PARABOLIC KAZHDAN-LUSZTIG BASIS, SCHUBERT CLASSES, AND EQUIVARIANT ORIENTED COHOMOLOGY

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ABSTRACT. We study the equivariant oriented cohomology ring $\mathtt{h}_T(G/P)$ of partial flag varieties using the moment map approach. We define the right Hecke action on this cohomology ring, and then prove that the respective Bott-Samelson classes in $\mathtt{h}_T(G/P)$ can be obtained by applying this action to the fundamental class of the identity point, hence generalizing previously known results by Brion, Knutson, Peterson, Tymoczko and others.

We then focus on the equivariant oriented cohomology theory corresponding to the 2-parameter Todd genus. We give a new interpretation of Deodhar's construction of the parabolic Kazhdan-Lusztig basis. Based on it, we define the parabolic Kazhdan-Lusztig (KL) Schubert classes independently of a reduced word. We make a positivity conjecture, and a conjecture about the relationship of such classes with smoothness of Schubert varieties. We then prove several special cases.

CONTENTS

1. Introduction

Let G be a split semisimple linear algebraic group over a field k of characteristic 0, with $G \supset P \supset B \supset T$, where P is a parabolic subgroup, B a Borel subgroup, and T a maximal torus. The partial flag variety G/P has many remarkable geometric properties; for instance, it has a B-equivariant cellular filtration given by B-orbits $\mathcal{O}_w$. Basic examples of such varieties are projective spaces, Grassmannians, and smooth quadric hypersurfaces. In the present paper we study the cohomology ring $\mathbf{h}_T(G/P)$, where $\mathbf{h}_T$ is the T-equivariant oriented cohomology theory in the sense of Levine-Morel [30]. Observe that the Chow groups CH (or singular cohomology) and Grothendieck's K_0 are standard examples of such theories. The corresponding equivariant versions have been introduced and extensively studied by Brion [6], Totaro [38], Edidin-Graham [17], and others. A universal example of such a theory is given by the algebraic cobordism Ω of Levine-Morel [30]. Its T-equivariant version Ω_T has been extensively studied recently (see e.g. [20]). An interesting family of examples comes from formal group laws of (singular) elliptic curves, which give, for instance, stalk versions of the elliptic cohomology of Ginzburg-Kapranov-Vasserot (see e.g. [41]).

The fact that G/P has an equivariant cellular filtration immediately implies that $\mathbf{h}_T(G/P)$ is a free module over the coefficient ring $\mathbf{h}_T(\mathrm{pt})$, with a basis given by the classes of desingularizations of the orbit closures $X(w) = \overline{\mathcal{O}_w}$; the varieties $X(w)$ are the Schubert varieties. Hence, the problem of describing the ring $\mathbf{h}_T(G/P)$ (one of the major problems in generalized Schubert calculus) splits into two parts: (i) constructing a desingularization $\hat{X}(w)$ for each $X(w)$, and (ii) expressing the products $[\hat{X}(w)] \cdot [\hat{X}(v)]$ as linear combinations of the classes $[\hat{X}(u)]$. A fundamental result of Bressler-Evens [5] says that the classes $[\hat{X}(w)]$ do not depend on the choice of a desingularization if and only if the theory $\mathbf{h}$ is obtained by specialization from connective K-theory (a universal birational theory). In the latter case (e.g. for Chow groups, singular cohomology, and usual K-theory), the situation drastically simplifies, as we can replace $[\hat{X}(w)]$ with the class of the Schubert variety $[X(w)]$ itself, hence avoiding the problem (i). In all other cases (e.g. for cobordism or for an elliptic cohomology), this leads to a natural question of constructing canonical classes in $\mathbf{h}_T(G/P)$ which serve as replacements for $[X(w)]$.

In the present paper we address this question by introducing and applying a new tool which we call a *right Hecke action* on $\mathbf{h}_T(G/P)$.

Historically, the classical left Hecke action (denoted '$\bullet$') was used by Demazure [14, 15], by Bernstein-Gelfand-Gelfand [3] to construct Schubert classes for ordinary Chow groups, by Arabia [1, 2] for T-equivariant Chow groups, and by Kostant-Kumar [25, 24] for T-equivariant Chow groups and T-equivariant K-theory. Namely, for Chow groups one sets $[X(w)] := Y_J \bullet (Y_{w^{-1}} \bullet [\mathrm{pt}_J])$, where J is the subset of simple roots corresponding to P, $[\mathrm{pt}_J]$ is the fundamental class of the identity point in $CH_T(G/P)$, $Y_{w^{-1}}$ is the push-pull element in the nil-Hecke ring corresponding to $\mathcal{O}_w$, and Y_J is the averaging element. However, the main computational disadvantage of the $\bullet$-action is that

it does not generate the Schubert classes (from the class of a point) directly (without applying the averaging element), because $Y_v \bullet [\text{pt}_J]$ does not produce all Schubert classes. In order to fix this problem (in the context of Chow groups), a new Hecke action was introduced by Brion [6], Peterson [32] and Knutson [39]. It was later studied by Tymoczko in [39, 40] (sometimes it is called the Tymoczko action). Geometrically speaking, it coincides with a natural left action of the Weyl group W on the equivariant Chow groups $CH_T(G/P) = CH_G(G/T \times G/P)$ induced by the right G-equivariant action of W on $G/T \times G/P$ via $(gT, x) \cdot \sigma T = (gT\sigma, x)$.

The next step was done in [10] and [11], where the classical $\bullet$-action was generalized to an arbitrary oriented theory $\mathbf{h}_T(G/P)$ to construct classes of the Bott-Samelson desingularizations $\zeta_{I_w}^J$ of the Schubert varieties $X(w)$. Namely, one sets $\zeta_{I_w}^J := Y_J \bullet (Y_{I_w^{-1}} \bullet [\text{pt}_J])$, where $Y_{I_w^{-1}}$ is the push-pull element in the respective formal affine Demazure algebra, which depends on the choice of a reduced word I_w for w.

In the present paper (see §3), we generalize the Brion-Peterson-Knutson-Tymoczko action to an arbitrary oriented theory $\mathbf{h}$ (we denote it by '$\odot$' and call it the right Hecke action), hence, completing the picture. Our arguments are based on the following geometric observation: Consider the convolution product on $CH_T(G/B) \cong CH_G(G/B \times G/B)$; taken from the left it gives the usual $\bullet$-action, and taken from the right it gives the Tymoczko action. It follows immediately that these two actions commute. Let $W_P \subset W$ be the subgroup corresponding to $P \supset B$. By identifying $CH_T(G/P)$ as the W_P-invariant subring via the $\bullet$-action, the Tymoczko action then restricts to an action on $CH_T(G/P)$. From this interpretation, the latter naturally extends to any equivariant oriented cohomology theory via $\mathbf{h}_T(G/B) \cong \mathbf{h}_G(G/B \times G/B)$.

As in the case of Chow groups, the $\odot$-action satisfies the following key property (see Theorem 3.12).

All parabolic Bott-Samelson classes $\zeta_{I_w}^J$ in $\mathbf{h}_T(G/P)$ can be obtained as $Y_{I_w} \odot [\text{pt}_J]$.

Using basic properties of the '$\odot$'-action (proven in §3.1-3.3), we provide several explicit formulas (see §3.4 and §3.5), which give previously know results for Chow groups (Example 3.20), and new results for K-theory (Example 3.21).

The $\odot$-action turns out to be a natural tool to generalize some of Deodhar's fundamental work in [16] to oriented cohomology theories; this is discussed in the second part of the paper.

For instance, using only the basic properties of the '$\odot$'-action, one can generalize the Deodhar acyclic complex for the Iwahori-Hecke algebra [16, §5] (see §3.7).

In section 4 we give a new interpretation of Deodhar's modules over Iwahori-Hecke algebras in terms of the equivariant K-theory of flag varieties. All the results stated here are known, but we provide proofs in the new setup (using the '$\odot$'-action), which give new insight and are sometimes simpler. Moreover, the basis in Deodhar's modules discussed in §4 is closely related to the *stable basis* in the K-theory of the Springer resolution defined by Maulik and Okounkov [31]; an investigation of the latter is contained in [37].

In the last section we give new interpretation of the Kazhdan-Lusztig (KL) theory in the context of oriented cohomology corresponding to a 2-parameter Todd genus (e.g., multiplicative and generic hyperbolic). Most of the results here are new. As a major application, we define the parabolic KL-Schubert classes (see §5.2) which do not depend on the choice of reduced decompositions. We discuss their functorial properties (see §5.3), and recover classical relationship between Kazhdan-Lusztig basis and Deodhar's parabolic version in 5.6. We state several conjectures (smoothness §5.4 and positivity §5.5), and prove some special cases. Our main motivation was Conjecture 5.14, which says that if the Schubert variety $X(w)$ in G/P is smooth, then the Schubert class $[X(w)]$ whose restriction formula is given in (23) coincides with the normalized parabolic KL-Schubert class indexed by w. In §5.9 we compute the parabolic KL-Schubert classes and prove this conjecture for projective spaces.

2. Preliminaries and notation

2.1. **The root datum.** We consider a semi-simple root datum $(\Lambda, \Sigma, \Sigma \hookrightarrow \Lambda^\vee)$ following [35, Exp. XXI] (cf. [9, §2]). Here Λ is a free abelian group of finite rank, $\Lambda^\vee$ is its dual, Σ is a finite non-empty subset of Λ, called the set of roots and $\Sigma \hookrightarrow \Lambda^\vee$, $\alpha \mapsto \alpha^\vee$ is an embedding satisfying standard axioms. The subgroup Λ_r generated by Σ is called the root lattice, and the subgroup

$$\Lambda_w = \{\lambda \in \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Sigma\}$$

is called the weight lattice. By definition, we have $\Lambda_r \subseteq \Lambda \subseteq \Lambda_w$.

