{-w_\circ r_1(-\omega_1)}[\mathcal{O}_{s_1s_2}] + x^{-w_\circ r_1 r_2(-\omega_1)}[\mathcal{O}_{s_2}].$$

We can write this expression as

$$e^{[1,0,0]}\cdot[\mathcal{O}_{w_\circ}] = x^{[0,0,1]}[\mathcal{O}_{w_\circ}] + x^{[0,1,0]}[\mathcal{O}_{s_1s_2}] + x^{[1,0,0]}[\mathcal{O}_{s_2}].$$

The character of the irreducible representation V_{ω_1} is obtained from the right-hand side of this expression by replacing each term $x^\mu[\mathcal{O}_w]$ with e^μ:

$$ch(V_{\omega_1}) = e^{[0,0,1]} + e^{[0,1,0]} + e^{[1,0,0]}.$$

Let us give a less trivial example.

Example 15.7. Suppose $n = 3$ and $\lambda = 2\,\omega_1 + \omega_2 = [3,1,0]$. The path

$$[3,2,1] \xrightarrow{-\alpha_{12}} [2,3,1] \xrightarrow{-\alpha_{13}} [1,3,2] \xrightarrow{-\alpha_{23}} [1,2,3]$$
$$\xrightarrow{-\alpha_{13}} [0,2,4] \xrightarrow{-\alpha_{12}} [-1,3,4] \xrightarrow{-\alpha_{13}} [-2,3,5]$$

from $\rho = [3,2,1]$ to $\rho - n\lambda = [-2,3,5]$ gives the λ-chain

$$(\beta_1,\dots,\beta_6) = (\alpha_{12},\ \alpha_{13},\ \alpha_{23},\ \alpha_{13},\ \alpha_{12},\ \alpha_{13}),$$

which is associated with the reduced decomposition $v_{-\lambda} = s_1s_2s_1s_0s_1s_2$ in the affine Weyl group. We have

$$R^{[\lambda]} = R_{\beta_6}\cdots R_{\beta_1} = R_{13}\,R_{12}\,R_{13}\,R_{23}\,R_{13}\,R_{12} = R^{[\omega_1]}\,R^{[\omega_2]}\,R^{[\omega_1]}.$$

The corresponding λ-chain of reflections is

$$(r_1,\dots,r_6) = (s_{\alpha_{12},0},\ s_{\alpha_{13},0},\ s_{\alpha_{23},0},\ s_{\alpha_{13},-1},\ s_{\alpha_{12},-1},\ s_{\alpha_{13},-2}).$$

Suppose that $u = s_2s_1$. There are five saturated chains in Bruhat order descending from u: (empty chain), $(u \gtrdot us_{\alpha_{12}} = s_2)$, $(u \gtrdot us_{\alpha_{13}} = s_1)$, $(u \gtrdot us_{\alpha_{12}} \gtrdot us_{\alpha_{12}}s_{\alpha_{23}} = 1)$, $(u \gtrdot us_{\alpha_{13}} \gtrdot us_{\alpha_{13}}s_{\alpha_{12}} = 1)$. Thus, the expansion of $e^\lambda\cdot[\mathcal{O}_u]$ is given by the sum over the following subsequences in the λ-chain $(\beta_1,\dots,\beta_6)$:

$$\text{(empty subsequence)},\ (\alpha_{12}),\ (\alpha_{13}), (\alpha_{12},\,\alpha_{23}),\ (\alpha_{13},\,\alpha_{12}).$$

The sequence $(\beta_1,\dots,\beta_6)$ contains one empty subsequence, two subsequences of the form (α_{12}), three subsequences of the form (α_{13}), one subsequence of the form $(\alpha_{12},\alpha_{23})$, and two subsequence of the form $(\alpha_{13},\alpha_{12})$. Hence, we have

$$\begin{aligned} e^{\lambda}\cdot[\mathcal{O}_{s_2s_1}] &= x^{-u(-\lambda)}\,[\mathcal{O}_{s_2s_1}] + \left(x^{-ur_1(-\lambda)}+x^{-ur_5(-\lambda)}\right)[\mathcal{O}_{s_2}] + \\ &+\ \left(x^{-ur_2(-\lambda)}+x^{-ur_4(-\lambda)}+x^{-ur_6(-\lambda)}\right)[\mathcal{O}_{s_1}] + \\ &+ x^{-ur_1r_3(-\lambda)}\,[\mathcal{O}_1] + \left(x^{-ur_2r_5(-\lambda)}+x^{-ur_4r_5(-\lambda)}\right)[\mathcal{O}_1]. \end{aligned}$$

We can explicitly write this expression as

$$\begin{aligned} e^{[3,1,0]}\cdot[\mathcal{O}_{s_2s_1}] &= x^{[1,0,3]}\,[\mathcal{O}_{s_2s_1}] + \left(x^{[3,0,1]}+x^{[2,0,2]}\right)[\mathcal{O}_{s_2}] + \\ &+\ \left(x^{[1,3,0]}+x^{[1,2,1]}+x^{[1,1,2]}\right)[\mathcal{O}_{s_1}] + x^{[3,1,0]}\,[\mathcal{O}_1] + \left(x^{[2,2,0]}+x^{[2,1,1]}\right)[\mathcal{O}_1]. \end{aligned}$$

The corresponding Demazure character is

$$\begin{aligned} &ch(V_{[3,1,0],s_2s_1}) = \\ &\quad e^{[1,0,3]}+e^{[3,0,1]}+e^{[2,0,2]}+e^{[1,3,0]}+e^{[1,2,1]}+e^{[1,1,2]}+e^{[3,1,0]}+e^{[2,2,0]}+e^{[2,1,1]}. \end{aligned}$$

16. Examples for Other Types

For an arbitrary root system, we can use the explicit construction of the λ-chain of reflections $(r_1,\dots,r_l)$ and the λ-chain of roots $(\beta_1,\dots,\beta_l)$ given by Proposition 6.7.

