EQUIVARIANT K-CHEVALLEY RULES FOR KAC-MOODY FLAG MANIFOLDS

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ABSTRACT. Explicit combinatorial cancellation-free rules are given for the product of an equivariant line bundle class with a Schubert class in the torus-equivariant K-theory of a Kac-Moody flag manifold. The weight of the line bundle may be dominant or antidominant, and the coefficients may be described either by Lakshmibai-Seshadri paths or by the λ-chain model of the first author and Postnikov [LP, LP1]. For Lakshmibai-Seshadri paths, our formulas are the Kac-Moody generalizations of results of Griffeth and Ram [GR] and Pittie and Ram [PR] for finite dimensional flag manifolds. A gap in the proofs of the mentioned results is addressed.

1. INTRODUCTION

1.1. **Chevalley formulas.** The Chevalley formula [Che] is a multiplication formula in the cohomology ring of a generalized flag variety G/B, where G is a complex semisimple Lie group, and B is a Borel subgroup. It expresses the action of multiplication by the class of a line bundle L^λ with respect to the basis of Schubert classes, and it implies a rule for multiplication by a divisor class, known as Monk's rule in type A. The surjectivity of Borel's homomorphism in this setting implies that the ring structure is determined by this rule.

One may consider the K-theory of G/B, equivariant with respect to a maximal torus $T \subset B$, and the multiplication by a line bundle class on the basis of equivariant classes of structure sheaves of Schubert varieties. By forgetting equivariance and applying the Chern map, one may recover the original Chevalley formula from an equivariant K-theoretic one. The multiplication in this ring is again determined by a rule for the product by line bundles: indeed, the equivariant Atiyah-Hirzebruch homomorphism surjects.

The first such equivariant K-Chevalley formula, due to Pittie and Ram [PR], is described using the combinatorics of Lakshmibai-Seshadri (LS) paths [LS]. It applies to a dominant weight λ, and it is a positive formula (i. e., it has no cancellations). It was derived using a commutation relation in the K-theoretic affine nilHecke algebra of Kostant and Kumar [KK]. The same approach was applied to an antidominant weight λ by Griffeth and Ram [GR], and the resulting formula is also cancellation-free. The Pittie-Ram formula was explained geometrically by Littelmann and Seshadri [LiS], based on standard monomial theory [LS]. Willems [Wil] gave a K-theory Chevalley formula for an arbitrary weight λ, which has cancellations even when λ is dominant or antidominant.

A K-Chevalley formula which works for an arbitrary weight λ, and which is cancellation-free when λ is dominant or antidominant, was given by the first author and Postnikov [LP] in terms of their alcove or λ-chain model (cf. also [LP1]). Besides allowing λ to be an arbitrary weight, the λ-chain model formula has other advantages over the previous formulas. For example, the

Deodhar lifts (from W/W_λ to W, where W_λ is the stabilizer of λ), which are required in the Pittie-Ram formula, and which are given by a nontrivial recursive procedure [Deo] [LS2], are completely avoided. The only computations necessary for the λ-chain model involve Bruhat covers, together with a simple lex order.

We give four cancellation-free Chevalley formulas for the torus-equivariant K-theory of Kashiwara's Kac-Moody thick flag manifold [Kas]: for dominant and antidominant weights, both cases being described in terms of LS paths and the λ-chain model. Note that both LS paths and the λ-chain model were already available in the Kac-Moody setting (as the term "alcove model" is only appropriate in the finite case, we will not use it here). Although it is known that the equivariant Borel and Atiyah-Hirzebruch maps need not be surjective outside of the finite-dimensional case [Kum], strong evidence has arisen that knowing the product by line bundles still determines the ring structure in equivariant K-theory. For thick Kac-Moody flag schemes associated with root systems of affine type, it is shown in [KS] that, after inverting some scalars which multiply trivially against Schubert classes, the equivariant Atiyah-Hirzebruch map is surjective.

1.2. Approach to proofs.

The LS path formulas are proved first, by extending the Pittie-Ram approach to the Kac-Moody setting. The first problem to address is the absence of a top Schubert class, which previously played a crucial role in the passage from the commutation relation in the Hecke algebra to the Chevalley formula. This is circumvented using equivariant localization in K-theory and the action of the K-theoretic nilHecke ring on the torus equivariant K-theory of a Kac-Moody flag manifold. This method of computation was pioneered by Kostant and Kumar [KK] for Kac-Moody flag ind-varieties, but it is perhaps most naturally interpreted in the geometry of the thick Kac-Moody flag schemes [Kas, KS].

Secondly, while working in the Kac-Moody setup, it is necessary to address a gap in the proofs in [PR, GR]. The formula in [PR], which also appears as the first formula in [GR, Theorem 3.5], is correct, but the gap in its proof in both papers was not addressed so far; a complete but different proof, based on geometric arguments, was given later in [LiS]. The second formula in [GR, Theorem 3.5] is incorrect, and the error is due to a similar gap in its proof; the correct formula is (3.5) in Theorem 3.4. These proofs are complete only when λ is regular (i. e., when W_λ and the lifts are trivial), as there is no treatment of the rather subtle interaction between the crystal graph structure on LS paths [Lit1, Lit2] and the Deodhar lifts. The study of this interaction constitutes the technical core of the proof of the LS path Chevalley formulas.

The λ-chain model Chevalley formulas are then derived from the LS path versions. In [LP1] a crystal graph isomorphism was given between LS paths and the λ-chain model; it was defined by perturbing an LS path and taking a limit. We refine this bijection to establish a bijection between the subcollections of LS paths and objects in the λ-chain model that are relevant to the Chevalley formulas. The refined bijection depends on a new description of the bijection in [LP1]. Some crucial ingredients in the refined bijection are the notion of EL-shellability of the Bruhat order on a Coxeter group based on Dyer's reflection orders [Dyer], and the description of LS paths using a weaker version of the Bruhat order called the b-Bruhat order [LS] (see also [Ste]). These considerations completely clarify the λ-chain model due to the first author and Postnikov [LP, LP1].

1.3. **Additional combinatorial consequences and examples.** As a byproduct, we obtain formulations in terms of the λ-chain model for the Demazure and opposite Demazure subcrystals of highest weight crystals for Kac-Moody algebras. For Demazure crystals, the λ-chain formulation was previously known in [LP] in the case of finite root systems.

Let us now point out some combinatorial features of LS paths and the λ-chain model regarding Demazure and opposite Demazure crystals. LS paths are the paragon for realizing the crystal graphs of highest weight modules over quantum groups. One of their historically well-known bonus features is the immediate detection of membership, via initial and final directions, in a Demazure (resp. opposite Demazure) subcrystal; the latter is the crystal subgraph whose underlying module is a submodule of a highest weight module generated from an extremal weight vector by the action of an upper (resp. lower) triangular subalgebra $U_q(\mathfrak{b}_+)$ (resp. $U_q(\mathfrak{b}_-)$) of the quantum group $U_q(\mathfrak{g})$. Another feature of LS paths is the efficient construction of Demazure crystals using stringwise crystal graph operations starting from the highest weight vector [Kas2].

By realizing the crystal in terms of the λ-chain model, one can equip it with some operations which are fundamental to this model, but which, to the authors' knowledge, were not previously part of the crystal graph technology in the literature. More precisely, in the λ-chain model, the vertices of the crystal are indexed by certain saturated chains in the Bruhat order on the Weyl group; the mentioned new operations correspond to appending a Bruhat cover to such a chain, subject to certain conditions (see below). It is these operations which make the λ-chain model more efficient than LS paths in calculating K-Chevalley coefficients, in particular, by avoiding the Deodhar lifts mentioned above.

In the λ-chain model, the most natural way to generate the Demazure crystal is to construct a rooted tree which starts at the *lowest* weight vector. More precisely, the root vertex is labeled by any Weyl group element that sends the highest weight of the Demazure module to its lowest weight, and a child of a tree vertex v is determined by a Weyl group element covered by v together with an integer decoration, subject to a pruning condition that compares this edge of the tree with the one coming down to v. The objects in the model are given by the paths from the root to any vertex.

For the λ-chain model, the opposite Demazure crystal is generated in an entirely similar manner to the Demazure crystal. Instead, one starts at the highest extremal weight vector, and generates decorated Bruhat covers using a different but equally simple pruning condition for branches. In contrast, the authors are unaware of a simple way to efficiently generate the opposite Demazure crystal using crystal graph operations.

We conclude the paper with some examples. These include a Chevalley formula for the K-theory of the type A affine Grassmannian Gr_{SL_n}, which uses the n-restricted partitions in the Misra-Miwa model [MM].

1.4. **Future work.** Since the λ-chain model K-Chevalley formula works in a uniform manner for an arbitrary weight λ in the finite-dimensional setting, we speculate that such a formula exists in Kac-Moody generality. We believe that the general approach in [LP] can be extended in spite of some obvious obstacles. New ideas are required for the definitions associated with a general weight λ, especially for weights outside the Tits cone, such as weights of level zero for root systems of affine type. Note that the LS path formula, used here as a starting point, is currently available only for dominant and antidominant λ.

1.5. **Computer implementation.** We have implemented the various Kac-Moody Chevalley rules in Sage [Sage], an open source mathematics software system, using the `sage-combinat` extension [SC]. After fine-tuning and a period of peer review, these programs will be incorporated into `sage-combinat` for free public distribution.

2. The Kac-Moody thick flag manifold and equivariant K-theory

The computations in this section directly apply to the classical flag manifold as well; indeed, we only need to note that the Bruhat stratification of a Schubert variety used here is labeled upside down compared to the classical case.

2.1. **Equivariant K-Chevalley coefficients.** Let X be a *Kac-Moody thick flag manifold* [Kas] over $\mathbb{C}$ with Dynkin node set I. If the underlying Lie algebra $\mathfrak{g}$ is infinite-dimensional then X is a scheme of infinite type (as opposed to the flag ind-scheme studied in [KK, Kum]). It contains a canonical point x_0, and it has an action of the Borel group B and each of the minimal parabolic subgroups P_i. Let W be the Weyl group. For $w \in W$, let $X_w^\circ := B \cdot wx_0$ be the Schubert cell. X_w° is the Spec of a polynomial ring (with a countably infinite number of generators if $\dim \mathfrak{g} = \infty$). Its Zariski closure is the *Schubert variety X_w*. It has codimension $\ell(w)$ in X. There are cell decompositions

$$(2.1) \qquad X = \bigsqcup_{w \in W} X_w^\circ \qquad\qquad X_w = \bigsqcup_{\substack{v \in W \\ v \geq w}} X_v^\circ$$

where $\geq$ is the Bruhat order on W. Let $S \subset W$ be a nonempty finite Bruhat order ideal (if $w \in S$ and $v \leq w$ then $v \in S$). Let

$$(2.2) \qquad \Omega_S := \bigcup_{w \in S} w \cdot X_e^\circ = \bigsqcup_{w \in S} X_w^\circ.$$

Let $K^B(\Omega_S)$ be the Grothendieck group of B-equivariant coherent sheaves on the B-stable quasi-compact open subset Ω_S. For $w \in W$ the structure sheaf $\mathcal{O}_{X_w}$ is a B-equivariant coherent $\mathcal{O}_X$-module which by restriction defines a class $[\mathcal{O}_{X_w}] \in K^B(\Omega_S)$ provided that $w \in S$. Define

$$(2.3) \qquad K^B(X) = \varprojlim_S K^B(\Omega_S).$$

There is an isomorphism $K^T(X) \cong K^B(X)$, where $T \subset B$ is the maximal torus. We have [KS]

$$(2.4) \qquad K^T(X) = \prod_{w \in W} K^T(\mathrm{pt})[\mathcal{O}_{X_w}].$$

In particular, the product of two Schubert classes $[\mathcal{O}_{X_u}]$ and $[\mathcal{O}_{X_v}]$ in $K^T(X)$ may be an infinite $K^T(\mathrm{pt})$-linear combination of classes $[\mathcal{O}_{X_w}]$ if X is infinite-dimensional.

For a weight λ, let L^λ denote the T-equivariant line bundle on X of weight λ. Define the *equivariant K-Chevalley coefficients* $a_{v\lambda}^w \in K^T(\text{pt})$ by

$$(2.5) \qquad [L^\lambda][\mathcal{O}_{X_v}] = \sum_{w \in W} a_{v\lambda}^w [\mathcal{O}_{X_w}].$$

Our main result, Corollary 3.7, gives explicit cancellation-free combinatorial formulas in terms of Lakshmibai-Seshadri (LS) paths for the Chevalley multiplicities $a_{v\lambda}^w$ if λ is dominant or antidominant. In Theorem 4.8, we express these Chevalley rules in terms of the λ-chains of the first author and Postnikov [LP, LP1].

2.2. The nilHecke ring and Chevalley coefficients.
The coefficients $a_{v\lambda}^w$ may be computed using the K-theoretic *nilHecke* ring of Kostant and Kumar [KK], which we recall here.

There are isomorphisms $K^T(\text{pt}) \cong R(T) \cong \mathbb{Z}[\Lambda]$, where $R(T)$ is the representation ring of the torus, Λ is the weight lattice, and $\mathbb{Z}[\Lambda]$ is its group algebra (with a $\mathbb{Z}$-basis of formal exponents e^λ, for $\lambda \in \Lambda$, and multiplication $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$). We shall use these identifications without additional mention in the sequel. Let $\{\alpha_i^\vee \mid i \in I\} \subset \Lambda^* = \text{Hom}_\mathbb{Z}(\Lambda, \mathbb{Z})$ be the simple coroots, and $\langle \cdot, \cdot \rangle : \Lambda^* \times \Lambda \to \mathbb{Z}$ be the evaluation pairing. The Weyl group W acts on Λ by $s_i \cdot \lambda = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i$ for $i \in I$ and $\lambda \in \Lambda$. This induces an action of W on $R(T)$ and $Q(T) = \text{Frac}(R(T))$. For $\lambda \in \Lambda$, let $e^\lambda \in R(T)$ denote the isomorphism class of the one-dimensional T-module of weight λ.

For $i \in I$, define the operator T_i on the fraction field $Q(T) = \text{Frac}(R(T))$ by

$$(2.6) \qquad T_i = (1 - e^{\alpha_i})^{-1}(s_i - 1).$$

For $\lambda \in \Lambda$, we have [LSS]

$$(2.7) \qquad T_i \cdot e^\lambda = \frac{e^\lambda - e^{s_i\lambda}}{e^{\alpha_i} - 1} = \begin{cases} e^\lambda(e^{-\alpha_i} + e^{-2\alpha_i} \cdots + e^{(-\langle \alpha_i^\vee, \lambda \rangle)\alpha_i}) & \text{if } \langle \alpha_i^\vee, \lambda \rangle > 0 \\ 0 & \text{if } \langle \alpha_i^\vee, \lambda \rangle = 0 \\ -e^\lambda(1 + e^{\alpha_i} + \cdots + e^{(-1-\langle \alpha_i^\vee, \lambda \rangle)\alpha_i}) & \text{if } \langle \alpha_i^\vee, \lambda \rangle < 0. \end{cases}$$

It follows that T_i acts on $R(T)$. The *Demazure operator* [Dem] is $D_i = 1 + T_i$. The T_i satisfy the braid relations and $T_i^2 = -T_i$ for all $i \in I$. Therefore we may define $T_w = T_{i_1} \cdots T_{i_N}$ for any reduced decomposition $w = s_{i_1} \cdots s_{i_N}$, where $i_1, \ldots, i_N \in I$. Let $\mathbb{K}_0$ be the 0-Hecke algebra, which is the subring of $\text{End}(R(T))$ generated by $\{T_i \mid i \in I\}$. We have the following identity of operators on $R(T)$ [LSS, (2.6)], where e^λ denotes the operator of left multiplication by $e^\lambda \in R(T)$:

$$(2.8) \qquad T_i e^\lambda = (T_i \cdot e^\lambda) + e^{s_i\lambda} T_i.$$

The nilHecke ring $\mathbb{K}$ is by definition the smash product of $\mathbb{K}_0$ and $R(T)$ (acting on itself by left multiplication). By (2.8) we have

$$(2.9) \qquad \mathbb{K} = \bigoplus_{w \in W} R(T) T_w.$$

Let $b_{v\lambda}^w \in K^T(\text{pt})$ be defined by the following relation in $\mathbb{K}$:

$$(2.10) \qquad T_w e^\lambda = \sum_{v \in W} b_{v\lambda}^w T_v.$$

Lemma 2.1. *We have*

$$(2.11) \qquad a_{v\lambda}^w = b_{v\lambda}^w.$$

2.3. **Localization and the proof of Lemma 2.1.** Lemma 2.1 may be proved using localization. We use the notation of [LSS]. The computations here are essentially due to [KK] but technically we are connecting the computations in [KK] to the thick Kac-Moody flag geometry.

