ALCOVE PATH AND NICHOLS-WORONOWICZ MODEL OF THE EQUIVARIANT K-THEORY OF GENERALIZED FLAG VARIETIES

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Abstract. Fomin and Kirillov initiated a line of research into the realization of the cohomology and K-theory of generalized flag varieties G/B as commutative subalgebras of certain noncommutative algebras. This approach has several advantages, which we discuss. This paper contains the most comprehensive result in a series of papers related to the mentioned line of research. More precisely, we give a model for the T-equivariant K-theory of a generalized flag variety $K_T(G/B)$ in terms of a certain braided Hopf algebra called the Nichols-Woronowicz algebra. Our model is based on the Chevalley-type multiplication formula for $K_T(G/B)$ due to the first author and Postnikov; this formula is stated using certain operators defined in terms of so-called alcove paths (and the corresponding affine Weyl group). Our model is derived using a type-independent and concise approach.

Dedicated to Professor Kenji Ueno on the occasion of his sixtieth birthday

Introduction

Schubert calculus evolved from the calculus of enumerative geometry to the study of geometric, algebraic, and combinatorial aspects related to various cohomology settings for algebraic homogeneous spaces. An important problem in Schubert calculus is to find a combinatorial description of the cohomology of generalized flag varieties G/B (where G is a semisimple Lie group and B a Borel subgroup); more general algebras were also considered, such as K-theory and quantum cohomology.

Fomin and Kirillov [5] initiated a line of research into the realization of the mentioned algebras as commutative subalgebras of certain noncommutative algebras. The idea is to map certain cohomology classes to elements of the noncommutative algebra which are thought of as multiplication operators acting on cohomology. This approach has several advantages. First of all, it allows us to recursively construct the basis of Schubert classes by using a bottom-up approach, rather than the traditional top-down approach based on *divided difference operators* (see [5]). Secondly, given that these constructions have certain multiplication operators (related to the *Chevalley multiplication* formula) built into them, they are readily amenable to deriving more general multiplication formulas. This approach was succesfully used in [20], where a *Pieri-type multiplication* formula in the cohomology and quantum cohomology of the flag variety Fl_n of type A was derived using the Fomin-Kirillov construction. Thirdly, in certain cases, the divided difference operators in cohomology correspond to some natural operators acting on the noncommutative algebra (e.g., certain twisted derivations in the case of the Nichols-Woronowicz algebra, see below).

We will now give more details about the line of research initiated by Fomin and Kirillov. They constructed a combinatorial model for the cohomology of the flag variety Fl_n as a subalgebra of a certain noncommutative algebra $\mathcal{E}_n$ defined by quadratic relations. More precisely, $\mathcal{E}_n$ is an algebra generated over $\mathbf{Z}$ by the generators $[i,j]$, $1 \le i \ne j \le n$, subject to the following relations:

(0) $[i,j] = -[j,i]$,
(1) $[i,j]^2 = 0$,

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(2) $[i,j][k,l]=[k,l][i,j]$, if $\{i,j\}\cap\{k,l\}=\emptyset$,
(3) $[i,j][j,k]+[j,k][k,i]+[k,i][i,j]=0$.

Fomin and Kirillov defined the commuting family of *Dunkl elements* $\theta_1,\ldots,\theta_n$ in $\mathcal{E}_n$ by

$$\theta_i := \sum_{j\neq i}[i,j],$$

and proved that the subalgebra generated by them is isomorphic to the cohomology ring of Fl_n. It is remarkable that the algebra $\mathcal{E}_n$ has a natural quantum deformation. The deformed algebra $\widetilde{\mathcal{E}}_n$ is an algebra over the polynomial ring $\mathbb{Z}[q_1,\ldots,q_{n-1}]$ obtained by replacing relation (1) above by the relation

($1'$) $[i,j]^2=0$, for $j>i+1$, and $[i,i+1]^2=q_i$.

It was conjectured in [5] that the subalgebra of $\widetilde{\mathcal{E}}_n$ generated by the Dunkl elements is isomorphic to the quantum cohomology ring $QH^*(Fl_n)$. This conjecture was proved by Postnikov [20]. A generalization to other root systems of the Fomin-Kirillov quadratic algebra and of the realization of $H^*(Fl_n)$ inside it is given in [9]. A realization of the K-theory of Fl_n as a commutative subalgebra of $\mathcal{E}_n$ was conjectured in [15, 17] and proved in [11]. This realization is based on the so-called multiplicative Dunkl elements in [15, 17, 21]; the latter were proved, in [14], to describe the Chevalley-type (or Monk-type) multiplication formula for *Grothendieck polynomials* (which are polynomial representatives for Schubert classes in $K(Fl_n)$).

The Hopf algebra structure related to the algebra $\mathcal{E}_n$ was studied by Fomin and Procesi [6]. The relationship between the algebra $\mathcal{E}_n$ and a braided Hopf algebra called the *Nichols-Woronowicz algebra* was pointed out by Milinski and Schneider [19]. Conjecturally, the algebra $\mathcal{E}_n$ is isomorphic to the Nichols-Woronowicz algebra $\mathcal{B}(V_{S_n})$ associated to a *Yetter-Drinfeld module* V_{S_n} over the symmetric group S_n. More generally, given an arbitary finite Coxeter group W, one can consider the Nichols-Woronowicz algebra $\mathcal{B}(V_W)$ associated to the Yetter-Drinfeld module V_W over W.

