AFFINE CRYSTALS, MACDONALD POLYNOMIALS, AND COMBINATORIAL MODELS

EXTENDED ABSTRACT

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Abstract. Crystals are colored directed graphs encoding information about Lie algebra representations. Kirillov-Reshetikhin (KR) crystals correspond to certain finite-dimensional representations of affine Lie algebras. We present a combinatorial model which realizes tensor products of (column shape) KR crystals uniformly across untwisted affine types. A corollary states that the Macdonald polynomials (which generalize the irreducible characters of simple Lie algebras), upon a certain specialization, coincide with the graded characters of tensor products of KR modules. Some computational applications, as well as related work based on the present one, are also discussed.

1. Introduction

Kashiwara's *crystals* [Kas91] are colored directed graphs encoding the structure of certain bases (called crystal bases) for certain representations of quantum groups $U_q(\mathfrak{g})$ as q goes to zero; more precisely, the edges of the crystal, given by the so-called crystal operators, encode the action of the Chevalley generators of $U_q(\mathfrak{g})$ on the crystal basis. The first author and A. Postnikov [LP07, LP08] defined the so-called *alcove model* for highest weight crystals associated to a semisimple Lie algebra $\mathfrak{g}$ (in fact, the model was defined more generally, for symmetrizable Kac-Moody algebras $\mathfrak{g}$). This model can be viewed as a discrete counterpart of the celebrated *Littelmann path model* [Lit94, Lit95].

We present a brief survey of our work [LNS+15a, LNS+14, LNS+15b]. One of the main objectives of this work is to show that a generalization of the alcove model, constructed in [LL15a] and called the *quantum alcove model*, uniformly describes tensor products of column shape *Kirillov-Reshetikhin (KR) crystals* [KR90], for all untwisted affine Lie types. (KR crystals correspond to certain finite-dimensional representations of affine algebras.) More precisely, the model realizes the crystal operators on the mentioned tensor product, and also gives an efficient formula for the corresponding *energy function* [HKO+99]. (The energy comes from solvable lattice models in statistical mechanics, and can be viewed as an affine grading on a tensor product of KR crystals, as explained below.) Furthermore, in [LL15b] the quantum alcove model is used to give a uniform realization of the *combinatorial R-matrix* (that is, the unique affine crystal isomorphism commuting factors in a tensor product of KR crystals).

The quantum alcove model is based on enumerating paths in the so-called *quantum Bruhat graph* of the corresponding finite Weyl group. This graph originates in the quantum cohomology theory for flag varieties [FW04], and was first studied in [BFP99]. The path enumeration is determined by the choice of a certain sequence of alcoves (an alcove path or, equivalently, a λ-chain of roots), like in the classical alcove model. If we restrict to paths in the Hasse diagram of the Bruhat order, we recover the classical alcove model. In fact, in [LNS+14] we present a second uniform model for tensor products of column shape KR crystals: the so-called *quantum Lakshmibai-Seshadri (LS) path model*, which is based on piecewise-linear paths and the parabolic analogue of the quantum Bruhat graph. A crystal isomorphism between the two models is also given in [LNS+14].

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We note that combinatorial models for all nonexceptional KR crystals were given in [FOS09] in terms of tableau models; these are type-specific, and are based on certain fillings of Young diagrams. The advantage of the quantum alcove model lies in the fact that it is a uniform model. Moreover, the tableau models are more involved beyond A, and certain computations (related to the energy function and the combinatorial R-matrix) were only worked out in special cases, whereas they are now available in the quantum alcove model in full generality. However, the tableau models have an advantage too: they are more explicit. Thus, it is important to relate the two types of models, by making the corresponding crystal isomorphism explicit. In this way, we can translate certain information (for instance, a statistic expressing the energy function, see below) from the quantum alcove model to the tableau models. Such isomorphisms/translations have been exhibited in types A, C, and B in [Len12, LS13, LL15a, BL]. Below we describe in detail the specialization of the quantum alcove model to type A, and the construction of the affine crystal isomorphism between this specialization and the corresponding tableau model.

Our work has an important application to the theory of symmetric *Macdonald polynomials* $P_\lambda(x;q,t)$, which are a vast generalization of the ireducible characters of simple Lie algebras depending on two parameters q,t. More precisely, we prove that the graded character of a tensor product of column shape KR modules, denoted $X_\lambda(x;q)$, concides with the specialization $P_\lambda(x;q,t=0)$ of the corresponding Macdonald polynomial. This is a corollary of our realization of KR cystals in terms of the quantum alcove model and the Ram-Yip formula for Macdonald polynomials [RY11]. An extension of the $P=X$ result to the nonsymmetric Macdonald polynomials $E_\mu(x;q,t)$ is also presented.

The context of the above $P=X$ result has its origins in Ion's observation [Ion03] that, when the affine simple root α_0 is short (which includes the duals of untwisted affine root systems), $P_\lambda(x;q,0)$ is the character of a *Demazure module* for an affine Lie algebra. (Demazure modules are submodules of highest weight ones determined by a Borel subalgebra acting on an extremal weight vector.) On the other hand, Fourier and Littelmann [FL06] showed that, for simply-laced untwisted affine Lie algebras, these Demazure characters are graded characters of tensor products of KR modules. Combining [Ion03] and [FL06], one deduces the equality $P=X$ in the simply-laced untwisted cases. While the extension of both cited results to types $B_n^{(1)}$, $C_n^{(1)}$, $F_4^{(1)}$, $G_2^{(1)}$ is problematic, we prove the mentioned equality for all untwisted types by different methods.

We also mention interesting connections that our work highlights, as well as recent developments it has led to. These are related to: q-Whittaker functions, the quantum K-theory of flag varieties, various properties of KR crystals, Weyl modules for current algebras, and a categorification of Macdonald polynomials.

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2. Background

2.1. **Root systems.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra, whose rank is r. Let $\Phi\subset\mathfrak{h}^*$ be the corresponding irreducible *root system*, $\mathfrak{h}^*_{\mathbb{R}}\subset\mathfrak{h}$ the real span of the roots, and $\Phi^+\subset\Phi$ the set of positive roots. Let $\Phi^-:=\Phi\backslash\Phi^+$. For $\alpha\in\Phi$ we will say that $\alpha>0$ if $\alpha\in\Phi^+$, and $\alpha<0$ if $\alpha\in\Phi^-$. The sign of the root α, denoted $\mathrm{sgn}(\alpha)$, is defined to be 1 if $\alpha\in\Phi^+$, and -1 otherwise. Let $|\alpha|=\mathrm{sgn}(\alpha)\alpha$. Let $\rho:=\frac{1}{2}(\sum_{\alpha\in\Phi^+}\alpha)$. Let $\alpha_1,\ldots,\alpha_r\in\Phi^+$ be the corresponding *simple roots*, and $s_i:=s_{\alpha_i}$ the corresponding simple reflections. We denote $\langle\cdot,\cdot\rangle$ the nondegenerate scalar product on $\mathfrak{h}^*_{\mathbb{R}}$ induced by the Killing form. Given a root α, we consider the corresponding *coroot* $\alpha^\vee:=2\alpha/\langle\alpha,\alpha\rangle$ and reflection s_α. If $\alpha=\sum_i c_i\alpha_i$, then the *height* of α, denoted by $\mathrm{ht}(\alpha)$, is given by $\mathrm{ht}(\alpha):=\sum_i c_i$. We denote by $\widetilde{\alpha}$ the highest root in Φ^+; we let $\theta=\alpha_0:=-\widetilde{\alpha}$ and $s_0:=s_{\widetilde{\alpha}}$.

Let W be the corresponding *Weyl group* and $w_\circ$ its longest element. The length function on W is denoted by $\ell(\cdot)$. The *Bruhat order* on W is defined by its covers $w \lessdot ws_\alpha$, for $\alpha \in \Phi^+$, if $\ell(ws_\alpha) = \ell(w) + 1$. Define $w \lhd ws_\alpha$, for $\alpha \in \Phi^+$, if $\ell(ws_\alpha) = \ell(w) - 2\mathrm{ht}(\alpha^\vee) + 1$. The *quantum Bruhat graph* [FW04] is the directed graph on W with edges labeled by positive roots

$$w \xrightarrow{\alpha} ws_\alpha \quad \text{for } w \lessdot ws_\alpha \text{ or } w \lhd ws_\alpha\,; \tag{1}$$

see Example 5.2.