Restriction to T-fixed points yields an injective ring homomorphism res : $K^T(X) \to K^T(X^T)$. Let Ψ be the image of res. Since $X^T \cong W$, one may view $K^T(X^T)$ as the $K^T(\mathrm{pt})$-algebra $\mathrm{Fun}(W, R(T))$ of functions $W \to K^T(\mathrm{pt})$.

Let $\psi^v = \mathrm{res}([\mathcal{O}_{X_v}])$ for $v \in W$. Then (2.4) translates to

$$\tag{2.12} \Psi = \prod_{v \in W} R(T)\psi^v.$$

The functions ψ^v may be characterized as follows. Let $\mathbb{K}_Q := Q(T) \otimes_{R(T)} \mathbb{K} = Q(T) \otimes_{\mathbb{Q}} \mathbb{Q}[W]$, in light of (2.6). Let $\mathrm{Hom}_{Q(T)}(\mathbb{K}_Q, Q(T))$ be the $\mathbb{Q}$-vector space of left $Q(T)$-module homomorphisms $\mathbb{K}_Q \to Q(T)$. A function $\psi \in \mathrm{Fun}(W, Q(T))$ may be regarded as an element of $\mathrm{Hom}_{Q(T)}(\mathbb{K}_Q, Q(T))$ by

$$\tag{2.13} \psi(\sum_{w \in W} a_w w) = \sum_w a_w \psi(w),$$

where $a = \sum_{w \in W} a_w w \in Q(T) \otimes_{\mathbb{Q}} \mathbb{Q}[W]$. Evaluation defines a perfect pairing

$$\langle \cdot, \cdot \rangle : \mathbb{K}_Q \times \mathrm{Hom}_{Q(T)}(\mathbb{K}_Q, Q(T)) \to Q(T).$$

This restricts to a perfect pairing $\mathbb{K} \times \Psi \to R(T)$, and with respect to this pairing, $\{T_w \mid w \in W\}$ and $\{\psi^v \mid v \in W\}$ are dual bases as left modules over $R(T)$ [LSS]:

$$\tag{2.14} \langle T_w, \psi^v \rangle = \delta_{vw} \qquad \text{for } v, w \in W.$$

By abuse of notation, for $\lambda \in \Lambda$, let L^λ denote the image under res of the class $[L^\lambda] \in K^T(X)$. We have

$$\tag{2.15} L^\lambda(w) = e^{w\lambda} \qquad \text{for all } w \in W.$$

Lemma 2.2. *For any $\lambda \in \Lambda$, $a \in \mathbb{K}$, and $\psi \in \Psi$ we have*

$$\tag{2.16} \langle a, L^\lambda \psi \rangle = \psi(ae^\lambda).$$

Proof. Write $a \in \mathbb{K}$ as $a = \sum_{w \in W} a_w w$ for $a_w \in Q(T)$. We have

$$\begin{aligned}
\langle a, L^\lambda \psi \rangle &= (L^\lambda \psi)(a) \\
&= (L^\lambda \psi)(\sum_w a_w w) \\
&= \sum_w a_w (L^\lambda \psi)(w) \\
&= \sum_w a_w L^\lambda(w) \psi(w) \\
&= \sum_w a_w e^{w\lambda} \psi(w).
\end{aligned}$$

On the other hand,

$$\psi(ae^\lambda) = \psi(\sum_w a_w w e^\lambda)$$

$$= \psi(\sum_w a_w e^{w\lambda} w)$$

$$= \sum_w a_w e^{w\lambda} \psi(w).$$

$\square$

Proof of Lemma 2.1. We have

$$\begin{aligned}
a_{v\lambda}^w &= \langle T_w \,, L^\lambda \psi^v \rangle && \text{by (2.5) and (2.14)} \\
&= (L^\lambda \psi^v)(T_w) \\
&= \psi^v(T_w e^\lambda) && \text{by Lemma 2.2} \\
&= \psi^v(\sum_u b_{u\lambda}^w T_u) && \text{by (2.10)} \\
&= \sum_u b_{u\lambda}^w \psi^v(T_u) \\
&= \sum_u b_{u\lambda}^w \delta_{uv} && \text{by (2.14)} \\
&= b_{v\lambda}^w.
\end{aligned}$$

$\square$

2.4. Recurrence.

A recurrence for the coefficients $b_{z\lambda}^w$ is obtained as follows. Assuming that $s_i w < w$, we have

$$\sum_{z \in W} b_{z\lambda}^w T_z = T_w e_\lambda$$

$$= T_i T_{s_i w} e^\lambda$$

$$= T_i \sum_y b_{y\lambda}^{s_i w} T_y$$

$$= \sum_y \left(T_i \cdot b_{y\lambda}^{s_i w} + s_i(b_{y\lambda}^{s_i w}) T_i \right) T_y \,.$$

Using

$$(2.17) \qquad\qquad T_i T_y = \begin{cases} T_{s_i y} & \text{if } s_i y > y \\ -T_y & \text{if } s_i y < y \end{cases}$$

and taking coefficients of T_z on both sides, we have

$$(2.18) \qquad\qquad b_{z\lambda}^w = T_i \cdot b_{z\lambda}^{s_i w} + \chi(s_i z < z)\, s_i(b_{s_i z, \lambda}^{s_i w} - b_{z\lambda}^{s_i w}).$$

Summarizing, for $s_i w < w$ and $z < s_i z$ we have

$$(2.19) \qquad\qquad b_{z\lambda}^w = T_i \cdot b_{z\lambda}^{s_i w}$$

$$(2.20) \qquad\qquad b_{s_i z, \lambda}^w = T_i \cdot b_{s_i z, \lambda}^{s_i w} + s_i(b_{z\lambda}^{s_i w} - b_{s_i z, \lambda}^{s_i w}) \,.$$

By iterating this recurrence, one may obtain an explicit formula for $b_{v\lambda}^w$ which typically has a lot of cancellation. Fix a reduced word $a_1 a_2 \cdots a_N$ of w. Let $E(v,a)$ be the set of sequences $\epsilon = (\epsilon_1, \ldots, \epsilon_N) \in [0,1]^N$ such that

$$\prod_{\substack{i \\ \epsilon_i=1}} T_{a_i} = \text{sgn}(\epsilon) T_v \,,$$

where $\text{sgn}(\epsilon) \in \{\pm 1\}$, and the product is ordered from left to right by increasing i. Here note that $T_i^2 = -T_i$. For $\epsilon \in E(v,a)$, let $A_\epsilon = A_1 A_2 \cdots A_N$, where

$$A_i = \begin{cases} s_{a_i} & \text{if } \epsilon_i = 1 \\ T_{a_i} & \text{if } \epsilon_i = 0. \end{cases}$$

Then we have

(2.21) $$b_{v\lambda}^w = \left(\sum_{\epsilon \in E(v,a)} \text{sgn}(\epsilon) A_\epsilon \right) \cdot e^\lambda.$$

Example 2.3. In type A_2 consider $w = s_1 s_2 s_1$, $v = s_1$, and $\lambda = 2\omega_1 + \omega_2 = (310)$. Using the above reduced word a for w, we have $E(v,a) = \{(100), (001), (101)\}$ and

$$b_{v\lambda}^w = (s_1 T_2 T_1 + T_1 T_2 s_1 - s_1 T_2 s_1) \cdot e^{310}$$
$$= e^{022} + e^{013}.$$

3. The Chevalley formula in terms of LS paths

3.1. **The main result.** We recall the definition of *Deodhar lifts* in Coxeter groups. For more details the reader may see Sections 3.2 and 3.3.

Let W be a Coxeter group and let $S = \{s_i \mid i \in I\}$ be the set of simple reflections. A reflection in W is a W-conjugate of a simple reflection. The length $\ell = \ell(w)$ of $w \in W$ is the minimum ℓ such that $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ for $i_1, \ldots, i_\ell \in I$. The (strong) Bruhat order $\leq$ on W is the partial order with covering relation $v \lessdot w$ if $\ell(w) = \ell(v) + 1$ and there is a reflection $r \in W$ such that $w = vr$.

For a subset $J \subset I$, let W_J denote the subgroup of W generated by s_i for $i \in J$. Let W^J be the set of minimum length coset representatives in W/W_J. The set W^J inherits the Bruhat order from W. The Bruhat order $\leq$ on W/W_J is defined by declaring that the bijection $W^J \to W/W_J$ given by $w \mapsto wW_J$ is an isomorphism of posets.

Proposition 3.1. [Deo] [LS2, Lemma 4.4, 4.4'] (1) *Let $\tau \in W/W_J$ and $v \in W$ be such that $vW_J \leq \tau$ in W/W_J. Then the set*

$$\{w \in W \mid v \leq w \text{ and } wW_J = \tau\}$$

has a Bruhat-minimum, which will be denoted by $\text{up}(v, \tau)$.

(2) *Let $\tau \in W/W_J$ and $w \in W$ be such that $wW_J \geq \tau$ in W/W_J. Then the set*

$$\{v \in W \mid w \geq v \text{ and } vW_J = \tau\}$$

has a Bruhat-maximum, which will be denoted by $\text{dn}(w, \tau)$. The symbol dn is an abbreviation for "down".

Fix a dominant weight $\lambda \in \Lambda^+$; its stabilizer W_λ is the parabolic subgroup W_J with $J = \{i \in I \mid s_i \cdot \lambda = \lambda\}$, and we identify the quotient W/W_J with W^J and the orbit $W \cdot \lambda$. We will now define the set $\mathcal{T}^\lambda$ of *Lakshmibai-Seshadri (LS) paths* of shape λ [LS]. We start with the definition of a suborder of the Bruhat order, called the *b-Bruhat order*, where b is a fixed rational number.

Definition 3.2. [LS, Ste] The *b-Bruhat order* $\leq_b$ on W is defined by those Bruhat covers $w \lessdot ws_\alpha$ in which the positive root α satisfies $b\langle \alpha^\vee, \lambda \rangle \in \mathbb{Z}$.

Clearly, if b is an integer, then $\leq_b$ is the entire Bruhat order.

Definition 3.3. [LS, Ste] An LS path $p \in \mathcal{T}^\lambda$ is given by two sequences

$$(3.1) \quad 0 = b_1 < b_2 < b_3 < \ldots < b_m < b_{m+1} = 1 \qquad \phi(p) = \sigma_1 <_{b_2} \sigma_2 <_{b_3} \ldots <_{b_m} \sigma_m = \iota(p) \,,$$

where $b_i \in \mathbb{Q}$ and $\sigma_i \in W/W_\lambda \cong W \cdot \lambda$. The elements σ_i are called the directions of p, while $\iota(p)$ and $\phi(p)$ are the initial and final directions, respectively.

This data encodes the sequence of vectors

$$(3.2) \qquad v_m := (b_{m+1} - b_m)\sigma_m \cdot \lambda \,, \quad \ldots \,, \quad v_2 := (b_3 - b_2)\sigma_2 \cdot \lambda \,, \quad v_1 := (b_2 - b_1)\sigma_1 \cdot \lambda \,.$$

We can view the LS path p as a piecewise-linear path given by the sequence of points

$$0 \,, \quad v_m \,, \quad v_{m-1} + v_m \,, \quad \ldots \,, \quad v_1 + \ldots + v_m \,.$$

There is also a standard way to express p as a map $p : [0,1] \to \mathfrak{h}^*_\mathbb{R}$ with $p(0) = 0$ (where $\mathfrak{h}^*_\mathbb{R}$ is the real part of the dual Cartan subalgebra), but we do not need this here. The endpoint of the path is $p(1) = v_1 + \ldots + v_m$.

Given an LS path $p \in \mathcal{T}^\lambda$ and $z \in W$ such that $zW_\lambda \leq \phi(p)$, let $\mathrm{up}(z, p) \in W$ be defined as follows. If the directions of p are as in (3.1), we define the sequence of Weyl group elements

$$z = w_0 \leq w_1 \leq \ldots \leq w_m = \mathrm{up}(z, p)$$

recursively by $w_i := \mathrm{up}(w_{i-1}, \sigma_i)$ for i from 1 to m. Similarly, given $w \in W$ such that $\iota(p) \leq wW_\lambda$, define

$$w = w_{m+1} \geq w_m \geq \ldots \geq w_1 = \mathrm{dn}(w, p)$$

by $w_i := \mathrm{dn}(w_{i+1}, \sigma_i)$ for i from m down to 1.

For $z, w \in W$, define

$$(3.3) \qquad \mathcal{U}^\lambda_{w,z} = \{\, p \in \mathcal{T}^\lambda \mid \phi(p) \geq zW_\lambda \text{ and } \mathrm{up}(z, p) = w \,\} \,,$$

$$(3.4) \qquad \mathcal{D}^\lambda_{w,z} = \{\, p \in \mathcal{T}^\lambda \mid \iota(p) \leq wW_\lambda \text{ and } \mathrm{dn}(w, p) = z \,\} \,.$$

Theorem 3.4. *We have*

$$(3.5) \qquad T_w \, e^\lambda = \sum_{z \leq w} \sum_{p \in \mathcal{U}^\lambda_{w,z}} e^{p(1)} \, T_z \,,$$

$$(3.6) \qquad T_w \, e^{-\lambda} = \sum_{\substack{p \in \mathcal{T}^\lambda \\ \iota(p) \leq wW_\lambda}} (-1)^{\ell(w) - \ell(\mathrm{dn}(w,p))} e^{-p(1)} \, T_{\mathrm{dn}(w,p)} \,.$$

Remark 3.5. The first formula in [GR, Theorem 3.5] implies (3.6) by reversing the order of terms (given the symmetry of the relations defining the K-theoretic nilHecke ring) and by applying the algebra involution defined by $e^\lambda \mapsto e^{-\lambda}$ (as left multiplication operators) and

$T_i \mapsto -1 - T_i$ (or, equivalently, $D_i \mapsto 1 - D_i$). The second formula in [GR, Theorem 3.5] is rewritten as follows, simply by reversing the order of terms:

$$(3.7) \qquad T_w \, e^\lambda = \sum_{\substack{p \in \mathcal{T}^\lambda \\ \iota(p)=wW_\lambda}} \sum_{\substack{\zeta \in W/W_\lambda \\ \zeta \leq \phi(p)}} e^{p(1)} \, T_{\mathrm{dn}(\mathrm{dn}(w,p),\zeta)} \, .$$

We claim that this formula is incorrect, by comparing it with (3.5). Indeed, for $z \leq w$, formula (3.5) expresses the coefficient $b_{z\lambda}^w$ (see (2.10)) as a sum over $\mathcal{U}_{w,z}^\lambda$. By contrast, (3.7) expresses the same coefficient as a sum over the set

$$\widetilde{\mathcal{D}}_{w,z}^\lambda = \{ \, p \in \mathcal{T}^\lambda \mid \phi(p) \geq zW_\lambda, \, \iota(p) = wW_\lambda, \text{ and } \mathrm{dn}(\mathrm{dn}(w,p),zW_\lambda) = z \, \} \, .$$

Note that the condition $\iota(p) = wW_\lambda$ in the above definition is implicit in the definition of $\mathcal{U}_{w,z}^\lambda$. The following simple example shows that, in general, even in finite types, we have $\mathcal{U}_{w,z}^\lambda \neq \widetilde{\mathcal{D}}_{w,z}^\lambda$. As we can see, the error is essentially due to the fact that the two Deodhar lifts, "up" and "dn", are not inverses (as maps between two comparable cosets).