The root lattice Λ_r has a $\mathbb{Z}$-basis $\Pi = \{\alpha_1, \dots, \alpha_n\}$, called the set of simple roots. Here n is the rank of the root datum. Any root $\alpha \in \Sigma$ can be written as a linear combination of simple roots with coefficients either all positive or all negative. So we have a decomposition $\Sigma = \Sigma^+ \amalg \Sigma^-$ into positive and negative roots.

For any $\alpha \in \Sigma$, the $\mathbb{Z}$-linear operator

$$s_\alpha \colon \Lambda \to \Lambda, \quad \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$$

is called a reflection. The Weyl group W of the root datum is generated by s_α, $\alpha \in \Sigma$; it is also generated by simple reflections s_i, $i \in [n]$, where $s_i := s_{\alpha_i}$ and $[n] = 1, \dots, n$. Sometimes we abuse notation and write Π instead of $[n]$. The length of $w \in W$ is denoted by $\ell(w)$, and if $w = s_{i_1} \cdots s_{i_l}$ is a reduced decomposition into a product of simple reflections of length l, we say that $I_w := (i_1, \dots, i_l)$ is a reduced sequence for w. The longest element of W with respect to the length is denoted by $w_\circ$.

Let $J \subseteq [n]$ and let W_J be the subgroup of W generated by s_i, $i \in J$. Then $W_\emptyset = \{e\}$ is the trivial group and $W_{[n]} = W$. We denote by $W^J = \{v \in W \mid \ell(vs_i) > \ell(v) \text{ for all } i \in J\}$ the subset of minimal left coset representatives of W/W_J. We say that a set of reduced sequences $\{I_w\}_{w \in W}$ is J-compatible if $w = uv$, $u \in W^J$, $v \in W_J$ implies that I_w is the concatenation of I_u and I_v. We set

$$\Sigma_J = \{\alpha \in \Sigma \mid s_\alpha \in W_J\}, \quad \Sigma_J^+ = \Sigma_J \cap \Sigma^+ \text{ and } \Sigma_J^- = \Sigma_J \cap \Sigma^-.$$

We will extensively use the following classical fact, sometimes implicitly.

Lemma 2.1. [16, Lemma 2.1] *For any $i \in [n]$ and $v \in W^J$, exactly one of the following three possibilities occurs:*

(i) $\ell(s_i v) < \ell(v)$, *in which case $s_i v \in W^J$ as well;*
(ii) $\ell(s_i v) > \ell(v)$ *and $s_i v \in W^J$;*
(iii) $\ell(s_i v) > \ell(v)$ *and $s_i v \notin W^J$. In this case $s_i v = v s_j$ for some $j \in J$.*

Let $w_J \in W_J$ denote the longest element of W_J, so $w_{[n]} = w_\circ$. For a subset $J' \subset J$, we set $\Sigma_{J/J'}^+ = \Sigma_J^+ \backslash \Sigma_{J'}^+$, $\Sigma_{J/J'}^- = \Sigma_J^- \backslash \Sigma_{J'}^-$ and we denote by $w_{J/J'}$ the longest minimal coset representative of $W_J/W_{J'}$, so $\ell(w_{J/J'}) = \ell(w_J) - \ell(w_{J'})$.

2.2. **Formal group laws.** Following [33, IV.2], a one-parameter commutative *formal group law* *(FGL) F* over a commutative unital ring $\mathbf{R}$ is a formal power series $F(x, y) \in \mathbf{R}[[x, y]]$ that satisfies $F(x, F(y, z)) = F(F(x, y), z)$, $F(x, 0) = x$ and $F(x, y) = F(y, x)$. It can be written as

$$F(x, y) = x + y + \sum_{i,j \geq 1} a_{ij} x^i y^j \quad \text{with } a_{ij} \in \mathbf{R}.$$

Basic examples are the additive FGL $F_a(x, y) = x+y$, the multiplicative FGL $F_m(x, y) = x+y-xy$ and the universal FGL where the coefficient ring $\mathbf{R}$ is the Lazard ring.

2.3. Hyperbolic formal group laws.

Consider an elliptic curve given in Weierstrass coordinates by the equation

$$y^2 + \mu_1 xy = x^3 + \mu_2 x^2$$

over $\mathbf{R} = \mathbb{Z}[\mu_1, \mu_2]$. Observe that it is a generic equation for a singular elliptic curve (e.g. see the proof of [33, III, Proposition 2.5]). The associated formal group law is given by

(1) $$F_{\mu_1,\mu_2}(x, y) = \frac{x+y-\mu_1 xy}{1+\mu_2 xy}.$$

Such a formal group law will be called hyperbolic (cf. [28, §2.2] and [10, Example 2.2]).

One of the key results of cobordism theory says that the following are in bijection: formal group laws over $\mathbf{R}$ and *Hirzebruch genera* with values in $\mathbf{R}$. The FGL F_{μ_1,μ_2} corresponds to the 2-parameter Todd genus $T_{\alpha,\beta}$, where $\mu_1 = \alpha + \beta$ and $\mu_2 = -\alpha\beta$. The latter was introduced and studied in [26] (see also [7]). The particular cases of $T_{\alpha,\beta}$ are the celebrated Hirzebruch genera: the Todd genus $\mu_2 = 0$, the signature $\mu_1 = 0$, and the Eulerian characteristic $\alpha = \beta$. Observe that the only singular point of our curve is a cusp in the Eulerian case, and it is a node otherwise.

The fundamental result by Quillen and Levine-Morel [30] establishes a correspondence between FGLs and algebraic oriented cohomology theories. If $\mathtt{h}$ denotes the oriented cohomology theory corresponding to F_{μ_1,μ_2} with the coefficient ring $\mathbf{R} = \mathtt{h}(\mathrm{pt})$, then the Chow group $\mathtt{h} = CH$ (or usual singular cohomology $H(X, \mathcal{O}_X)$ over $\mathbb{C}$) corresponds to $F_a = F_{0,0}$, and the K-theory $\mathtt{h} = K_0$ corresponds to $F_m = F_{1,0}$.

2.4. The generic hyperbolic formal group law.

In the present paper we will deal with the generic version of F_{μ_1,μ_2}, that is the formal group law $F_t = F_{\mu_1,\mu_2}$ over the coefficient ring $\mathbf{R} = \mathbb{Z}[t, t^{-1}, (t + t^{-1})^{-1}]$, $t \neq 0$ parametrized as $\alpha = \frac{t}{t+t^{-1}}$ and $\beta = \frac{t^{-1}}{t+t^{-1}}$, hence

(2) $$\mu_1 = 1 \text{ and } \mu_2 = -\frac{1}{(t+t^{-1})^2} \,;$$

for simplicity, we denote $u := -\mu_2 = (t + t^{-1})^{-2}$. Such a formal group law together with the respective cohomology theory will be called *generic hyperbolic*.

Observe that as $t \to 1$, $F_t(x, y) \to F_1(x, y) = \frac{x+y-xy}{1-xy/4}$ over $\mathbb{Z}\left[\frac{1}{2}\right]$, which corresponds to the Eulerian genus. If $t \to 0$, then $\mu_2 \to 0$, so $F_t(x, y) \to x + y - xy$ over $\mathbb{Z}$, which corresponds to the Todd genus (K-theory). The latter suggests that the cohomology corresponding to F_t shares more common properties with K-theory than with the Chow group or singular cohomology.

2.5. Equivariant oriented cohomology.

Let G be a split semi-simple linear algebraic group over a field k of characteristic 0 with a split maximal torus T contained in a Borel subgroup B. Consider the associated T-equivariant oriented cohomology $\mathtt{h}_T$ defined on the category of smooth algebraic T-varieties following [20] (see also [11, §2]). By the main result of [11], the T-equivariant cohomology ring $\mathtt{h}_T(G/B)$ of the variety of Borel subgroups G/B can be identified with the dual $\mathbf{D}_F^\star$ of the so-called *formal affine Demazure algebra* $\mathbf{D}_F$. Moreover, in the generic hyperbolic case $F = F_t$, the algebra $\mathbf{D}_F$ contains the classical *Iwahori-Hecke algebra* H (see [10, §9] and [27]). We briefly recall the definition of the algebra $\mathbf{D}_F$ and the details of the above identification.

2.6. Twisted group algebras.

First, we define the *formal group algebra*

$$\mathbf{S} = \mathbf{R}[[\Lambda]]_F := \mathbf{R}[[x_\lambda]]_{\lambda \in \Lambda}/(x_0, \ x_{\lambda+\nu} - F(x_\lambda, x_\nu)),$$

cf. [8, Definition 2.4]; we have $\mathbf{S} \simeq \mathtt{h}_T(\mathrm{pt})$. Let $\mathbf{Q}$ be the localization of $\mathbf{S}$ at all roots, i.e. $\mathbf{Q} = \mathbf{S}\left[\frac{1}{x_\alpha} \mid \alpha \in \Sigma\right]$. The action of W on Λ extends to actions on $\mathbf{S}$ and on $\mathbf{Q}$. Let $\mathbf{S}_W$ and $\mathbf{Q}_W$ denote the respective twisted group algebras of W spanned by δ_w, $w \in W$, that is

$$\mathbf{S}_W = \mathbf{S} \rtimes_{\mathbf{R}} \mathbf{R}[W] \text{ and } \mathbf{Q}_W = \mathbf{Q} \rtimes_{\mathbf{R}} \mathbf{R}[W]$$

and multiplication is given by $\delta_w q = w(q)\delta_w$, $q \in \mathbf{Q}$, $w \in W$; we view $\mathbf{S} \subset \mathbf{S}_W, \mathbf{Q} \subset \mathbf{Q}_W$ via the map $q \mapsto q\delta_e$.