The *weight lattice* Λ is given by

$$\Lambda := \left\{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \,:\, \langle \lambda, \alpha^\vee\rangle \in \mathbb{Z} \text{ for any } \alpha \in \Phi\right\}. \tag{2}$$

The weight lattice Λ is generated by the *fundamental weights* $\omega_1, \ldots \omega_r$, which form the dual basis to the basis of simple coroots, i.e., $\langle \omega_i, \alpha_j^\vee\rangle = \delta_{ij}$. The set Λ^+ of *dominant weights* is given by

$$\Lambda^+ := \left\{\lambda \in \Lambda \,:\, \langle \lambda, \alpha^\vee\rangle \geq 0 \text{ for any } \alpha \in \Phi^+\right\}. \tag{3}$$

Let $\mathbb{Z}[\Lambda]$ be the group algebra of Λ. It has a $\mathbb{Z}$-basis of formal exponents $\{x^\lambda \,:\, \lambda \in \Lambda\}$ with multiplication $x^\lambda \cdot x^\mu := x^{\lambda+\mu}$, i.e., $\mathbb{Z}[\Lambda] = \mathbb{Z}[x^{\pm\omega_1}, \cdots, x^{\pm\omega_r}]$ is the algebra of Laurent polynomials in r variables.

Given $\alpha \in \Phi$ and $k \in \mathbb{Z}$, we denote by $s_{\alpha,k}$ the reflection in the affine hyperplane

$$H_{\alpha,k} := \left\{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \,:\, \langle \lambda, \alpha^\vee\rangle = k\right\}. \tag{4}$$

These reflections generate the *affine Weyl group* W_{aff} for the *dual root system* $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$. The hyperplanes $H_{\alpha,k}$ divide the real vector space $\mathfrak{h}^*_{\mathbb{R}}$ into open regions, called *alcoves*. The *fundamental alcove* $A_\circ$ is given by

$$A_\circ := \left\{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid 0 < \langle \lambda, \alpha^\vee\rangle < 1 \text{ for all } \alpha \in \Phi^+\right\}. \tag{5}$$

2.2. **Kirillov-Reshetikhin (KR) crystals.** A $\mathfrak{g}$-*crystal* (for a symmetrizable Kac-Moody algebra $\mathfrak{g}$) is a nonempty set B together with maps $e_i, f_i : B \to B \cup \{\mathbf{0}\}$ for $i \in I$ (I indexes the simple roots, as usual, and $\mathbf{0} \notin B$), and $\mathrm{wt} : B \to \Lambda$. We require $b' = f_i(b)$ if and only if $b = e_i(b')$, and $\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i$. The maps e_i and f_i are called *crystal operators* and are represented as arrows $b \to b' = f_i(b)$ colored i; thus they endow B with the structure of a colored directed graph. For $b \in B$, we set $\varepsilon_i(b) := \max\{k \mid e_i^k(b) \neq \mathbf{0}\}$, and $\varphi_i(b) := \max\{k \mid f_i^k(b) \neq \mathbf{0}\}$. Given two $\mathfrak{g}$-crystals B_1 and B_2, we define their tensor product $B_1 \otimes B_2$ as follows. As a set, $B_1 \otimes B_2$ is the Cartesian product of the two sets. For $b = b_1 \otimes b_2 \in B_1 \otimes B_2$, the weight function is simply $\mathrm{wt}(b) := \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$. The crystal operators are given by

$$f_i(b_1 \otimes b_2) := \begin{cases} f_i(b_1) \otimes b_2 & \text{if } \varepsilon_i(b_1) \geq \varphi_i(b_2) \\ b_1 \otimes f_i(b_2) & \text{otherwise,} \end{cases} \tag{6}$$

and similarly for e_i. The *highest weight crystal* $B(\lambda)$ of highest weight $\lambda \in \Lambda^+$ is a certain crystal with a unique element u_λ such that $e_i(u_\lambda) = \mathbf{0}$ for all $i \in I$ and $\mathrm{wt}(u_\lambda) = \lambda$. It encodes the structure of the crystal basis of the $U_q(\mathfrak{g})$-irreducible representation with highest weight λ as q goes to 0.

A *Kirillov-Reshetikhin (KR) crystal* [KR90] is a finite crystal $B^{r,s}$ for an affine algebra, associated to a rectangle of height r and width s, where $r \in I \setminus \{0\}$ and s is any positive integer. We refer, throughout, to the untwisted affine types $A^{(1)}_{n-1} - G^{(1)}_2$, and only consider column shape KR crystals $B^{r,1}$.

We now describe the tableau model for KR crystals $B^{r,1}$ of type $A^{(1)}_{n-1}$, where $r \in \{1, 2, \ldots, n-1\}$. As a classical type A_{n-1} crystal, the KR crystal $B^{r,1}$ is isomorphic to the corresponding $B(\omega_r)$. Therefore, we can use the corresponding tableau model, as mentioned below.

In type $A^{(1)}_{n-1}$, an element $b \in B^{r,1}$ is represented by a strictly increasing filling of a height r column, with entries in $[n] := \{1, \ldots, n\}$. We will now describe the crystal operators on a tensor product of type $A^{(1)}_{n-1}$ KR crystals $B^{r,1}$ in terms of the so-called signature rule, which is just a translation of the tensor product rule (6). To apply f_i (or e_i) on $b := b_1 \otimes \cdots \otimes b_k$ in

$B^{i_1,1} \otimes \cdots \otimes B^{i_k,1}$, consider the word with letters i and $i+1$, if $1 \leq i \leq n-1$ (resp., the letters n and 1, if $i = 0$) formed by recording these letters in $b_1, \ldots, b_k$, which are scanned from left to right and bottom to top; we make the convention that if $i = 0$ and a column contains both 1 and n, then we discard this column. We replace the letter i with the symbol $+$ and the letter $i+1$ with $-$ (resp., n with $+$ and 1 with $-$, if $i = 0$). Then, we remove from our binary word adjacent pairs $-+$, as long as this is possible. At the end of this process, we are left with a word

$$\rho_i(b) = \underbrace{+ + \ldots +}_{x} \underbrace{- - \ldots -}_{y}, \tag{7}$$

called the i-signature of b.

Definition 2.1. (1) If $y > 0$, then $e_i(b)$ is obtained by replacing in b the letter $i+1$ which corresponds to the leftmost $-$ in $\rho_i(b)$ with the letter i (resp., the letter 1 with n, after which we sort the column, if $i = 0$). If $y = 0$, then $e_i(b) = \mathbf{0}$.

(2) If $x > 0$, then $f_i(b)$ is obtained by replacing in b the letter i which corresponds to the rightmost $+$ in $\rho_i(b)$ with the letter $i+1$ (resp., the letter n with 1, after which we sort the column, if $i = 0$). If $x = 0$, then $f_i(b) = \mathbf{0}$.

Example 2.2. Let $n = 3$, $b = \begin{array}{|c|c|c|}\hline 2&1&1\\\hline 3&2\\\cline{1-2}\end{array} \mapsto \boxed{2} \otimes \boxed{1} \otimes \boxed{1}$, and has $+--$ as its 0-signature. So we have $f_0\left(\begin{array}{|c|c|c|}\hline 2&1&1\\\hline 3&2\\\cline{1-2}\end{array}\right) = \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2\\\cline{1-2}\end{array}$.

We refer again to (column shape) KR crystals of arbitrary (untwisted) type. Given a composition $\mathbf{p} = (p_1, \ldots, p_k)$, we define the tensor product of KR crystals

$$B = B^{\otimes \mathbf{p}} := \bigotimes_{i=1}^{k} B^{p_i,1}. \tag{8}$$

Remarks 2.3. (1) It is known that B is connected as an affine crystal, but disconnected as a classical crystal (i.e., with the 0-arrows removed); see Example 2.8.

(2) Let $\mathbf{p}'$ be a composition obtained from $\mathbf{p}$ by permuting its parts. There is an affine crystal isomorphism betwee $B^{\otimes \mathbf{p}}$ and $B^{\otimes \mathbf{p}'}$, which is unique by the previous remark. This isomorphism is called the *combinatorial R-matrix*.

We need to distinguish certain arrows in B, which are related to affine Demazure crystals, as we shall explain.

Definition 2.4. An arrow $b \to f_i(b)$ in B is called a *Demazure arrow* if $i \neq 0$, or $i = 0$ and $\varepsilon_0(b) \geq 1$. An arrow $b \to f_i(b)$ in B is called a *dual Demazure arrow* if $i \neq 0$, or $i = 0$ and $\varphi_i(b) \geq 2$.

Remarks 2.5. (1) By Fourier-Littelmann [FL06], in simply-laced types, the tensor product of KR crystals B is isomorphic, as a classical crystal (discard the affine 0-arrows) with a certain *Demazure crystal* for the corresponding affine algebra. (Recall that Demazure modules are submodules of highest weight ones determined by a Borel subalgebra acting on an extremal weight vector.) Moreover, by [FSS07], the 0-arrows in the latter correspond precisely to the Demazure arrows in B.

(2) In the case when all of the tensor factors in B are *perfect* crystals [HK00], B remains connected upon removal of the non-Demazure (resp. non-dual Demazure) 0-arrows.