Example 3.6. Consider type A_2 and $\lambda = \omega_2 = (110)$, so $W_\lambda = \langle s_1 \rangle$. Let $z = 1$ and $w = s_1 s_2$. The LS paths in $\mathcal{T}^\lambda$ all consist only of one direction, which can be any of the 3 elements of W/W_λ (as the corresponding crystal consists only of the extremal weight vertices). The only LS path p which can be in $\mathcal{U}_{w,z}^\lambda$ and $\widetilde{\mathcal{D}}_{w,z}^\lambda$ is the one with direction $w = s_1 s_2$. This is indeed in $\mathcal{U}_{w,z}^\lambda$ as $\mathrm{up}(z,p) = \mathrm{up}(z,wW_\lambda) = w$. However, p is not in $\widetilde{\mathcal{D}}_{w,z}^\lambda$ because $\mathrm{dn}(\mathrm{dn}(w,p),zW_\lambda) = \mathrm{dn}(w,zW_\lambda) = s_1 \neq z$.

By Lemma 2.1 we obtain the following result.

Corollary 3.7. *We have*

$$(3.8) \qquad [L^\lambda] \, [\mathcal{O}_z] = \sum_{\substack{p \in \mathcal{T}^\lambda \\ \phi(p) \geq zW_\lambda}} e^{p(1)} [\mathcal{O}_{\mathrm{up}(z,p)}] \, ,$$

$$(3.9) \qquad [L^{-\lambda}] \, [\mathcal{O}_z] = \sum_{w \geq z} \sum_{p \in \mathcal{D}_{w,z}^\lambda} (-1)^{\ell(w)-\ell(z)} \, e^{-p(1)} \, [\mathcal{O}_w] \, .$$

Remark 3.8. We have described the antidominant formula (3.6) using the combinatorics of the highest weight crystal $\mathcal{T}^\lambda$ with λ dominant. Conceptually it is better to use the lowest weight crystal $\mathcal{T}^{-\lambda}$; for infinite-dimensional Lie algebras these are not highest weight crystals, as they are in the finite-dimensional setting. The natural parametrization of the directions in an LS path in $\mathcal{T}^{-\lambda}$ gives a sequence that is increasing in the Bruhat order. We find it convenient to work with the more familiar objects in $\mathcal{T}^\lambda$. This is achieved by applying the crystal antiautomorphism $\mathcal{T}^{-\lambda} \to \mathcal{T}^\lambda$; upon viewing an LS path as the sequence of vectors $(v_m, \dots, v_1)$ in (3.2), this antiautomorphism is defined by reversing the sequence of vectors and negating them, that is, by sending it to the sequence $(-v_1, \dots, -v_m)$. This contragredient duality map has the effect of negating weights and reversing colored arrows. We do not see a way to deduce the antidominant formula from the dominant one.

Proposition 3.9 below is the crucial result needed to prove relation (3.5). To state it, recall that $\mathcal{T}^\lambda$ is a model for the *crystal graph* of the irreducible $U_q(\mathfrak{g})$-module of highest weight λ where $U_q(\mathfrak{g})$ is the quantum group. As such, $\mathcal{T}^\lambda$ is a graph with vertex set $\mathcal{T}^\lambda$ and directed edges colored by the Dynkin node set I. The connected components of the restriction of the

graph to arrows labeled i for a fixed i, are finite directed paths called *i-strings*. For more details on the crystal graph structure on LS paths we refer to Section 3.4.

Fix an i-string $\mathcal{S}$ in $\mathcal{T}^\lambda$. Let h be the head or source of the string $\mathcal{S}$, t its tail, and $m = \mathcal{S}\backslash\{h,t\}$ its middle. Provisionally, for all $w,z \in W$, we write $\mathcal{U}_{w,z}(\mathcal{S}) = \mathcal{U}_{w,z}^\lambda \cap \mathcal{S}$ and $s = s_i$.

Proposition 3.9. *Let $z \in W$ be such that $sz > z$ and $zW_\lambda \leq \phi(h)$ and write $w = \mathrm{up}(z,h)$. Then $sw > w$ and*

$$(3.10) \qquad \{\,\mathrm{up}(z,p) \mid p \in \mathcal{S}\,\} \subseteq \{w,sw\} \text{ and } \phi(p) \geq zW_\lambda \text{ for } p \in \mathcal{S}$$

In terms of $\mathrm{up}(sz,\cdot)$ on $\mathcal{S}$, we have the following three disjoint cases:

$(3.11) \quad \phi(p) \geq szW_\lambda \text{ for } p \in \mathcal{S} \text{ and } \{\mathrm{up}(sz,p) \mid p \in \mathcal{S}\} \subseteq \{w,sw\}$

$(3.12) \quad \phi(t) \geq szW_\lambda, \ \mathrm{up}(sz,t) \in \{w,sw\}, \text{ and } \phi(p) \not\geq szW_\lambda \text{ for } p \in \mathcal{S} \setminus \{t\}$

$(3.13) \quad \phi(p) \geq szW_\lambda \text{ for } p \in \mathcal{S}, \ \mathrm{up}(sz,t) \in \{w,sw\}, \text{ and } \{\mathrm{up}(sz,p) \mid p \in \mathcal{S} \setminus \{t\}\} \subseteq \{\tilde{w},s\tilde{w}\}$

where $\tilde{w} \notin \{w,sw\}$ and $s\tilde{w} > \tilde{w}$. More precisely, the following chart gives the pairwise disjoint possibilities in terms of arbitrary x with $sx > x$.

(3.14)

$\mathcal{U}_{x,z}(\mathcal{S})$	$\mathcal{S}$			h			$\emptyset$	
$\mathcal{U}_{sx,z}(\mathcal{S})$	$\emptyset$			$\mathcal{S}\setminus h$			$\emptyset$	
$\mathcal{U}_{x,sz}(\mathcal{S})$	t	$\mathcal{S}$	$\{h,t\}$	$\emptyset$	h	$\mathcal{S}\setminus t$	$\mathcal{S}\setminus t$	h
$\mathcal{U}_{sx,sz}(\mathcal{S})$	$\emptyset$	$\emptyset$	m	t	$\mathcal{S}\setminus h$	t	$\emptyset$	m
$|\mathcal{S}|$	≥ 1	≥ 2	≥ 3	≥ 1	≥ 2	≥ 3	≥ 2	≥ 3
Cases	U.1.1	U.1.2	U.1.3	U.2.1	U.2.2	U.2.3	U.3.1	U.3.2

Cases U.a.b for $a \in \{1,2\}$ and $b \in \{2,3\}$ correspond to (3.11) when $x = w$. Cases U.1.1 and U.2.1 correspond to (3.12) and (3.13) when $x = w$. Cases U.3. correspond to (3.13) when $x = \tilde{w}$.*

Given Proposition 3.9 one may deduce the dominant weight LS-path Chevalley rule (3.5).

Proof of relation (3.5). Given a set of paths $\mathcal{P}$, let

$$\Sigma(\mathcal{P}) := \sum_{p \in \mathcal{P}} e^{p(1)}\,.$$

Let $x,z \in W$ be such that $sx > x$ and $sz > z$. It suffices to show that the coefficients in (3.5) satisfy the recurrence relations (2.19) and (2.20) for the Chevalley coefficients, namely,

$$\Sigma(\mathcal{U}_{sx,z}) = T_i \cdot \Sigma(\mathcal{U}_{x,z}) \qquad \Sigma(\mathcal{U}_{sx,sz}) = T_i \cdot \Sigma(\mathcal{U}_{x,sz}) + s\left(\Sigma(\mathcal{U}_{x,z}) - \Sigma(\mathcal{U}_{x,sz})\right).$$

Since $\mathcal{T}^\lambda$ is partitioned into i-strings, it suffices to establish these relations for every i-string $\mathcal{S}$ with $\mathcal{U}_{u,v}$ replaced by $\mathcal{U}_{u,v}(\mathcal{S})$ for all u,v. In every column of the above table, it is easy to verify these relations. For instance, in Case U.1.1, the first relation $T_i \cdot \Sigma(\mathcal{S}) = 0$ follows by acting with T_i on

$$\Sigma(\mathcal{S}) = T_i \cdot e^{h(1)} + e^{h(1)}\,,$$

which is (2.7). The second relation, in the same case, amounts to

$$T_i \cdot e^{s\,h(1)} + \Sigma(\mathcal{S}) - e^{h(1)} = 0\,,$$

which is also (2.7). All the other relations follow in a similar way. $\qquad \square$

The proof of Proposition 3.9 occupies the next several subsections.

Remark 3.10. If λ is regular (that is, $W_\lambda = \{1\}$) then the Deodhar lifts are trivial. For such λ, by Proposition 3.19 only cases U.a.b for $a, b \in \{1, 2\}$ can arise.

3.2. **The Bruhat order.** We recall some basic properties of the Bruhat order on a Coxeter group, beginning with the well-known *Z-property*.

Proposition 3.11. [Hum] *Let $sv < v$ and $sw < w$. Then $v \leq w$ if and only if $sv \leq w$ if and only if $sv \leq sw$.*

Proposition 3.12. (1) [Bou][p. 37] *We have $W^J = \{w \in W \mid ws > w \text{ for all } s \in S \cap W_J\}$.*
  (2) [Bou][p. 37] *Given $w \in W$ there exist unique $w^J \in W^J$ and $w_J \in W_J$ such that $w = w^J w_J$. Moreover $\ell(w) = \ell(w^J) + \ell(w_J)$.*
  (3) [Deo] *Given $w \in W^J$ and $s \in S$, exactly one of the following occurs:*
    (a) *$sw < w$. Then $sw \in W^J$ and $swW_J < wW_J$.*
    (b) *$sw > w$ and $sw \in W^J$. Then $swW_J > wW_J$.*
    (c) *$sw > w$ and $sw \notin W^J$. Then $sw = ws'$ for some $s' \in S \cap W_J$ and $swW_J = wW_J$.*
  (4) [Deo] *For $v^J, w^J \in W^J$ and $v_J, w_J \in W_J$, if $v = v^J v_J \leq w = w^J w_J$ then $v^J \leq w^J$ and $vW_J \leq wW_J$.*

Lemma 3.13. *Consider σ, τ in W/W_J and $s \in S$. Suppose $s\sigma \leq \sigma$ and $s\tau \leq \tau$. Then $\sigma \leq \tau$ if and only if $s\sigma \leq \tau$ if and only if $s\sigma \leq s\tau$.*

Proof. Let $v, w \in W^J$ be such that $vW_J = \sigma$ and $wW_J = \tau$. By Proposition 3.12 (3), if $s\sigma < \sigma$ then $sv < v$ and $sv \in W^J$, while if $s\tau < \tau$ then $sw < w$ and $sw \in W^J$. If both of these cases hold, the result is immediate by Proposition 3.11. If $s\sigma = \sigma$ and $s\tau < \tau$, then $sv > v$, by Proposition 3.12 (3). It suffices to show that $\sigma \leq \tau$ implies $\sigma \leq s\tau$, i.e., $v \leq w$ implies $v \leq sw$; but this follows from Proposition 3.11. Finally, assume that $s\sigma < \sigma$ and $s\tau = \tau$. Like above, by Proposition 3.12 (3), we have $w < sw = ws'$ for $s' \in W_J$. It now suffices to show that $s\sigma \leq \tau$ implies $\sigma \leq \tau$; but this follows since $sv \leq w$ implies $v \leq sw = ws'$, by Proposition 3.11, which in turn implies $v \leq w$, by Proposition 3.12 (4). $\square$

3.3. **Deodhar lifts.** The setup is still that of Coxeter groups. Recall from Proposition 3.1 (2) the definition of the Deodhar lifts $\mathrm{up}(v, \tau)$.

Lemma 3.14. *Let $v \in W$ and $\tau \in W/W_J$ be such that $vW_J \leq \tau$. Let $s \in S$ and $w := \mathrm{up}(v, \tau)$. If $s\tau > \tau$ (resp. $s\tau < \tau$), then $sw > w$ (resp. $sw < w$). If $\tau = s\tau$ and $sv > v$, then $sw > w$.*

Proof. Since $wW_J = \tau$, the first statement is clear by Proposition 3.12 (4). Now let us consider the case $\tau = s\tau$. Suppose $sw < w$. Since $v \leq w$ and $sv > v$, we have $v \leq sw$, by Proposition 3.11. But $swW_J = s\tau = \tau$. By the minimality of w we obtain the contradiction $w \leq sw$. Therefore $sw > w$, as required. $\square$

Lemma 3.15. *Suppose $v \in W$ and $\tau \in W/W_J$ are such that $sv > v$ and $vW_J \leq s\tau < \tau$. Letting $y = \mathrm{up}(v, s\tau)$ and $w = \mathrm{up}(v, \tau)$, we have $y = sw < w$.*

Proof. By Lemma 3.14, we have $sy > y$ and $w > sw$. Since $sy > y \geq v$ and $syW_J = wW_J$, by the minimality of w we have $w \leq sy$. Then Proposition 3.11 gives $sw \leq y$.

On another hand, since $v \leq w$, we have $v \leq sw$, by Proposition 3.11. Since $swW_J = yW_J$, by the minimality of y we have $y \leq sw$. Therefore $y = sw$. $\square$

Lemma 3.16. *Let $v \in W$, $s \in S$, and $\tau \in W/W_J$ be such that $v < sv$, $svW_J \leq \tau$, and $s\tau \leq \tau$. Let $y = \mathrm{up}(sv, \tau)$ and $w = \mathrm{up}(v, \tau)$. Then $y = w$ or $y = sw > w$, and the latter holds only if $s\tau = \tau$.*

Proof. Since $v < sv \leq y$ and $yW_J = wW_J$, by the minimality of w we have $w \leq y$. Suppose first that $s\tau < \tau$. By Lemma 3.14, we have $sw < w$. Since $v \leq w$, we have $sv \leq w$, by Proposition 3.11. The minimality of y implies $y \leq w$, so $y = w$. Now consider the case $s\tau = \tau$. By Lemma 3.14, we have $sw > w$. Proposition 3.11 gives $sv \leq sw$. We have $swW_J = s\tau = \tau$, so the minimality of y implies $y \leq sw$. Thus we proved $w \leq y \leq sw$, as required. $\square$

Example 3.17. If $v < sv$ but $\tau < s\tau$, then y can be something completely different. Take type A_3, $J = \{1\}$, $v = s_3$, $s = s_1$, and $\tau = s_2 s_3 W_J$. Then $w = s_2 s_3$ and $y = s_2 s_3 s_1$.

3.4. **LS paths and lifts.** We revert to the setup of root systems for symmetrizable Kac-Moody algebras. For a detailed description of LS paths we refer to [LS, Ste]. The set $\mathcal{T}^\lambda$ of LS paths of shape λ can be characterized as the set generated by crystal operators f_i, starting from the straight-line path from 0 to λ [Lit2]. The non-recursive description of LS paths is given in Section 4.5.

Let $i \in I$ be a fixed Dynkin node. Recall the decomposition of the crystal graph $\mathcal{T}^\lambda$ into i-strings. The crystal operator f_i (resp. e_i) on $\mathcal{T}^\lambda$ is the partial operator on $\mathcal{T}^\lambda$ (function mapping from a subset of $\mathcal{T}^\lambda$ into $\mathcal{T}^\lambda$) that sends a vertex $p \in \mathcal{T}^\lambda$ to the next (resp. previous) vertex on its i-string if that vertex exists, and is otherwise undefined on p. We now recall the explicit definitions of f_i and e_i on $\mathcal{T}^\lambda$.

Let $p \in \mathcal{T}^\lambda$. Let the i-height of a point v be $\langle \alpha_i^\vee, v \rangle$. The following hold [Lit2]:

- The maximum drop in i-height for an initial segment of p, is a nonnegative integer; call it $\varepsilon_i(p)$. (This is also the number of steps along the i-string of p back to its beginning.)
- The maximum jump in i-height for a final segment of p, is a nonnegative integer; call it $\varphi_i(p)$. (This is also the number of steps along the i-string of p to the end.)

The element $p \in \mathcal{T}^\lambda$ may be viewed as a sequence of vectors (the vectors v_j in (3.2)) and the path proceeds by walking along the successive vectors (from 0 to v_m to $v_m + v_{m-1}$, etc.) Every vector in p changes the i-height by a rational number. Let N be a positive integer that is an integer multiple of all the denominators of those rational numbers.