2.7. **Formal affine Demazure algebras.** Let $\mathbf{D}_F$ be the subalgebra of $\mathbf{Q}_W$ generated by $\mathbf{S}$ and by the *push-pull elements* $Y_\alpha = \frac{1}{x_{-\alpha}} + \frac{1}{x_\alpha}\delta_{s_\alpha}$ (or, equivalently, by the *Demazure elements* $X_\alpha = \frac{1}{x_\alpha}\delta_{s_\alpha} - \frac{1}{x_\alpha}$) for all simple roots α. We denote $X_i = X_{\alpha_i}$ and $Y_i = Y_{\alpha_i}$. Observe that for the Demazure elements we use the same sign as in [28, Example 2.3]. There is a canonical action of $\mathbf{Q}_W$ on $\mathbf{Q}$, defined by

$$(3) \qquad (q\delta_w) \cdot p = qw(p), \ p, q \in \mathbf{Q}, w \in W,$$

which reduces to an action of $\mathbf{D}_F$ on $\mathbf{S}$.

If $I_w = (i_1, \ldots, i_k)$ is a reduced sequence of w, i.e. $w = s_{i_1}s_{i_2} \ldots s_{i_k}$, denote $I_w^{-1} = (i_k, \ldots, i_1)$, and set $X_{I_w} = X_{i_1}X_{i_2}\cdots X_{i_k}$, $Y_{I_w} = Y_{i_1}Y_{i_2}\cdots Y_{i_k}$. These products depend on the choice of I_w unless F is the additive or a multiplicative formal group law. By the results of [21] and [9], the elements $\{Y_{I_w}\}_{w \in W}$ (resp. $\{X_{I_w}\}_{w \in W}$) form an $\mathbf{S}$-basis of $\mathbf{D}_F$. Moreover, we know the complete set of relations in $\mathbf{D}_F$ (see [21, Proposition 5.8] for a general F and [27, Example 4.12] in the hyperbolic case). For instance, the quadratic relations

$$(4) \qquad X_\alpha^2 = -\kappa_\alpha X_\alpha, \quad Y_\alpha^2 = \kappa_\alpha Y_\alpha, \quad \text{where } \kappa_\alpha = \frac{1}{x_\alpha} + \frac{1}{x_{-\alpha}} \in \mathbf{S},$$

will play an essential role in the sequel.

2.8. **The moment map.** Finally, consider the left $\mathbf{S}$-module and the left $\mathbf{Q}$-module duals

$$\mathbf{S}_W^\star = Hom_\mathbf{S}(\mathbf{S}_W, \mathbf{S}), \quad \mathbf{Q}_W^* = Hom_\mathbf{Q}(\mathbf{Q}_W, \mathbf{Q}) \ \text{and} \ \mathbf{D}_F^\star := Hom_\mathbf{S}(\mathbf{D}_F, \mathbf{S}).$$

Let $\{f_v\}_{v \in W}$ be the canonical bases of $\mathbf{S}_W^\star$ and $\mathbf{Q}_W^*$, i.e., $f_v(w) = \delta_{v,w}$, $v, w \in W$. There is a coordinate-wise product on $\mathbf{Q}_W^*$, defined by $(pf_v)(qf_w) = pq\delta_{v,w}f_v$ where the multiplicative identity is given by $\mathbf{1} = \sum_{v \in W} f_v$. The ring $\mathbf{Q}_W^*$ is isomorphic to localization of the cohomology of the T-fixed points set

$$\mathbf{S}_W^\star = Hom(W, \mathbf{S}) \cong \mathtt{h}_T(W)$$

and $\mathbf{D}_F^\star \cong \mathtt{h}_T(G/B)$ by [11, Theorem 8.11]. Under these isomorphisms the natural embedding $\mathbf{D}_F^\star \hookrightarrow \mathbf{S}_W^\star \hookrightarrow \mathbf{Q}_W^*$ coincides with the so-called moment map (see [10, §10]).

2.9. **The parabolic case.** In the present paper we will mostly deal with the parabolic version of the above construction (see [10] and [11] for details). That is, for any $J \subset [n]$ we consider a natural projection $p_J \colon \mathbf{Q}_W \to \mathbf{Q}_{W/W_J}$, $\delta_w \mapsto \delta_{\overline{w}}$, of [10, §11]; here $\mathbf{Q}_{W/W_J}$ is spanned by $\delta_{\overline{w}}$ for $\overline{w} \in W/W_J$. Let $\mathbf{D}_{F,J}$ denote the image of $\mathbf{D}_F$ in $\mathbf{Q}_{W/W_J}$. Observe that there is a coproduct on $\mathbf{Q}_{W/W_J}$ which restricts to $\mathbf{D}_{F,J}$. Let $\mathbf{D}_{F,J}^\star = Hom_S(\mathbf{D}_{F,J}, \mathbf{S})$ be the dual. By [10, Lemma 15.1], the embedding $p_J^\star \colon \mathbf{D}_{F,J}^\star \hookrightarrow \mathbf{D}_F^\star$ identifies $\mathbf{D}_{F,J}^\star$ with $(\mathbf{D}_F^\star)^{W_J}$, where W_J acts on $\mathbf{D}_F^\star$ via the Hecke •-action (see below). Finally, the main result of [11] says that $\mathbf{D}_{F,J}^\star$ is isomorphic to $\mathtt{h}_T(G/P_J)$, where P_J is a standard parabolic subgroup corresponding to J and G/P_J is the respective projective homogeneous variety.

3. Two Hecke actions

In this section we give the central definition of this paper, i.e., the right Hecke action.

3.1. **Left and right Hecke actions.** We first recall several definitions and facts. From [10, Lemma 10.3] and [11, Corollary 6.4] the class of the identity T-fixed point in $\mathbf{D}_F^\star \cong \mathbf{h}_T(G/B)$ is given by

$$\text{(5)} \qquad \text{pt} = x_\Pi f_e, \text{ where } x_\Pi = \prod_{\alpha \in \Sigma^-} x_\alpha \,.$$

Following [10, §13] there is an anti-involution ι of $\mathbf{Q}_W$ defined by

$$\text{(6)} \qquad \iota(q\delta_w) = \delta_{w^{-1}} q \frac{w(x_\Pi)}{x_\Pi}.$$

Its restriction to $\mathbf{Q}$ is the identity map, and it restricts to an involution of $\mathbf{D}_F$ that maps X_i to X_i (and hence Y_i to Y_i). Thus, we have $\iota(X_I) = X_{I^{-1}}$ and $\iota(Y_I) = Y_{I^{-1}}$, where I^{-1} is the sequence obtained from I by reversing the order.

Following [10, §4] there is an action of $\mathbf{Q}_W$ on $\mathbf{Q}$, defined by

$$(z \bullet f)(z') = f(z'z), \quad z, z' \in \mathbf{Q}_W, f \in \mathbf{Q}_W^*.$$

From [10, Corollary 12.1], it restricts to an action of $\mathbf{D}_F$ on $\mathbf{D}_F^\star$. Moreover, we have the following result.

Theorem 3.1. [10, Theorem 10.13] *Via the $\bullet$-action, $\mathbf{D}_F^\star$ (resp. $\mathbf{Q}_W^*$) is a free $\mathbf{D}_F$-module (resp. $\mathbf{Q}_W$) of rank 1, and pt is their basis, i.e.,*

$$\mathbf{D}_F^\star = \mathbf{D}_F \bullet \text{pt}, \quad \mathbf{Q}_W^* = \mathbf{Q}_W \bullet \text{pt} \,.$$

So the elements of $\mathbf{D}_F^\star$ (resp. $\mathbf{Q}_W^*$) can be uniquely written as $z \bullet \text{pt}$ with $z \in \mathbf{D}_F$ (resp. $z \in \mathbf{Q}_W$). For example, it follows from [10, Lemma 4.2] that

$$\text{(7)} \qquad q f_v = v^{-1} \left(\frac{q}{x_\Pi} \right) \delta_{v^{-1}} \bullet \text{pt}, \quad q \in \mathbf{Q}.$$

We now introduce a key notion of the present paper.

Definition 3.2. Consider a right action of $\mathbf{Q}_W$ on $\mathbf{Q}_W^*$, defined as follows:

$$(z' \bullet \text{pt}) \star z := z'z \bullet \text{pt}, \quad z, z' \in \mathbf{Q}_W.$$

Observe that it restricts to a right action of $\mathbf{D}_F$ on $\mathbf{D}_F^\star$. Moreover, if composed with the involution ι, it gives left actions of $\mathbf{D}_F$ on $\mathbf{D}_F^\star$ and $\mathbf{Q}_W$ on $\mathbf{Q}_W^*$, respectively, which will be denoted by $\odot$. That is,

$$\text{(8)} \qquad z \odot (z' \bullet pt) := (z' \bullet \text{pt}) \star \iota(z) = z'\iota(z) \bullet \text{pt}, \quad z, z' \text{ belongs to } \mathbf{D}_F \text{ or } \mathbf{Q}_W.$$

We call $\bullet$ the *left Hecke action* and $\odot$ the *right Hecke action*, respectively.

By definition the left Hecke action is $\mathbf{Q}$-linear, but the right Hecke action is not.

Remark 3.3. (i) The $\odot$-action depends on the choice of a class of a T-fixed point pt of $\mathbf{D}_F^\star$. If one chooses a class of a different T-fixed point from [10, Theorem 10.13], one gets a different version of the $\odot$-action.

(ii) The terminology is slightly misleading as both the left and right Hecke actions are both left actions (which will be implicitly used throughout). Indeed, both actions are induced by the multiplication on $\mathbf{D}_F$: the left Hecke action comes from multiplication from the left in $\mathbf{D}_F$, and the right Hecke action comes from multiplication from the right. From the geometric point of view, there is an isomorphism $\mathbf{h}_T(G/B) \cong \mathbf{h}_G(G/B \times G/B)$, where the right hand side has a convolution product that defines the product in $\mathbf{D}_F$. So convolution from the left gives the $\bullet$-action, and convolution from the right gives the $\odot$-action.