(3) In classical types, $B^{k,1}$ is perfect as follows: in types $A^{(1)}_{n-1}$ and $D^{(1)}_n$ for all k, in type $B^{(1)}_n$ only for $k \neq n$, and in type $C^{(1)}_n$ only for $k = n$ (using the standard indexing of the Dynkin diagram); in other words, for all the Dynkin nodes in simply-laced types, and only for the nodes corresponding to the long roots in non-simply-laced types, see [FOS10]. It was conjectured in [HKO$^+$99] that the same is true in the exceptional types. In type $G^{(1)}_2$ this was confirmed in [Yam98]. For the other exceptional types, see Section 7.

The *energy function* $D = D_B$ is a function from B to the integers, defined by summing the so-called local energies of all pairs of tensor factors; it is used to express one-dimensional configuration sums in statistical mechanics [HKO+02, HKO+99]. We will only refer here to the so-called *left energy* [LNS+14], so we will not make this specification. (There are two conventions in defining the local energy of a pair of tensor factors: commuting the right one towards the head of the tensor product, or the left one towards the tail; the left energy corresponds to the second choice, and the *right energy* to the first.) We will only need the following property of the energy function, which defines it as an affine grading on B.

Theorem 2.6. [NS08, ST12] *The energy is preserved by the classical crystal operators f_i, i.e., $i \neq 0$. If $b \to f_0(b)$ is a dual Demazure arrow, then $D(f_0(b)) = D(b) - 1$.*

Remark 2.7. Theorem 2.6 shows that the energy is determined up to a constant on the connected components of the subgraph of the affine crystal B containing only the dual Demazure arrows. See also Remark 2.5 (2).

Example 2.8. The crystal $B^{\otimes(1,1,1)} = (B^{1,1})^{\otimes 3}$ in type $A_2^{(1)}$ is plotted in Figure 1, using the tableau model. All the arrows labeled 1 and 2 are displayed, but only some arrows labeled 0; the dotted arrows are non-dual Demazure arrows, as they are at the end of a 0-string. One can see that the $3^3 = 27$ vertices of the crystal are divided into four classical components, in which the energy is 0, -1, -2, and -3, by Theorem 2.6.

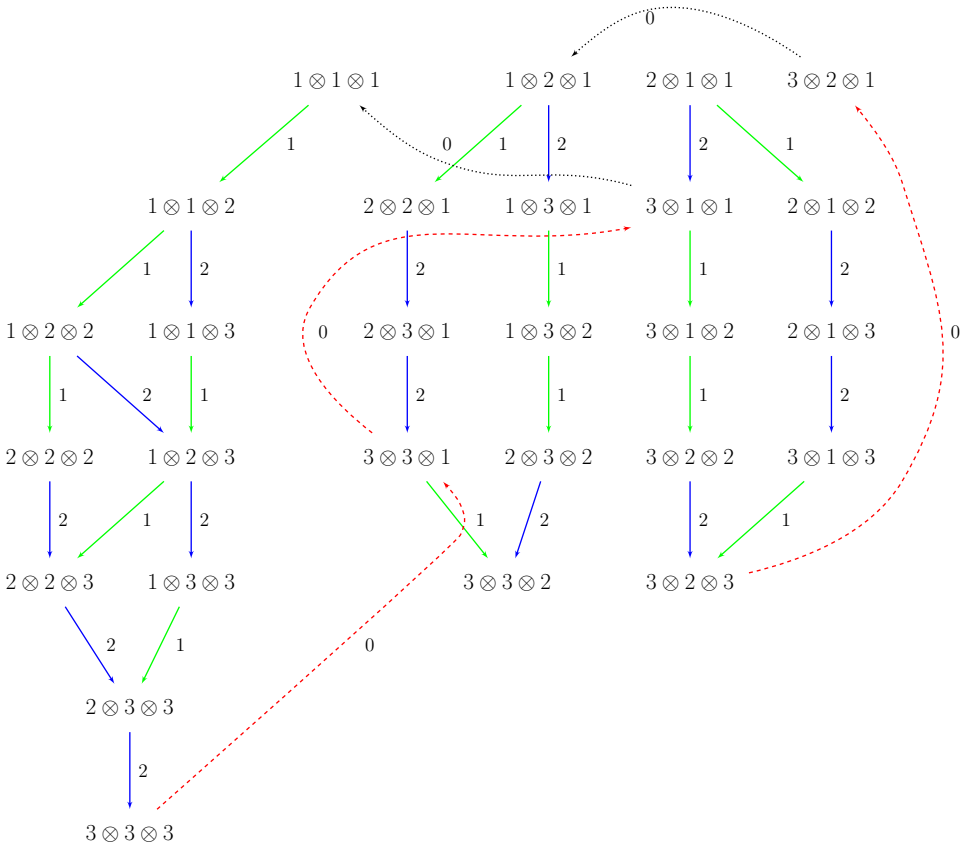


FIGURE 1. The crystal $B^{\otimes(1,1,1)}$

3. The quantum alcove model

In this section we recall from [LL15a] the construction of the quantum alcove model and its main properties.

3.1. **λ-chains and admissible subsets.** We say that two alcoves are *adjacent* if they are distinct and have a common wall. Given a pair of adjacent alcoves A and B, we write $A \xrightarrow{\beta} B$ if the common wall is of the form $H_{\beta,k}$ and the root $\beta \in \Phi$ points in the direction from A to B.

Definition 3.1. [LP07] An *alcove path* is a sequence of alcoves $(A_0, A_1, \dots, A_m)$ such that A_{j-1} and A_j are adjacent, for $j = 1, \dots m$. We say that an alcove path is *reduced* if it has minimal length among all alcove paths from A_0 to A_m.

Let $A_\lambda = A_\circ + \lambda$ be the translation of the fundamental alcove $A_\circ$ by the weight λ.

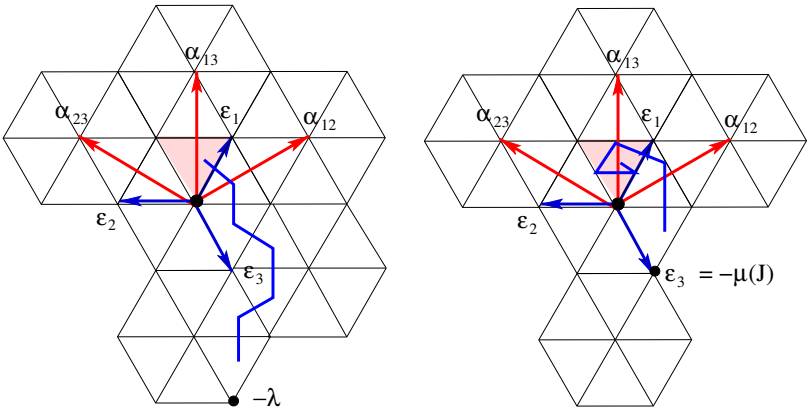
Definition 3.2. [LP07] The sequence of roots $(\beta_1, \beta_2, \dots, \beta_m)$ is called a λ-*chain* if

$$A_0 = A_\circ \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_m} A_m = A_{-\lambda}$$

is a reduced alcove path.

We now fix a dominant weight λ and an alcove path $\Pi = (A_0, \dots, A_m)$ from $A_0 = A_\circ$ to $A_m = A_{-\lambda}$. Note that Π is determined by the corresponding λ-chain $\Gamma := (\beta_1, \dots, \beta_m)$, which consists of positive roots. A specific choice of a λ-chain, called a *lex* λ-*chain*, is given in [LP08, Proposition 4.2]; this choice depends on a total order on the simple roots. We let $r_i := s_{\beta_i}$, and let $\widehat{r}_i$ be the affine reflection in the hyperplane containing the common face of A_{i-1} and A_i, for $i = 1, \dots, m$; in other words, $\widehat{r}_i := s_{\beta_i, -l_i}$, where $l_i := |\{j < i\,;\, \beta_j = \beta_i\}|$. We define $\widetilde{l}_i := \langle \lambda, \beta_i^\vee \rangle - l_i = |\{j \geq i\,;\, \beta_j = \beta_i\}|$.

Example 3.3. Consider the dominant weight $\lambda = 3\varepsilon_1 + 2\varepsilon_2$ in the root system A_2 (cf. Section 5 and the notation therein). A λ-chain is $\Gamma = (\alpha_{23}, \alpha_{13}, \alpha_{23}, \alpha_{13}, \alpha_{12}, \alpha_{13})$. The corresponding l_i are $(0, 0, 1, 1, 0, 2)$ and $\widetilde{l}_i$ are $\{2, 3, 1, 2, 1, 1\}$. The alcove path is shown in Figure 2a; here $A_\circ$ is shaded, and $A_{-\lambda}$ is the alcove at the end of the path.