Bazlov [2] constructed a model for the coinvariant algebra of a finite Coxeter group W as a subalgebra of the corresponding Nichols-Woronowicz algebra $\mathcal{B}(V_W)$. The natural braided differential operators acting on the Nichols-Woronowicz algebra play a key role in his argument. In [10], Kirillov and the second author constructed a model for the small quantum cohomology ring $QH^*(G/B)$ in terms of the Nichols-Woronowicz algebra. In [11], Kirillov and the second author constructed a model for $K(G/B)$ in $\mathcal{B}(V_W)$ when the Lie group G is of classical type or of type G_2.

The purpose of this paper is to construct a model for the T-equivariant K-theory of an arbitrary generalized flag variety $K_T(G/B)$ as a subalgebra of the Nichols-Woronowicz algebra $\mathcal{B}(V_W)$ associated to the Yetter-Drinfeld module V_W over the corresponding Weyl group W. Our model is based on the Chevalley-type multiplication formula for $K_T(G/B)$ in [16]; this formula is stated using certain operators defined in terms of so-called *alcove paths* (and the corresponding affine Weyl group). We define the *multiplicative Dunkl elements* in $\mathcal{B}(V_W)$ based on the path operators in [16], and show that the commutative subalgebra generated by them is isomorphic to the K-ring $K(G/B)$; this result is then extended to equivariant K-theory. Hence, the present paper extends the results in [2, 5, 9, 11], and contains the most comprehensive result related to the cohomology and K-theory of G/B in the line of research initiated by Fomin and Kirillov. Being based on the Nichols-Woronowicz algebra and the path operators in [16], our model uses a type-independent approach, which also has the advantage of being concise. The quantization of our model, which is expected to give a model for the *quantum K-ring* of the flag variety G/B [7, 13], will be discussed in [12].

We now offer a preview of the path operators on which our construction is based. Let P be the weight lattice in the Cartan subalgebra $\mathfrak{h}$ of the Lie algebra of the simple Lie group G. Denote by T the maximal torus in G, and by $R(T)$ the representation ring of T. The K-ring $K(G/B)$ has a presentation as a quotient of the group algebra of the weight lattice P; a similar description can be given for the T-equivariant K-theory $K_T(G/B)$, as follows. Let us consider an algebra isomorphism

$\iota : \mathbb{Z}[P] = \bigoplus_{\lambda\in P} \mathbb{Z}\cdot e^{\lambda} \to R(T)$ such that $\iota(e^{\lambda})$ is the character χ^{λ} corresponding to the weight λ. The T-equivariant K-algebra $K_T(G/B)$ is isomorphic to the quotient algebra $R(T)\otimes\mathbb{Z}[P]/J$, where J is the ideal $(1\otimes f - \iota(f)\otimes 1,\ \ f\in\mathbb{Z}[P]^W)$. The first author and Postnikov [16] introduced the path operator $R^{[\lambda]}$ acting on $K_T(G/B)$ in order to derive a Chevalley-type formula in $K_T(G/B)$ which describes the multiplication by the class of the line bundle $\mathcal{L}_\lambda$ on G/B associated to the weight λ. The path operator $R^{[\lambda]}$ is defined by using the alcove path $A_0 \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_l} A_l$ which connects the fundamental alcove $A_\circ$ with its translation $A_{-\lambda} := A_\circ - \lambda$. The operator $R^{[\lambda]}$ is defined as the composition

$$R^{[\lambda]} = (1+B_{\beta_l})\cdots(1+B_{\beta_1}),$$

where B_β is the *Bruhat operator* studied by Brenti, Fomin and Postnikov [3]. Our construction of the model for $K_T(G/B)$ in $\mathcal{B}(V_W)$ is based on the operators $R^{[\lambda]}$.

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1. Nichols-Woronowicz algebra

The *Nichols-Woronowicz algebra* associated to a braided vector space is an analog of the polynomial ring in a braided setting. For details on the Nichols-Woronowicz algebra, see [1, 2, 18].

Let (V,ψ) be a braided vector space, i.e. a vector space equipped with a linear isomorphism $\psi : V\otimes V \to V\otimes V$ such that the braid relation $\psi_i\psi_{i+1}\psi_i = \psi_{i+1}\psi_i\psi_{i+1}$ is satisfied on $V^{\otimes n}$; here ψ_i is the linear endomorphism on $V^{\otimes n}$ obtained by applying ψ on the i-th and $(i+1)$-st components. Throughout this paper, we assume that V is a finite dimensional $\mathbb{Q}$-vector space. Let $w = s_{i_1}\dots s_{i_l}$ be a reduced decomposition of an element $w\in S_n$, where $s_i = (i, i+1)$ is an adjacent transposition. Then the linear map $\Psi_w := \psi_{i_1}\dots\psi_{i_l}$ on $V^{\otimes n}$ is independent of the choice of a reduced decomposition of w due to the braid relations. We define the *Woronowicz symmetrizer* on $V^{\otimes n}$ by $\sigma_n(\psi) := \sum_{w\in S_n}\Psi_w$. Such a definition of the braided analog of the symmetrizer (or anti-symmetrizer) is due to Woronowicz [22].