Lemma 3.4. *We have*

(i) *The left and right Hecke actions commute with each other.*

(ii) $z \odot \text{pt} = \iota(z) \bullet \text{pt}$. *In particular, $X_I \odot \text{pt} = X_{I^{-1}} \bullet \text{pt}$ and $Y_I \odot \text{pt} = Y_{I^{-1}} \bullet \text{pt}$.*

(iii) *Viewing $\mathbf{D}_F^\star$ as a left $\mathbf{D}_F$-module via the $\bullet$-action, there is an isomorphism of rings*

$$\pi\colon \mathbf{D}_F \to \operatorname{End}_{\mathbf{D}_F}(\mathbf{D}_F^\star), \quad \pi(z)\colon f \mapsto z \odot f.$$

Proof. (i) and (ii). Both follow from the definition of the $\odot$-action.

(iii). It follows from part (i) that π is well defined. Since ι is an anti-isomorphism and the $\bullet$-action is effective, π is injective. Moreover, given any $\phi \in \operatorname{End}_{\mathbf{D}_F}(\mathbf{D}_F^\star)$, by Theorem 3.1, ϕ is determined by $\phi(\mathrm{pt})$, which can be written as $z \bullet \mathrm{pt}, z \in \mathbf{D}_F$, then $\phi(\mathrm{pt}) = z \bullet \mathrm{pt} = \iota(z) \odot \mathrm{pt} = \pi(\iota(z))(\mathrm{pt})$, so $\phi = \pi(\iota(z))$. Therefore, π is an isomorphism. $\qquad\square$

Lemma 3.5. *The two Hecke actions satisfy*

$$(p\delta_w) \bullet (qf_v) = qvw^{-1}(p)f_{vw^{-1}}, \quad (p\delta_w) \odot (qf_v) = pw(q)f_{wv}, \quad p, q \in \mathbf{Q}, w, v \in W.$$

Proof. The first identity is proved in [10, Lemma 4.2]. For the second one, we have

$$(p\delta_w) \odot (qf_v) \overset{(7)}{=} (p\delta_w) \odot \left(v^{-1}\left(\frac{q}{x_\Pi}\right) \delta_{v^{-1}} \bullet \mathrm{pt} \right) \overset{(8)}{=} \left(v^{-1}\left(\frac{q}{x_\Pi}\right) \delta_{v^{-1}} \iota(p\delta_w) \right) \bullet \mathrm{pt}$$

$$\overset{(6)}{=} \left(v^{-1}\left(\frac{q}{x_\Pi}\right) \delta_{v^{-1}} \delta_{w^{-1}} p \frac{w(x_\Pi)}{x_\Pi} \right) \bullet \mathrm{pt} \overset{\sharp_1}{=} \left(\delta_{v^{-1}w^{-1}} \frac{pw(qx_\Pi)}{w(x_\Pi)x_\Pi} \right) \bullet (x_\Pi f_e)$$

$$\overset{\sharp_2}{=} \delta_{v^{-1}w^{-1}} \bullet (pw(q)f_e) = pw(q)f_{wv},$$

where $\sharp_1$ follows from the product in $\mathbf{Q}_W$, and $\sharp_2$ follows from the first identity of this lemma. $\qquad\square$

Lemma 3.6. *We have*

(i) $Y_\alpha \odot \left(\sum_v p_v f_v\right) = \sum_v \left(\frac{p_v}{x_{-\alpha}} + s_\alpha \left(\frac{p_{s_\alpha v}}{x_{-\alpha}} \right) \right) f_v,$

(ii) $X_\alpha \odot \left(\sum_v p_v f_v\right) = \sum_v \frac{s_\alpha(p_{s_\alpha v}) - p_v}{x_\alpha} f_v.$

Proof. We just check the first identity. We have

$$Y_\alpha \odot p f_v = \left(\frac{1}{x_{-\alpha}} + \frac{1}{x_\alpha}\delta_\alpha \right) \odot p f_v = \frac{p}{x_{-\alpha}} f_v + s_\alpha \left(\frac{p}{x_{-\alpha}} \right) f_{s_\alpha v}. \qquad\square$$

3.2. **Hecke actions and the characteristic map.** We now look at the behavior of the two Hecke actions with respect to the characteristic map. Consider the equivariant characteristic map

$$c_{\mathbf{S}}\colon \mathbf{S} \to \mathbf{D}_F^\star \simeq \mathrm{h}_T(G/B) \text{ given by } s \mapsto s \bullet \mathbf{1}$$

(or, equivalently, by evaluation of the respective operator from $\mathbf{D}_F$ at $s \in \mathbf{S}$, see [9, §11]). After tensoring $\mathbf{D}_F^\star$ with the augmentation $\epsilon\colon \mathbf{S} \to \mathbf{R}$, it turns into the classical characteristic map of ordinary cohomology $c_{\mathbf{R}}\colon \mathbf{S} \to \epsilon \mathbf{D}_F^\star \simeq \mathrm{h}(G/B)$. Consider the Borel map

$$\rho\colon \mathbf{S} \otimes_{\mathbf{S}^W} \mathbf{S} \to \mathbf{D}_F^\star \text{ given by } q_1 \otimes q_2 \mapsto q_1 c_S(q_2) = q_1(q_2 \bullet \mathbf{1}) = q_2 \bullet (q_1 \mathbf{1}).$$

By [9, Theorem 11.4] it is an isomorphism if the characteristic map $c_{\mathbf{R}}$ is surjective (for example, if the torsion index is 1, or when $F = F_{1,0}$ is the multiplicative formal group law).

Using the canonical action (3) of $\mathbf{D}_F$ on $\mathbf{S}$, we obtain the following result.

Lemma 3.7. *For any $z \in \mathbf{D}_F$, we have the following commutative diagrams.*

$$
\begin{array}{ccc}
\mathbf{S} \otimes_{\mathbf{S}^W} \mathbf{S} & \overset{\rho}{\longrightarrow} & \mathbf{D}_F^\star \\
{\scriptstyle (z\cdot\_)\otimes 1}\downarrow & & \downarrow{\scriptstyle z\odot\_} \\
\mathbf{S} \otimes_{\mathbf{S}^W} \mathbf{S} & \overset{\rho}{\longrightarrow} & \mathbf{D}_F^\star
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{S} \otimes_{\mathbf{S}^W} \mathbf{S} & \overset{\rho}{\longrightarrow} & \mathbf{D}_F^\star \\
{\scriptstyle 1\otimes(z\cdot\_)}\downarrow & & \downarrow{\scriptstyle z\bullet\_} \\
\mathbf{S} \otimes_{\mathbf{S}^W} \mathbf{S} & \overset{\rho}{\longrightarrow} & \mathbf{D}_F^\star
\end{array}
$$

Proof. By direct computation using Lemma 3.5 and the definition of ρ. $\qquad\square$

Remark 3.8. Using the element Y_Π introduced in the next section, the lemma can be also deduced from the following identities:

$$\iota(Y_\Pi) = Y_\Pi, \quad Y_\Pi \bullet \mathrm{pt} = \mathbf{1} \quad \text{and} \quad zpY_\Pi = (z \cdot p)Y_\Pi \text{ for any } z \in \mathbf{D}_F, \ p \in \mathbf{S}.$$

Moreover, using (10) below, we have

$$Y_\alpha \odot \mathbf{1} = Y_\alpha \odot Y_\Pi \bullet \mathrm{pt} = Y_\Pi Y_\alpha \bullet \mathrm{pt} = \kappa_\alpha \mathbf{1}, \quad Y_\alpha \bullet \mathbf{1} = Y_\alpha Y_\Pi \bullet \mathrm{pt} = \kappa_\alpha \bullet \mathbf{1},$$
$$X_\alpha \odot \mathbf{1} = Y_\Pi X_\alpha \bullet \mathrm{pt} = 0, \quad X_\alpha \bullet \mathbf{1} = X_\alpha Y_\Pi \bullet \mathbf{1} = 0.$$

The following Proposition was proved in [6, page 253] for equivariant Chow groups, and in [8, Lemma 13.4] for ordinary oriented cohomology.

Proposition 3.9. *(cf. also* [10, Lemma 13.3]*) For any $z \in \mathbf{Q}_W$, $q \in \mathbf{Q}$, we have*

$$A_\Pi((z \odot \mathrm{pt})c_{\mathbf{S}}(q)) = (z \cdot q)\mathbf{1},$$

where $A_\Pi = Y_\Pi \bullet \_$ is the pushforward to the point map (see the definition in the next section).

3.3. **Hecke actions on cohomology.** We now study the Hecke actions on the cohomology ring $\mathtt{h}_T(G/P_J)$, where P_J is the parabolic subgroup corresponding to a subset $J \subset \Pi$.