(A) Γ for $\lambda = 3\varepsilon_1 + 2\varepsilon_2$ (B) $\Gamma(J)$ for $J = \{1, 2, 3, 5\}$

FIGURE 2. Unfolded and folded λ-chain

Let $J = \{j_1 < j_2 < \cdots < j_s\}$ be a subset of $[m]$. The elements of J are called *folding positions*. We fold Π in the hyperplanes corresponding to these positions and obtain a folded path, see Example 3.6 and Figure 2b. Like Π, the folded path can be recorded by a sequence of roots, namely $\Gamma(J) = (\gamma_1, \gamma_2, \dots, \gamma_m)$, where

$$\gamma_k := r_{j_1} r_{j_2} \dots r_{j_p}(\beta_k)\,, \tag{9}$$

with j_p the largest folding position less than k. We define $\gamma_\infty := r_{j_1} r_{j_2} \dots r_{j_s}(\rho)$. Upon folding, the hyperplane separating the alcoves A_{k-1} and A_k in Π is mapped to

$$H_{|\gamma_k|, -l_k^J} = \widehat{r}_{j_1} \widehat{r}_{j_2} \dots \widehat{r}_{j_p}(H_{\beta_k, -l_k})\,, \tag{10}$$

for some l_k^J, which is defined by this relation.

Given $i \in J$, we say that i is a *positive folding position* if $\gamma_i > 0$, and a *negative folding position* if $\gamma_i < 0$. We denote the positive folding positions by J^+, and the negative ones by J^-. We call $\mathrm{wt}(J) := -\widehat{r}_{j_1}\widehat{r}_{j_2}\dots\widehat{r}_{j_s}(-\lambda)$ the *weight* of J. We define the *height* and *coheight* of J by

$$\mathrm{ht}(J) := \sum_{j\in J^-} \widetilde{l}_j\,, \qquad \mathrm{cht}(J) := \sum_{j\in J^-} l_j\,. \tag{11}$$

Definition 3.4. A subset $J = \{j_1 < j_2 < \cdots < j_s\} \subseteq [m]$ (possibly empty) is an *admissible subset* if we have the following path in the quantum Bruhat graph on W:

$$1 \xrightarrow{\beta_{j_1}} r_{j_1} \xrightarrow{\beta_{j_2}} r_{j_1}r_{j_2} \xrightarrow{\beta_{j_3}} \cdots \xrightarrow{\beta_{j_s}} r_{j_1}r_{j_2}\cdots r_{j_s} =: \phi(J)\,. \tag{12}$$

We call $\Gamma(J)$ an *admissible folding*, and $\phi(J)$ its *final direction*. We let $\mathcal{A}(\Gamma)$ be the collection of admissible subsets.

Remarks 3.5. (1) Positive and negative folding positions correspond to up and down steps (in Bruhat order) in the chain (12), respectively.

(2) If we restrict to admissible subsets for which the path (12) has no down steps, we recover the classical alcove model in [LP07, LP08].

Example 3.6. We continue Example 3.3. Let $J = \{1, 2, 3, 5\}$, then $\Gamma(J) = (\alpha_{23}, \alpha_{12}, \alpha_{31}, \alpha_{23}, \alpha_{21}, \alpha_{13})$. The folded path is shown in Figure 2b. We have $J^+ = \{1, 2\}$, $J^- = \{3, 5\}$, $\mathrm{wt}(J) = -\varepsilon_3$, $\mathrm{ht}(J) = \widetilde{l}_3 + \widetilde{l}_5 = 1 + 1 = 2$, and $\mathrm{cht}(J) = l_3 + l_5 = 1 + 0 = 1$. In Section 5 we will describe an easy way to verify that J is admissible.

3.2. **Crystal operators.** In this section we define the crystal operators on $\mathcal{A}(\Gamma)$. Given $J \subseteq [m]$ and $\alpha \in \Phi$, we will use the following notation:

$$I_\alpha = I_\alpha(J) := \{i \in [m] \mid \gamma_i = \pm\alpha\}\,, \qquad \widehat{I}_\alpha = \widehat{I}_\alpha(J) := I_\alpha \cup \{\infty\}\,,$$

and $l_\alpha^\infty := \langle \mathrm{wt}(J), \mathrm{sgn}(\alpha)\alpha^\vee\rangle$. The following graphical representation of the heights l_i^J for $i \in I_\alpha$ and l_α^∞ is useful for defining the crystal operators. Let

$$\widehat{I}_\alpha = \{i_1 < i_2 < \cdots < i_n < i_{n+1} = \infty\} \text{ and } \varepsilon_i := \begin{cases} 1 & \text{if } i \notin J \\ -1 & \text{if } i \in J \end{cases}.$$

If $\alpha > 0$, we define the continuous piecewise linear function $g_\alpha : [0, n + \frac{1}{2}] \to \mathbb{R}$ by

$$g_\alpha(0) = -\frac{1}{2}, \quad g'_\alpha(x) = \begin{cases} \mathrm{sgn}(\gamma_{i_k}) & \text{if } x \in (k-1, k-\frac{1}{2}),\ k = 1,\dots,n \\ \varepsilon_{i_k}\mathrm{sgn}(\gamma_{i_k}) & \text{if } x \in (k-\frac{1}{2}, k),\ k = 1,\dots,n \\ \mathrm{sgn}(\langle \gamma_\infty, \alpha^\vee\rangle) & \text{if } x \in (n, n+\frac{1}{2}). \end{cases} \tag{13}$$

If $\alpha < 0$, we define g_α to be the graph obtained by reflecting $g_{-\alpha}$ in the x-axis. By [LP08, Propositions 5.3 and 5.5], for any α we have

$$\mathrm{sgn}(\alpha) l_{i_k}^J = g_\alpha\left(k - \frac{1}{2}\right), k = 1,\dots,n, \text{ and } \mathrm{sgn}(\alpha) l_\alpha^\infty := \langle \mathrm{wt}(J), \alpha^\vee\rangle = g_\alpha\left(n + \frac{1}{2}\right). \tag{14}$$

Example 3.7. We continue Example 3.6. The graphs of g_{α_2} and g_θ are given in Figure 3.

Let J be an admissible subset. Let $\delta_{i,j}$ be the Kronecker delta function. Fix p in $\{0,\dots,r\}$, so α_p is a simple root if $p > 0$, or θ if $p = 0$. Let M be the maximum of g_{α_p}. Let m be the minimum index i in $\widehat{I}_{\alpha_p}$ for which we have $\mathrm{sgn}(\alpha_p) l_i^J = M$. It turns out that, if $M \geq \delta_{p,0}$, then we have either $m \in J$ or $m = \infty$; furthermore, if $M > \delta_{p,0}$, then m has a predecessor k in $\widehat{I}_{\alpha_p}$, and we have $k \notin J$. We define

$$f_p(J) := \begin{cases} (J \setminus \{m\}) \cup \{k\} & \text{if } M > \delta_{p,0} \\ \mathbf{0} & \text{otherwise}\,. \end{cases} \tag{15}$$

Now we define e_p. Again let $M := \max g_{\alpha_p}$. Assuming that $M > \langle \mathrm{wt}(J), \alpha_p^\vee\rangle$, let k be the maximum index i in I_{α_p} for which we have $\mathrm{sgn}(\alpha_p) l_i^J = M$, and let m be the successor of k

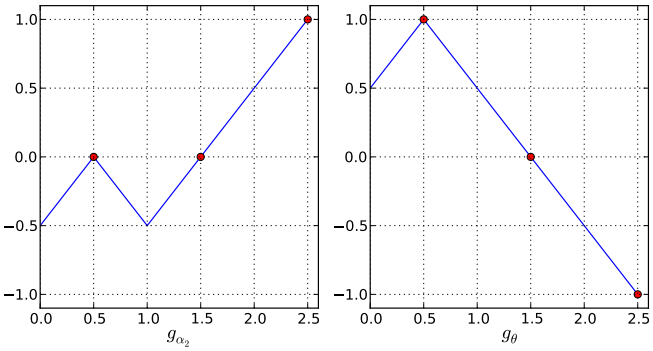


FIGURE 3

in $\widehat{I}_{\alpha_p}$. Assuming also that $M \geq \delta_{p,0}$, it turns out that we have $k \in J$, and either $m \notin J$ or $m = \infty$. Define

$$e_p(J) := \begin{cases} (J \setminus \{k\}) \cup \{m\} & \text{if } M > \langle \mathrm{wt}(J), \alpha_p^\vee \rangle \text{ and } M \geq \delta_{p,0} \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{16}$$

In the above definitions, we use the convention that $J \setminus \{\infty\} = J \cup \{\infty\} = J$.