Definition 1.1. (see [2] and [18]) *The Nichols-Woronowicz algebra $\mathcal{B}(V)$ associated to the braided vector space (V,ψ) is defined as the quotient of the tensor algebra $T(V)$ by the ideal $\bigoplus_{n\geq 0}\mathrm{Ker}(\sigma_n(\psi))$.*

Remark 1.2. For a more systematic treatment, we need to work in a fixed braided category $\mathcal{C}$ of vector spaces. If the braided vector space (V,ψ) is an object in the braided category $\mathcal{C}$, the tensor algebra $T(V)$ has a natural braided Hopf algebra structure in $\mathcal{C}$. It is known that the kernel $\bigoplus_{n\geq 0}\mathrm{Ker}(\sigma_n(\psi))$ is a Hopf ideal of $T(V)$. Hence, $\mathcal{B}(V)$ is also a braided Hopf algebra in $\mathcal{C}$.

The following is the alternative definition of the Nichols-Woronowicz algebra due to Andruskiewitsch and Schneider [1]. In [1], the algebra $\mathcal{B}(V)$ is called the Nichols algebra.

Definition 1.3. [1] *The graded braided Hopf algebra $\mathcal{B}(V)$ is called the Nichols-Woronowicz algebra if it satisfies the following conditions:*

1. $\mathcal{B}_0(V) = \mathbb{Q}$,
2. $V = \mathcal{B}_1(V) = \{x\in\mathcal{B}(V) \mid \triangle(x) = x\otimes 1 + 1\otimes x\}$,
3. *$\mathcal{B}(V)$ is generated by $\mathcal{B}_1(V)$ as a $\mathbb{Q}$-algebra.*

We use a particular braided vector space called the *Yetter-Drinfeld module* in the subsequent construction. Let Γ be a finite group.

Definition 1.4. *A $\mathbb{Q}$-vector space V is called a Yetter-Drinfeld module over Γ if*

1. *V is a Γ-module,*
2. *V is Γ-graded, i.e. $V = \bigoplus_{g\in\Gamma} V_g$, where V_g is a linear subspace of V,*
3. *for $h\in\Gamma$ and $v\in V_g$, we have $h(v)\in V_{hgh^{-1}}$.*

The category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ of the Yetter-Drinfeld modules over a fixed group Γ is naturally braided. The tensor product of the objects V and V' of ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ is again a Yetter-Drinfeld module with the Γ-action $g(v \otimes w) = g(v) \otimes g(w)$ and the Γ-grading $(V \otimes V')_g = \bigoplus_{h,h' \in \Gamma,\, hh'=g} V_h \otimes V'_{h'}$. The braiding between V and V' is defined by $\psi_{V,V'}(v \otimes w) = g(w) \otimes v$, for $v \in V_g$ and $w \in V'$.

Fix a Borel subgroup B in a simple Lie group G. Let Δ be the set of roots, and Δ_+ the set of the positive roots corresponding to B. We define a Yetter-Drinfeld module

$$V_W := \bigoplus_{\alpha \in \Delta} \mathbb{Q} \cdot [\alpha]/([\alpha] + [-\alpha])$$

over the Weyl group W. The W-action on V_W is given by $w([\alpha]) = [w(\alpha)]$. The W-degree of the symbol $[\alpha]$ is the reflection s_α. Note that $[\alpha]^2 = 0$ for all $\alpha \in \Delta$, in the associated Nichols-Woronowicz algebra $\mathcal{B}(V_W)$. It is also easy to see that $[\alpha][\beta] = [\beta][\alpha]$ when $s_\alpha s_\beta = s_\beta s_\alpha$. The following proposition can be shown by checking the quadratic relations in $\mathcal{B}(V_{S_n})$ via direct computation of the symmetrizer.

Proposition 1.5. *Fix the standard orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{Q}^n$. Let $\Delta = \{e_i - e_j \mid 1 \le i \ne j \le n\}$ be the root system of type A_{n-1}. Then there exists a surjective algebra homomorphism*

$$\begin{array}{rccc} \eta: & \mathcal{E}_n & \to & \mathcal{B}(V_{S_n}) \\ & [i,j] & \mapsto & [e_i - e_j]. \end{array}$$

Conjecture 1.6. The algebra homomorphism η is an isomorphism.

This conjecture is now confirmed up to $n = 6$.