Example 16.1. Suppose that the root system Φ is of type G_2. Let us find λ-chains for $\lambda=\omega_1$ and $\lambda=\omega_2$ using Proposition 6.7. The positive roots are $\gamma_1=\alpha_1,\ \gamma_2=3\alpha_1+\alpha_2,\ \gamma_3=2\alpha_1+\alpha_2,\ \gamma_4=3\alpha_1+2\alpha_2,\ \gamma_5=\alpha_1+\alpha_2,\ \gamma_6=\alpha_2$. The corresponding coroots are $\gamma_1^\vee=\alpha_1^\vee,\ \gamma_2^\vee=\alpha_1^\vee+\alpha_2^\vee,\ \gamma_3^\vee=2\alpha_1^\vee+3\alpha_2^\vee,\ \gamma_4^\vee=\alpha_1^\vee+2\alpha_2^\vee,\ \gamma_5^\vee=\alpha_1^\vee+3\alpha_2^\vee,\ \gamma_6^\vee=\alpha_2^\vee$.

Suppose that $\lambda=\omega_1$. The set $\mathcal{R}_{\omega_1}$ of affine reflections with respect to the hyperplanes separating the alcoves $A_\circ$ and $A_{-\omega_1}$ is

$$\mathcal{R}_{\omega_1}=\{s_{\gamma_1,0},\ s_{\gamma_2,0},\ s_{\gamma_3,0},\ s_{\gamma_3,-1},\ s_{\gamma_4,0},\ s_{\gamma_5,0}\}.$$

The map $h:\mathcal{R}_{\omega_1}\to\mathbb{R}^{r+1}$ given by (6.2) sends these affine reflections to the vectors

$$(0,1,0),\ (0,1,1),\ (0,1,\tfrac{3}{2}),\ (\tfrac{1}{2},1,\tfrac{3}{2}),\ (0,1,2),\ (0,1,3),$$

respectively. The lexicographic order on vectors in $\mathbb{R}^3$ induces the following total order on the set $\mathcal{R}_{\omega_1}$:

$$s_{\gamma_1,0}<s_{\gamma_2,0}<s_{\gamma_3,0}<s_{\gamma_4,0}<s_{\gamma_5,0}<s_{\gamma_3,-1}\,.$$

Suppose now that $\lambda=\omega_2$. The set $\mathcal{R}_{\omega_2}$ of affine reflections with respect to the hyperplanes separating $A_\circ$ and $A_{-\omega_2}$ is

$$\mathcal{R}_{\omega_2}=\{s_{\gamma_2,0},\ s_{\gamma_3,0},\ s_{\gamma_3,-1},\ s_{\gamma_3,-2},\ s_{\gamma_4,0},\ s_{\gamma_4,-1},\ s_{\gamma_5,0},\ s_{\gamma_5,-1},\ s_{\gamma_5,-2},s_{\gamma_6,0}\}.$$

The map $h:\mathcal{R}_{\omega_2}\to\mathbb{R}^{r+1}$ sends these affine reflections to the vectors

$$\begin{aligned} &(0,1,1),\ (0,\tfrac{2}{3},1),\ (\tfrac{1}{3},\tfrac{2}{3},1),\ (\tfrac{2}{3},\tfrac{2}{3},1),\ (0,\tfrac{1}{2},1),\ (\tfrac{1}{2},\tfrac{1}{2},1),\\ &\qquad\qquad (0,\tfrac{1}{3},1),\ (\tfrac{1}{3},\tfrac{1}{3},1),\ (\tfrac{2}{3},\tfrac{1}{3},1),\ (0,0,1), \end{aligned}$$

respectively. The lexicographic order on vectors in $\mathbb{R}^3$ induces the following total order on $\mathcal{R}_{\omega_2}$:

$$s_{\gamma_6,0}<s_{\gamma_5,0}<s_{\gamma_4,0}<s_{\gamma_3,0}<s_{\gamma_2,0}<s_{\gamma_5,-1}<s_{\gamma_3,-1}<s_{\gamma_4,-1}<s_{\gamma_5,-2}<s_{\gamma_3,-2}\,.$$

The total orders on $\mathcal{R}_{\omega_1}$ and $\mathcal{R}_{\omega_2}$ correspond to the ω_1-chain $(\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_3)$ and the ω_2-chain $(\gamma_6,\gamma_5,\gamma_4,\gamma_3,\gamma_2,\gamma_5,\gamma_3,\gamma_4,\gamma_5,\gamma_3)$. Thus, the operators of multiplication by the classes e^{ω_1} and e^{ω_2} in $K_T(G/B)$ are given by

$$\begin{aligned} R^{[\omega_1]} &= R_{\gamma_3}\,R_{\gamma_5}\,R_{\gamma_4}\,R_{\gamma_3}\,R_{\gamma_2}\,R_{\gamma_1}\,,\\ R^{[\omega_2]} &= R_{\gamma_3}\,R_{\gamma_5}\,R_{\gamma_4}\,R_{\gamma_3}\,R_{\gamma_5}\,R_{\gamma_2}\,R_{\gamma_3}\,R_{\gamma_4}\,R_{\gamma_5}\,R_{\gamma_6}\,. \end{aligned}$$

By Lemma 15.3, the element $v_{-\omega_k}$ belongs to the (nonaffine) Weyl group W, for all fundamental weights ω_k in type A. Let us show that a similar phenomenon occurs for minuscule weights in other types as well. Recall that a dominant weight λ is *minuscule* if the set of weights in the G-module V_λ is in the orbit $W \cdot \lambda$ of the Weyl group.