Example 3.8. We continue Example 3.7. We find $f_2(J)$ by noting that $\widehat{I}_{\alpha_2} = \{1, 4, \infty\}$. From g_{α_2} in Figure 3 we can see that the heights l_i^J and $l_{\alpha_2}^\infty$ corresponding to these positions are $0, 0, 1$, so $k = 4$, $m = \infty$, and $f_2(J) = J \cup \{4\} = \{1, 2, 3, 4, 5\}$. We can see from Figure 3 that the maximum of $g_\theta = 1$, hence $f_0(J) = \mathbf{0}$. To compute $e_0(J)$ observe that $\widehat{I}_\theta = \{3, 6\}$ with $k = 3$ and $m = 6$. So $e_0(J) = (J \setminus \{k\}) \cup \{m\} = \{1, 2, 5, 6\}$.

The following theorem is the main result of [LL15a].

Theorem 3.9. [LL15a]

(1) *If J is an admissible subset and if $f_p(J) \neq \mathbf{0}$, then $f_p(J)$ is also an admissible subset. Similarly for $e_p(J)$. Moreover, $f_p(J) = J'$ if and only if $e_p(J') = J$.*

(2) *We have $\mathrm{wt}(f_p(J)) = \mathrm{wt}(J) - \alpha_p$. Moreover, if $M \geq \delta_{p,0}$, then*

$$\varphi_p(J) = M - \delta_{p,0}\,, \quad \varepsilon_p(J) = M - \langle \mathrm{wt}(J), \alpha_p^\vee \rangle\,,$$

while otherwise $\varphi_p(J) = \varepsilon_p(J) = 0$.

4. Main results

We summarize the main results in [LNS+14], cf. also [LNS+15a, LL15b]. The setup is that of untwisted affine root systems.

Theorem 4.1. [LNS+14, LL15b] *Consider a composition $\mathbf{p} = (p_1, \ldots, p_k)$ and the corresponding crystal $B := \bigotimes_{i=1}^k B^{p_i,1}$. Let $\lambda := \omega_{p_1} + \ldots + \omega_{p_k}$, and let Γ be any λ-chain (see Section 3).*

(1) The (combinatorial) crystal $\mathcal{A}(\Gamma)$ is isomorphic to the subgraph of B consisting of the dual Demazure arrows, via a specific bijection which preserves the weights of the vertices.

(2) If the vertex b of B corresponds to J under the isomorphism in part (1), then the energy is given by $D_B(b) - D_B^{\mathrm{ext}} = -\mathrm{ht}(J)$, where D_B^{ext} is a global constant.

The proof proceeds as follows. Based on earlier work of Naito and Sagaki [NS05, NS06, NS08] on crystal bases for tensor products of column-shape KR modules (also called *level-zero fundamental representations*), we first derive a combinatorial model for the entire crystal B above (including the dual Demazure arrows), in terms of so-called *quantum Lakshmibai-Seshadri (LS) paths*; these are piecewise-linear paths constructed in terms of the parabolic analogue of the quantum Bruhat graph. In order to achieve this, we also rely on the results in [LNS+15a] related to the quantum Bruhat graph and its parabolic analogue. Then we exhibit a crystal isomorphism from the quantum alcove model to the quantum LS path model based on the lex λ-chain. In fact, this map is a very natural one, as it is the "forgetful map" on the quantum

alcove model (whose structure is richer than that of quantum LS paths). The passage between the quantum alcove models based on the lex λ-chain and an arbitrary λ-chain is investigated in [LL15b], and is discussed below.

Remarks 4.2. (1) Based on the above crystal isomorphism between the quantum LS path model and the quantum alcove model, we can construct the non-dual Demazure arrows in the latter. However, this construction is considerably more involved than (15)-(16).

(2) Although the quantum alcove model so far misses the non-dual Demazure arrows, it has the advantage of being a discrete model. Therefore, combinatorial methods are applicable, for instance in the realization of the combinatorial R-matrix, see below. This should be compared with the continuous arguments used for the similar purpose in the Littelmann path model [Lit95].

(3) In a similar way to Theorem 4.1 (2), the right energy (see Section 2.2) is given by the coheight statistic. These two expressions have the advantage of using only the local combinatorial data indexing a crystal vertex, whereas the recursive calculation of energy in Theorem 2.6 is less efficient, especially for large crystals.

In [LL15b] we enhance the quantum alcove model in order to give a uniform realization of the combinatorial R-matrix. The construction is based on certain combinatorial moves called *quantum Yang-Baxter moves*, which generalize their alcove model versions defined in [Len07]. These moves are explicitly described in all Lie types by reduction to rank 2 root subsystems. Note that, as far as existing realizations of the combinatorial R-matrix are concerned, they are limited in scope and type-specific. For instance, in terms of the tableau model, there is a construction in type A based on Schützenberger's *jeu de taquin* (sliding algorithm) on two columns [Ful97], whereas the extensions of this procedure to types B and C are involved and not transparent. By contrast, our construction is easy to formulate, and is related to more general concepts (especially the *shellability property* of the quantum Bruhat graph [BFP99]). We also show that, like the alcove model, its quantum generalization does not depend on the choice of a λ-chain, cf. Theorem 4.1 (1); in fact, we identify $\mathcal{A}(\Gamma)$ and $\mathcal{A}(\Gamma')$ for two λ-chains Γ and Γ' based on quantum Yang-Baxter moves.

5. The quantum alcove model in type A

In this section we specialize the quantum alcove model to type A, and construct an affine crystal isomorphism between this specialization and the usual tableau model for the tensor products of type A KR crystals (see Section 2.2).

We start with the basic facts about the root system of type A_{n-1}. We can identify the space $\mathfrak{h}^*_{\mathbb{R}}$ with the quotient $V := \mathbb{R}^n/\mathbb{R}(1,\dots,1)$, where $\mathbb{R}(1,\dots,1)$ denotes the subspace in $\mathbb{R}^n$ spanned by the vector $(1,\dots,1)$. Let $\varepsilon_1,\dots,\varepsilon_n \in V$ be the images of the coordinate vectors in $\mathbb{R}^n$. The root system is $\Phi = \{\alpha_{ij} := \varepsilon_i - \varepsilon_j \ : \ i \neq j,\ 1 \leq i,j \leq n\}$. The simple roots are $\alpha_i = \alpha_{i,i+1}$, for $i = 1,\dots,n-1$. The highest root $\widetilde{\alpha} = \alpha_{1n}$. We let $\alpha_0 = \theta = \alpha_{n1}$. The weight lattice is $\Lambda = \mathbb{Z}^n/\mathbb{Z}(1,\dots,1)$. The fundamental weights are $\omega_i = \varepsilon_1 + \ldots + \varepsilon_i$, for $i = 1,\dots,n-1$. A dominant weight $\lambda = \lambda_1\varepsilon_1 + \ldots + \lambda_{n-1}\varepsilon_{n-1}$ is identified with the partition $(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq \lambda_n = 0)$ having at most $n-1$ parts. Note that $\rho = (n-1, n-2, \dots, 0)$. Considering the Young diagram of the dominant weight λ as a concatenation of columns, whose heights are $\lambda'_1, \lambda'_2, \dots$, corresponds to expressing λ as $\omega_{\lambda'_1} + \omega_{\lambda'_2} + \dots$ (as usual, λ' is the conjugate partition to λ).

The Weyl group W is the symmetric group S_n, which acts on V by permuting the coordinate vectors $\varepsilon_1,\dots,\varepsilon_n$. Permutations $w \in S_n$ are written in one-line notation $w = w(1)\dots w(n)$. For simplicity, we use the same notation (i,j) with $1 \leq i < j \leq n$ for the root α_{ij} and the reflection $s_{\alpha_{ij}}$, which is the transposition t_{ij} of i and j. We recall a criterion for the edges of the type A quantum Bruhat graph. We need the circular order $\prec_i$ on $[n]$ starting at i, namely $i \prec_i i+1 \prec_i \dots \prec_i n \prec_i 1 \prec_i \dots \prec_i i-1$. It is convenient to think of this order in terms of the numbers $1,\dots,n$ arranged on a circle clockwise. We make the convention that, whenever we write $a \prec b \prec c \prec \dots$, we refer to the circular order $\prec = \prec_a$.

Proposition 5.1. [Len12] *For $1 \leq i < j \leq n$, we have an edge $w \xrightarrow{(i,j)} w(i,j)$ if and only if there is no k such that $i < k < j$ and $w(i) \prec w(k) \prec w(j)$.*

Example 5.2. The quantum Bruhat graph of type A_2, i.e., on the symmetric group S_3, is indicated in Figure 4.