Take the standard realization of a root system of rank two with respect to an orthonormal basis $\{e_i\}$ as follows:

$(A_1 \times A_1) : \Delta_+^{A_1 \times A_1} = \{a_1 = e_1,\ a_2 = e_2\}$,

$(A_2) : \Delta_+^{A_2} = \{a_1 = e_1 - e_2,\ a_2 = e_1 - e_3,\ a_3 = e_2 - e_3\}$,

$(B_2) : \Delta_+^{B_2} = \{a_1 = e_1 - e_2,\ a_2 = e_1,\ a_3 = e_1 + e_2,\ a_4 = e_2\}$,

$(C_2) : \Delta_+^{C_2} = \{a_1 = e_1 - e_2,\ a_2 = 2e_1,\ a_3 = e_1 + e_2,\ a_4 = 2e_2\}$,

$(G_2) : \Delta_+^{G_2} = \{a_1 = e_1 - e_2,\ a_2 = e_1 - 2e_2 + e_3,\ a_3 = -e_2 + e_3,\ a_4 = -e_1 - e_2 + 2e_3,\ a_5 = -e_1 + e_3,\ a_6 = -2e_1 + e_2 + e_3\}$.

If the set Δ of roots contains a subset of the form Δ_+^X, where $X = A_1 \times A_1$, A_2, B_2, C_2, or G_2, then one can check that the following relations are satisfied in the algebra $\mathcal{B}(V_W)$, respectively (see also [11]).

$(A_1 \times A_1) : [a_1][a_2] = [a_2][a_1]$,

$(A_2) : [a_1][a_2] + [a_2][a_3] = [a_3][a_1]$,

$(B_2, C_2) : [a_1][a_2] + [a_2][a_3] + [a_3][a_4] = [a_4][a_1]$,

$$[a_1][a_2][a_3][a_2] + [a_2][a_3][a_2][a_1] + [a_3][a_2][a_1][a_2] + [a_2][a_1][a_2][a_3] = 0,$$

$$[a_2][a_3][a_4][a_3] + [a_3][a_4][a_3][a_2] + [a_4][a_3][a_2][a_3] + [a_3][a_2][a_3][a_4] = 0,$$

$$[a_1][a_2][a_3][a_4] = [a_4][a_3][a_2][a_1],$$

$$
\begin{aligned}
(G_2):\ & [a_1][a_2]+[a_2][a_3]+[a_3][a_4]+[a_4][a_5]+[a_5][a_6]=[a_6][a_1],\\
& [a_1][a_2][a_1][a_3]+[a_1][a_3][a_1][a_2]+[a_1][a_3][a_2][a_3]=\\
& =[a_2][a_1][a_3][a_1]+[a_3][a_2][a_3][a_1]+[a_3][a_1][a_2][a_1],\\
& [a_6][a_5][a_6][a_4]+[a_6][a_4][a_6][a_5]+[a_6][a_4][a_5][a_4]=\\
& =[a_5][a_6][a_4][a_6]+[a_4][a_5][a_4][a_6]+[a_4][a_6][a_5][a_6],\\
& [a_1][a_2][a_4][a_5]+[a_2][a_3][a_4][a_5]+[a_2][a_3][a_5][a_6]+[a_5][a_3][a_4][a_2]=\\
& =[a_3][a_4][a_2][a_3]+[a_3][a_4][a_3][a_4]+[a_4][a_5][a_3][a_4]+[a_6][a_4][a_3][a_1],\\
& [a_5][a_4][a_2][a_1]+[a_5][a_4][a_3][a_2]+[a_6][a_5][a_3][a_2]+[a_2][a_4][a_3][a_5]=\\
& =[a_3][a_2][a_4][a_3]+[a_4][a_3][a_4][a_3]+[a_4][a_3][a_5][a_4]+[a_1][a_3][a_4][a_6],\\
& [a_1][a_2][a_3][a_4][a_5][a_6]=[a_6][a_5][a_4][a_3][a_2][a_1].
\end{aligned}
$$

The complete set of the independent defining relations for the algebra $\mathcal{B}(V_W)$ has not yet been determined in general. The relations listed above imply the following.

Proposition 1.7. *The elements $h_\alpha := 1 + [\alpha]$, $\alpha \in \Delta$, satisfy the Yang-Baxter equations, i.e., if Δ contains a subset Δ' of the form Δ_+^X, where $X = A_1 \times A_1$, A_2, B_2, C_2, or G_2, then the elements h_α, $\alpha \in \Delta'$, satisfy the following equations, respectively:*

$(A_1 \times A_1):\ h_{a_1}h_{a_2} = h_{a_2}h_{a_1}$,

$(A_2):\ h_{a_1}h_{a_2}h_{a_3} = h_{a_3}h_{a_2}h_{a_1}$,

$(B_2, C_2):\ h_{a_1}h_{a_2}h_{a_3}h_{a_4} = h_{a_4}h_{a_3}h_{a_2}h_{a_1}$,

$(G_2):\ h_{a_1}h_{a_2}h_{a_3}h_{a_4}h_{a_5}h_{a_6} = h_{a_6}h_{a_5}h_{a_4}h_{a_3}h_{a_2}h_{a_1}$.

For a general braided vector space V, the elements $v \in V$ act on the algebra $\mathcal{B}(V^*)$ as braided differential operators. In the subsequent construction, we use the braided differential operator D_α for a positive root α, whose action on the algebra $\mathcal{B}(V_W)$ is determined by the following conditions:

(0) $D_\alpha(t) = 0$, for $t \in \mathcal{B}_0(V_W) = \mathbb{Q}$,
(1) $D_\alpha([\beta]) = \delta_{\alpha,\beta}$, for $\alpha, \beta \in \Delta_+$,
(2) $D_\alpha(FF') = D_\alpha(F)F' + s_\alpha(F)D_\alpha(F')$ for $F, F' \in \mathcal{B}(V_W)$.