Lemma 16.2. *Let $\lambda \in \Lambda^+$. Then $v_{-\lambda} \in W$ if and only if λ is a minuscule weight.*

Proof. Let $(\beta_1, \dots, \beta_l)$ be a reduced λ-chain of roots, and let $(r_1, \dots, r_l)$ be the corresponding λ-chain of reflections. According to Lemmas 5.3 and 6.2, the following statements are equivalent: (1) $v_{-\lambda} \in W$; (2) $r_1, \dots, r_l \in W$; (3) all (positive) roots $\beta_1, \dots, \beta_l$ are distinct; (4) $(\lambda, \alpha^\vee) = 0$ or 1, for any $\alpha \in \Phi^+$. According to Corollary 6.6, the condition $r_1, \dots, r_l \in W$ implies that all weights in V_λ are in the W-orbit $W \cdot \lambda$ and, thus, λ is minuscule. On the other hand, if λ is minuscule, then $(\lambda, \alpha^\vee) = 0$ or 1, for any $\alpha \in \Phi^+$. Otherwise, if $(\lambda, \alpha^\vee) \geq 2$, then V_λ contains the weight $\lambda - \alpha \notin W \cdot \lambda$. □

The last two examples concern minuscule weights in types B and C. Recall that the element $v_{-\lambda}$ is uniquely defined by the condition $v_{-\lambda}(\rho/h) = \rho/h - \lambda$. If $v_{-\lambda} \in W$. We can rewrite this condition as $v_{-\lambda}(\rho) = \rho - h\,\lambda$.

Example 16.3. Suppose that Φ is of type C_r. This root system can be embedded into $\mathbb{R}^r$ as follows: $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm 2\varepsilon_i \mid i \neq j\}$, where $\varepsilon_1, \dots, \varepsilon_r$ are the coordinate vectors in $\mathbb{R}^r$. The simple roots are $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, $\dots \alpha_{r-1} = \varepsilon_{r-1} - \varepsilon_r$, $\alpha_r = 2\varepsilon_r$. The Weyl group W is the semidirect product of S_r and $(\mathbb{Z}/2\mathbb{Z})^r$. It acts on $\mathbb{R}^r$ by permuting the coordinates and changing their signs. The fundamental weights are $\omega_k = \varepsilon_1 + \cdots + \varepsilon_k$, $k = 1, \dots, r$. We have $\rho = (r, \dots, 1) \in \mathbb{R}^r$, and the Coxeter number is $h = (\rho, \theta^\vee) + 1 = 2r$.

Suppose that $\lambda = \omega_1$. Then $\rho - h\omega_1 = (-r, r-1, r-2, \dots, 1) \in \mathbb{R}^r$. This weight is obtained from ρ by applying the Weyl group element $s_{2\varepsilon_1}$ that changes the sign of the first coordinate. Thus $v_{-\omega_1} = s_{2\varepsilon_1} \in W \subset W_{\text{aff}}$. The only reduced decomposition of this element is $v_{-\omega_1} = s_1 \cdots s_{r-1}\, s_r\, s_{r-1} \cdots s_1$, so $\ell(v_{-\omega_1}) = 2r - 1$. This reduced decomposition corresponds to the ω_1-chain

$$
\begin{aligned}
&(\alpha_1,\ s_1(\alpha_2),\ s_1 s_2(\alpha_3),\ \dots,\ s_1 \dots s_{r-1}(\alpha_r),\ \dots,\ s_1 \dots s_r \dots s_2(\alpha_1)) = \\
&(\varepsilon_1 - \varepsilon_2,\ \varepsilon_1 - \varepsilon_3,\ \cdots,\ \varepsilon_1 - \varepsilon_r,\ 2\varepsilon_1,\ \varepsilon_1 + \varepsilon_r,\ \cdots,\ \varepsilon_1 + \varepsilon_3,\ \varepsilon_1 + \varepsilon_2),
\end{aligned}
$$

cf. Definition 5.4. The operator $R^{[\omega_1]}$ is given by

$$R^{[\omega_1]} = R_{\varepsilon_1+\varepsilon_2} R_{\varepsilon_1+\varepsilon_3} \cdots R_{\varepsilon_1+\varepsilon_r} R_{2\varepsilon_1} R_{\varepsilon_1-\varepsilon_r} \cdots R_{\varepsilon_1-\varepsilon_3} R_{\varepsilon_1-\varepsilon_2}\,.$$

Example 16.4. Suppose that Φ is of type B_r. This root system can be embedded into $\mathbb{R}^r$ as follows: $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i \mid i \neq j\}$, where $\varepsilon_1, \dots, \varepsilon_r$ are the coordinate vectors in $\mathbb{R}^r$. The simple roots are $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, $\dots \alpha_{r-1} = \varepsilon_{r-1} - \varepsilon_r$, $\alpha_r = \varepsilon_r$. The Weyl group W and its action on $\mathbb{R}^r$ are the same as in type C_r. The fundamental weights are $\omega_k = \varepsilon_1 + \cdots + \varepsilon_k$, $k = 1, \dots, r-1$, and $\omega_r = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r)$. We have $\rho = (r - \frac{1}{2}, \dots, 1 - \frac{1}{2}) \in \mathbb{R}^r$, and $h = (\rho, \theta^\vee) + 1 = 2r$.