Let $d : \mathcal{T}^\lambda \to \mathcal{T}^{N\lambda}$ be the map that dilates every vector by N. This map leaves the data (3.1) of an element p unchanged, so d is obviously well-defined and injective. It clearly satisfies $\varphi_i(d(p)) = N\varphi_i(p)$. Since $\varphi_i(p) \in \mathbb{Z}$, the following are equivalent: $\varphi_i(d(p)) \geq N$; $\varphi_i(p) \geq 1$; $\varphi_i(p) > 0$; $\varphi_i(d(p)) > 0$. Similar statements hold for ε_i.

So to compute $f_i(p)$, first p is dilated to $d(p)$. Then one applies the signature rule (explained below) for f_i on $d(p)$. If $d(p)$ does not admit f_i then neither does p. If $d(p)$ admits f_i then $d(p)$ admits f_i^N and $f_i(p)$ is obtained by shrinking the path $f_i^N(d(p))$ by a factor of N.

We now explain the process of applying f_i^N to $d(p)$. By the choice of N, each vector in $d(p)$ changes i-height by an integer. Replace each vector v in $d(p)$ of i-height h, by $|h|$ copies of the vector $(1/|h|)v$. Then each vector in $d(p)$ changes i-height by -1, 0, or 1. We now treat this sequence of vectors as tensor factors in a tensor product of crystals and use the *signature rule*. The i-signature is the word $h_1 h_2 \ldots$, where h_j is the height of the j-th vector. Each -1 (resp. 1) in the i-signature is treated as a left (resp. right) parenthesis, and 0s are ignored. Pairing parentheses as usual, the unpaired subsequence consists of some number (which can

be shown to be $\varepsilon_i(d(p))$ of right parentheses followed by some number (which can be shown to be $\varphi_i(d(p))$) of left parentheses. To apply f_i (resp. e_i), the reflection s_i is applied to the vector corresponding to the first unmatched right (resp. last unmatched left) parenthesis. This negates the height of the changed vector, and has the effect of changing the i-signature by negating the corresponding entry and changing the kind of parenthesis at that spot.

One may instead apply this process directly to p instead of $d(p)$, except that each vector (after appropriate subdivision) should change height by $1/N$, 0, or $-1/N$, and that to achieve $f_i(p)$ (resp. $e_i(p)$) one must apply s_i to each of the first (resp. last) N vectors corresponding to unmatched right (resp. left) parentheses. An important point is that if there is one unmatched right (resp. left) parenthesis then there must automatically be N of them.

In the sequel, when discussing the i-string structure on $\mathcal{T}^\lambda$, we will abuse notation by replacing each suitably subdivided vector in p by its direction, i.e., the corresponding element of $W/W_\lambda \cong W \cdot \lambda$.

Example 3.18. Consider the root system of type A_2, with $\lambda = \omega_1 + \omega_2$. We use GL_3 notation $\alpha_1 = (1, -1, 0)$, $\alpha_2 = (0, 1, -1)$, $\omega_1 = (1, 0, 0)$ and $\omega_2 = (1, 1, 0)$. Then $\lambda = (2, 1, 0)$. The highest weight vector $v_\lambda \in \mathcal{T}^\lambda$ consists of the single vector $v_\lambda = ((2, 1, 0))$. We apply f_1. We have $\langle \alpha_1^\vee, (2, 1, 0) \rangle = 2 - 1 = 1$ so the 1-signature of v_λ is $(+1)$. We apply s_1 to this step, so that $f_1 v_\lambda = (((1, 2, 0)))$. Let us apply f_2 to $f_1 v_\lambda$. We have $\langle \alpha_2^\vee, (1, 2, 0) \rangle = 2 - 0 = 2$, so we subdivide the path to obtain $((1/2, 1, 0), (1/2, 1, 0))$. The path $f_1 v_\lambda$ has 2-signature $(+1, +1)$. Applying f_2 has the effect of applying s_2 to the vector for the first $+1$, so we get $f_2 f_1 v_\lambda = ((1/2, 0, 1), (1/2, 1, 0))$. If we now try to apply f_1, note that the changes in 1-height are $1/2, -1/2$. By definition $\varphi_1(f_2 f_1 v_\lambda)$ is the maximum sum among tails of the sequence $(1/2, -1/2)$; this is 0. Therefore f_1 cannot be applied.

Let $\mathcal{S} \subset \mathcal{T}^\lambda$ be a fixed i-string. It may be written

$$(3.15) \qquad \mathcal{S} = \{h = p_0, \, p_1, \, \ldots, \, p_m = t\}.$$

where h is the head, t is the tail, and $p_k = f_i^k(h)$. Let

$$(3.16) \qquad \phi_k := \phi(p_k), \qquad \iota_k := \iota(p_k),$$

for $0 \le k \le m$. The following standard result is a consequence of the signature rule.

Proposition 3.19. *If* $|\mathcal{S}| \ge 2$, *we have*

$$(3.17) \qquad \iota_0 = \iota_1 = \ldots = \iota_m < s_i \iota_0 \qquad or \qquad \iota_0 < \iota_1 = \ldots = \iota_m = s_i \iota_0$$

$$(3.18) \qquad s_i \phi_m < \phi_0 = \ldots = \phi_{m-1} = \phi_m \qquad or \qquad s_i \phi_m = \phi_0 = \ldots = \phi_{m-1} < \phi_m.$$

If $|\mathcal{S}| = 1$, *we have* $s_i \iota_0 \ge \iota_0$ *and* $s_i \phi_0 \le \phi_0$.

We now study the way the lifts of LS paths change along the string $\mathcal{S}$ in (3.15).

Lemma 3.20. *Let* $z \in W$ *be such that* $z W_\lambda \le \phi(h)$, *and set* $u = \mathrm{up}(z, h)$. *Assume that* $|\mathcal{S}| \ge 2$ *or* $z < s_i z$. *Then we have* $u < s_i u$.

Proof. Let $h = (\sigma_1 \le \sigma_2 \le \cdots \le \sigma_\ell)$. Let j be largest such that $\sigma_j \ne s_i \sigma_j$, assuming that such an index exists. In particular, this happens when $|\mathcal{S}| \ge 2$. Since h is the head, by the signature rule we have $\sigma_j < s_i \sigma_j$. Let $w_0 = z$ and $w_k = \mathrm{up}(w_{k-1}, \sigma_k)$ for $1 \le k \le \ell$. By applying Lemma 3.14 repeatedly, we deduce first $w_j < s_i w_j$, and then $u < s_i u$.

We are left with the case when $\sigma_k = s_i\sigma_k$ for all k. But then $|\mathcal{S}| = 1$, so we can use the assumption $z = w_0 < s_iw_0$. We conclude the proof as above, by applying Lemma 3.14 repeatedly. $\quad\square$

Lemma 3.21. *Let $p, p' \in \mathcal{S}$ with $f_i(p) = p'$. Let $z \in W$ be such that $zW_\lambda \leq \phi(p)$, so we can define $u := \mathrm{up}(z, p)$ and $u' := \mathrm{up}(z, p')$. Suppose that $p' \neq t$ or $z < s_iz$. Then $u' = u$ or $u' = s_iu > u$, and the latter occurs only if $p = h$.*

Proof. Denote the directions of p as in (3.1) by

$$\phi(p) = \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_\ell = \iota(p).$$

We know that p' is obtained from p by replacing a step σ_j by $s_i\sigma_j > \sigma_j$. Setting $z = w_0 = w'_0$, let $w_k = \mathrm{up}(w_{k-1}, \sigma_k)$ for $1 \leq k \leq \ell$, and define w'_k similarly for p' instead of p. Then $u = w_\ell$ and $u' = w'_\ell$.

Let $1 \leq r \leq j$ be smallest such that $s_i\sigma_k = \sigma_k$ for all $r \leq k < j$. If $r > 1$ then the signature rule implies $\sigma_{r-1} < s_i\sigma_{r-1}$. By applying Lemma 3.14 repeatedly, we deduce first

$$(3.19) \qquad\qquad\qquad\qquad w_{r-1} < s_iw_{r-1},$$

and then $w_{j-1} < s_iw_{j-1}$. Alternatively, if $r = 1$, then $p' = t$ by the signature rule, so we must have $z < s_iz$. But this means that (3.19) again holds, so we can deduce $w_{j-1} < s_iw_{j-1}$ like above. On another hand, we clearly have $w'_k = w_k$ for $0 \leq k < j$. Lemma 3.15 then gives $w'_j = s_iw_j > w_j$.

Let $j \leq q \leq \ell$ be largest such that $s_i\sigma_k = \sigma_k$ for $j < k \leq q$. By repeated applications of Lemma 3.16, we deduce that either $w'_q = w_q$, so that $w'_\ell = w_\ell$ and we are done, or (as we shall assume) $w'_q = s_iw_q > w_q$. If $q = \ell$ then $w'_\ell = s_iw_\ell > w_\ell$; moreover, as there were no steps σ_k with negative i-height for $k > j$, we have $p = h$. Otherwise, we have $q < \ell$ and, by the signature rule, $\sigma_{q+1} > s_i\sigma_{q+1}$. By Lemma 3.16, we have $w'_{q+1} = w_{q+1}$, and therefore $w'_\ell = w_\ell$. $\quad\square$

Lemma 3.22. *Consider $z \in W$ satisfying $z < s_iz$ and $s_izW_\lambda \leq \phi(t)$. Let $u = \mathrm{up}(z, t)$ and $u' = \mathrm{up}(s_iz, t)$. If $|\mathcal{S}| \geq 2$, we always have $u' = u$. If $|\mathcal{S}| = 1$, then we have $u' = u < s_iu$ or $u' = s_iu > u$.*

Proof. Let $t = (\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_\ell)$, $w_0 = z$, $w'_0 = s_iz$, $w_k = \mathrm{up}(w_{k-1}, \sigma_k)$, and $w'_k = \mathrm{up}(w'_{k-1}, \sigma_k)$, for $1 \leq k \leq \ell$. We have $u = w_\ell$ and $u' = w'_\ell$.

Let $0 \leq j \leq \ell$ be largest such that $\sigma_k = s_i\sigma_k$ for $1 \leq k \leq j$. By repeated applications of Lemma 3.16, we have $w'_j = w_j$, which implies $u' = u$, or $w'_j = s_iw_j > w_j$. If $j = \ell$, then the relationship between u' and u is established. Otherwise, in the latter case, we have $\sigma_{j+1} > s_i\sigma_{j+1}$, by the signature rule. Lemma 3.16 then gives $w'_{j+1} = w_{j+1}$, and therefore $u' = u$ once again.

Note that $j = \ell$ only if $|\mathcal{S}| = 1$, so $|\mathcal{S}| \geq 2$ implies $j < \ell$. In the case $|\mathcal{S}| = 1$, we also need to show that $u < s_iu$ in general (this is already known if $u' = s_iu$). This follows from Lemma 3.20, since now $t = h$. $\quad\square$

We called $\mathcal{S} \setminus \{h, t\}$ the middle of the i-string $\mathcal{S}$.

Lemma 3.23. *If $\mathcal{U}^\lambda_{w,z} \cap \mathcal{S}$ contains any element in the middle of $\mathcal{S}$, then it contains the entire middle of $\mathcal{S}$.*

Proof. This follows immediately from Proposition 3.19 and Lemma 3.21. $\quad\square$

3.5. **The proof of Proposition 3.9.** We retain the notation from the previous subsection, in particular (3.15) and (3.16). Let $z \in W$ be such that

$$(3.20) \qquad\qquad\qquad z < s_i z.$$

For the rest of this section we assume that at least one of the lifts $\mathrm{up}(z, p)$ or $\mathrm{up}(s_i z, p)$ are defined for some p in $\mathcal{S}$. That is, at least one of the following statements holds for some k:

$$(3.21) \qquad\qquad\qquad z W_\lambda \leq \phi_k, \qquad s_i z W_\lambda \leq \phi_k.$$

Lemma 3.24. *We have one of the following two cases, which are merged into a single case if $|\mathcal{S}| = 1$: (1) both statements in (3.21) hold for all k; (2) the first statement holds for all k, while the second one only holds for $k = m$.*

Proof. We start by assuming that $|\mathcal{S}| \geq 2$, and by showing that the first statement either holds for all k or for no k. Based on (3.18), this amounts to showing that, if $\phi_0 = s_i \phi_m$ (which is the second case in the mentioned relation), then $z W_\lambda \leq \phi_m$ implies $z W_\lambda \leq \phi_0 = s_i \phi_m$. This implication follows from Lemma 3.13, since $s_i \phi_m < \phi_m$ and $s_i z > z$, so $s_i z W_\lambda \geq z W_\lambda$. To complete the proof, it suffices to show that $z W_\lambda \leq \phi_m$ implies $s_i z W_\lambda \leq \phi_m$, for any $m \geq 0$. This is again justified by Lemma 3.13, since we have $s_i \phi_m \leq \phi_m$ in all possible cases. $\qquad\square$

Now let us consider the following Deodhar lifts, whenever the corresponding inequality in (3.21) holds:

$$u_k := \mathrm{up}(z, p_k), \qquad u'_k := \mathrm{up}(s_i z, p_k).$$

By Lemma 3.24, all u_k are defined, and either all u'_k are defined or only u'_m. We will implicitly use this fact below.

Proof of Proposition 3.9. The main idea is to analyze systematically all possibilities regarding the relationships between u_k and u'_l, for all k and l. For each case, we indicate the corresponding case in the table (3.14).

By Proposition 3.19, there are three main cases.

- *Case* S.0: $|\mathcal{S}| = 1$.
- *Case* S.1: $|\mathcal{S}| \geq 2$ and $u_0 W_\lambda = u_1 W_\lambda = \cdots = u_m W_\lambda < s_i u_0 W_\lambda$.
- *Case* S.2: $|\mathcal{S}| \geq 2$ and $u_0 W_\lambda < u_1 W_\lambda = \cdots = u_m W_\lambda = s_i u_0 W_\lambda$.

<u>*Case* S.0.</u> We have the following two cases, by Lemma 3.22.

Case S.0.1: $u_0 = u'_0 < s_i u_0$. This case leads to Case U.1.1.

Case S.0.2: $u'_0 = s_i u_0 > u_0$. This case leads to Case U.2.1.

We now analyze cases S.1 and S.2, and start with some general observations. We have $u_0 < s_i u_0$ and $u'_0 < s_i u'_0$, by Lemma 3.20. This implies that $u'_0 \neq s_i u_0$, which is implicitly used several times below. By Lemma 3.23, we have $u_1 = u_2 = \cdots = u_{m-1}$, and (when they exist) $u'_1 = u'_2 = \cdots = u'_{m-1}$. In addition, if $|\mathcal{S}| \geq 3$, then Lemma 3.21 gives $u_{m-1} = u_m$. Finally, Lemma 3.22 gives $u_m = u'_m$, so we always have $u_1 = u_2 = \cdots = u_m = u'_m$ in cases S.1 and S.2.

<u>*Case* S.1.</u> By Lemma 3.21, we have either $u_1 = u_0$ or $u_1 = s_i u_0 > u_0$. Suppose the latter. Then $s_i u_1 < u_1$. But $u_1 = u_m$, so $s_i u_m < u_m$, contradicting the assumption of Case S.1. Therefore $u_1 = u_0$ and we have

$$u_0 = u_1 = \cdots = u_m = u'_m < s_i u_0.$$

Case S.1.1: $u_0', \ldots, u_{m-1}'$ are not defined. This leads to Case U.1.1.

We may now assume that $u_0', \ldots, u_{m-1}'$ are defined. We have the following two cases.

Case S.1.2: $u_1' = u_0'$ or $|\mathcal{S}| = 2$. It follows that $u_0' = u_1' = \ldots = u_{m-1}'$. If $u_0' = u_0$, then all u_k and u_l' coincide, and Case U.1.2 occurs. Otherwise, Case U.1.1 occurs for $x = u_0$, and Case U.3.1 for $x = u_0'$.