We first look at the $\bullet$-action. Given a subset $J' \subset J$ we define

$$x_{J/J'} = \prod_{\alpha \in \Sigma^-_{J/J'}} x_\alpha, \quad x_J = x_{J/\emptyset}.$$

Note that the definition of x_Π in (5) is a special case of the one above. Given a set of left coset representatives $W_{J/J'}$ of $W_J/W_{J'}$, we define a push-pull element

$$Y_{J/J'} := \left(\sum_{w \in W_{J/J'}} \delta_w \right) \frac{1}{x_{J/J'}}, \quad Y_J = Y_{J/\emptyset} = \left(\sum_{w \in W_J} \delta_w \right) \frac{1}{x_J}.$$

Clearly, if $J = \{i\}$, we recover the definition of Y_i. The $\bullet$-action of $Y_{J/J'}$ on $\mathbf{Q}^*_W$ is denoted by $A_{J/J'}$. It follows from [10, Corollary 12.1] that $Y_J \in \mathbf{D}_F$, so A_J is an endomorphism of $\mathbf{D}^\star_F$, which corresponds to the composition of the pushforward and pullback maps in cohomology

$$A_J \colon \mathtt{h}_T(G/B) \to \mathtt{h}_T(G/P_J) \to \mathtt{h}_T(G/B).$$

Let $J'' \subset J' \subset J$, and let $W_{J/J'}$ and $W_{J'/J''}$ be some sets of coset representatives of $W_J/W_{J'}$ and $W_{J'}/W_{J''}$, respectively. Let $W_{J/J''} = W_{J/J'}W_{J'/J''}$, which is a set of coset representatives of $W_J/W_{J''}$. We then have (see [10, Lemma 5.7])

(9) $$Y_{J/J''} = Y_{J/J'}Y_{J'/J''} \quad \text{and} \quad A_{J/J''} = A_{J/J'} \circ A_{J'/J''}.$$

If $j \in J$, then (see [10, Lemma 5.9])

(10) $$Y_j Y_J = \kappa_j Y_J, \ Y_J Y_j = Y_J \kappa_j \quad \text{and} \quad X_j Y_J = Y_J X_j = 0.$$

Define $\mathbf{D}^\star_{F,J} = A_J(\mathbf{D}^\star_F)$. It follows from [10, Corollary 14.6, Lemma 15.1] and [11, Theorem 8.11] that

$$\mathbf{D}^\star_{F,J} = (\mathbf{D}^\star_F)^{W_J} \cong \mathtt{h}_T(G/P_J).$$

Here the superscript W_J means the W_J-invariant subset via the $\bullet$-action of $W_J \subset W \subset \mathbf{D}_F$ on $\mathbf{D}^\star_F$. Moreover, they also imply that the endomorphism $A_{J/J'}$ of $\mathbf{Q}^*_W$ induces a surjective map $\mathbf{D}^\star_{F,J'} \to \mathbf{D}^\star_{F,J}$, which corresponds to the pushforward map $\mathtt{h}_T(G/P_{J'}) \to \mathtt{h}_T(G/P_J)$.

Remark 3.10. The definition of $A_{J/J'}$ depends on the choice of the coset representatives of $W_J/W_{J'}$, but its restriction to $\mathbf{D}^\star_{F,J'}$ does not (see [10, Lemma 6.5]).

The element $A_J(\mathrm{pt}) \in \mathbf{D}_{F,J}^\star$ corresponds to the class of the identity point in $\mathtt{h}_T(G/P_J)$, so we denote it by pt_J. Based on definitions, this can be expressed more explicitly as follows:

$$(11) \qquad\qquad \mathrm{pt}_J = x_{\Pi/J} \sum_{w \in W_J} f_w \,.$$

By using $w(x_{\Pi/J}) = x_{\Pi/J}$ for $w \in W_J$, we have $\iota(Y_J) = Y_J$, so

$$(12) \qquad\qquad \mathrm{pt}_J = Y_J \bullet \mathrm{pt} = Y_J \odot \mathrm{pt} \,.$$

Lemma 3.11. *The $\odot$-action of $\mathbf{D}_F$ on $\mathbf{D}_F^\star$ restricts to an action on $\mathbf{D}_{F,J}^\star$, making the latter into a $\mathbf{D}_F$-module.*

Proof. This follows from the definition of $\mathbf{D}_{F,J}^\star$ and Lemma 3.4.(i). $\qquad\qquad\qquad\qquad \square$

By definition, the $\odot$-action of $\mathbf{D}_F$ on $\mathbf{D}_{F,J}^\star$ can be expressed as follows:

$$\mathbf{D}_F \to \mathbf{D}_{F,J}^\star, \quad z \mapsto z \odot \mathrm{pt}_J = z \odot A_J(\mathrm{pt}) = A_J(z \odot \mathrm{pt}) = A_J(\iota(z) \bullet \mathrm{pt}) \,.$$

Thus, the *parabolic Bott-Samelson classes* inside $\mathbf{D}_{F,J}^\star \cong \mathtt{h}_T(G/P_J)$ are determined by

$$(13) \qquad\qquad \zeta_{I_w}^J := Y_{I_w} \odot \mathrm{pt}_J = A_J(Y_{I_w^{-1}} \bullet \mathrm{pt}), \quad w \in W^J \,.$$

Similarly, one can look at the classes determined by the X-operators, namely at

$$(14) \qquad\qquad \hat{\zeta}_{I_w}^J := X_{I_w} \odot \mathrm{pt}_J = A_J(X_{I_w^{-1}} \bullet \mathrm{pt}), \quad w \in W^J \,.$$

We have $\hat{\zeta}_{I_e} = \zeta_{I_e} = \mathrm{pt}$ and $\hat{\zeta}_{I_e}^J = \zeta_{I_e}^J = \mathrm{pt}_J$. It follows from [10, Theorem 14.3] that the sets $\{\hat{\zeta}_{I_w}^J\}_{w \in W^J}$ and $\{\zeta_{I_w}^J\}_{w \in W^J}$ are bases of $\mathbf{D}_{F,J}^\star$. Therefore, the following theorem holds:

Theorem 3.12. *All parabolic Bott-Samelson classes $\zeta_{I_w}^J$, $w \in W^J$, in $\mathbf{D}_{F,J}^\star \simeq \mathtt{h}_T(G/P)$ can be obtained by applying the operators $Y_i \odot \_$, $i \in [n]$ to the class of the point $\zeta_{I_e}^J = \mathrm{pt}_J$. The same holds for the classes $\hat{\zeta}_{I_w}^J$ if we use the operators $X_i \odot \_$ instead.*

Remark 3.13. (i) By (13) and (14), the parabolic Bott-Samelson classes can be calculated both via the $\odot$- and the $\bullet$-actions. The advantage of the $\odot$-action is that it leads to a recursive computation which stays inside $\mathbf{D}_{F,J}^\star$. More precisely, given a reduced sequence $I_w = (i_1, i_2, \ldots, i_k)$ of $w \in W^J$, we know from Lemma 2.1 that $v := s_{i_1} w < w$ is in W^J; so $I_v := (i_2, \ldots, i_k)$ is a reduced sequence of v, and we have

$$(15) \qquad\qquad \zeta_{I_w}^J = Y_{i_1} \odot \zeta_{I_v}^J, \quad \hat{\zeta}_{I_w}^J = X_{i_1} \odot \hat{\zeta}_{I_v}^J \,.$$

(ii) Also by (13) and (14), the parabolic Bott-Samelson classes are obtained from the ones corresponding to G/B via the pushforward map, namely

$$\zeta_{I_w}^J = A_J(\zeta_{I_w}), \quad \hat{\zeta}_{I_w}^J = A_J(\hat{\zeta}_{I_w}), \quad w \in W^J \,.$$

On the other hand, in ordinary cohomology and K-theory, it is well-known that the pullback map sends Schubert classes to Schubert classes, see e.g. [18, §10.6]. This can be seen easily in our algebraic setting. Indeed, for $F = F_a$ or F_m, it follows from similar argument as in Proposition 4.10.(ii) that $Y_J = Y_{w_J}$. So

$$\zeta_w^J = A_J(\zeta_w) = Y_{w_J} \bullet Y_{w^{-1}} \bullet \mathrm{pt} = Y_{(ww_J)^{-1}} \bullet \mathrm{pt} = \zeta_{ww_J}, \quad w \in W^J \,.$$

For general F, we may not have $Y_J = Y_{w_J}$. However, see Theorem 5.12 below for a similar situation.

3.4. **Hecke actions on parabolic Bott-Samelson classes.** We now study the action of $A_{J/J'}$ on the classes $\zeta_{I_w}^J$ and $\hat{\zeta}_{I_w}^J$, with the goal of extending Remark 3.13 (ii). For $J' \subset J$, we have $W^J \subset W^{J'}$, and by definition, the **S**-linear homomorphism $A_{J/J'}$ induced by $Y_{J/J'}$ commutes with the $\odot$-action of $\mathbf{D}_F$ on $\mathbf{D}_{F,J}^{\star}$.

Lemma 3.14. *Let $\{I_w\}_{w \in W}$ be a set of reduced sequences. Let $J' \subset J$, and $v \in W^{J'}$, then*

$$A_{J/J'}(\hat{\zeta}_{I_v}^{J'}) = \begin{cases} \hat{\zeta}_{I_v}^J, & \text{if } v \in W^J; \\ \displaystyle\sum_{u \in W^J, \ell(u) < \ell(v)} d_u \hat{\zeta}_{I_u}^J, & \text{if } v \notin W^J. \end{cases}$$

The same conclusion holds if one replaces $\hat{\zeta}_{I_v}^{J'}$ by $\zeta_{I_v}^{J'}$.

Proof. By (9), we have

$$A_{J/J'}(\hat{\zeta}_{I_v}^{J'}) = A_{J/J'} \circ A_{J'}(X_{I_v} \odot \text{pt}) = A_J(X_{I_v} \odot \text{pt}) = A_J(X_{I_v^{-1}} \bullet \text{pt}).$$

If $v \in W^J$, it follows that $A_{J/J'}(\hat{\zeta}_{I_v}^{J'}) = \hat{\zeta}_{I_v}^J$.

If $v \notin W^J$, then $v = uw$ with $u \in W^J, w \in W_J$ and $\ell(u) < \ell(v)$. Hence, the concatenation of I_u and I_w is another reduced sequence of v. We prove the conclusion by induction on $\ell(u)$. If $\ell(u) = 0$, then $v = w$ and $A_J(X_{I_w^{-1}} \bullet \text{pt}) = \delta_{w,e}\hat{\zeta}_{I_e}^J$, by (10). So the conclusion follows.

Now suppose $\ell(u) \geq 1$. By [9, Lemma 7.1], we have

$$X_{I_v} - X_{I_u}X_{I_w} = \sum_{v' \in W, \ell(v') < \ell(v)} d_{v'} X_{I_{v'}}.$$

So

$$A_J(X_{I_v} \odot \text{pt}) = A_J(X_{I_u}X_{I_w} \odot \text{pt}) + \sum A_J((d_{v'}X_{I_{v'}}) \odot \text{pt}).$$

By induction,

$$A_J((d_{v'}X_{I_{v'}}) \odot \text{pt}) = A_J(X_{I_{v'}^{-1}}d_{v'} \bullet \text{pt}) = d_{v'} A_J(X_{I_{v'}^{-1}} \bullet \text{pt})$$

is a linear combination of $\hat{\zeta}_{I_{u'}}^J$ with $u' \in W^J$ and $\ell(u') \leq \ell(v') < \ell(v)$. Moreover,

$$A_J(X_{I_u}X_{I_w} \odot \text{pt}) = A_J(X_{I_w^{-1}}X_{I_u^{-1}} \bullet \text{pt}) = \delta_{w,e}\hat{\zeta}_{I_u}^J,$$

by (10). So the conclusion follows.