Suppose that $\lambda = \omega_r$ is the last fundamental weight. Then $\rho - h\omega_r = (-\frac{1}{2}, -1 - \frac{1}{2}, -2 - \frac{1}{2}, \dots, -r + \frac{1}{2}) \in \mathbb{R}^r$. This weight is obtained from ρ by applying the Weyl group element $v_{-\omega_r} \in W \subset W_{\text{aff}}$ that reverses the order of all coordinates and changes their signs. The element $v_{-\omega_r} \in W$ has length $\ell(v_{-\omega_r}) = r(r+1)/2$. One of the reduced decompositions for this element is

$$v_{-\omega_r} = (s_r)(s_{r-1}\, s_r)(s_{r-2}\, s_{r-1}\, s_r) \cdots (s_2 \cdots s_r)(s_1 \cdots s_r).$$

The associated ω_r-chain is $(\alpha_r,\ s_r(\alpha_{r-1}),\ s_r s_{r-1}(\alpha_r),\ s_r s_{r-1} s_r(\alpha_{r-2}),\ \dots)$. We can explicitly find the roots in this ω_r-chain and write the operator $R^{[\omega_r]}$ as

$$
\begin{aligned}
R^{[\omega_r]} = (R_{\varepsilon_1}\, R_{\varepsilon_1+\varepsilon_2}\, R_{\varepsilon_1+\varepsilon_3} \cdots R_{\varepsilon_1+\varepsilon_r})(R_{\varepsilon_2}\, R_{\varepsilon_2+\varepsilon_3}\, R_{\varepsilon_2+\varepsilon_4} \cdots R_{\varepsilon_2+\varepsilon_r}) \cdots \\
\cdots (R_{\varepsilon_{r-2}}\, R_{\varepsilon_{r-2}+\varepsilon_{r-1}}\, R_{\varepsilon_{r-2}+\varepsilon_r})(R_{\varepsilon_{r-1}}\, R_{\varepsilon_{r-1}+\varepsilon_r})(R_{\varepsilon_r}).
\end{aligned}
$$

17. Quantum K-theory

In this section, we conjecture a natural Chevalley-type formula in the *quantum K-theory* of G/B. The quantum K-theory, which is a K-theoretic version of *quantum cohomology*, was introduced by Lee [Lee]. The quantum K-theory of flag varieties, in particular, has been first studied by Givental and Lee [GiLe]. We recall a few basic facts below.

Let us denote by $QK(G/B)$ the quantum K-theory of G/B. In order to describe it, we associate a variable q_i to each simple root α_i, and let $\mathbb{Z}[q] = \mathbb{Z}[q_1, \ldots, q_r]$ be the polynomial ring in the q_i. Given a collection of nonnegative integers $d = (d_1, \ldots, d_r)$, called multidegree, we let $q^d := q_1^{d_1} \ldots q_r^{d_r}$. As a $\mathbb{Z}[q]$-module, the quantum K-theory is defined as $QK(G/B) := K(G/B) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$. Let $[w]$ denote the class of the structure sheaf of the Schubert variety $X_{w_\circ w}$. Then the classes of $[w]$ form a $\mathbb{Z}[q]$-basis of $QK(G/B)$. The multiplication in $QK(G/B)$ is a deformation of the classical multiplication:

$$[u] \circ [v] = \sum_d q^d \sum_{w \in W} N_{uv}^w(d)\, [w]\,,$$

where the first sum is over all multidegrees d, and $N_{uv}^w(d)$ is the *quantum K-invariant of Gromov-Witten type* for $[u]$, $[v]$, and the quantum dual of $[w]$. As defined in [Lee], this invariant is the K-theoretic push-forward to $\operatorname{Spec} \mathbb{C}$ of some natural vector bundle on the moduli space $\overline{M}_{3,0}(G/B, d)$ (via the orientation defined by the virtual structure sheaf). The associativity of the quantum K-product was established in [Lee], based on a sheaf-theoretic version of an argument of WDVV-type.

Let us recall the Chevalley-type formula for the small quantum cohomology ring $QH^*(G/B)$ of G/B. For type A, this formula was first proved in [FGP]. In general type, it was proved by D. Peterson (unpublished) and by Fulton and Woodward [FuWo] (who, in fact, obtained a more general formula for G/P). Again, as a $\mathbb{Z}[q]$-module, $QH^*(G/B) := H^*(G/B) \otimes \mathbb{Z}[q]$. Thus, the quantum cohomology ring has a $\mathbb{Z}[q]$-basis basis given by the cohomology classes of $X_{w_\circ w}$, which we denoted by $\langle w \rangle$.

The Chevalley-type formula in $QH^*(G/B)$ can be stated using the *quantum Bruhat operators* defined in [BFP]. These are operators on the group algebra $\mathbb{Z}[q][W]$ of the Weyl group W over $\mathbb{Z}[q]$. For each positive root α, the quantum Bruhat operator Q_α is defined by

$$Q_\alpha(w) = \begin{cases} w s_\alpha & \text{if } \ell(w s_\alpha) = \ell(w) + 1, \\ q^{d(\alpha)}\, w s_\alpha & \text{if } \ell(w s_\alpha) = \ell(w) - 2\,\mathrm{ht}(\alpha^\vee) + 1, \\ 0 & \text{otherwise}, \end{cases}$$

where $\mathrm{ht}(\alpha^\vee) = (\rho, \alpha^\vee)$ is the height of the coroot $\alpha^\vee$, and $q^{d(\alpha)} = q_1^{d_1} \cdots q_r^{d_r}$, for $\alpha^\vee = d_1 \alpha_1^\vee + \cdots + d_r \alpha_r^\vee$, i.e., $d_i = (\omega_i, \alpha^\vee)$. Also define $Q_\alpha := -Q_{-\alpha}$ if α is a negative root. It was proved in [BFP] that the operators Q_α satisfy the Yang-Baxter equation.