Case S.1.3: $u_1' \neq u_0'$ and $|\mathcal{S}| \geq 3$. By Lemma 3.21, we have $u_1' = s_i u_0' > u_0'$. If $u_0' = u_0$, then we have Case U.1.3. Otherwise, we have Case U.1.1 for $x = u_0$, and Case U.3.2 for $x = u_0'$.

<u>*Case* S.2.</u> By Lemma 3.21 and the Case S.2 assumption, we have $u_0 < u_1 = s_i u_0$. Therefore, we have

$$u_0 < u_1 = \ldots = u_m = u_m' = s_i u_0 \, .$$

Case S.2.1: $u_0', \ldots, u_{m-1}'$ are not defined. This leads to Case U.2.1.

We may now assume that $u_0', \ldots, u_{m-1}'$ are defined. We have the following two cases.

Case S.2.2: $u_1' \neq u_0'$ or $|\mathcal{S}| = 2$. If $|\mathcal{S}| \geq 3$, then Lemma 3.21 gives $u_1' = s_i u_0' > u_0'$. Assume first $u_0' = u_0$. The above facts imply $u_1 = u_2 = \ldots = u_m = u_1' = u_2' = \ldots = u_m' = s_i u_0$, so Case U.2.2 occurs. Now assume $u_0' \neq u_0$. Then Case U.2.1 occurs for $x = u_0$. Alternatively, for $x = u_0'$, we have Case U.3.2 if $|\mathcal{S}| \geq 3$, and Case U.3.1 if $|\mathcal{S}| = 2$.

Case S.2.3: $u_1' = u_0'$ and $|\mathcal{S}| \geq 3$. If $u_0' = u_0$, then Case U.2.3 occurs. If $u_0' \neq u_0$, then we have Case U.2.1 for $x = u_0$, and Case U.3.1 for $x = u_0'$. $\qquad\qquad \square$

3.6. **On the proof of** (3.6). The proof of the antidominant line bundle Chevalley rule (3.6) is omitted as it is entirely analogous to that of (3.5), but does not appear to formally follow from it. The following is the analogue of Proposition 3.9. Note that the two propositions are related as follows: up switches with dn, ϕ switches with ι, the strings are reversed, and the Bruhat relations are dualized. See Remark 3.8.

Proposition 3.25. *Let $w \in W$ be such that $sw < w$ and $wW_\lambda \geq \iota(t)$. Let $z = \mathrm{dn}(w, t)$. Then $sz < z$ and*

(3.22) $$\{\, \mathrm{dn}(w, p) \mid p \in \mathcal{S} \,\} \subseteq \{z, sz\} \text{ and } \iota(p) \leq wW_\lambda \text{ for } p \in \mathcal{S}.$$

In terms of $\mathrm{dn}(sw, \cdot)$ on $\mathcal{S}$, we have the following three disjoint cases:

(3.23) $\iota(p) \leq swW_\lambda$ *for* $p \in \mathcal{S}$ *and* $\{\mathrm{dn}(sw, p) \mid p \in \mathcal{S}\} \subseteq \{z, sz\}$

(3.24) $\iota(h) \leq swW_\lambda$, $\mathrm{dn}(sw, h) \in \{z, sz\}$, *and* $\iota(p) \not\leq swW_\lambda$ *for* $p \in \mathcal{S} \setminus \{h\}$

(3.25) $\iota(p) \leq swW_\lambda$ *for* $p \in \mathcal{S}$, $\mathrm{dn}(sw, h) \in \{z, sz\}$, *and* $\{\mathrm{dn}(sw, p) \mid p \in \mathcal{S} \setminus \{h\}\} \subseteq \{\tilde{z}, s\tilde{z}\}$

where $\tilde{z} \notin \{z, sz\}$ and $s\tilde{z} < \tilde{z}$.

More precisely, the following chart gives the pairwise disjoint possibilities in terms of arbitrary x with $sx < x$.

(3.26)

$\mathcal{D}_{w,x}(\mathcal{S})$	$\mathcal{S}$			t			$\emptyset$	
$\mathcal{D}_{w,sx}(\mathcal{S})$	$\emptyset$			$\mathcal{S} \setminus t$			$\emptyset$	
$\mathcal{D}_{sw,x}(\mathcal{S})$	h	$\mathcal{S}$	$\{h,t\}$	$\emptyset$	t	$\mathcal{S} \setminus h$	$\mathcal{S} \setminus h$	t
$\mathcal{D}_{sw,sx}(\mathcal{S})$	$\emptyset$	$\emptyset$	m	h	$\mathcal{S} \setminus t$	h	$\emptyset$	m
$|\mathcal{S}|$	≥ 1	≥ 2	≥ 3	≥ 1	≥ 2	≥ 3	≥ 2	≥ 3
Cases	D.1.1	D.1.2	D.1.3	D.2.1	D.2.2	D.2.3	D.3.1	D.3.2

Cases D.*a*.*b* *for* $a \in \{1, 2\}$ *and* $b \in \{2, 3\}$ *correspond to* (3.23) *when* $x = z$. *Cases* D.1.1 *and* D.2.1 *correspond to* (3.24) *and* (3.25) *when* $x = z$. *Cases* D.3.* *correspond to* (3.25) *when* $x = \tilde{z}$.

This concludes the proof of Theorem 3.4.

4. The Chevalley formula in terms of the λ-chain model

In this section the Chevalley rules in Theorem 3.4 are formulated in terms of the alcove or λ-chain model of the first author and Postnikov [LP, LP1]; as mentioned in the Introduction, we will avoid here the term "alcove model" because it is only appropriate in the finite case. As a by-product, we obtain combinatorial descriptions of Demazure crystals and opposite Demazure crystals [Kas2] in terms of the λ-chain model, in Kac-Moody generality. In a mild difference of notation, in the definition of a λ-chain we use coroots consistently instead of roots, as in [LP, LP1].

4.1. **The λ-chain model.** Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra defined over $\mathbb{C}$. Fix a dominant weight λ. By a root or coroot, we always mean a real root or coroot unless explicitly stated otherwise.

An *integral hyperplane* in $\Lambda_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \Lambda$ is one of the form

$$(4.1) \qquad\qquad H_{\alpha,k} = \{x \in \Lambda_{\mathbb{R}} \mid \langle \alpha, x \rangle = k\}$$

where α is a positive coroot and $k \in \mathbb{Z}$.

Definition 4.1. A λ-*hyperplane* is an integral hyperplane $H_{\alpha,k}$ such that

$$(4.2) \qquad\qquad 0 \le k < \langle \alpha, \lambda \rangle.$$

By abuse of language, we will use the term "λ-hyperplane" to refer either to the pair (α, k) or the actual hyperplane $H_{\alpha,k}$. We call k the height of $H_{\alpha,k}$.

Remarks 4.2. (1) Consider the straight-line path in $\Lambda_{\mathbb{R}}$ from 0 to λ. The λ-hyperplanes are precisely the integral hyperplanes that touch this path but do not contain the endpoint λ.

(2) If $\mathfrak{g}$ is infinite-dimensional then there are infinitely many positive coroots, so that there are typically infinitely many λ-hyperplanes.

Definition 4.3. A λ-*chain* is a total order on the set of λ-hyperplanes such that the following hold.

(1) If $(\alpha, k), (\alpha, k')$ are λ-hyperplanes with $k < k'$, then $(\alpha, k) < (\alpha, k')$.
(2) Given a λ-hyperplane $h = (\beta, k)$, a positive coroot $\alpha \neq \beta$, and an integer m such that $\gamma = \alpha + m\beta$ is a positive coroot, we have

$$N_{<h}(\gamma) = N_{<h}(\alpha) + mN_{<h}(\beta)$$

where $N_{<h}(\eta)$ is the number of λ-hyperplanes less than h with coroot η.

Remark 4.4. The original definition of a λ-chain in [LP, LP1] is obtained by forgetting the integer k in each pair (α, k). More precisely, it involves the corresponding sequence of roots, while the counting condition uses the associated coroots. The integers k are easily recovered by labeling the copies of the coroot α by 0 through $\langle \alpha, \lambda \rangle - 1$ in order of their appearance in the sequence of coroots.

In [LP1] a particular λ-chain is constructed. It is described in the following proposition. In particular λ-chains exist.

Proposition 4.5. [LP1] *Given a total order $I = \{1 < 2 < \cdots < r\}$ on the set of Dynkin nodes, one may express a coroot $\alpha = \sum_{i=1}^{r} c_i \alpha_i^{\vee}$ in the $\mathbb{Z}$-basis of simple coroots. Consider the total order on the set of λ-hyperplanes defined by the lexicographic order on their images in $\mathbb{Q}^{r+1}$ under the map*

$$(4.3) \qquad (\alpha, k) \mapsto \frac{1}{\langle \alpha, \lambda \rangle}(k, c_1, \ldots, c_r).$$

This map is injective, thereby endowing the set of λ-hyperplanes with a total order, which is a λ-chain. We call it the lexicographic (lex) λ-chain.

For a finite root system, the definition of a λ-chain may be simplified. The following is a characterization of the sequence of coroots (with repetition) obtained from a λ-chain in the above sense, when the height k of a λ-hyperplane (α, k) is forgotten. The heights are easily recovered due to condition (1) of Definition 4.3.

Proposition 4.6. [LP1] *Consider a finite root system and a finite sequence $(\beta_1, \beta_2, \ldots, \beta_\ell)$ of positive coroots. Then the following are equivalent:*

(1) *The sequence of coroots is a λ-chain.*
(2) *Each positive coroot α occurs exactly $\langle \alpha, \lambda \rangle$ times in the sequence, and for each triple of positive coroots (α, β, γ) with $\gamma = \alpha + \beta$, the subsequence restricted to copies of α, β, and γ is a concatenation of pairs (α, γ) and (β, γ) (in any order).*
(3) *There exists a reduced alcove path $A_0 = A_\circ \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_l} A_l = A_{-\lambda}$, in the sense of [LP].*

Recall the notation $A \xrightarrow{\beta} A'$, which means that the alcoves A and A' are separated by a hyperplane orthogonal to the coroot β, which points in the direction from A to A'; $A_\circ$ is the fundamental alcove, and A_μ is its translation by μ. Note that the first two properties are not equivalent in the Kac-Moody case. The reason is that there are broken β-strings of real roots through α. Indeed, for an affine root system, consider the positive roots $\alpha = \overline{\alpha} + k\delta$ and $\beta = -\overline{\alpha} + m\delta$ with $k, m > 0$, and $\overline{\alpha}$ a root of the corresponding non-affine root system. Note that $\alpha + 2\beta$ is a positive real root, but $\alpha + \beta$ is an imaginary root.

In the sequel we will treat λ-chains either as sequences of positive coroots or as sequences of λ-hyperplanes, passing between the two definitions without further mention.

4.2. **The Chevalley formula.** Let us fix a dominant integral weight λ and an arbitrary λ-chain. For $\gamma \in \Lambda$, let t_γ be the operator on Λ given by translation by γ. Note that we are not working in an affine Weyl group but inside the group of automorphisms of the lattice Λ. For a coroot α, let $\alpha^{\vee}$ be the associated root. By definition, if $\alpha = w\alpha_i^{\vee}$ for $w \in W$ and $i \in I$, then $\alpha^{\vee} = w\alpha_i$. For a coroot α, let s_α act on Λ by the reflection

$$(4.4) \qquad s_\alpha \cdot \mu = \mu - \langle \alpha, \mu \rangle \alpha^{\vee}.$$

For a λ-hyperplane $h = (\alpha, k)$, we use the notation

$$k_h := k \tag{4.5}$$

$$m_h := \langle \alpha, \lambda \rangle - k \tag{4.6}$$

$$s_h := s_\alpha \tag{4.7}$$

$$\widehat{s}_h := t_{k\alpha^\vee} s_\alpha \tag{4.8}$$

$$\widetilde{s}_h := t_{m_h \alpha^\vee} s_\alpha. \tag{4.9}$$

The quantity k_h is the number of hyperplanes with the same coroot α before h in the given λ-chain. Note that $\widehat{s}_h$ is the reflection in Λ across the affine hyperplane $H_{\alpha,k}$.

Definition 4.7. For $z, w \in W$ with $z \leq w$, we say that a sequence of λ-hyperplanes $h_1, h_2, \ldots, h_q$ (not necessarily increasing in some λ-chain) is $[z, w]$-*adapted* if the coroots of the hyperplanes are the associated coroots for a saturated Bruhat chain from z to w:

$$z \lessdot zs_{h_1} \lessdot zs_{h_1}s_{h_2} \lessdot \ldots \lessdot zs_{h_1}s_{h_2} \cdots s_{h_q} = w. \tag{4.10}$$

We say that a sequence of λ-hyperplanes is z-*adapted* if it is $[z, w]$-adapted for some $w \geq z$.

Theorem 4.8. *Let λ be a dominant weight. We have*

$$[L^\lambda][\mathcal{O}_z] = \sum_{\substack{(h_1 < \cdots < h_q) \\ z\text{-adapted}}} e^{z\widehat{s}_{h_1} \cdots \widehat{s}_{h_q}(\lambda)} [\mathcal{O}_{zs_{h_1}\ldots s_{h_q}}], \tag{4.11}$$

$$[L^{-\lambda}][\mathcal{O}_z] = \sum_{\substack{(h_1 > \cdots > h_q) \\ z\text{-adapted}}} (-1)^q \, e^{-z\widetilde{s}_{h_1} \cdots \widetilde{s}_{h_q}(\lambda)} [\mathcal{O}_{zs_{h_1}\ldots s_{h_q}}], \tag{4.12}$$

where the order on the hyperplanes h_i is the lex λ-chain (cf. Proposition 4.5).

There are also analogues of the two commutation formulas (3.5) and (3.6) in terms of the λ-chain model (see Example 5.2 and 5.5), which are similar to (4.11) and (4.12). Formula (4.11) is proved in Sections 4.3, 4.4, and 4.5, based on the corresponding formula (3.8) in terms of LS paths. The proof of (4.12) is based on (3.9), and is completely similar, cf. Remark 4.17 and Section 4.6.

Remark 4.9. Theorem 4.8 yields a formula for multiplying by $[\mathcal{O}_{X_{s_i}}]$ by noting that

$$[\mathcal{O}_{X_{s_i}}] = 1 - e^{\Lambda_i}[L^{-\Lambda_i}], \tag{4.13}$$

where Λ_i is the i-th fundamental weight.

Conjecture 4.10. *Given an arbitrary fixed dominant weight λ, Theorem 4.8 holds for any λ-chain.*

Note that it sometimes makes sense to consider other λ-chains beside the lexicographic one. For instance, writing λ as a sum of fundamental weights ω_i, we can consider the corresponding concatenation of ω_i-chains; in the finite classical types, the ω_i-chains were explicitly constructed, and the above λ-chains allow one to exhibit a nice correspondence with tableau models; see, e.g., [Len].

Furthermore, as the finite-type K-Chevalley formula in [LP] works for an arbitrary weight λ, Theorem 4.8 should extend to an arbitrary λ as well. This would require the corresponding generalization of the concept of a λ-chain in the Kac-Moody case, which is non-trivial (in the finite case, we can use condition (3) in Proposition 4.6 as a definition, and this is straightforward

to extend to an arbitrary λ). In particular, a λ-chain will now have positive and negative roots, and the condition that the coroot α appears $\langle \alpha, \lambda \rangle$ times must be replaced by the requirement that the number of occurrences of an arbitrary root α is the maximum of 0 and $\langle \alpha, \lambda \rangle$. For instance, we can define a $w(\lambda)$-chain, for λ dominant and w in the Weyl group (so that $w(\lambda)$ is in the Tits cone) essentially by applying w to a λ-chain for dominant λ. Also note that the negative reverse of a λ-chain should be a $(-\lambda)$-chain.

Assuming an arbitrary λ, in order to extend the proof techniques in [LP], it is not enough to consider only the λ-chains mentioned above. It turns out that it is necessary to uniformly prove a Chevalley formula in which the adapted sequences are chosen from a more general set of hyperplanes; these are obtained from the λ-chains above via a "folding" procedure, see [LP1, Section 5]. In such a "folded λ-chain", the same hyperplane can appear several times. Also note that, for general λ, a Chevalley formula will have cancellations, even if it is based on a minimal λ-chain (i.e., having no repeated hyperplanes). This is not the case when λ is dominant or antidominant, as the formulas in Theorem 4.8 have no cancellations.