We set $D_\alpha := -D_{-\alpha}$ if α is a negative root. The following is a key lemma in the proof of the main theorem.

Lemma 1.8. (see [19, Proposition 2.4] and [2, Criterion 3.2]) *We have*

$$\bigcap_{\alpha\in\Delta_+} \mathrm{Ker}(D_\alpha) = \mathcal{B}_0(V_W) = \mathbb{Q}.$$

2. Alcove path and multiplicative Chevalley elements

In this section, we define a family of elements Ξ^λ, $\lambda \in P$, in the Nichols-Woronowicz algebra $\mathcal{B}(V_W)$ based on the construction of the path operators due to the first author and Postnikov [16].

Let W_{aff} be the affine Weyl group of the dual root system $\Delta^\vee := \{\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle \mid \alpha \in \Delta\}$. The affine Weyl group W_{aff} is generated by the affine reflections $s_{\alpha,k}$, $\alpha \in \Delta$, $k \in \mathbb{Z}$, with respect to the affine hyperplanes $H_{\alpha,k} := \{\lambda \in \mathfrak{h}^* \mid \langle\lambda, \alpha^\vee\rangle = k\}$. The connected components of $\mathfrak{h}^* \setminus \bigcup_{\alpha\in\Delta, k\in\mathbb{Z}} H_{\alpha,k}$ are called alcoves. The fundamental alcove $A_\circ$ is the alcove defined by the inequalities $0 < \langle\lambda, \alpha^\vee\rangle < 1$, for all $\alpha \in \Delta_+$. We now recall some concepts from [16].

Definition 2.1. [16] (1) *A sequence $(A_0, \dots, A_l)$ of alcoves A_i is called an* alcove path *if A_i and A_{i+1} have a common wall and $A_i \neq A_{i+1}$.*

(2) *An alcove path* $(A_0, \ldots, A_l)$ *is called reduced if the length* l *of the path is minimal among all alcove paths connecting* A_0 *and* A_l.

(3) *We use the symbol* $A_i \xrightarrow{\beta} A_{i+1}$ *when* A_i *and* A_{i+1} *have a common wall of the form* $H_{\beta,k}$ *and the direction of the root* β *is from* A_i *to* A_{i+1}.

Let $\{\alpha_1, \ldots, \alpha_r\} \subset \Delta_+$ be the set of the simple roots. Let ω_i be the fundamental weight corresponding to the simple root α_i, i.e. $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$. Take an alcove path $A_0 \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_l} A_l$ connecting $A_0 = A_\circ$ and $A_l = A_{-\lambda} := A_\circ - \lambda$. The sequence of roots $(\beta_1, \ldots, \beta_l)$ appearing here is called a λ-*chain*.

Definition 2.2. *We define the elements* $\Xi^{[\lambda]}$ *in* $\mathcal{B}(V_W)$ *for* $\lambda \in P$ *by the formula*

$$\Xi^{[\lambda]} = h_{\beta_l} \ldots h_{\beta_1}.$$

We call the elements $\Xi_i := \Xi^{\omega_i}$ *the multiplicative Chevalley elements.*

The element $\Xi^{[\lambda]}$ is independent of the choice of an alcove path from $A_\circ$ to $A_{-\lambda}$. Indeed, the elements h_α satisfy the Yang-Baxter equations and $h_\alpha h_{-\alpha} = 1$, so the argument of [16, Lemma 9.3], which is implicitly given by Cherednik [4], is applicable to our case.

We use the following results from [16].

Lemma 2.3. ([16, Lemmas 12.3 and 12.4]) *Let* $A_0 \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_l} A_l$ *be an alcove path from* $A_0 = A_\circ$ *to* $A_l = A_{-\lambda}$.

(1) *The sequence* $(\alpha_i, s_{\alpha_i}(\beta_1), \ldots, s_{\alpha_i}(\beta_l), -\alpha_i)$ *is an* $s_{\alpha_i}(\lambda)$*-chain for* $i = 1, \ldots, r$.

(2) *Asume that* $\beta_j = \pm\alpha_i$ *for some* $1 \le j \le l$ *and* $1 \le i \le r$. *Denote by* s *the reflection with respect to the common wall of* A_{l-j} *and* A_{l-j+1}. *Then the sequence* $(\alpha_i, s_{\alpha_i}(\beta_1), \ldots, s_{\alpha_i}(\beta_{j-1}), \beta_{j+1}, \ldots, \beta_l)$ *is an* $s(\lambda)$*-chain.*

Proposition 2.4. ([16, Proposition 12.2]) *We have* $\Xi^{[\lambda]} \cdot \Xi^{[\lambda']} = \Xi^{[\lambda'+\lambda]}$, *for* $\lambda, \lambda' \in P$. *In particular, the elements* $\Xi^{[\lambda]}$ *and* $\Xi^{[\lambda']}$ *commute.*

3. Main result

Consider the *Demazure operator* $\pi_i : \mathbb{Z}[P] \to \mathbb{Z}[P]$ corresponding to the simple root α_i, which is defined by the formula

$$\pi_i(f) := \frac{f - s_{\alpha_i}(f)}{e^{\alpha_i} - 1}.$$

The operator π_i is characterized by the following conditions:

(1) $\pi_i(e^{\omega_j}) = \delta_{i,j} e^{\omega_i - \alpha_i}$,
(2) $\pi_i(fg) = \pi_i(f)g + s_{\alpha_i}(f)\pi_i(g)$.