The map $w \mapsto \langle w \rangle$ extends linearly to the isomorphism $\mathbb{Z}[q][W] \to QH^*(G/B)$ of $\mathbb{Z}[q]$-modules, for which we use the same notation $a \to \langle a \rangle$. Similarly, we extend the map $w \mapsto [w]$. The Chevalley formula in quantum cohomology can now be stated, as follows, see [FuWo, BFP].

$$\langle s_i \rangle * \langle w \rangle = \sum_{\alpha \in \Phi^+} (\omega_i, \alpha^\vee)\, \langle Q_\alpha(w) \rangle\,, \tag{17.1}$$

where s_i is a simple reflection and $*$ denotes the product in $QH^*(G/B)$.

Based on Corollary 1.2 and (17.1), we formulate the following conjecture.

Conjecture 17.1. Fix a simple reflection s_i. Let $(\beta_1, \ldots, \beta_l)$ be a $(-\omega_i)$-chain of roots. Then we have

$$[s_i] \circ [w] = [(1 - (1 - Q_{\beta_1}) \cdots (1 - Q_{\beta_l}))(w)]\,,$$

where $\circ$ denotes the product in the ring $QK(G/B)$.

The conjectured formula in $QK(G/B)$ specializes to Corollary 1.2, upon setting $q_1 = \cdots = q_r = 0$. It also specializes to QH-Chevalley formula (17.1), upon taking the linear terms in the expansion of the operator $1-(1-Q_{\beta_1})\cdots(1-Q_{\beta_l})$, cf. Remark 13.4. One can extend this conjecture to the quantum T-equivariant K-theory of G/B, see [Lee] for the definition of the ring $QK_T(G/B)$. In order to do this, one has to consider the operator $R_q^{[-\omega_i]}$ obtained from $R^{[-\omega_i]}$ by replacing all Bruhat operators B_β with the quantum Bruhat operators Q_β, cf. Theorem 13.1. It is not hard to extend the above conjecture to generalized partial flag varieties G/P, as well.

In [LeMa2], we define quantum Grothendieck polynomials, which we conjecture to represent Schubert classes in the quantum K-theory [Lee] of the classical flag variety SL_n/B. In support of our conjecture, we prove that the quantum Grothendieck polynomials satisfy (and, in fact, are determined by) the type A specialization of our conjectured Chevalley-type formula for quantum K-theory.

18. Appendix: Foldings of Galleries, LS-galleries, and LS-paths

In this appendix, we introduce admissible foldings of galleries, and use this notion to reformulate our model for the characters of the irreducible representations (Corollary 6.6) and for the Demazure characters (Corollary 6.5). For regular weights, admissible foldings of galleries are similar, but not equivalent, to the LS-galleries of Gaussent and Littelmann [GaLi]. We clarify this relationship by showing that it is based on Dyer's theorem [Dyer] about the EL-shellability of the Bruhat order.

18.1. **Admissible Foldings.**

Definition 18.1. A *gallery* is a sequence $\gamma=(F_0,A_0,F_1,A_1,F_2,\ldots,F_l,A_l,F_{l+1})$ such that $A_0,\ldots,A_l$ are alcoves; F_j is a codimension 1 common face of the alcoves A_{j-1} and A_j, for $j=1,\ldots,l$; F_0 is a vertex of the first alcove A_0; and F_{l+1} is a vertex of the last alcove A_l. Furthermore, we require that $F_0=\{0\}$ and $F_{l+1}=\{\mu\}$ for some weight $\mu\in\Lambda$, which is called the *weight* of the gallery. We say that a gallery is *unfolded* if $A_{j-1}\neq A_j$, for $j=1,\ldots,l$.

These galleries are special cases of the generalized galleries in [GaLi].

In this subsection, we will consider only galleries such that $A_0=A_\circ$ is the fundamental alcove. Unfolded galleries of weight μ with $A_0=A_\circ$ are in one-to-one correspondence with alcove paths $(A_\circ,\ldots,A_l)$ such that $\mu\in A_l$. Indeed, F_j should be the unique common wall of two adjacent alcoves A_{j-1} and A_j, for $j=1,\ldots,l$.

Definition 18.2. Let us say that a gallery γ of weight μ is *reduced* if $A_0=A_\circ$, and γ has has minimal length among all galleries of weight μ with $A_0=A_\circ$. Clearly, every reduced gallery is unfolded.