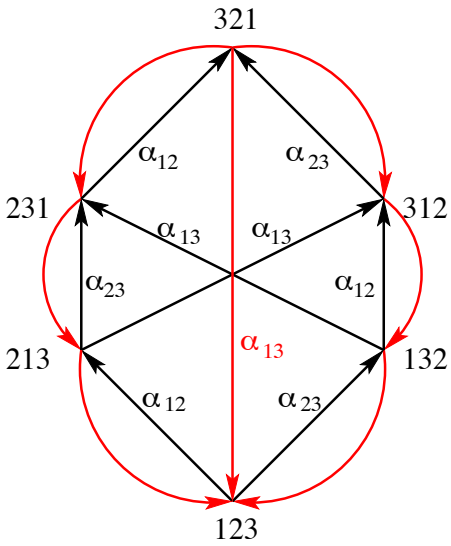


FIGURE 4. The quantum Bruhat graph for S_3

We now consider the specialization of the quantum alcove model to type A. For any $k = 1, \ldots, n-1$, we have the following ω_k-chain, from $A_\circ$ to $A_{-\omega_k}$, denoted by $\Gamma(k)$ [LP07]:

$$
\begin{array}{llll}
((k,k+1), & (k,k+2) & \ldots, & (k,n), \\
(k-1,k+1), & (k-1,k+2) & \ldots, & (k-1,n), \\
\vdots & \vdots & & \vdots \\
(1,k+1), & (1,k+2) & \ldots, & (1,n)) \,.
\end{array} \tag{17}
$$

Example 5.3. We specialize (17) to $n = 4$ and $k = 1, 2, 3$. It is best to visualize $\Gamma(k)$ based on a column of height n broken into two pieces, with the top part of height k and the bottom one of height $n-k$; then $\Gamma(k)$ is obtained by pairing row numbers in the top and bottom parts, in the prescribed order.

$$
\begin{array}{|c|}\hline 1 \\ \hline \end{array}
\qquad
\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}
\,, \quad \Gamma(1) = ((1,2),(1,3),(1,4)) \,; \qquad
\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}
\qquad
\begin{array}{|c|}\hline 4 \\ \hline \end{array}
\,, \quad \Gamma(3) = ((3,4),(2,4),(1,4)) \,;
$$

$$
\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}
\qquad
\begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline \end{array}
\,, \quad \Gamma(2) = ((2,3),(2,4),(1,3),(1,4)) \,.
$$

Fix a dominant weight/partition λ for the remainder of this section. We construct a λ-chain $\Gamma = (\beta_1, \beta_2, \ldots, \beta_m)$ as the concatenation $\Gamma := \Gamma^1 \ldots \Gamma^{\lambda_1}$, where $\Gamma^j = \Gamma(\lambda'_j)$. Let $J = \{j_1 < \cdots < j_s\}$ be a set of folding positions in Γ, not necessarily admissible, and let T be the corresponding list of roots of Γ, also viewed as transpositions. The factorization of Γ induces a factorization of T as $T = T^1 T^2 \ldots T^{\lambda_1}$, and of $\Delta = \Gamma(J)$ as $\Delta = \Delta^1 \ldots \Delta^{\lambda_1}$. Recalling that the roots in Δ were denoted γ_k, we use the notation $\gamma_k \in \Delta^q$ to indicate that the kth root in Δ

falls in the segment Δ^q (rather than the fact that Δ^q contains a root equal to γ_k). We denote by $T^1 \dots T^j$ the permutation obtained by composing the transpositions in $T^1, \dots, T^j$ left to right. For $w \in W$, let $w_i = w(i)$. For w written in one-line notation as $w = w_1 w_2 \dots w_n$, let $w[i,j] = w_i \dots w_j$.

We now recall from [Len12] the construction of the correspondence between the type A quantum alcove model and tableau model.

Definition 5.4. Let $\pi_j = \pi_j(T) := T^1 \dots T^j$. We define the *filling map*, which associates with each $J \subseteq [m]$ a filling of the Young diagram λ, by

$$\text{fill}(J) = \text{fill}(T) := C_1 \dots C_{\lambda_1}, \quad \text{where } C_i := \pi_i[1, \lambda_i'], \tag{18}$$

see the notation above. We define the *sorted filling map* sfill(J) by sorting ascendingly the columns of fill(J).

In other words, the ith column C_i of fill(J) consists of the first λ_i' entries of the permutation π_i, written in one-line notation; see Example 5.5.

Example 5.5. Let $n = 3$ and $\lambda = (4,3,0)$, which is identified with $4\varepsilon_1 + 3\varepsilon_2 = 3\omega_2 + \omega_1$, and corresponds to the Young diagram $\ydiagram{4,3}$. We have

$$\Gamma = \Gamma^1\Gamma^2\Gamma^3\Gamma^4 = \Gamma(2)\Gamma(2)\Gamma(2)\Gamma(1) = (\underline{(2,3)}, \underline{(1,3)} \mid \underline{(2,3)}, (1,3) \mid \underline{(2,3)}, (1,3) \mid \underline{(1,2)}, (1,3)),$$

where we underlined the roots in positions $J = \{1,2,3,5,7\}$. Then

$$T = ((2,3), (1,3) \mid (2,3) \mid (2,3) \mid (1,2)), \text{ and}$$

$$\Gamma(J) = \Delta = \Delta^1\Delta^2\Delta^3\Delta^4 = (\underline{(2,3)}, \underline{(1,2)} \mid \underline{(3,1)}, (2,3) \mid \underline{(1,3)}, (2,1) \mid \underline{(2,3)}, (3,1)), \tag{19}$$

where we again underlined the folding positions, and indicated the factorizations of T and Δ by vertical lines. It is easy to check that J is admissible; indeed, the sequence of permutations (12) corresponding to J is a path in the quantum Bruaht graph, cf. Proposition 5.1 and Example 5.2:

$$\begin{array}{ccccccccccccccc} \boxed{1}\!\boxed{\mathbf{2}}\!\boxed{\mathbf{3}} & \lessdot & \boxed{\mathbf{1}}\!\boxed{3}\!\boxed{\mathbf{2}} & \lessdot & \boxed{2}\!\boxed{3}\!\boxed{1} & | & \boxed{2}\!\boxed{\mathbf{3}}\!\boxed{\mathbf{1}} & \lhd & \boxed{2}\!\boxed{1}\!\boxed{3} & | & \boxed{2}\!\boxed{\mathbf{1}}\!\boxed{\mathbf{3}} & \lessdot & \boxed{2}\!\boxed{3}\!\boxed{1} & | & \boxed{\mathbf{2}}\!\boxed{\mathbf{3}}\!\boxed{1} & \lessdot & \boxed{3}\!\boxed{2}\!\boxed{1} & | \end{array} \tag{20}$$

Here each permutation in (12) is written vertically in one-line notation, with the entries in bold to be transposed; moreover, if the transposition to be applied lies in Γ^i, then the corresponding permutation is represented as a broken column with the top part of height λ_i'; see the structure of Γ^i in (17) and Example 5.3. By considering the top part of the last column in each segment and by concatenating these columns left to right, we obtain fill(J), i.e., fill$(J) = \begin{array}{|c|c|c|c|}\hline 2&2&2&3\\\hline 3&1&3\\\cline{1-3}\end{array}$.

We now state the main result of this section.

Theorem 5.6. [LL15a] *The map* sfill *is an affine crystal isomorphism between* $\mathcal{A}(\Gamma)$ *and the subgraph of* $B^{\otimes \lambda'}$ *consisting of the dual Demazure arrows. In other words, given* sfill$(J) = b$, *there is a dual Demazure arrow* $b \to f_p(b)$ *if and only if* $f_p(J) \neq \mathbf{0}$, *and we have* $f_i(b) =$ sfill$(f_i(J))$.

Remarks 5.7. (1) The affine crystal isomorphism in Theorem 5.6 is unique, by Remarks 2.5 (2), (3).

(2) In [Len11] it was proved that the map sfill preserves weights. Furthermore, in [Len12] it was shown this map translates the height statistic to the Lascoux-Schützenberger *charge statistic* [LS79], which is known to express the energy function in the tableau model. This should be compared with Theorem 4.1 (2), where the constant D_B^{ext} is 0 in this case.

(3) A similar affine crystal isomorphism in type C (between the quantum alcove model and the corresponding tableau model, based on *Kashiwara-Nakashima columns* [KN94]) is also given in [LL15a]. The height statistic is translated to a type C analogue of the charge statistic in [Len12, LS13]. Type B is under investigation in [BL].

6. Macdonald polynomials

The *Macdonald polynomials* [Ma95, Ma03] are a remarkable family of orthogonal polynomials associated to a finite root system, which depend on two parameters q, t; more precisely, they are polynomials in the group algebra of the weight lattice whose coefficients are rational functions in q, t. There are two families of Macdonald polynomials: the symmetric ones (under the Weyl group action) $P_\lambda(x; q, t)$, and the nonsymmetric ones $E_\mu(x; q, t)$; here λ is a dominant weight (i.e., a partition in type A), and μ is an arbitrary weight (i.e., a composition in type A) – a convention we adopt for the rest of this paper. The symmetric Macdonald polynomials specialize to the *Hall-Littlewood polynomials* (or spherical functions for a Chevalley group over a p-adic field) upon setting $q = 0$. They further specialize to the corresponding irreducible characters upon setting $q = t = 0$. By contrast, the nonsymmetric Macdonald polynomials become the characters of Demazure modules when $q = t = 0$.