We have an algebra homomorphism

$$\begin{array}{cccc} \varphi: & \mathbb{Z}[P] & \to & \mathbb{Z}[\Xi_1, \ldots, \Xi_r] \\ & e^{\omega_i} & \mapsto & \Xi_i. \end{array}$$

Proposition 3.1. *Let* f *be an element in* $\mathbb{Z}[P]$. *We have*

$$\Pi_i(\varphi(f)) = \varphi(\pi_i f),$$

where $\Pi_i = h_{\alpha_i}^{-1} \circ D_{[\alpha_i]}$.

Proof. It is enough to check that the operator Π_i satisfies the following conditions:

(1) $\Pi_i(\Xi_j) = \delta_{i,j} \Xi^{[\omega_i - \alpha_i]}$,

(2) $\Pi_i(FF') = \Pi_i(F)F' + s_{\alpha_i}(F)\Pi_i(F')$, for $F, F' \in \mathbb{Z}[\Xi_1, \ldots, \Xi_r]$.

Let $t_i = t_{-\omega_j} \in W_{\mathrm{aff}}$ be the translation by $-\omega_i$. Since the hyperplane of the form $H_{\alpha_j,k}$, $j \neq i$, does not separate the alcoves $A_\circ$ and $t_i^{-1}(A_\circ)$, the roots $\pm\alpha_j$, $j \neq i$, cannot appear as a component of the ω_i-chain $(\beta_1, \ldots, \beta_l)$ corresponding to a reduced alcove path $A_0 \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_l} A_l$ from $A_0 = A_\circ$ to $A_l = t_i(A_\circ)$ (see [8, Chapter 4]). Hence, we have $\Pi_i(\Xi_j) = 0$ if $j \neq i$. Based on Lemma 2.3 (2), we also have $\Pi_i(\Xi_i) = \Xi^{[\omega_i - \alpha_i]}$, so condition (1) follows.

Pick an element $f \in \mathbb{Z}[P]$ such that $\varphi(f) = F$. We have

$$\Pi_i(FF') = h_{\alpha_i}^{-1} D_{[\alpha]}(F)F' + h_{\alpha_i}^{-1} \cdot \varphi(s_{\alpha_i}(f)) \cdot h_{\alpha_i} \cdot h_{\alpha_i}^{-1} D_{[\alpha]}(F')$$

by applying the twisted Leibniz rule for $D_{[\alpha]}$. From Lemma 2.3 (1), one can see that $h_{\alpha_i}^{-1} \cdot \varphi(s_{\alpha_i}(f)) \cdot h_{\alpha_i} = s_{\alpha_i}(F)$. So condition (2) is satisfied. □

Theorem 3.2. *The subalgebra $\mathbb{Z}[\Xi_1, \ldots, \Xi_r]$ generated by the multiplicative Chevalley elements in the Nichols-Woronowicz algebra $\mathcal{B}(V_W)$ is isomorphic to the K-ring $K(G/B)$.*

Proof. Define the algebra homomorphism $\varepsilon : \mathbb{Z}[P] \to \mathbb{Z}$ by the assignment $e^\lambda \mapsto 1$, for all $\lambda \in P$. Let $I \subset \mathbb{Z}[P]$ be the ideal generated by the elements of the form $f - \varepsilon(f)$, $f \in \mathbb{Z}[P]^W$. Then the K-ring $K(G/B)$ is isomorphic to the quotient algebra $\mathbb{Z}[P]/I$. Pick an element $g \in I$. We have $\Pi_i(\varphi(g)) = \varphi(\pi_i(g)) = 0$ for $i = 1, \ldots, r$, by Proposition 3.1. Since $w \circ D_{[\alpha]} \circ w^{-1} = D_{[w(\alpha)]}$, we obtain $D_{[\alpha]}(\varphi(g)) = 0$, for all $\alpha \in \Delta_+$, and therefore $\varphi(g) = 0$ by Lemma 1.8. If $g \notin I$, there exists an operator ϖ on $\mathbb{Z}[P]$, written as a linear combination of the composites of the multiplication operators and the operators π_i, such that the constant term of $\varpi(g)$ is nonzero. We conclude that $\mathrm{Im}(\varphi) \cong \mathbb{Z}[P]/I \cong K(G/B)$. □

Remark 3.3. (1) The idea of the proof of the above theorem is used in [11, Sections 5 and 6] for the root systems of classical type and of type G_2. The multiplicative Dunkl elements $\Theta_i := \Xi^{[e_i]}$ corresponding to the components of the orthonormal basis $\{e_i\}$ are used in [11]. The multiplicative Dunkl elements in the Fomin-Kirillov quadratic algebra $\mathcal{E}_n$ appear in [15, 17, 21].