Lemma 18.3. *Let λ be a dominant weight. Then the last alcove in a reduced gallery of weight $-\lambda$ is $A_l=A_{-\lambda}$. Hence, reduced galleries with an antidominant weight $-\lambda$ are in one-to-one correspondence with reduced alcove paths from $A_\circ$ to $A_{-\lambda}$, which, in turn, correspond to reduced decompositions of $v_{-\lambda}\in W_{\mathrm{aff}}$.*

Proof. The number of hyperplanes $H_{\alpha,k}$ that separate the point $E=\{-\lambda\}$ from the fundamental alcove $A_\circ$ is $m=\sum_{\alpha\in\Phi^+}(\lambda,\alpha^\vee)$. Thus, the length of any alcove path from $A_\circ$ to an alcove A_l with vertex E should be at least m. The number m is precisely the length of a reduced alcove path from $A_\circ$ to $A_{-\lambda}$. On the other hand, for any other alcove $A'\neq A_{-\lambda}$ such that E is a vertex of A', the number of hyperplanes that separate A' from $A_\circ$ is strictly greater than m. □

For a gallery $\gamma=(F_0,A_0,F_1,\ldots,F_l,A_l,F_{l+1})$, let $r_1,\ldots,r_l\in W_{\mathrm{aff}}$ denote the affine reflections with respect to the affine hyperplanes containing the faces $F_1,\ldots,F_l$. For $j=1,\ldots,l$, let the j-th *tail-flip operator* f_j be the operator that sends the gallery $\gamma=(F_0,A_0,F_1,\ldots,F_l,A_l,F_{l+1})$ to the gallery $f_j(\gamma)$ given by

$$f_j(\gamma):=(F_0,A_0,F_1,A_1,\ldots,A_{j-1},F_j'=F_j,A_j',F_{j+1}',A_{j+1}',\ldots,A_l',F_{l+1}'),$$

where $A_i':=r_j(A_i)$ and $F_i':=r_j(F_i)$, for $i=j,\ldots,l+1$. In other words, the operator f_j leaves the initial segment of the gallery from A_0 to A_{j-1} intact and reflects the remaining tail by r_j. Clearly, the operators f_j commute. Hence, they determine an action of the group $(\mathbb{Z}/2\mathbb{Z})^l$ on galleries. Every gallery is obtained from an unfolded gallery by applying several tail-flips. Equivalently, using the operators f_j, one can always transform (unfold) an arbitrary gallery into a uniquely defined unfolded gallery.

Lemma 18.4. *If γ is a gallery of weight μ, then $f_{j_1}\cdots f_{j_s}(\gamma)$ is a gallery of weight $r_{j_1}\cdots r_{j_s}(\mu)$, for any $1\le j_1<\cdots<j_s\le l$.*

Proof. First, let us apply f_{j_s} to γ. We obtain a gallery of weight $r_{j_s}(\mu)$. Applying the tail-flip $f_{j_{s-1}}$ to $f_{j_s}(\gamma)$ changes its weight to $r_{j_{s-1}}r_{j_s}(\mu)$, etc. □

Definition 18.5. Let γ be an unfolded gallery, and let $r_1,\dots,r_l$ be the affine reflections with respect to the faces of γ. An *admissible folding* of γ is a gallery of the form $f_{j_1}\cdots f_{j_s}(\gamma)$ for some $1\le j_1<\cdots<j_s\le l$ such that

$$1\lessdot \bar r_{j_1}\lessdot \bar r_{j_1}\bar r_{j_2}\lessdot\cdots\lessdot \bar r_{j_1}\bar r_{j_2}\cdots\bar r_{j_s}$$

is a saturated increasing chain in the Bruhat order on the Weyl group W. More generally, for $u\in W$, a *u-admissible folding* of γ is a gallery of the form $f_{j_1}\cdots f_{j_s}(\gamma)$ for some $1\le j_1<\cdots<j_s\le l$ such that

$$u\gtrdot u\,\bar r_{j_1}\gtrdot u\,\bar r_{j_1}\bar r_{j_2}\gtrdot\cdots\gtrdot u\,\bar r_{j_1}\bar r_{j_2}\cdots\bar r_{j_s}$$

is a saturated decreasing chain in the Bruhat order on the Weyl group W. We allow $s=0$, so the gallery γ itself is an admissible (u-admissible) folding of γ. Notice that admissible foldings are precisely $w_\circ$-admissible foldings.

We can also give the following intrinsic characterization of u-admissible foldings.

Lemma 18.6. *Let $\gamma'=(A_0',F_1',\dots,F_l',A_l',E')$ be a gallery, and $r_1',\dots,r_l'$ be the affine reflections with respect to the faces $F_1',\dots,F_l'$. Let $\{j_1<\cdots<j_s\}:=\{j\in\{1,\dots,l\}\mid A_{j-1}'=A_j'\}$. Then the gallery γ' is a u-admissible folding of some unfolded gallery γ if and only if*

$$u^{-1}\gtrdot \bar r_{j_1}'\,u^{-1}\gtrdot \bar r_{j_1}'\bar r_{j_2}'\,u^{-1}\gtrdot\cdots\gtrdot \bar r_{j_1}'\bar r_{j_2}'\cdots\bar r_{j_s}'\,u^{-1}$$

is a saturated decreasing chain in the Bruhat order on the Weyl group W.

Proof. We have $\gamma'=f_{j_1}\cdots f_{j_s}(\gamma)$. Let $r_1,\dots,r_l$ be the reflections with respect to the faces of the unfolded gallery γ. Then

$$r_{j_1}'=r_{j_1},\ r_{j_2}'=r_{j_1}r_{j_2}r_{j_1},\ r_{j_3}'=r_{j_1}r_{j_2}r_{j_3}r_{j_2}r_{j_1},\ \dots$$

This implies $r_{j_1}'r_{j_2}'\cdots r_{j_i}'=(r_{j_1}r_{j_2}\cdots r_{j_i})^{-1}$, for $i=1,\dots,s$. Now the lemma follows from Definition 18.5. □

Corollaries 6.5 and 6.6 are equivalent to the following claim. Let weight(γ) denote the weight of a gallery γ.

Corollary 18.7. *Let λ be a dominant weight, and let γ be a reduced gallery with* weight$(\gamma)=-\lambda$.