The importance of Macdonald polynomials is due to their deep connections to many areas of mathematics, such as: p-adic and real reductive groups, Kac-Moody algebras, double affine Hecke algebras, Hilbert schemes, integrable quantum systems, conformal field theory, harmonic analysis, special functions, multivariate statistics etc.

Ram and Yip [RY11] gave a combinatorial formula for both the symmetric and the nonsymmetric Macdonald polynomials in terms of alcove walks. The specialization of this formula upon setting $t = 0$ was worked out in [Len12] and [OS13], in the symmetric and nonsymmetric cases, respectively. More precisely, in the symmetric case we have the following formula in terms of the quantum alcove model, where Γ is any λ-chain for a dominant weight λ.

Theorem 6.1. [RY11, Len12] *We have*

$$P_\lambda(x; q, 0) = \sum_{J \in \mathcal{A}(\Gamma)} q^{\mathrm{ht}(J)} x^{\mathrm{wt}(J)}\,.$$

Note also that, for $t = 0$, the symmetric Macdonald polynomial coincides with a particular nonsymmetric one (see [LNS+14]):

$$P_\lambda(x; q, 0) = E_{w_\circ \lambda}(x; q, 0)\,. \tag{21}$$

Now define the graded character corresponding to the KR crystal B in (8) (see for example [HKO+02, HKO+99]) by

$$X_\lambda(x; q) := \sum_{b \in B} q^{D_B(b) - D_B^{\mathrm{ext}}} x^{\mathrm{wt}(b)}, \tag{22}$$

where $\mathrm{wt}(b)$ is the weight of the crystal element b. From Theorems 4.1 and 6.1, we immediately derive one of our main results.

Corollary 6.2. [LNS+14] *We have*

$$P_\lambda(x; q^{-1}, 0) = X_\lambda(x; q)\,.$$

We will now present another result which follows from our work (and the result in [OS13] mentioned above), namely a combinatorial formula for the specialization of a nonsymmetric Macdonald polynomial at $t = 0$ in terms of the quantum alcove model. This formula can be viewed as the nonsymmetric analogue of Theorem 6.1. Furthermore, it generalizes the formula for Demazure characters in terms of the alcove model in [Len07, Theorem 6.3].

Let λ be a dominant weight, whose stabilizer is denoted by W_λ. Let $\mu = v(\lambda)$, where v is assumed to be a lowest coset representative modulo W_λ. Given a Weyl group element u, we denote by $\lfloor u \rfloor$ the lowest coset representative of uW_λ. Consider also an arbitrary λ-chain Γ, and recall the relevant notation from Section 3.

Theorem 6.3. [LNS+15b] *With the above notation, we have*

$$E_\mu(x; q, 0) = \sum_{\substack{J \in \mathcal{A}(\Gamma) \\ \lfloor \phi(J) \rfloor \leq v}} q^{\mathrm{cht}(J)} x^{\mathrm{wt}(J)}\,.$$

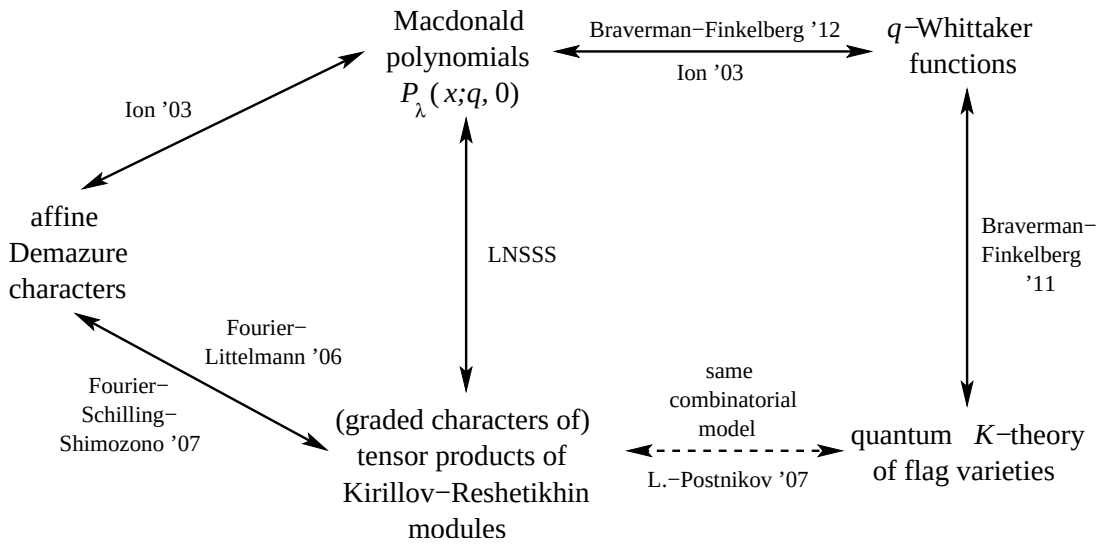


FIGURE 5. Connections highlighted by our work

A similar formula for $E_\mu(x;q,0)$ in terms of quantum LS paths is also given in [LNS+15b]. Furthermore, it is shown that this specialized nonsymmetric Macdonald polynomial can be interpreted as the graded character of a Demazure-type submodule of the tensor product of KR modules whose crystal is B in (8).

7. Other applications and related developments

In this section, we summarize other applications of our work, some interesting connections that it highlights, as well as recent developments it has led to.

Braverman and Finkelberg [BF14b] have recently shown that, for simply-laced untwisted affine root systems, the characters of the duals of certain current algebra modules, called *global Weyl modules*, coincide with the characters $\Psi_\lambda(x;q)$ of the spaces of global sections of line bundles on quasi-maps spaces; in this case, it is also shown that the function $\Psi_\lambda(x;q)$ is equal to $P_\lambda(x;q,0)$ times an explicit product of geometric series whose ratios are powers of q, and these functions are called *q-Whittaker functions* due to their appearance in the quantum group version of the Kostant-Whittaker reduction of Etingof and Sevostyanov for the *q-Toda integrable system*. More precisely, the functions $\Psi_\lambda(x;q)$ are eigenfunctions of the q-Toda difference operators, and their generating function yields the *K-theoretic J-function* of Givental and Lee [BF14a]. Note, however, that in the non-simply-laced untwisted cases, the situation differs considerably: indeed, the proof in [BF14b] of the equality between $\Psi_\lambda(x;q)$ and $P_\lambda(x;q,0)$ times the explicit product above does not carry over; this is mainly because $X_\lambda(x;q)$ is not a single affine Demazure character. Finally, the quantum alcove model arises in Lenart and Postnikov's conjectural description of the quantum product by a divisor in the *quantum K-theory* of flag varieties G/B [LP07] (quantum K-theory is a K-theory analogue of quantum cohomology). Note that, in principle, one can derive the structure constants in the quantum K-theory of G/B from the K-theoretic J-function mentioned above, although this is hard. In Figure 5 we summarize the connections discussed above, as well as the related work of Ion [Ion03] and Fourier-Littelmann [FL06] mentioned in Section 1.

The quantum LS path model and the quantum alcove model were implemented in the computer algebra system `Sage` [Sage]. Using this implementation, we verified some conjectures related to KR crystals in the exceptional types (except for two Dynkin nodes for type $E_8^{(1)}$), see [LNS+14]; these conjectures, which had been previously proved only in the classical types in [FOS10], are concerned with the perfectness property of KR crystals [HKO+02], and with their graded classical decompositions [HKO+99].

There have been several developments related to our work. Starting from our results, a combinatorial realization of the crystal basis of a level 0 extremal weight module and the corresponding Demazure modules (over a quantum affine algebra) is exhibited in [INS14, NS14], in terms of so-called *semi-infinite LS paths*. (An extremal weight module is generated by an extremal weight vector of some affine weight λ; while for λ of positive or negative level, this is just the corresponding integrable highest, respectively lowest weight module, for λ of level 0 its structure is much more complicated.) As a corollary, a crystal-theoretic interpretation of the relation between local and global Weyl modules is given in [NS14].

A formula for the specialized Macdonald polynomial $E_{w_\circ\lambda}(x;q,t=\infty)$ in terms of quantum LS paths is given in [NNS15], by analogy with the one for $E_\mu(x;q,0)$ mentioned in Section 6. Furthermore, a representation-theoretic interpretation of $E_{w_\circ\lambda}(x;q,\infty)$ is given in terms of a Demazure submodule of the level 0 extremal weight module mentioned above. This is again somewhat similar to the representation-theoretic interpretation of $E_\mu(x;q,0)$ discussed in Section 6.