(2) For an arbitrary parabolic subgroup $P \supset B$, the K-ring $K(G/P)$ of the homogeneous space G/P is a subalgebra of $K(G/B)$. Hence, the algebra $\mathcal{B}(V_W)$ also contains $K(G/P)$ as a commutative subalgebra.

Bazlov [2] has proved that the subalgebra in $\mathcal{B}(V_W)$ generated by the elements $[\alpha]$ corresponding to the simple roots α is isomorphic to the nil-Coxeter algebra

$$NC_W := \mathbb{Z}\langle u_1, \ldots, u_r \rangle / (u_i^2,\ (u_i u_j)^{[m_{ij}/2]} u_i^{\nu_{ij}} - (u_j u_i)^{[m_{ij}/2]} u_j^{\nu_{ij}},\ i = 1, \ldots, r),$$

where m_{ij} is the order of $s_{\alpha_i} s_{\alpha_j}$ in W, $[m_{ij}/2]$ stands for the integer part of $m_{ij}/2$, and $\nu_{ij} := m_{ij} - 2[m_{ij}/2]$. In our case, we can show the following.

Corollary 3.4. *The subalgebra in* $\mathrm{End}(\mathcal{B}(V_W))$ *generated by the operators* $\Pi_1, \ldots, \Pi_r$ *is isomorphic to the nil-Hecke algebra*

$$NH_W := \mathbb{Z}\langle T_1, \ldots, T_r \rangle / (T_i^2 + T_i,\ (T_i T_j)^{[m_{ij}/2]} T_i^{\nu_{ij}} - (T_j T_i)^{[m_{ij}/2]} T_j^{\nu_{ij}},\ i = 1, \ldots, r)$$

via the map given by $T_i \mapsto \Pi_i$.

Proof. One can check that the operators Π_i satisfy the defining relations of NH_W by direct computations. Since the map given by $T_i \mapsto \pi_i$ defines a faithful representation of NH_W on $\mathbb{Z}[P]/I$, the subalgebra generated by Π_i, $i = 1, \ldots, r$ is isomorphic to NH_W. □

4. Model of the equivariant K-ring

The results in the previous section are generalized to the case of the T-equivariant K-ring $K_T(G/B)$. Our construction of the model for $K_T(G/B)$ is also parallel to the approach in [16].

Since the Nichols-Woronowicz algebra $\mathcal{B}(V_W)$ is a braided Hopf algebra in the category of the Yetter-Drinfeld modules over W, it is W-graded. Denote by w_x the W-degree of a W-homogeneous element $x \in \mathcal{B}(V_W)$. Let h be the Coxeter number and $P' := h^{-1} \cdot P \subset \mathfrak{h}^*$. The Weyl group W acts on the group algebra $\mathbb{Z}[P'] = \bigoplus_{\lambda \in P'} X^\lambda$ by $w(X^\lambda) = X^{w(\lambda)}$, $w \in W$. The twist map

$$\begin{array}{rccc} c: & \mathcal{B}(V_W) \otimes \mathbb{Z}[P'] & \to & \mathbb{Z}[P'] \otimes \mathcal{B}(V_W) \\ & x \otimes X & \mapsto & w_x(X) \otimes x \end{array}$$

gives an associative multiplication map m on $\mathcal{B}(V_W)\langle P' \rangle := \mathbb{Z}[P'] \otimes \mathcal{B}(V_W)$ as follows:

$$\begin{aligned} m : \mathbb{Z}[P'] \otimes \mathcal{B}(V_W) \otimes \mathbb{Z}[P'] \otimes \mathcal{B}(V_W) &\xrightarrow{1 \otimes c \otimes 1} \mathbb{Z}[P'] \otimes \mathbb{Z}[P'] \otimes \mathcal{B}(V_W) \otimes \mathcal{B}(V_W) \\ &\xrightarrow{m_{\mathbb{Z}[P']} \otimes m_{\mathcal{B}}} \mathbb{Z}[P'] \otimes \mathcal{B}(V_W), \end{aligned}$$

where $m_{\mathbb{Z}[P']}$ and $m_{\mathcal{B}}$ are the multiplication maps on the algebras $\mathbb{Z}[P']$ and $\mathcal{B}(V_W)$, respectively. The algebras $\mathbb{Z}[P']$ and $\mathcal{B}(V_W)$ are considered as subalgebras of $\mathcal{B}(V_W)\langle P' \rangle$. We have the commutation relation

$$[\alpha] \cdot X^\lambda = X^{s_\alpha(\lambda)} \cdot [\alpha], \ \ \alpha \in \Delta, \ \lambda \in P',$$

in $\mathcal{B}(V_W)\langle P' \rangle$.