The statement for $\zeta_{I_v}^J$ can be proved similarly. $\qquad\square$

We will now state a more explicit version of Lemma 3.14 in a special case.

Definition 3.15. For $J' \subset J$, we say a set of reduced sequences $\{I_w\}_{w \in W}$ is $J' \subset J$-compatible if for any $w = v_1 v_2 v_3$ such that $v_1 \in W^J, v_2 \in W_J, v_3 \in W_{J'}$ and $v_1 v_2 \in W^{J'}$, the sequence I_w is the concatenation of I_{v_1}, I_{v_2}, and I_{v_3} in this order.

Lemma 3.16. *If $\{I_w\}_{w \in W}$ is $J' \subset J$-compatible, then for any $v_1 \in W^J, v_2 \in W_J$ such that $v_1 v_2 \in W^{J'}$, we have*

$$A_{J/J'}(\hat{\zeta}_{I_{v_1 v_2}}^{J'}) = \delta_{v_2,e}\hat{\zeta}_{I_{v_1}}^J.$$

If κ_α is a constant κ for any α, then

$$A_{J/J'}(\zeta_{I_{v_1 v_2}}^{J'}) = \kappa^{\ell(v_2)}\zeta_{I_{v_1}}^J.$$

Proof. The proof is similar to that of Lemma 3.14. $\qquad\square$

If F is F_a or F_m, that is, in the case of the Chow group or the Grothendieck group, both conditions in the above lemma are satisfied.

3.5. **Recursive formulas.** Now assume that $\kappa_\alpha = \kappa$ is a constant, and that W^J is *fully commutative*; this means that for each $w \in W^J$, any two reduced sequences I, I' of w can be related by means of only braid relations involving commuting generators. Thus, $X_I = X_{I'}$ and $Y_I = Y_{I'}$, so we can write X_w instead. Note that, although w^{-1} might not be in W^J, it still has the fully commutative property, so $X_{w^{-1}}$ is defined and $\iota(X_w) = X_{w^{-1}}$.

Remark 3.17. For irreducible finite Weyl groups W, the fully commutative parabolic quotients W^J were classified in [36, Theorem 6.1]. They are the irreducible *minuscule quotients*, which are also maximal quotients (so the complement of J consists of a single element); for the explicit list, which includes the type A Grassmannians, see also [36].

Proposition 3.18. *Under the above hypothesis, we have*

$$X_j \odot \hat{\zeta}_v^J = \begin{cases} -\kappa\hat{\zeta}_v^J, & \text{if } \ell(s_jv) < \ell(v), \text{in which case } s_jv \in W^J; \\ \hat{\zeta}_{s_jv}^J, & \text{if } \ell(s_jv) > \ell(v) \text{ and } s_jv \in W^J; \\ \displaystyle\sum_{u \in W^J, \ell(u) \le \ell(v)-1} d_{v,u}^J \hat{\zeta}_u^J, & \text{if } \ell(s_jv) > \ell(v) \text{ and } s_jv = vs_i \text{ for some } i \in J, \end{cases}$$

where $d_{v,u}^J \in \mathbf{S}$.

Proof. By definition we have $X_j \odot \hat{\zeta}_v^J = X_j X_v \odot \mathrm{pt}_J$.

If $\ell(s_jv) < \ell(v)$, then $X_v = X_j X_{s_jv}$, and $X_j X_v = X_j^2 X_{s_jv} = -\kappa X_j X_{s_jv} = -\kappa X_v$. So $X_j \odot \hat{\zeta}_v^J = -\kappa\hat{\zeta}_v^J$.

If $\ell(s_jv) > \ell(v)$ and $s_jv \in W^J$, then Remark 3.13 (i) applies.

If $\ell(s_jv) > \ell(v)$ and $s_jv = vs_i$ with $i \in J$, then by [9, Lemma 7.1],

$$X_j X_v - X_v X_i = \sum_{w \in W, \ell(w) \le \ell(v)-1} \tilde{d}_{v,w} X_{I_w}, \ \tilde{d}_{v,w}^J \in \mathbf{S}.$$

So

$$X_j X_v \odot \mathrm{pt}_J = X_v X_i \odot \mathrm{pt}_J + \sum_{w \in W, \ell(w) \le \ell(v)-1} \tilde{d}_{v,w} X_{I_w} \odot \mathrm{pt}_J.$$

We have

$$X_v X_i \odot \mathrm{pt}_J \overset{(12)}{=} X_v X_i \odot Y_J \odot \mathrm{pt} = X_v \odot (X_i Y_J) \odot \mathrm{pt} \overset{(10)}{=} 0.$$

Moreover, we know that

$$\tilde{d}_{v,w} X_{I_w} \odot \mathrm{pt}_J = A_J(\iota(\tilde{d}_{v,w} X_{I_w}) \bullet \mathrm{pt}) = A_J(X_{I_w^{-1}} \tilde{d}_{v,w} \bullet \mathrm{pt}) = \tilde{d}_{v,w} A_J(X_{I_w^{-1}} \bullet \mathrm{pt})$$

is a linear combination of $\hat{\zeta}_u^J$ with $u \in W^J$ and $\ell(u) \le \ell(w) \le \ell(v) - 1$, by the $\mathbf{S}$-linearity of the $\bullet$-action and the basis property of $\mathbf{D}_{F,J}^\star$. $\qquad\square$

Similarly, using (4) and (10), we have the following result.

Proposition 3.19. *Under the above hypothesis, we have*

$$Y_j \odot \zeta_v^J = \begin{cases} \kappa\zeta_v^J, & \text{if } \ell(s_jv) < \ell(v), \text{in which case } s_jv \in W^J; \\ \zeta_{s_jv}^J, & \text{if } \ell(s_jv) > \ell(v) \text{ and } s_jv \in W^J; \\ \kappa\zeta_v^J + \displaystyle\sum_{u \in W^J, \ell(u) \le \ell(v)-1} \hat{d}_{v,u}^J \zeta_u^J, & \text{if } \ell(s_jv) > \ell(v) \text{ and } s_jv = vs_i \text{ for some } i \in J \end{cases}$$

where $\hat{d}_{v,u}^J \in \mathbf{S}$.

In the case of an additive F_a or a multiplicative F_m formal group law the proofs of Propositions 3.18 and 3.19 work for any flag variety G/P_J (as the respective X_{I_v} and Y_{I_v} do not depend on the choice of a reduced sequence I_v of v). So we obtain the following formulas.

Example 3.20. In the case $F = F_a$, we have $X_i = Y_i$, $\kappa_\alpha = 0$, and

$$
X_j \odot \zeta_v^{F_a,J} = \begin{cases} 0, & \text{if } \ell(s_j v) < \ell(v), \text{in which case } s_j v \in W^J; \\ \zeta_{s_j v}^{F_a,J}, & \text{if } \ell(s_j v) > \ell(v) \text{ and } s_j v \in W^J; \\ 0, & \text{if } \ell(s_j v) > \ell(v) \text{ and } s_j v = v s_i \text{ for some } i \in J. \end{cases}
$$

This formula was known before to Brion [6], Knutson [23], and Tymoczko [40].

The next two formulas, which correspond to the case of K-theory, seem to be new; here we use $\kappa_\alpha = 1$.

Example 3.21. We have

$$
X_j \odot \hat{\zeta}_v^{F_m,J} = \begin{cases} -\hat{\zeta}_v^{F_m,J}, & \text{if } \ell(s_j v) < \ell(v), \text{in which case } s_j v \in W^J; \\ \hat{\zeta}_{s_j v}^{F_m,J}, & \text{if } \ell(s_j v) > \ell(v) \text{ and } s_j v \in W^J; \\ 0, & \text{if } \ell(s_j v) > \ell(v) \text{ and } s_j v = v s_i \text{ for some } i \in J; \end{cases}
$$

$$
Y_j \odot \zeta_v^{F_m,J} = \begin{cases} \zeta_v^{F_m,J}, & \text{if } \ell(s_j v) < \ell(v), \text{in which case } s_j v \in W^J; \\ \zeta_{s_j v}^{F_m,J}, & \text{if } \ell(s_j v) > \ell(v) \text{ and } s_j v \in W^J; \\ \zeta_v^{F_m,J}, & \text{if } \ell(s_j v) > \ell(v) \text{ and } s_j v = v s_i \text{ for some } i \in J. \end{cases}
$$

3.6. **An example.** We give an example for the calculation of the parabolic Bott-Samelson classes in the case of the hyperbolic formal group law $F_{1,-u}$. Consider the Grassmannian of 2-planes in $\mathbb{C}^4$, so $W = S_4$, $J = \{1, 3\}$, and

$$
W^J = \{e,\ s_2,\ s_3 s_2,\ s_1 s_2,\ s_1 s_3 s_2,\ s_2 s_1 s_3 s_2\}.
$$

This Grassmannian contains a singular Schubert variety, namely the one indexed by $s_1 s_3 s_2$. Recall that, since W^J is fully commutative, we can index the parabolic Bott-Samelson classes by the elements of W^J.