(1) *The character $ch(V_\lambda)$ is equal to the sum*

$$ch(V_\lambda)=\sum_{\gamma'}e^{-\mathrm{weight}(\gamma')}$$

over all admissible foldings γ' of the gallery γ.

(2) *Let $u\in W$. The Demazure character $ch(V_{\lambda,u})$ is equal to the sum*

$$ch(V_{\lambda,u})=\sum_{\gamma'}e^{-u(\mathrm{weight}(\gamma'))}$$

over all u-admissible foldings γ' of the gallery γ.

18.2. **LS-galleries.** In this section, we discuss the relationship between admissible foldings and LS-galleries of Gaussent and Littelmann in case of a *regular* weight λ. We show that LS-galleries can be associated with admissible foldings of some *special* reduced galleries.

We start by recalling some terminology from [GaLi]. Let us fix a dominant regular weight λ. Let us say that a gallery γ of weight λ is *minimal* if γ crosses only the hyperplanes strictly separating 0 and λ. Note that in such a gallery we have $A_0 = A_\circ$, and the last alcove A_l is $w_\circ(A_\circ) + \lambda = -A_\circ + \lambda$.

Recall that the facets of the fundamental alcove are $H_i = H_{\alpha_i,0}$, for $i = 1, \ldots, r$; and $H_0 = H_{\alpha_0,-1}$. If F is a face of the fundamental alcove $A_\circ$, we define its *type* by

$$\mathrm{type}(F) = \{i \mid F \subset H_i,\ i = 0, 1, \ldots, r\}\,.$$

For instance, $\mathrm{type}(\{0\}) = \{1, \ldots, r\}$ and $\mathrm{type}(A_\circ) = \emptyset$. For an arbitrary face F, its type is defined as $\mathrm{type}(F')$, where F' is the unique face of $A_\circ$ such that $F = w(F')$ for some w in W_{aff}. The type of a gallery $\gamma = (F_0, A_0, F_1, \ldots, A_l, F_{l+1})$ is defined as $\mathrm{type}(\gamma) = (\mathrm{type}(F_0), \mathrm{type}(A_0), \ldots, \mathrm{type}(F_{l+1}))$.

For a gallery $\gamma = (F_0, A_0, F_1, \ldots, A_l, F_{l+1})$, let $\{j_1 < \ldots < j_s\} = \{j \mid A_{j-1} = A_j\}$, and let r_j be the reflections with respect to the hyperplanes containing the faces F_j. The *companion* of γ is the sequence $(u_0, \ldots, u_s)$ of elements in W, where $u_0 \in W$ is the unique element such that $u(A_\circ) = A_0$; and $u_i = \bar{r}_{j_i} u_{i-1}$, for $i = 1, \ldots, s$.

Definition 18.8. [GaLi] For a minimal gallery γ of a (dominant regular) weight λ, the set $\Gamma_{LS}(\gamma)$ of *LS-galleries* associated with γ is the set of all galleries γ' such that (1) $\mathrm{type}(\gamma') = \mathrm{type}(\gamma)$; and (2) the companion $(u_0, \ldots, u_s)$ of γ' is a saturated decreasing chain in the Bruhat order on W.

The general definition of LS-galleries given is [GaLi] for arbitrary dominant weights λ is more complicated. They are defined as certain collections of faces of alcoves that satisfy several conditions, including some positivity and dimension conditions. The companion of such a gallery is a chain in the Bruhat order on the quotient W/W_λ. For regular weights, the definition of LS-galleries from [GaLi] is equivalent to the simplified definition above.

It was shown in [GaLi] that, for a minimal gallery γ of weight λ,

$$ch(V_\lambda) = \sum_{\gamma' \in \Gamma_{LS}(\gamma)} e^{\mathrm{weight}(\gamma')}.$$

Let us now clarify the relationship between Corollary 18.7.(1) and this statement.

Let us say that a gallery of $\gamma = (F_0, A_0, F_1, \ldots, A_l, F_{l+1})$ is *special* if $l \geq N = |\Phi^+|$ (the number of positive roots) and all alcoves $A_0, \ldots, A_N$ and faces $F_1, \ldots, F_N$ are adjacent to the origin 0. Let us define the transformation

$$t\,:\,\{\text{special galleries of weight } -\mu\} \longrightarrow \{\text{galleries of weight } \mu\}.$$

For a special gallery $\gamma = (F_0, A_0, F_1, \ldots, A_l, F_{l+1})$ of weight $-\mu$, the gallery $t(\gamma)$ is defined as follows: (1) remove the first N alcoves $A_0, \ldots, A_{N-1}$ from the gallery γ together with the faces $F_1, \ldots, F_N$; (2) translate all remaining alcoves and faces by the weight μ; (3) reverse the sequence of alcoves and faces in the gallery. In other words,

$$t\,:\,(F_0, A_0, \ldots, A_l, F_{l+1}) \longmapsto (F_{l+1} + \mu, A_l + \mu, \ldots, F_{N+1} + \mu, A_N + \mu, F_0 + \mu),$$

If $\gamma = (F_0, A_0, F_1, \ldots, A_l, F_{l+1})$ is a special reduced gallery of weight $-\lambda$ (Definition 18.2), then $A_N = w_\circ(A_\circ)$ and $F_i \subset H_{\beta_i,0}$, for $i = 1, \ldots, N$. All foldings of γ are also special. The image $t(\gamma)$ of γ is a minimal gallery of weight λ. Moreover, all minimal galleries are of this form. Notice that, for a regular weight λ, we can always find a special reduced gallery of weight $-\lambda$.