On another hand, our work was used in [CSS+14] to provide the character of a stable level-one Demazure module associated to type $B_n^{(1)}$ as an explicit combination of suitably specialized Macdonald polynomials. In addition, our results were used in a crucial way by Chari and Ion in [CI15, Theorem 4.2] to show that Macdonald polynomials at $t=0$ are characters of *local Weyl modules* for current algebras. Based on this, they prove a *Bernstein-Gelfand-Gelfand (BGG) reciprocity* theorem for the category of representations of a current algebra. In related work, Khoroshkin [Kho13] exhibits a categorification of Macdonald polynomials, by realizing them as the Euler characteristic of bigraded characters for certain complexes of modules over a current algebra. This realization simplifies considerably if BGG reciprocity holds (the mentioned complexes become actual modules concentrated in homological degree zero).

References

[BF14a] A. Braverman and M. Finkelberg. Semi-infinite Schubert varieties and quantum K-theory of flag manifolds. *J. Amer. Math. Soc.*, 27:1147-1168, 2014.

[BF14b] A. Braverman and M. Finkelberg. Weyl modules and q-Whittaker functions. *Math. Ann.*, 359:45-59, 2014.

[BFP99] F. Brenti, S. Fomin, and A. Postnikov. Mixed Bruhat operators and Yang-Baxter equations for Weyl groups. *Int. Math. Res. Not.*, 8:419–441, 1999.

[BL] C. Briggs and C. Lenart. A charge statistic in type B. In preparation.

[CI15] V. Chari and B. Ion. BGG reciprocity for current algebras. *Compos. Math.*, 151:1265–1287, 2015.

[CSS+14] V. Chari, L. Schneider, P. Shereen, and J. Wand. Modules with Demazure flags and character formulae. *SIGMA*, 10, 032, 16pp, 2014.

[FL06] G. Fourier and P. Littelmann. Tensor product structure of affine Demazure modules and limit constructions. *Nagoya Math. J.*, 182:171–198, 2006.

[FOS09] G. Fourier, M. Okado, and A. Schilling. Kirillov-Reshetikhin crystals for nonexceptional types. *Adv. Math.*, 222:1080–1116, 2009.

[FOS10] G. Fourier, M. Okado, and A. Schilling. Perfectness of Kirillov-Reshetikhin crystals for nonexceptional types. *Contemp. Math.*, 506:127–143, 2010.

[FSS07] G. Fourier, A. Schilling, and M. Shimozono. Demazure structure inside Kirillov-Reshetikhin crystals. *J. Algebra*, 309:386–404, 2007.

[Ful97] W. Fulton. *Young Tableaux*. Cambridge University Press, 1997.

[FW04] W. Fulton and C. Woodward. On the quantum product of Schubert classes. *J. Algebraic Geom.*, 13:641–661, 2004.

[HKO+02] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Z. Tsuboi. Paths, crystals and fermionic formulae. In *MathPhys Odyssey 2001. Integrable Models and Beyond* (M. Kashiwara and T. Miwa, Eds.), Prog. Math. Phys. Vol. 23, pp. 205–272, Birkhäuser, Boston, 2002.

[HKO+99] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Y. Yamada. Remarks on fermionic formula. In *Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998)*, volume 248 of *Contemp. Math.*, pages 243–291. Amer. Math. Soc., Providence, RI, 1999.

[HK00] J. Hong and S.J. Kang. *Introduction to Quantum Groups and Crystal Bases*, volume 42 of *Graduate Studies in Mathematics*. Amer. Math. Soc., 2000.

[Ion03] B. Ion. Nonsymmetric Macdonald polynomials and Demazure characters. *Duke Math. J.*, 116:299–318, 2003.

[INS14] M. Ishii, S. Naito, and D. Sagaki. Semi-infinite Lakshmibai-Seshadri path model for level-zero extremal weight modules over quantum affine algebras. *Adv. Math.*, to appear (`arXiv:1402.3884`).

[Kas91] M. Kashiwara. On crystal bases of the q-analogue of universal enveloping algebras. *Duke Math. J.*, 63:465–516, 1991.

[KN94] M. Kashiwara and T. Nakashima. Crystal graphs for representations of the q-analogue of classical Lie algebras. *J. Algebra*, 165:295–345, 1994.

[Kho13] A. Khoroshkin. Highest weight categories and Macdonald polynomials. `arXiv:1312.7053`.

[KR90] A. Kirillov and N. Reshetikhin. Representations of Yangians and multiplicities of the inclusion of the irreducible components of the tensor product of representations of simple Lie algebras. *J. Sov. Math.*, 52:3156–3164, 1990.

[LS79] A. Lascoux and M.-P. Schützenberger. Sur une conjecture de H. O. Foulkes. *C. R. Acad. Sci. Paris Sér. I Math.*, 288:95–98, 1979.

[Len07] C. Lenart. On the combinatorics of crystal graphs, I. Lusztig's involution. *Adv. Math.*, 211:324–340, 2007.

[Len11] C. Lenart. Hall-Littlewood polynomials, alcove walks and fillings of Young diagrams. *Discrete Math.*, 311:258–275, 2011.

[Len12] C. Lenart. From Macdonald polynomials to a charge statistic beyond type A. *J. Combin. Theory Ser. A*, 119:683–712, 2012.

[LL15a] C. Lenart and A. Lubovsky. A generalization of the alcove model and its applications. *J. Algebraic Combin.*, 41:751–783, 2015.

[LL15b] C. Lenart and A. Lubovsky. A uniform realization of the combinatorial R-matrix. `arXiv:1503.01765`.

[LNS+14] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono. A uniform model for Kirillov-Reshetikhin crystals II: Alcove model, path models and $P = X$. *Int. Math. Res. Not.*, to appear (`arXiv:1402.2203`).

[LNS+15a] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono. A uniform model for Kirillov-Reshetikhin crystals I: Lifting the parabolic quantum Bruhat graph. *Int. Math. Res. Not.*, no. 7, 1848–1901, 2015.

[LNS+15b] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono. A uniform model for Kirillov-Reshetikhin crystals III: Nonsymmetric Macdonald polynomials at $t = 0$ and Demazure characters. `arXiv:1511.00465`.

[LP07] C. Lenart and A. Postnikov. Affine Weyl groups in K-theory and representation theory. *Int. Math. Res. Not.*, pages 1–65, 2007. Art. ID rnm038.

[LP08] C. Lenart and A. Postnikov. A combinatorial model for crystals of Kac-Moody algebras. *Trans. Amer. Math. Soc.*, 360:4349–4381, 2008.

[LS13] C. Lenart and A. Schilling. Crystal energy functions via the charge in types A and C. *Math. Z.*, 273:401–426, 2013.

[Lit94] P. Littelmann. A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras. *Invent. Math.*, 116:329–346, 1994.

[Lit95] P. Littelmann. Paths and root operators in representation theory. *Ann. of Math.* (2), 142:499–525, 1995.

[Ma95] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford Mathematical Monographs. Oxford University Press, Oxford, second edition, 1995.

[Ma03] I. G. Macdonald. *Affine Hecke algebras and orthogonal polynomials*. Cambridge Tracts in Mathematics, 157. Cambridge University Press, Cambridge, 2003.

[NNS15] S. Naito, F. Nomoto, and D. Sagaki. Specialization of nonsymmetric Macdonald polynomials at $t = \infty$ and Demazure submodules of level-zero extremal weight modules. arXiv:1511.07005.

[NS05] S. Naito and D. Sagaki. Crystal of Lakshmibai-Seshadri paths associated to an integral weight of level zero for an affine Lie algebra. *Int. Math. Res. Not.*, 14:815–840, 2005.

[NS06] S. Naito and D. Sagaki. Path model for a level-zero extremal weight module over a quantum affine algebra. II. *Adv. Math.*, 200:102–124, 2006.

[NS08] S. Naito and D. Sagaki. Lakshmibai-Seshadri paths of level-zero weight shape and one-dimensional sums associated to level-zero fundamental representations. *Compos. Math.*, 144:1525–1556, 2008.

[NS14] S. Naito and D. Sagaki. Demazure submodules of level-zero extremal weight modules and specializations of Macdonald polynomials. *Math. Zeit.*, to appear (arXiv:1404.2436).

[OS13] D. Orr and M. Shimozono. Specializations of nonsymmetric Macdonald-Koornwinder polynomials. arXiv:1310.0279.

[RY11] A. Ram and M. Yip. A combinatorial formula for Macdonald polynomials. *Adv. Math.*, 226:309–331, 2011.

[Sage] Stein, W. A., and others. *Sage Mathematics Software (Version 7.2)*. The Sage Development Team, 2016. http://www.sagemath.org.

[ST12] A. Schilling and P. Tingley. Demazure crystals, Kirillov-Reshetikhin crystals, and the energy function. *Electronic J. Combin.*, 19, P2, 2012.

[Yam98] Y. Yamane. Perfect crystals of $U_q(G_2^{(1)})$. *J. Algebra*, 210:440–486, 1998.

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