We summarize the recursive calculation (11) of the parabolic Bott-Samelson classes via the $\odot$-action in the following diagram; here the edges are labeled by the simple reflections used in the corresponding left multiplication.
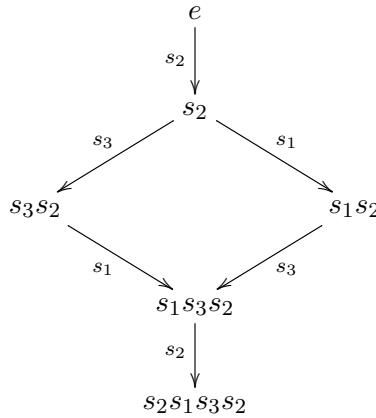
For simplicity, we use the notation $[ij] := x_{-\alpha_{ij}}$, where α_{ij} is the root $\varepsilon_i - \varepsilon_j$. Writing $\zeta_v^J = \sum_{w \in W} q_w f_w$, we note that it is enough to record the coefficients q_w for $w \in W^J$ (because ζ_v^J can be viewed as elements of $(\mathbf{D}_F^\star)^{W_J}$, see [10, Lemma 6.1]); we do this by also using the above diagram.

Recalling the expression of ζ_e^J in (15), we proceed as follows.

$$
\zeta_e^J = \quad
\begin{array}{c}
[13][14][23][24] \\
\downarrow \\
0 \\
0 \swarrow \quad \searrow 0 \\
\searrow \quad \swarrow \\
0 \\
\downarrow \\
0
\end{array}
\quad \xrightarrow{\;Y_2\;} \quad
\begin{array}{c}
[13][14][24] \\
\downarrow \\
[12][14][34] \\
0 \swarrow \quad \searrow 0 \\
\searrow \quad \swarrow \\
0 \\
\downarrow \\
0
\end{array}
\quad \xrightarrow{\;Y_1\;}
$$

$$
\begin{array}{c}
[14][24](1 + u[13][23]) \\
\downarrow \\
[14][34] \\
0 \swarrow \quad \searrow [24][34] \\
\searrow \quad \swarrow \\
0 \\
\downarrow \\
0
\end{array}
$$

$$
\zeta_{s_2}^J = \quad
\begin{array}{c}
[13][14][24] \\
\downarrow \\
[12][14][34] \\
0 \swarrow \quad \searrow 0 \\
\searrow \quad \swarrow \\
0 \\
\downarrow \\
0
\end{array}
\quad \xrightarrow{\;Y_3\;} \quad [12][13]
\begin{array}{c}
[13][14](1 + u[23][24]) \\
\downarrow \\
[12][14] \\
\swarrow \quad \searrow 0 \\
\searrow \quad \swarrow \\
0 \\
\downarrow \\
0
\end{array}
\quad \xrightarrow{\;Y_1\;}
$$

$$[14] + [23] - [14][23] + u[14][23]([13] + [24] + [13][24])$$

$$\downarrow$$

$$[14]$$

$$[13] \qquad\qquad [24] \qquad \xrightarrow{Y_2}$$

$$[23]$$

$$\downarrow$$

$$0$$

$$1 + u[14]([13] + [24] + [13][24])$$

$$\downarrow$$

$$1 + u[14]([12] + [34] + [12][34])$$

$$1 + u[12][13] \qquad\qquad\qquad 1 + u[24][34]$$

$$1$$

$$\downarrow$$

$$1$$
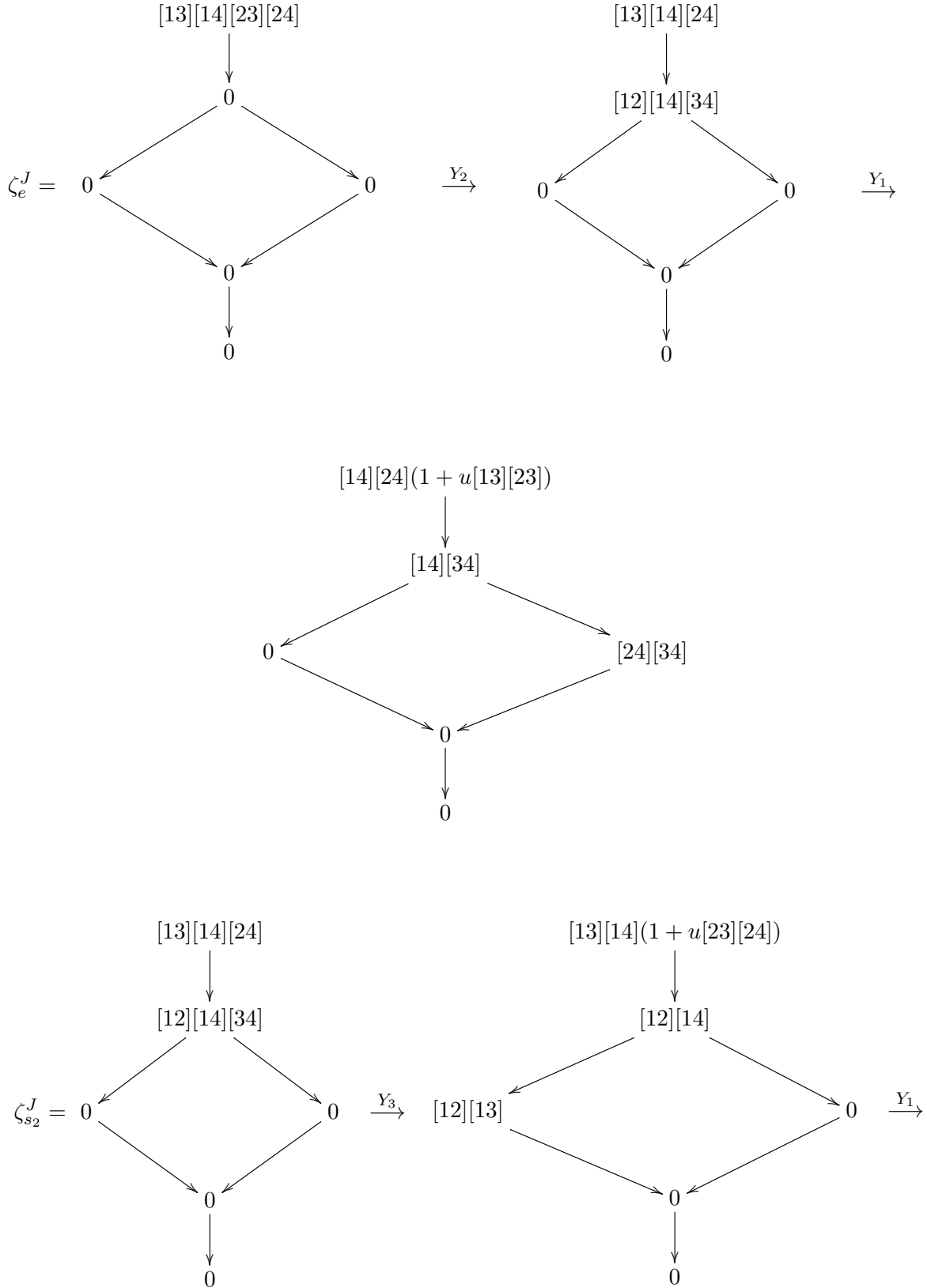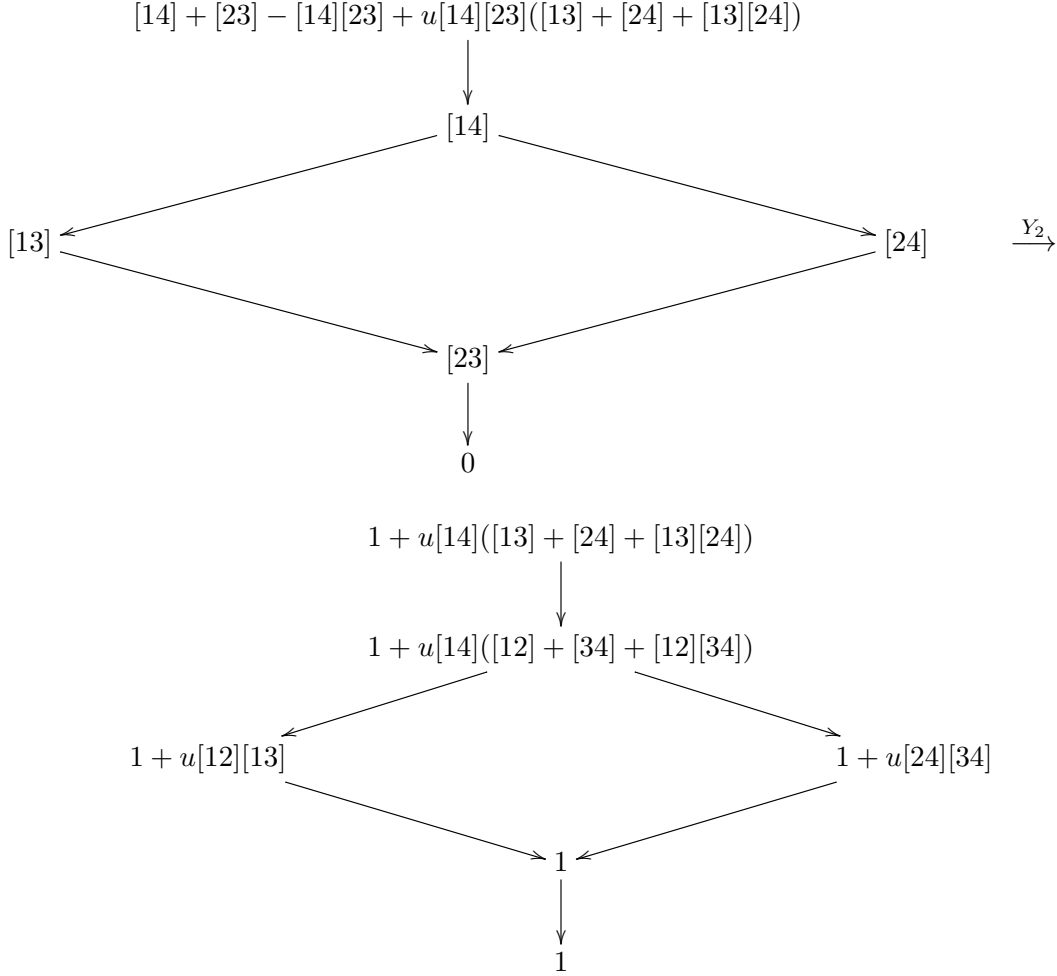
In the above calculations, we repeatedly used the following identities in the corresponding formal group algebra **S**:

$$(16) \qquad \frac{x_{\alpha+\beta}}{x_\alpha} + \frac{x_\beta}{x_{-\alpha}} = 1 + u x_\beta x_{\alpha+\beta}, \qquad \frac{x_{\alpha+\beta} - x_\beta}{x_\alpha} = 1 - x_\beta + u x_\beta x_{\alpha+\beta}.$$

The first identity is part of [29, Lemma 4.2]; the second one follows in the same way, using the fact that, by the definition of the hyperbolic formal group law, we have

$$x_{\alpha+\beta} = x_\alpha + x_\beta - x_\alpha x_\beta + u x_\alpha x_\beta x_{\alpha+\beta}.$$

We also used $k_\alpha = 1$.