Proposition 18.9. *Let γ be a special reduced gallery of weight $-\lambda$, where λ is a regular weight. Then the map $\gamma' \mapsto t(\gamma')$ is a bijection between the set of admissible foldings of γ and the set $\Gamma_{LS}(t(\gamma))$ of LS-galleries associated with $t(\gamma)$. Moreover, we have* $\mathrm{weight}(t(\gamma')) = -\mathrm{weight}(\gamma')$.

The proof of this proposition is based on the following fundamental (and nontrivial) result, which expresses the *EL-shellability* of the Bruhat order on a Weyl group, and is closely related to the Verma theorem [Ver]. This result was proved for an arbitrary Coxeter group in [Dyer, Proposition 4.3]. We also refer to [BFP, Theorem 6.4] for a new approach and a different generalization. Recall that *reflection orderings* [Hum, Dyer] are total orders on roots in Φ^+ that are associated with reduced decompositions $w_\circ = s_{i_1} \dots s_{i_N}$ for $w_\circ$, as follows:

$$\alpha_{i_N} < s_{i_N}(\alpha_{i_{N-1}}) < \dots < s_{i_N} s_{i_{N-1}} \dots s_{i_2}(\alpha_{i_1})\,.$$

Proposition 18.10. [Dyer, BFP] *Fix a reflection ordering $\beta_1 < \cdots < \beta_N$. For any Weyl group element w, there is a unique saturated increasing chain in Bruhat order from 1 to w of the form*

$$1 \lessdot s_{\beta_{j_1}} \lessdot s_{\beta_{j_1}} s_{\beta_{j_2}} \lessdot \dots \lessdot s_{\beta_{j_1}} \dots s_{\beta_{j_p}} = w\,, \tag{18.1}$$

where $1 \le j_1 < \dots < j_p \le N$.

Proof of Proposition 18.9. Let γ' be an arbitrary admissible folding of γ. Every tail-flip operator f_j preserves the type of γ', that is, $\mathrm{type}(\gamma') = \mathrm{type}(f_j(\gamma'))$, and changes its weight by a multiple of a root. Hence, the transformation t applied to γ' can be viewed as a composition of the translation by λ with a translation by an element of the root lattice. Note that the second translation is an element of W_{aff}. Recalling that γ is mapped to $t(\gamma)$ via the translation by λ, we conclude that the gallery $t(\gamma')$ has the same type as $t(\gamma)$.

Let us now examine the companion of $t(\gamma')$. Let $r_1, \dots, r_l$ and $r'_1, \dots, r'_l$ be the affine reflections with respect to the faces of γ and γ', respectively. Let p be such that $j_p \le N$ and $j_{p+1} > N$. Assume that $\gamma' = f_{j_1} \cdots f_{j_s}(\gamma)$, where $j_1 < \cdots < j_s$, so

$$1 \lessdot \bar r_{j_1} \lessdot \bar r_{j_1} \bar r_{j_2} \lessdot \cdots \lessdot \bar r_{j_1} \bar r_{j_2} \cdots \bar r_{j_s}$$

is a saturated decreasing chain in the Bruhat order. The companion of $t(\gamma')$ is the sequence

$$(u_0 = \bar r_{j_1} \dots \bar r_{j_s},\, \bar r'_{j_s} u_0,\, \bar r'_{j_{s-1}} \bar r'_{j_s} u_0,\, \dots,\, \bar r'_{j_{p+1}} \dots \bar r'_{j_s} u_0)\,.$$

But since $r'_{j_1} r'_{j_2} \cdots r'_{j_i} = (r_{j_1} r_{j_2} \cdots r_{j_i})^{-1}$, for $i = 1, \dots, s$ (see the proof of Lemma 18.6), the companion of $t(\gamma')$ is the sequence

$$(\bar r_{j_1} \dots \bar r_{j_s},\, \bar r_{j_1} \dots \bar r_{j_{s-1}},\, \dots,\, \bar r_{j_1} \dots \bar r_{j_p})\,,$$

which is a saturated decreasing chain in Bruhat order. We have thus shown that the image of map t is contained in $\Gamma_{LS}(\gamma)$.

It suffices to construct the inverse map. Recall that the first N faces F_i of Γ satisfy $F_i \subset H_{\beta_i,0}$. This gives a reflection ordering $\beta_1 < \cdots < \beta_N$, according to Lemma 5.3. Given a gallery γ'' in $\Gamma_{LS}(t(\gamma))$, assume that its companion ends at some w in W. According to Proposition 18.10, there is a unique way of writing $w = s_{\beta_{j_1}} \dots s_{\beta_{j_p}}$ for $1 \le j_1 < \dots < j_p \le N$, such that (18.1) holds.

Let us now relabel the faces of γ'' as follows: $(F'_{l+1}, A'_l, F'_l, A'_{l-1}, F'_{l-1}, \dots)$. Let $\{j_{p+1} < \dots < j_s\} = \{j \mid A'_{j-1} = A'_j\}$. We associate with γ'' the gallery $f_{j_1} \dots f_{j_p} f_{j_{p+1}} \dots f_{j_s}(\gamma)$. The facts stated above imply that this construction gives the inverse map to t. □

Remark 18.11. (i) For a nonregular weight λ, it is not clear how to associate LS-galleries with our admissible foldings.

(ii) According to [GaLi], one can associate a collection of continuous piecewise-linear Littelmann paths with the set of LS-galleries $\Gamma_{LS}(\gamma)$ by connecting the centers of the lower dimensional faces in the galleries. In [LePo], we show that we do not obtain Littelmann paths by applying the same procedure (or similar ones) to our model.

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