4.3. **Reflection orders.** The proof of the model Chevalley formula begins with some considerations regarding Dyer's *reflection orders* [Dyer]. Let W be the Weyl group of a symmetrizable Kac-Moody algebra $\mathfrak{g}$.

For the entire proof the dominant weight λ is fixed. Let $J = \{i \in I \mid s_i \cdot \lambda = \lambda\}$. Then $W_\lambda = W_J$ is the stabilizer of λ. Let $W^\lambda = W^J$. Denote by $\Phi^{\vee +}$ the set of positive real coroots for $\mathfrak{g}$. The Bruhat graph on W is the graph with vertex set W and a directed edge from v to w if v is covered by w. This edge is labeled by the unique element $\alpha \in \Phi^{\vee +}$ such that $w = v s_\alpha$. This is denoted $v \xrightarrow{\alpha} w$.

Definition 4.11. [Dyer] A reflection order is a total order on $\Phi^{\vee +}$ satisfying the following property: for every $\alpha, \beta \in \Phi^{\vee +}$ and $a, b \in \mathbb{R}_{>0}$ such that $a\alpha + b\beta \in \Phi^{\vee +}$, we have

$$(4.14) \qquad \alpha < a\alpha + b\beta < \beta \quad \text{or} \quad \beta < a\alpha + b\beta < \alpha.$$

The above definition is one of several equivalent ones. The main result related to reflection orders is the following one, known as the *EL-shellability* of the Bruhat order; we state only the part of this result that we need.

Proposition 4.12. [Dyer] *Let $v \leq w$ in Bruhat order. Then for any reflection order, there exists a unique saturated Bruhat chain from v to w with labels which increase in the reflection order.*

We now define a total order $<_\lambda$ on $\Phi^{\vee +}$ which depends on λ. The bottom of the order $<_\lambda$ consists of the coroots $\alpha \in \Phi^{\vee +}$ such that $\langle \alpha, \lambda \rangle > 0$. For two such coroots α and β, define $\alpha < \beta$ if $(\alpha, 0) < (\beta, 0)$ in the lex λ-chain. This forms an *initial section* [Dyer] of $<_\lambda$. The top of the order $<_\lambda$ consists of the $\alpha \in \Phi^{\vee +}$ orthogonal to λ; such α form the positive coroots for the Weyl group W_λ, and one may use any reflection order for them.

Lemma 4.13. *The total order $<_\lambda$ is a reflection order on $\Phi^{\vee +}$.*

Proof. Consider $\alpha <_\lambda \beta$ in $\Phi^{\vee +}$, and assume that $a\alpha + b\beta \in \Phi^{\vee +}$ for some $a, b \in \mathbb{R}_{>0}$. Let $c_\alpha = \langle \alpha, \lambda \rangle$ and $c_\beta = \langle \beta, \lambda \rangle$. It suffices to show (4.14). This holds if $c_\alpha = c_\beta = 0$ since a reflection order was used for the positive coroots of W_λ. We represent vectors in the basis of

simple coroots as tuples of coordinates, using the chosen order on the simple coroots. Assume first that $c_\alpha, c_\beta > 0$. It suffices to show that

$$\frac{\alpha}{c_\alpha} < \frac{a\alpha + b\beta}{ac_\alpha + bc_\beta} < \frac{\beta}{c_\beta}$$

in lexicographic order; here the inequality between the first vector and the last one is known, as it expresses $\alpha <_\lambda \beta$. The proof is completed by noting that the middle fraction can be written

$$c\,\frac{\alpha}{c_\alpha} + (1-c)\,\frac{\beta}{c_\beta}\,, \qquad \text{where} \qquad c = \frac{ac_\alpha}{ac_\alpha + bc_\beta}$$

is in the interval $(0,1)$. The remaining case is $c_\alpha > 0$, $c_\beta = 0$. It suffices to show

$$\frac{\alpha}{c_\alpha} < \frac{a\alpha + b\beta}{ac_\alpha} = \frac{\alpha}{c_\alpha} + \frac{b}{ac_\alpha}\beta\,,$$

which is obvious. $\qquad\qquad\square$

4.4. **Properties of the b-Bruhat order.** Recall the b-Bruhat order in Definition 3.2, which depends on the fixed dominant weight λ and a fixed rational number b. Let

$$\Phi_b = \{\alpha \in \Phi^{\vee+} \mid b\,\langle\alpha\,,\lambda\rangle \in \mathbb{Z}\} \qquad\qquad \Phi_b^* = \Phi_b \setminus \{\alpha \in \Phi^{\vee+} \mid \langle\alpha\,,\lambda\rangle = 0\}.$$

With this notation, the covers $v \xrightarrow{\alpha} w$ of the b-Bruhat order are defined by the condition $\alpha \in \Phi_b$. For simplicity, we use the term Bruhat (resp. b-Bruhat) chain for a saturated chain in Bruhat (resp. b-Bruhat) order.

Lemma 4.14. *Suppose that $v \leq_b w$. Then every Bruhat chain from v to w is a b-Bruhat chain. In particular, the Bruhat interval $[v,w]$ coincides with the corresponding b-Bruhat interval.*

Proof. Björner and Wachs [BW] showed that the order complex of the open Bruhat interval (v,w) is a combinatorial sphere. It follows that any two Bruhat chains from v to w can be connected with a sequence of Bruhat chains from v to w such that any two adjacent chains differ in exactly one position. By hypothesis, there is a b-Bruhat chain from v to w. It must be shown that any Bruhat chain from v to w is a b-Bruhat chain. By the above connectedness and induction, it suffices to prove this under the assumption that the b-Bruhat chain and the Bruhat chain differ at exactly one position. This involves studying the relationships of labels on an interval of length 2 in a dihedral subgroup.

Consider a dihedral group with generators s_α, s_β. Consider an interval of length 2 with minimum u and two chains labeled by coroots (γ, δ) and (ε, ϕ). We may assume that (γ, δ) are b-Bruhat covers, and must show that (ε, ϕ) are b-Bruhat covers. It suffices to show that the latter coroots are integer linear combinations of the former; we denote this property by $(\gamma, \delta) \to (\varepsilon, \phi)$. For $k \geq 0$, let

$$(4.15) \qquad\qquad \gamma_{2k} = (s_\alpha s_\beta)^k(\alpha)\,, \quad \gamma_{2k+1} = (s_\alpha s_\beta)^k s_\alpha(\beta)\,.$$

The typical pairs (γ, δ), (ε, ϕ) are of the following form, where in each case we indicate the non-trivial property $(\gamma, \delta) \to (\varepsilon, \phi)$ to be proved:

- $(\alpha, s_\alpha(\gamma)) \to (\gamma, \alpha)$, where $u = \ldots s_\alpha s_\beta$;
- $(\gamma_i, \gamma_{i+1}) \to (\beta, \alpha)$, where $u = \ldots s_\beta s_\alpha$ and $\ell(u) = i$, for $i = 0, 1, \ldots$.

The first property is obvious. For the second one, note that the reflections s_{γ_i} and $s_{\gamma_{i+1}}$ generate the dihedral group (although they are not Coxeter generators). So, by (4.15), α and β can be obtained by applying a certain sequence consisting of the new generators to γ_i and another such sequence to γ_{i+1}. This concludes the proof. $\qquad\square$

Lemma 4.15. *Consider σ, τ in W^λ and w_λ, w'_λ in W_λ. We have $\sigma w_\lambda \leq_b \tau w'_\lambda$ if and only if $\sigma \leq_b \tau$ and $\mathrm{up}(\sigma w_\lambda, \tau W_\lambda) \leq \tau w'_\lambda$.*

Proof. For $b = 0$, this is just a restatement of Proposition 3.1 (2) combined with Proposition 3.12 (4). Assuming that the two equivalent statements hold for $b = 0$, it remains to show for an arbitrary b that $\sigma w_\lambda \leq_b \tau w'_\lambda$ if and only if $\sigma \leq_b \tau$. For the "if" statement, note that $\tau \leq_b \tau w'_\lambda$, since the labels of any corresponding Bruhat chain are orthogonal to λ. Thus $\sigma \leq_b \tau w'_\lambda$, and the conclusion follows from Lemma 4.14, since $\sigma \leq \sigma w_\lambda \leq \tau w'_\lambda$. The "only if" statement is completely similar. $\qquad\square$

Lemma 4.16. *Consider $\sigma, \tau \in W^\lambda$ and $w_\lambda, w'_\lambda \in W_\lambda$ such that $\sigma w_\lambda \leq \tau w'_\lambda$. We have $\sigma \leq_b \tau$ and $\tau w'_\lambda = \mathrm{up}(\sigma w_\lambda, \tau W_\lambda)$ if and only if the unique Bruhat chain from σw_λ to $\tau w'_\lambda$ with labels which are increasing with respect to the reflection order $<_\lambda$ has all its labels in Φ_b^*.*

Proof. For the "only if" statement, by Proposition 4.12 and Lemma 4.13 there is a unique saturated chain from σw_λ to $\tau w'_\lambda$ that has labels increasing with respect to the reflection order $<_\lambda$. By Lemma 4.15, we have $\sigma w_\lambda \leq_b \tau w'_\lambda$. So the above Bruhat chain has all its labels in Φ_b, by Lemma 4.14. If it had labels orthogonal to λ, all of them would have to appear at the end of the chain, cf. the definition of the reflection order $<_\lambda$. But then $\tau w'_\lambda$ cannot be smallest among the elements of τW_λ which are greater or equal to σw_λ, contradicting the minimality of the lift. For the "if" statement, note first that $\sigma \leq_b \tau$ and $\mathrm{up}(\sigma w_\lambda, \tau W_\lambda) \leq \tau w'_\lambda$, by Lemma 4.15. If the latter inequality is strict, construct a chain from σw_λ to $\tau w'_\lambda$ by concatenating the chains determined by $<_\lambda$ from σw_λ to $\mathrm{up}(\sigma w_\lambda, \tau W_\lambda)$ and from $\mathrm{up}(\sigma w_\lambda, \tau W_\lambda)$ to $\tau w'_\lambda$. The first chain has all its labels in Φ_b^*, by the "only if" statement just proved. Therefore, the constructed chain has increasing labels (with respect to $<_\lambda$), so it coincides with the one in the statement of the lemma (by the uniqueness property in Proposition 4.12). The fact that the latter chain has all its labels in Φ_b^* is thus contradicted. $\qquad\square$

Remark 4.17. There is a version of Lemma 4.16 for the Deodhar lift "down", which is used in the proof of (4.12), by analogy with the proof in Section 4.5. This version is based on a reflection order in which the roots corresponding to W_λ form an initial section, as opposed to being larger than the other roots, as in the case of $<_\lambda$.

4.5. **The proof of** (4.11). In the sequel, the total order on the λ-hyperplanes is the lex λ-chain. Let $\mathrm{Inc}_{w,z}^\lambda$ be the set of lex-increasing sequences of λ-hyperplanes which are $[z, w]$-adapted.

The dominant weight λ-chain model Chevalley formula (4.11) is an immediate consequence of the corresponding LS path formula (3.8) due to the following bijection. Refer to (3.3) and Definition 4.7 for the key definitions.

Proposition 4.18. *There is a bijection $\mathrm{Inc}_{w,z}^\lambda \cong \mathcal{U}_{w,z}^\lambda$ that preserves weights: if $H = (h_1 < h_2 < \cdots < h_q) \in \mathrm{Inc}_{w,z}^\lambda$ maps to $p \in \mathcal{U}_{w,z}^\lambda$, then*

$$(4.16) \qquad p(1) = \mathrm{wt}(H) := z\, \widehat{s}_{h_1} \widehat{s}_{h_2} \cdots \widehat{s}_{h_q} \cdot \lambda.$$

In the remainder of this subsection we fix a Bruhat interval $[z, w]$, construct a map $\mathrm{Inc}^\lambda_{w,z} \to \mathcal{U}^\lambda_{w,z}$, then a map $\mathcal{U}^\lambda_{w,z} \to \mathrm{Inc}^\lambda_{w,z}$, show that the maps are inverses, and finally that (4.16) is satisfied.

The proof starts by recalling Definition 3.3, describing an LS path as a pair of sequences: one of rational numbers $0 = b_1 < b_2 < b_3 < \ldots < b_m < b_{m+1} = 1$ and one of elements $\sigma_1, \ldots, \sigma_m$ in W/W_λ. Also recall the description (3.2) as a sequence of vectors, and the corresponding realization as a piecewise-linear path.

Let $H = (h_1 < h_2 < \cdots < h_q) \in \mathrm{Inc}^\lambda_{w,z}$. The *relative height* $\mathrm{rht}(h)$ and *relative coheight* $\overline{\mathrm{rht}}(h)$ of the λ-hyperplane $h = (\alpha, k)$ are defined by

$$(4.17) \qquad\qquad\qquad\qquad \mathrm{rht}(h) = \frac{k}{\langle \alpha \,,\, \lambda \rangle}$$

$$(4.18) \qquad\qquad\qquad\qquad \overline{\mathrm{rht}}(h) = 1 - \mathrm{rht}(h).$$

By the definition of the lex λ-chain, the relative heights of $h_1, h_2, \ldots, h_q$ form a weakly increasing sequence in $[0, 1) \cap \mathbb{Q}$. Let us call the sequence of distinct nonzero relative heights $0 < b_2 < b_3 < \cdots < b_m < 1$, and let $b_1 = 0$. Note that the relative height 0 is treated differently; in part, this is because the 0-Bruhat order is the same as the Bruhat order. For $j \geq 1$, let I_j be the subinterval of H consisting of the elements of relative height b_j; these sets are all nonempty, except perhaps I_1.

Since $H \in \mathrm{Inc}^\lambda_{w,z}$, there is a saturated Bruhat chain

$$(4.19) \qquad\qquad z \lessdot z s_{h_1} \lessdot z s_{h_1} s_{h_2} \lessdot \cdots \lessdot z s_{h_1} \cdots s_{h_q} = w.$$

We pick out some of the elements in the above chain by multiplying by groups of reflections given by the subsets I_j. For $0 \leq j \leq m$, define the Weyl group elements

$$(4.20) \qquad\qquad\qquad\qquad z_j = z \overrightarrow{\prod_{h \in I_1 \cup I_2 \cup \cdots \cup I_j}} s_h \,,$$

where the (non-commutative) product over h occurs from left to right in the order $h_1 < h_2 < \ldots$. In particular $z_0 = z$. Let

$$(4.21) \qquad\qquad\qquad \sigma_j = z_j W_\lambda \in W/W_\lambda \qquad \text{for } 1 \leq j \leq m.$$

The required map is

$$(4.22) \qquad\qquad\qquad H \mapsto (b_2, \ldots, b_m); (\sigma_1, \ldots, \sigma_m).$$

This map is clearly well-defined: the b_j-Bruhat condition is satisfied by the definition of relative height.

The inverse map is constructed using Deodhar lifts and the EL-shellability of the Bruhat order. We begin with an LS path $p \in \mathcal{U}^\lambda_{w,z}$ in the form (3.1). By definition, we have $zW_\lambda \leq \phi(p) = \sigma_1$. Letting $z_0 = z$, define the lifts

$$(4.23) \qquad\qquad\qquad z_j = \mathrm{up}(z_{j-1}, \sigma_j) \qquad \text{for } j = 1, 2, \ldots, m.$$

By definition we have $z_m = \mathrm{up}(z, p) = w$. By the "only if" part of Lemma 4.16, for each $1 \leq j \leq m$ there is a unique b_j-Bruhat chain from z_{j-1} to z_j with labels in $\Phi^*_{b_j}$, which are increasing with respect to the reflection order $<_\lambda$. Let us replace each label β in this chain with the pair $(\beta, b_j \langle \beta \,,\, \lambda \rangle)$, where the second component is in $\{0, 1, \ldots, \langle \beta \,,\, \lambda \rangle - 1\}$ as $\beta \in \Phi^*_{b_j}$. Thus,

each such pair is a λ-hyperplane. By concatenating these chains we obtain a Bruhat chain from z to w, together with a sequence of λ-hyperplanes H. This defines the map $p \mapsto H$.