The subalgebra $\mathbb{Z}[P]^W \subset \mathbb{Z}[P']$ is viewed as a subalgebra of the representation ring $R(T)$ of the maximal torus via the isomorphism

$$\begin{array}{rccc} \iota: & \mathbb{Z}[P] & \to & R(T) \\ & e^\lambda & \mapsto & \chi^\lambda. \end{array}$$

Let us consider an $R(T)$-algebra $\mathcal{B}_T(V_W) := R(T) \otimes_{\mathbb{Z}[P]^W} \mathcal{B}(V_W)\langle P' \rangle$. We introduce the elements

$$H_\alpha := X^{\rho/h} \cdot (X^{\alpha/h} + [\alpha]) \cdot X^{-\rho/h}, \ \ \alpha \in \Delta, \ \rho := \frac{1}{2}\left(\sum_{\beta \in \Delta_+} \beta \right),$$

in the algebra $\mathcal{B}_T(V_W)$. Since the argument in the proof of [16, Theorem 10.1] is applicable to our case, Proposition 1.7 implies the following.

Lemma 4.1. The elements H_α, $\alpha \in \Delta$, satisfy the Yang-Baxter equations in the algebra $\mathcal{B}_T(V_W)$.

Let $(\beta_1, \ldots, \beta_l)$ be a λ-chain for a weight $\lambda \in P$. Define the element

$$\Xi_{\mathrm{eq}}^{[\lambda]} := H_{\beta_l} \ldots H_{\beta_1}$$

in $\mathcal{B}_T(V_W)$. The element $\Xi_{\mathrm{eq}}^{[\lambda]}$ is independent of the choice of the λ-chain from Lemma 4.1. We also have $\Xi_{\mathrm{eq}}^{[\lambda]} \cdot \Xi_{\mathrm{eq}}^{[\lambda']} = \Xi_{\mathrm{eq}}^{[\lambda+\lambda']}$ by [16, Proposition 12.2].

The braided differential operators D_α are naturally extended as $R(T)$-linear operators on $\mathcal{B}_T(V_W)$, being determined by the following conditions:

(0) $D_\alpha(X) = 0$, for $X \in \mathbb{Z}[P']$,
(1) $D_\alpha([\beta]) = \delta_{\alpha,\beta}$, for $\alpha, \beta \in \Delta_+$,
(2) $D_\alpha(FF') = D_\alpha(F)F' + s_\alpha(F)D_\alpha(F')$ for $F, F' \in \mathcal{B}(V_W)\langle P' \rangle$.

Lemma 4.2. *In the algebra $\mathcal{B}_T(V_W)$, we have*

$$\bigcap_{\alpha \in \Delta_+} \mathrm{Ker}(D_\alpha) = R(T) \otimes_{\mathbb{Z}[P]^W} \mathbb{Q}[P'].$$

Proof. This follows immediately from Lemma 1.8. $\square$

The operator π_i defined in the previous section is extended $R(T)$-linearly to the group algebra $R(T)[P]$, being determined by the following conditions:

(1) $\pi_i(e^{\omega_j}) = \delta_{i,j}e^{\omega_i-\alpha_i}$,
(2) $\pi_i(fg) = \pi_i(f)g + s_{\alpha_i}(f)\pi_i(g)$.

Here, the action of W on $R(T)$ is assumed to be trivial.

Note that $K_T(G/B)$ is isomorphic to the quotient algebra $R(T)[P]/J$, where J is the ideal generated by the elements of form $f - \iota(f)$, $f \in \mathbb{Z}[P]^W$.

Theorem 4.3. *The subalgebra* $R(T)[\Xi_{\mathrm{eq}}^{[\lambda]},\ \lambda \in P]$ *of* $\mathcal{B}_T(V_W)$ *is isomorphic to the* T*-equivariant* K*-ring* $K_T(G/B)$.

Proof. Let us consider the homomorphism between $R(T)$-algebras

$$\begin{array}{cccc} \psi: & R(T)[P] & \to & R(T)[\Xi_{\mathrm{eq}}^{[\lambda]},\ \lambda \in P] \\ & e^{\lambda} & \mapsto & \Xi_{\mathrm{eq}}^{[\lambda]}. \end{array}$$

We can see that

$$X^{\rho/h}(X^{-\alpha_i/h} - [\alpha_i])X^{-s_{\alpha_i}(\rho)/h}D_{\alpha_i}(\psi(f)) = \psi(\pi_i(f)),\ f \in R(T)[P],$$

in the same manner as in the proof of Proposition 3.1. Therefore, if $f \in \mathbb{Z}[P]^W$, then $D_\alpha(\psi(f)) = 0$, for all $\alpha \in \Delta_+$. Based on Lemma 4.2, we have $\psi(f) \in R(T) \otimes_{\mathbb{Z}[P]^W} \mathbb{Z}[P']$. Here, the constant term of $\psi(f)$ for $f \in \mathbb{Z}[P]^W$ is in $\mathbb{Z}[P]^W$, and therefore equals $\iota(f)$ (see also [16, Proposition 14.5]). So we obtain $\psi(f) = 0$ for $f \in J$. On the other hand, if $f \notin J$, there exists an operator ϖ on $R(T)[P]$, written as a linear combination of the composites of the multiplication operators and the operators π_i, such that the constant term of $\varpi(f)$ is nonzero. We conclude that $\mathrm{Ker}(\psi) = J$. $\square$

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