Proof of Proposition 4.18. The forward map was seen to be well-defined. For the well-definedness of the inverse map, note first that the sequence of λ-hyperplanes is $[z, w]$-adapted by definition. It is also lex-increasing: the relative heights of the λ-hyperplanes weakly increase by construction, and within the same relative height the λ-hyperplanes increase because of the compatibility of the reflection order $<_\lambda$ on coroots with the lex λ-chain.

To show that the two maps are mutually inverse, the crucial fact to check is that the forward map composed with the backward one is the identity. This follows from the "if" part of Lemma 4.16, after recalling from the above discussion that the order of the λ-hyperplanes with fixed relative height is given by the reflection order $<_\lambda$ on the coroots.

Finally, we check the weight condition (4.16). If $h = (\alpha, k)$ is a λ-hyperplane of relative height b, we have

$$\widehat{s}_h \cdot b\lambda = k\alpha^\vee + b(\lambda - \langle \alpha, \lambda\rangle \alpha^\vee)$$
$$= b\lambda + (k - b\langle \alpha, \lambda\rangle)\alpha^\vee$$
$$= b\lambda$$

by the definitions (4.8) of $\widehat{s}$ and (4.17) of relative height. This given, using our previous conventions on ordered products and $b_{m+1} = 1$, we have

$$\left(\overrightarrow{\prod_{h\in I_m}} \widehat{s}_h\right) \cdot \lambda = b_m\lambda + (b_{m+1} - b_m)\left(\overrightarrow{\prod_{h\in I_m}} s_h\right) \cdot \lambda.$$

Writing $b_m\lambda = b_{m-1}\lambda + (b_m - b_{m-1})\lambda$ and applying $\prod_{h\in I_{m-1}} \widehat{s}_h$ to the above equation, we obtain

$$\left(\overrightarrow{\prod_{h\in I_{m-1}\cup I_m}} \widehat{s}_h\right) \cdot \lambda = b_{m-1}\lambda + (b_m - b_{m-1})\left(\overrightarrow{\prod_{h\in I_{m-1}}} s_h\right) \cdot \lambda + (b_{m+1} - b_m)\left(\overrightarrow{\prod_{h\in I_{m-1}\cup I_m}} s_h\right) \cdot \lambda.$$

Iterating this until $\prod_{h\in I_1} \widehat{s}_h$ has been applied, then applying z, and using the definition (4.20), we obtain

$$z\left(\overrightarrow{\prod_{h\in H}} \widehat{s}_h\right) \cdot \lambda = \sum_{j=1}^{m}(b_{j+1} - b_j)z_j \cdot \lambda$$
$$= \sum_{j=1}^{m}(b_{j+1} - b_j)\sigma_j \cdot \lambda$$
$$= p(1).$$

Here the last equality follows from [Ste, (8.3)], or the fact that p consists of following the vectors $(b_{j+1} - b_j)\sigma_j \cdot \lambda$ for j going from m down to 1. $\qquad \square$

Remark 4.19. Consider any increasing sequence of λ-hyperplanes H (not necessarily satisfying (4.19)), with weight $\mathrm{wt}(H)$ given by (4.16). The same construction as that in (4.22) associates with it a piecewise-linear path p. Since the proof of the weight preservation property $\mathrm{wt}(H) = p(1)$ in Proposition 4.18 does not depend on the condition (4.19), this property still holds. We will use this remark in future work.

Proof of (4.11). Immediate based on (3.8) and Proposition 4.18. $\qquad\square$

4.6. **Lex-decreasing analogue.** Let $\mathrm{Dec}^\lambda_{w,z}$ be the set of lex-decreasing sequences of λ-hyperplanes which are $[z, w]$-adapted.

Proposition 4.20. *There is a bijection* $\mathrm{Dec}^\lambda_{w,z} \cong \mathcal{D}^\lambda_{w,z}$ *that is weight preserving: if* $H = (h_1 > h_2 > \cdots > h_q) \in \mathrm{Dec}^\lambda_{w,z}$ *maps to* $p \in \mathcal{D}^\lambda_{w,z}$, *then*

$$(4.24) \qquad\qquad p(1) = \widetilde{\mathrm{wt}}(H) := z\, \widetilde{s}_{h_1} \widetilde{s}_{h_2} \cdots \widetilde{s}_{h_q} \cdot \lambda.$$

The proof and constructions are analogous to those in the lex-increasing case. Some details are given below.

Let $H = \{h_1 > h_2 > \cdots > h_q\} \in \mathrm{Dec}^\lambda_{w,z}$.

Since H is lex-decreasing, the sequence $\{\overline{\mathrm{rht}}(h_i)\}$ of relative *coheights* (see (4.18)) is a weakly increasing sequence in $(0, 1] \cap \mathbb{Q}$. Let their distinct values other than 1 be $0 < b_2 < b_3 < \cdots < b_m < 1$ and let $b_{m+1} = 1$. Let I_j be the elements in H of relative coheight b_j, for $2 \le j \le m+1$. Then I_j is nonempty, except possibly for I_{m+1}.

For $1 \le j \le m + 1$ let

$$(4.25) \qquad\qquad w_j = z \overrightarrow{\prod_{h \in I_2 \cup I_3 \cup \cdots \cup I_j}} s_h.$$

In particular, $w_1 = z$ and $w_{m+1} = w$. For $2 \le j \le m + 1$, the elements of I_j define a $(1 - b_j)$-Bruhat chain from w_{j-1} up to w_j. But by definition, the b-Bruhat order coincides with the $(1 - b)$-Bruhat order. So we have a b_j-Bruhat chain.

Define $\sigma_j = w_j W_\lambda$, for $1 \le j \le m$.

Then $\sigma_1 <_{b_2} \sigma_2 <_{b_3} \cdots <_{b_m} \sigma_m$ defines an element $p \in \mathcal{T}^\lambda$ and $\sigma_m \le wW_\lambda$.

Lemma 4.21. *We have* $\mathrm{dn}(w_{j+1}, \sigma_j) = w_j$ *for* $1 \le j \le m$.

It follows that $p \in \mathcal{D}^\lambda_{w,z}$.

The definition of the inverse map is similar to that for the lex-increasing case. One must use the fact that the dual of a reflection order is again a reflection order.

4.7. **Consequences of the bijections.** Let λ be a dominant weight and $z, w \in W$. The *Demazure module* of lowest weight $w\lambda$ is the module $U_q(\mathfrak{b})v_{w\lambda}$, where $U_q(\mathfrak{b})$ is the upper triangular part of the quantum universal enveloping algebra $U_q(\mathfrak{g})$, and $v_{w\lambda}$ is a vector of extremal weight $w\lambda$ in the highest weight $U_q(\mathfrak{g})$-module of highest weight λ. The *opposite Demazure module* of highest weight $z\lambda$ is the module $U_q(\mathfrak{b}_-)v_{z\lambda}$, where $U_q(\mathfrak{b}_-)$ is the lower triangular part of $U_q(\mathfrak{g})$.

Theorem 4.22. [Kas2, Lit1] (1) *The Demazure crystal of lowest weight* $w\lambda$ *is given by the subcrystal of* $\mathcal{T}^\lambda$ *consisting of paths* p *with* $\iota(p) \le w\lambda$.

(2) *The opposite Demazure crystal of highest weight* $z\lambda$ *is given by the subcrystal of* $\mathcal{T}^\lambda$ *consisting of paths* p *with* $\phi(p) \ge z\lambda$.

Via Propositions 4.18 and 4.20, we have the following realization of Demazure and opposite Demazure crystals in terms of the λ-chain model, whose crystal structure was defined in [LP1] in Kac-Moody generality.

$$(\lambda) \xrightarrow{\;\;1\;\;} (s_1\lambda) \xrightarrow{\;\;2\;\;} (s_2s_1\lambda)$$

with vertical arrows labeled 0 leading to:

$(s_0\lambda)$; $\left(\frac{1}{2}s_0s_1\lambda, \frac{1}{2}s_1\lambda\right)$; $\left(\frac{1}{3}s_0s_2s_1\lambda, \frac{2}{3}s_2s_1\lambda\right) = p_1$

then arrows labeled 0 leading to:

$(s_0s_1\lambda)$; $\left(\frac{2}{3}s_0s_2s_1\lambda, \frac{1}{3}s_2s_1\lambda\right) = p_2$

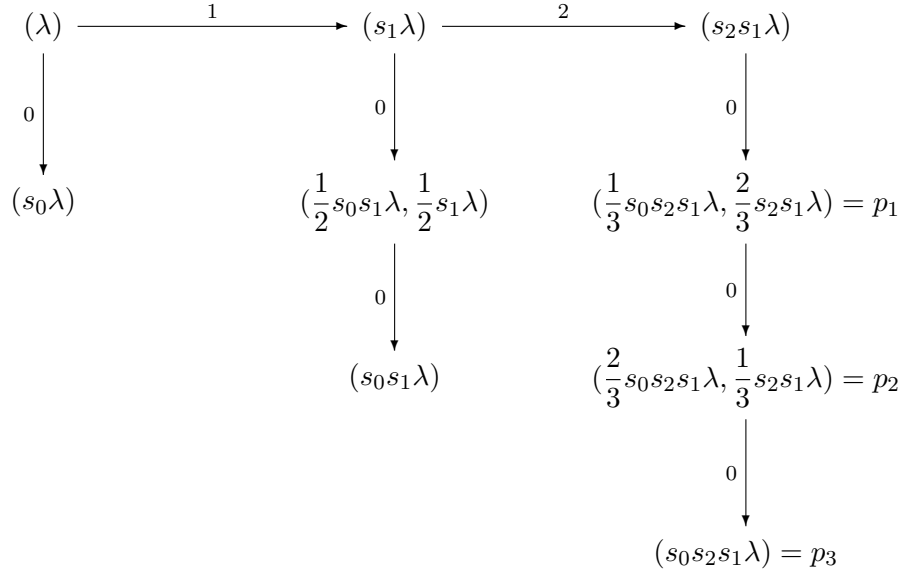then arrow labeled 0 leading to:

$(s_0s_2s_1\lambda) = p_3$

FIGURE 5.1. Demazure crystal

Corollary 4.23. (1) *The Demazure crystal of lowest weight $w\lambda$ is given by set of lex-decreasing sequences of λ-hyperplanes adapted to intervals of the form $[z, w]$, for some $z \leq w$.*

(2) *The opposite Demazure crystal of highest weight $z\lambda$ is given by the set of lex-increasing sequences of λ-hyperplanes which are z-adapted, that is, $[z, w]$-adapted for some $w \geq z$.*

Remark 4.24. In the case of finite root systems, the statement of Corollary 4.23 (1) appears in [LP][Corollary 6.5].

5. EXAMPLES

5.1. **Dominant weight.** Consider the affine Lie algebra $\mathfrak{g}$ of type $A_2^{(1)}$, the dominant weight $\lambda = \Lambda_0 + \Lambda_1$, and $w = s_0s_1s_2s_1 = s_0s_2s_1s_2$. In this subsection we will obtain all coefficients $b_{z\lambda}^w$ for λ and w fixed as above and varying $z \leq w$. This will be done first by LS paths and then by the λ-chain model.

Example 5.1. We illustrate the dominant weight LS path Chevalley formula (3.5). The stabilizer subgroup W_λ is W_J with $J = \{2\}$, and the representative of w in W^J is $s_0s_2s_1$.

The Demazure subcrystal of $\mathcal{T}^\lambda$ of lowest weight $w\lambda$ is given in Figure 5.1. The vertices are LS paths, which are written as sequences of vectors in $\mathbb{Q} \otimes \Lambda$. The crystal operators f_i are denoted by arrows labeled i. Recall that traversing an arrow i changes weight by $-\alpha_i$. In the K-Chevalley formula (3.5), any LS path $p \in \mathcal{U}_{w,z}^\lambda$ must have initial direction $w\lambda = s_0s_2s_1\lambda$. Here the set of such paths is $\{p_1, p_2, p_3\}$. For each of these paths p, the set of z such that $w = \mathrm{up}(z, p)$ is given as follows. For p_1 and p_2, we have $z \in \{s_1s_2, s_1s_2s_1\}$, and for p_3 we have $z \in \{s_1s_2, s_1s_2s_1, s_0s_1s_2, s_0s_1s_2s_1\}$. This yields a total of 8 pairs (z, p). We have

$$b_{z\lambda}^w = \begin{cases} e^{\mathrm{wt}(p_3)} & \text{for } z \in \{s_0s_1s_2, s_0s_1s_2s_1\} \\ e^{\mathrm{wt}(p_1)} + e^{\mathrm{wt}(p_2)} + e^{\mathrm{wt}(p_3)} & \text{for } z \in \{s_1s_2, s_1s_2s_1\}. \end{cases}$$
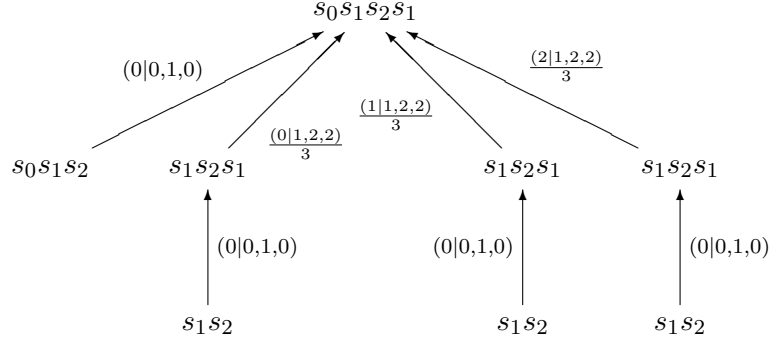
FIGURE 5.2. Dominant tree

Let us see for p_1 why $z = s_1 s_2$ works, and $z = s_1$ does not. With $z = s_1 s_2$, the Deodhar lifts (of the directions of the path p_1, with z at the front) are $(s_1 s_2; s_1 s_2 s_1, s_0 s_1 s_2 s_1)$, and this sequence ends at w. For $z = s_1$, the Deodhar lifts form the sequence $(s_1; s_2 s_1, s_0 s_2 s_1)$, which does not end at w.

Example 5.2. Using $\mathfrak{g}$, λ, and w as above, we work out the dominant weight λ-chain model Chevalley formula (4.11), using the lex λ-chain based on the order $0 < 1 < 2$ on the Dynkin node set.
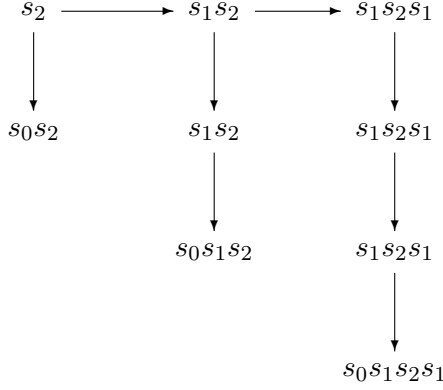
Let λ and w be fixed. We compute $b_{z\lambda}^w$ for varying $z \leq w$ using $\mathrm{Inc}_{w,z}^\lambda$. These sequences may be efficiently generated by a depth-first search, which generates the tree in Figure 5.2. This tree is rooted at w, its vertices are Weyl group elements (which can be repeated), its edges are labeled by λ-hyperplanes, and its branches going towards the root are lex-increasing. The vertices are in bijection with $\bigsqcup_{z \leq w} \mathrm{Inc}_{w,z}^\lambda$; given a vertex, the corresponding lex-increasing sequence is obtained by reading the edge labels going from the vertex to the root. We denote the λ-hyperplane (α, k) with coroot $\alpha = c_0 \alpha_0^\vee + c_1 \alpha_1^\vee + c_2 \alpha_2^\vee$ by $\frac{(k | c_0, c_1, c_2)}{\langle \alpha, \lambda \rangle}$, omitting the denominator if it is 1. The tree has 8 vertices, which agrees with the 8 terms in Example 5.1.

Consider the rightmost leaf vertex in the tree. It is labeled $z = s_1 s_2$, and it defines the sequence $H = (h_1, h_2) \in \mathrm{Inc}_{w,z}^\lambda$ where $h_1 = (0 | 0, 1, 0)$ and $h_2 = 1/3(2 | 1, 2, 2)$.

Let $\beta = \alpha_0^\vee + 2\alpha_1^\vee + 2\alpha_2^\vee$ and $\beta^\vee = \alpha_0 + 2\alpha_1 + 2\alpha_2$. Using the notation $\alpha(ijk)$ for the element $i\alpha_0 + j\alpha_1 + k\alpha_2$, the weight of $H \in \mathrm{Inc}_{w,z}^\lambda$ is computed by the right hand side of (4.16):

$$
\begin{aligned}
\mathrm{wt}(H) &= s_1 s_2 s_1 t_{2\beta^\vee} s_\beta (\Lambda_0 + \Lambda_1) \\
&= s_1 s_2 s_1 t_{2\beta^\vee} (\Lambda_0 + \Lambda_1 - 3\beta^\vee) \\
&= s_1 s_2 s_1 (\Lambda_0 + \Lambda_1 - \alpha(122)) \\
&= s_1 s_2 (\Lambda_0 + \Lambda_1 - \alpha(122)) \\
&= s_1 (\Lambda_0 + \Lambda_1 - \alpha(121)) \\
&= \Lambda_0 + \Lambda_1 - \alpha(111).
\end{aligned}
$$

Let us compute the LS path in $\mathcal{U}_{w,z}^\lambda$ corresponding to $H \in \mathrm{Inc}_{w,z}^\lambda$. We use the notation of Section 4.5. We have $m = 2$, $(b_1, b_2, b_3) = (0, 2/3, 1)$, $I_1 = \{h_1\}$, $I_2 = \{h_2\}$, $z_0 = z = s_1 s_2$, $z_1 = s_1 s_2 s_1$, and $z_2 = s_1 s_2 s_1 s_\beta = s_0 s_1 s_2 s_1$. Recalling that $s_2 \lambda = \lambda$, the directions of the LS path are given by $z_1 \lambda = s_2 s_1 \lambda$ and $z_2 \lambda = s_0 s_2 s_1 \lambda$. The relative distances traversed in these

$$
\begin{array}{ccc}
s_2 \longrightarrow & s_1 s_2 \longrightarrow & s_1 s_2 s_1 \\
\downarrow & \downarrow & \downarrow \\
s_0 s_2 & s_1 s_2 & s_1 s_2 s_1 \\
& \downarrow & \downarrow \\
& s_0 s_1 s_2 & s_1 s_2 s_1 \\
& & \downarrow \\
& & s_0 s_1 s_2 s_1
\end{array}
$$

FIGURE 5.3. Computations of dn

directions are given by $2/3 - 0$ and $1 - 2/3$, respectively. This is the path p_1 of Example 5.1. To check that $p_1 \in \mathcal{U}_{w,z}^\lambda$, we calculate the lifts $\mathrm{up}(s_1 s_2, s_2 s_1 W_\lambda) = s_1 s_2 s_1 = z_1$ and $\mathrm{up}(s_1 s_2 s_1, s_0 s_2 s_1 W_\lambda) = s_0 s_1 s_2 s_1 = z_2 = w$.

Remark 5.3. For the Chevalley rule for dominant weights, the λ-chain model is more efficient than the LS path model. The λ-chain model uses only the computation of Bruhat cocovers (i.e., elements covered by a given one) and comparisons of λ-hyperplanes with no wasted effort (if the set of λ-hyperplanes adapted to cocovers of a given Weyl group element are remembered). In contrast, the LS path formula requires the generation of the entire Demazure crystal just to obtain the paths $p \in \mathcal{T}^\lambda$ with a specified initial direction $w\lambda$. Then, for each such path p, one must compute the possible values of z such that $\mathrm{up}(z, p) = w$. For a fixed p, this can be done by starting with w, by considering which w' in the second step (i.e., coset) of p make the lift of the initial step σ equal to w, that is, $\mathrm{up}(w', \sigma) = w$, and by continuing recursively. Thus, one needs to compute many Deodhar lifts, which are given by a non-trivial recursive procedure [Deo].

5.2. **An antidominant example.** With $\mathfrak{g}$, λ, and w fixed as above, we compute the coefficients $b_{z,-\lambda}^w$ for all $z \leq w$, again both by LS paths and by the λ-chain model.

Example 5.4. We illustrate the antidominant weight LS path Chevalley formula (3.6).

The entire set of 9 LS paths given in Figure 5.1 must be used. These paths form the Demazure crystal of lowest weight $w\lambda$. In Figure 5.3, in the position of each path p in Figure 5.1, we place the element $\mathrm{dn}(w, p)$, where we recall that $w = s_0 s_1 s_2 s_1$. The coefficients $b_{z,-\lambda}^w$ can be read from this diagram. For example, if $z = s_1 s_2$ then the coefficient is $e^{-\mathrm{wt}(q_1)} + e^{-\mathrm{wt}(q_2)}$, where $q_1 = (s_1\lambda)$ and $q_2 = ((1/2)s_0 s_1 \lambda, (1/2)s_1 \lambda)$ are the top two paths in the second column of the diagram in Figure 5.1. We have $\mathrm{wt}(q_1) = \lambda - \alpha_1$ and $\mathrm{wt}(q_2) = \lambda - \alpha_0 - \alpha_1$.

Example 5.5. We now work out the antidominant weight λ-chain model Chevalley rule (4.12).

As in the dominant case we create a suitable tree; see Figure 5.4. This tree has nine vertices, as there are 9 LS paths in Example 5.4. As before, given a vertex in the tree, we consider the path from the vertex up to the root. Consider the sole vertex labeled $z = s_0 s_2$. The path to the root is given by $H = (h_1 > h_2) \in \mathrm{Dec}_{w,z}^\lambda$ where $h_1 = (0|0, 1, 1)$ and $h_2 = (0|0, 1, 0)$. We now find the corresponding LS path in $\mathcal{D}_{w,z}^\lambda$ where $z = s_1 s_2$. Since both h_1 and h_2 have
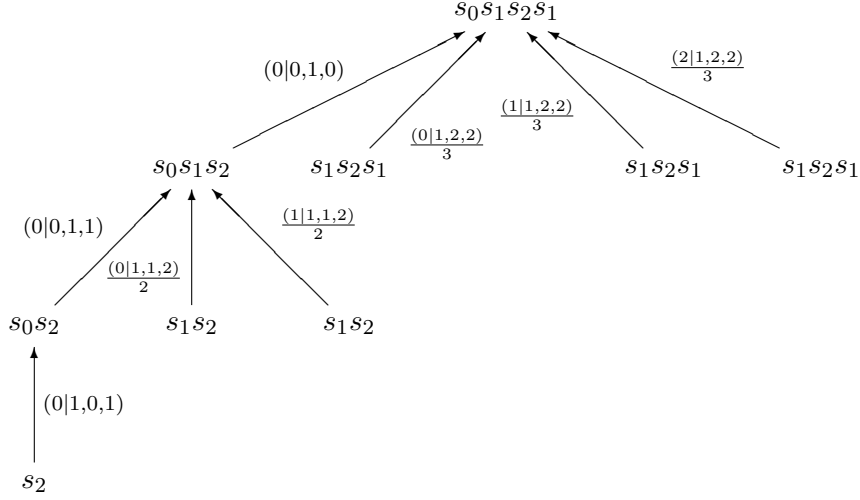
$$s_0s_1s_2s_1$$

FIGURE 5.4. Antidominant tree

relative coheight 1, in the notation of Section 4.6, we have $m = 1$, $b_2 = 1$, $w_1 = z = s_0s_2$, $w_2 = w = s_0s_1s_2s_1$, and $\sigma_1 = s_0s_2W_\lambda = s_0W_\lambda$. The path is given by the single vector $(s_0\lambda)$. Its endpoint is $s_0\lambda$. We have $\mathrm{dn}(s_0s_1s_2s_1, s_0W_\lambda) = s_0s_2 = z$. The required saturated Bruhat chain in the interval $[s_0s_2, s_0s_1s_2s_1]$ is uniquely specified by having increasing coroot labels with respect to the reflection order on $\Phi^{\vee+}$ dual to the one induced by $0 < 1 < 2$.

We compute $\widetilde{\mathrm{wt}}(H)$. Let $\gamma = \alpha_1^\vee + \alpha_2^\vee$. Then $\gamma = \alpha_1 + \alpha_2$, $\langle \gamma, \lambda \rangle = 1$, and

$$
\begin{aligned}
\widetilde{\mathrm{wt}}(H) &= s_0s_2 t_{\gamma^\vee} s_\gamma t_{\alpha_1} s_1(\Lambda_0 + \Lambda_1) \\
&= s_0s_2 t_{\gamma^\vee} s_\gamma t_{\alpha_1}(\Lambda_0 + \Lambda_1 - \alpha_1) \\
&= s_0s_2 t_{\gamma^\vee} s_\gamma(\Lambda_0 + \Lambda_1) \\
&= s_0s_2 t_{\gamma^\vee}(\Lambda_0 + \Lambda_1 - \gamma^\vee) \\
&= s_0s_2(\Lambda_0 + \Lambda_1) \\
&= s_0(\Lambda_0 + \Lambda_1)
\end{aligned}
$$

which agrees with the weight of the LS path.

5.3. **The affine Grassmannian.** Let us consider the K-theory of the affine Grassmannian Gr_{SL_n} of type A_{n-1}. It is known that we can index the vertices of the crystal of highest weight Λ_0 by the *n-restricted partitions* in the Misra-Miwa model [MM]. Note the concepts of "roof" and "base" of an n-restricted partition J, whose definitions use some subtle combinatorial constructions that are outside the scope of this paper [AKT]. The roof and ceiling lemmas in [AKT] state that $\mathrm{roof}(J)$ and $\mathrm{base}(J)$ are essentially the initial and final directions of the corresponding LS path, respectively. On another hand, it is known that the Schubert classes in the K-theory of Gr_{SL_n} are indexed by *stable n-restricted (or core) partitions*, see [LSS]. These correspond to lowest coset representatives in W/W_{Λ_0}, where W is the affine symmetric group and W_{Λ_0} is the symmetric group (on n letters).

Consider the Chevalley product $[L^{\Lambda_0}][\mathcal{O}_{X_I}]$, where I is a stable n-restricted partition. By the roof and ceiling lemmas, the K-Chevalley formula (3.8) becomes

$$(5.1) \qquad [L^{\Lambda_0}][\mathcal{O}_{X_I}] = \sum_{J \,:\, \mathrm{base}(J) \geq I} e^{\mathrm{wt}(J)}[\mathcal{O}_{X_{\mathrm{roof}(J)}}];$$

here the summation is over the corresponding n-restricted partitions, and for the definition of the weight $\mathrm{wt}(J)$ we also refer to [AKT]. Note that all the Deodhar lifts in (3.8) are trivial in this case.

Similarly, one can derive a formula for $[\mathcal{O}_{X_{s_0}}][\mathcal{O}_{X_I}]$ based on (3.9) and (4.13). More generally, Λ_0 and s_0 can be replaced by any Λ_i and s_i, using the corresponding notions in [AKT].

It would be interesting to connect the model of n-restricted partitions and the λ-chain model. As the analogues of the initial and final direction of an LS path are easy to read off in the λ-chain model, such a connection would lead to a more transparent construction of roof and base (the current construction is highly non-transparent).

References

[AKT]   S. Ariki, V. Kreiman, and S. Tsuchioka. On the tensor product of two basic representations of $U_v(\widehat{\mathfrak{sl}}_e)$. *Adv. Math.*, 218:28–86, 2008.

[BW]   A. Björner and M. Wachs. Bruhat order of Coxeter groups and shellability. *Adv. Math.*, 43:87–100, 1982.

[Bou]   N. Bourbaki. *Groupes et algèbres de Lie: Chapitres 4, 5 et 6*. Springer, 2006.

[Che]   C. Chevalley. Sur les décompositions cellulaires des espaces G/B. In *Algebraic Groups and Generalizations: Classical Methods*, volume 56 Part 1 of *Proceedings and Symposia in Pure Mathematics*, pages 1–23. Amer. Math. Soc., 1994.

[Dem]   M. Demazure. Désingularization des variétés de Schubert. *Annales E.N.S.*, 6:53–88, 1974.

[Deo]   V. V. Deodhar. A splitting criterion for the Bruhat orderings on Coxeter groups. *Comm. Algebra*, 15:1889–1894, 1987.

[Dyer] M. J. Dyer. Hecke algebras and shellings of Bruhat intervals. *Compositio Math.*, 89:91–115, 1993.

[GR]   S. Griffeth and A. Ram. Affine Hecke algebras and the Schubert calculus. *European J. Combin.*, 25:1263–1283, 2004.

[Hum]   J. E. Humphreys. *Reflection Groups and Coxeter Groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[Kas]   M. Kashiwara. The flag manifold of Kac-Moody Lie algebra. *Algebraic analysis, geometry, and number theory* (Baltimore, MD, 1988), 161–190, Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[Kas2] M. Kashiwara, The crystal base and Littelmann's refined Demazure character formula. *Duke Math. J.* 71 (1993), no. 3, 839–858.

[KS]   M. Kashiwara and M. Shimozono. Equivariant K-theory of affine flag manifolds and affine Grothendieck polynomials. *Duke Math. J.*, 148:501–538, 2009.

[KK]   B. Kostant and S. Kumar. T-equivariant K-theory of generalized flag varieties. *J. Diff. Geom.*, 32:549–603, 1990.

[Kum]   S. Kumar. *Kac-Moody groups, their flag varieties and representation theory*. Progress in Mathematics, 204, Birkhäuser Boston, Inc., Boston, MA, 2002. xvi+606 pp.

[LS2]   V. Lakshmibai and C. S. Seshadri. Geometry of G/P. V. *J. Algebra* 100:462–557, 1986.

[LS]   V. Lakshmibai and C. S. Seshadri. Standard monomial theory. In *Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989)*, pages 279–322, Madras, 1991. Manoj Prakashan.

[LSS]   T. Lam, A. Schilling, and M. Shimozono. K-theory Schubert calculus of the affine Grassmannian. *Compos. Math.*, 146:811–852, 2010.

[LP]   C. Lenart and A. Postnikov. Affine Weyl groups in K-theory and representation theory. *Int. Math. Res. Not.*, pages 1–65, 2007. Art. ID rnm038.

[LP1]   C. Lenart and A. Postnikov. A combinatorial model for crystals of Kac-Moody algebras. *Trans. Amer. Math. Soc.*, 360:4349–4381, 2008.

[Len]   C. Lenart. From Macdonald polynomials to a charge statistic beyond type A. *J. Combin. Theory Ser. A*, 119:683–712, 2012.

[Lit1] P. Littelmann. A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras. *Invent. Math.*, 116:329–346, 1994.

[Lit2] P. Littelmann. Paths and root operators in representation theory. *Ann. of Math. (2)*, 142:499–525, 1995.

[LiS] P. Littelmann and C. S. Seshadri. A Pieri-Chevalley type formula for $K(G/B)$ and standard monomial theory. In *Studies in Memory of Issai Schur (Chevaleret/Rehovot, 2000)*, volume 210 of *Progr. Math.*, pages 155–176. Birkhäuser Boston, Boston, MA, 2003.

[MM] K. Misra and T. Miwa. Crystal base for the basic representation of $U_q(\mathfrak{sl}(n))$. *Comm. Math. Phys.*, 134:79–88, 1990.

[PR] H. Pittie and A. Ram. A Pieri-Chevalley formula in the K-theory of a G/B-bundle. *Electron. Res. Announc. Amer. Math. Soc.*, 5:102–107, 1999.

[SC] The Sage-Combinat community, *Sage-Combinat*: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, `http://combinat.sagemath.org`, 2008.

[Sage] William A. Stein et. al. Sage Mathematics Software (Version 4.8), The Sage Development Team, 2012, `http://www.sagemath.org`.

[Ste] J. R. Stembridge. Combinatorial models for Weyl characters. *Adv. Math.*, 168:96–131, 2002.

[Wil] M. Willems. A Chevalley formula in equivariant K-theory. *J. Algebra*, 308:764-779, 2007.

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