In addition, in the calculation of $\zeta_{s_1 s_3 s_2}$, we used

$$\frac{[13][14]}{[12]} + \frac{[23][24]}{[21]} = [14]\frac{[13]-[23]}{[12]} + [23]\left(\frac{[14]}{[12]} + \frac{[24]}{[21]}\right)$$
$$= [14](1 - [23] + u[13][23]) + [23](1 + u[14][24]).$$

3.7. A generalized Deodhar's resolution.

We apply the Hecke actions to construct a resolution of $\mathbf{D}_F^\star$ using the algebras $\mathbf{D}_{F,J}^\star$, for $J \subset \Pi$ and an arbitrary formal group law F. This generalizes Deodhar's acyclic complex for the Iwahori-Hecke algebra [16, §5], and also offers a geometric interpretation of it.

Theorem 3.22. *We have a chain complex of $\mathbf{D}_F$-modules (also of free $\mathbf{S}$-modules)*

$$(17) \qquad 0 \longrightarrow \mathbf{D}_F^\star \xrightarrow{\ \partial_0\ } \bigoplus_{|J|=1} \mathbf{D}_{F,J}^\star \xrightarrow{\ \partial_1\ } \bigoplus_{|J|=2} \mathbf{D}_{F,J}^\star \xrightarrow{\ \partial_2\ } \cdots \xrightarrow{\ \partial_{n-1}\ } \mathbf{D}_{F,[n]}^\star \xrightarrow{\ \partial_n\ } 0 \ ,$$

where the maps are defined as follows:

$$\partial_r = \sum_{|J'|=|J|-1=r} \epsilon(J,J') A_{J/J'} \,,$$

with $\epsilon(J,J') = (-1)^{|\{i' \in J' | i' < i\}|}$ if $J = J' \cup \{i\}$. Moreover, its cohomology $H^r = 0$ for $r \geq 1$.

Proof. Although in a different context, the proof that this sequence is a chain complex follows Deodhar's proof of [16, Theorem 5.1 (i)] using the identity in (9). The only difference is concerned with the use of Lemma 3.14, where in the second case of the corresponding formula ($v \notin W^J$) the evaluation is just 0 in Deodhar's case. $\square$

Theorem 3.23. *If $F = F_a$ or F_m, then $\ker \partial_0$ is a free module of rank 1 over $\mathbf{S}$, with basis $\{\hat{\zeta}_{I_{w_\circ}}^\emptyset\}$ where $w_\circ$ is the longest element of W.*

Proof. In these cases, X_{I_v} and Y_{I_v} does not depend on the choice of I_v, so we can write X_v and Y_v respectively. If $\ell(v) \geq \ell(vs_i)$, then $X_{v^{-1}} = X_i X_{(vs_i)^{-1}}$, and hence, $A_i(X_{v^{-1}} \bullet \mathrm{pt}) = 0$. Let $g = \sum_{v \in W} d_v(X_{v^{-1}} \bullet \mathrm{pt}) \in \ker \partial_0$, then

$$0 = \partial_0(g) = \sum_{i=1}^n \left(\sum_{v \in W^{\{i\}}} A_i \left(d_v(X_{v^{-1}} \bullet \mathrm{pt}) + d_{vs_i}(X_i X_{v^{-1}} \bullet \mathrm{pt}) \right) \right) = \sum_{i=1}^n \left(\sum_{v \in W^{\{i\}}} d_v \hat{\zeta}_{I_v}^i \right) .$$

For any $v \neq w_\circ$, there exists i such that $v \in W^{\{i\}}$, so $d_v = 0$ and we obtain $g = d_{w_\circ}(X_{w_\circ} \bullet \mathrm{pt})$. $\square$

Example 3.24. Let $F = F_a$ and the root datum be of Dynkin type A_2. There are four $\mathbf{D}_F$-modules, $\mathbf{D}_F^\star, \mathbf{D}_{F,1}^\star, \mathbf{D}_{F,2}^\star, \mathbf{D}_{F,\Pi}^\star \cong \mathbf{S}$. Moreover, $\epsilon(1,\emptyset) = \epsilon(2,\emptyset) = \epsilon(\Pi,2) = 1, \epsilon(\Pi,1) = -1$, so the chain complex is

$$0 \longrightarrow \mathbf{D}_F^\star \xrightarrow{\ (A_1,A_2)\ } \mathbf{D}_{F,1}^\star \oplus \mathbf{D}_{F,2}^\star \xrightarrow{\ (-A_{\Pi/1}) \oplus A_{\Pi/2}\ } \mathbf{D}_{F,\Pi}^\star \longrightarrow 0.$$

We know that $A_{\Pi/1}(\zeta_v^1) = 0$, for $v \in W^1, v \neq e$ and $A_{\Pi/1}(\zeta_e^1) = \zeta_e^\Pi$, and similar conclusions hold if one replaces 1 by 2, so

$$\partial_1 \left(\sum_{v \in W^1} d_v^1 \zeta_v^1, \sum_{u \in W^2} d_u^2 \zeta_u^2 \right) = d_e^1 - d_e^2.$$

So $\mathrm{im}\, \partial_1 = \mathbf{D}_{F,\Pi}^\star$, and

$$\ker \partial_1 = \left\{ \left(\sum_{v \in W^1} d_v^1 \zeta_v^1, \sum_{u \in W^2} d_u^2 \zeta_u^2 \right) \mid d_e^1 = d_e^2 \right\}.$$

For any $z = \sum_{w \in W} d_w \zeta_w$,

$$\partial_0(z) = (A_1(z), A_2(z)) = \left(\sum_{w \in W^1} d_w \zeta_w^1, \sum_{w' \in W^2} d_{w'} \zeta_{w'}^2 \right),$$

so $\mathrm{im}\, \partial_0 = \ker \partial_1$. Moreover, $\ker \partial_0 = \{d\zeta_{w_\circ} \mid d \in \mathbf{S}\} \cong \mathbf{S}$.

4. Deodhar's parabolic Hecke modules revisited

The goal of this section is to give a new interpretation of Deodhar's modules over Iwahori-Hecke algebras in terms of the equivariant K-theory of flag varieties; a related dictionary is provided. All the results stated here are known, but we provide proofs in the new setup, which give new insights and are sometimes simpler.

4.1. **The case of a multiplicative formal group law.** In this section we consider only the multiplicative FGL $F_m = x + y - xy$. We apply definitions and results of §2 and §3 to the case of F_m. We set $R = \mathbb{Z}[t, t^{-1}]$ and $Q_m = R[\Lambda][\frac{1}{1-e^\alpha}|\alpha \in \Sigma]$. Let $Q_{W,m} = Q_m \rtimes R[W]$ be the twisted group algebra with basis δ_w^m indexed by $w \in W$. We define the divided difference elements

$$X_i^m = \frac{1}{1-e^{-\alpha_i}}(\delta_i^m - 1) \in Q_{W,m},$$

and define X_w^m as products of X_i^m's corresponding to reduced sequences. Let D_m be the subalgebra generated by X_i^m and $R[\Lambda]$. This is the affine 0-Hecke algebra defined in [24, Definition 2.8] (see also [9, Definition 5.7]). The set $\{X_w^m\}_{w \in W}$ is a basis of D_m as an $R[\Lambda]$-module. We can similarly define $D_m^\star$ as the dual of D_m, and define the $\bullet$ and $\odot$-actions of D_m on $D_m^\star$, which commute. Indeed, there is a canonical embedding of $D_m \subset \mathbf{D}_{F_m}$ (one can view $\mathbf{D}_{F_m}$ as a completion of D_m) induced by the embedding $R[\Lambda] \subset R[[\Lambda]] \cong R[[\Lambda]]_{F_m}$ sending $1 - e^{-\lambda}$ to x_λ.

Denote

$$Y_J^m = \left(\sum_{w \in W_J} \delta_w^m\right) \prod_{\alpha \in \Sigma_J^+} \frac{1}{1-e^\alpha} \in D_m, \quad \mathrm{pt}_J^m = Y_J^m \bullet \mathrm{pt}^m = Y_J^m \odot \mathrm{pt}^m,$$

where

$$\mathrm{pt}^m = \left(\prod_{\alpha > 0}(1 - e^\alpha)\right) f_e \in D_m^\star \cong K_T(G/B).$$

We similarly define $Y_{J/J'}^m$, and denote $A_{J/J'}^m = Y_{J/J'}^m \bullet \_ : D_m^\star \to D_m^\star$, which gives the composition $K_T(G/B) \to K_T(G/P_J) \to K_T(G/B)$. Note that $\mathrm{pt}_J^m \in K_T(G/P_J)$ is the class of the structure sheaf of eP_J/P_J. We will frequently use that $D_{m,J}^\star = Y_J^m \bullet D_m^\star$ and $Y_J^m \bullet \mathrm{pt}^m = Y_J^m \odot \mathrm{pt}^m$, see (12).

4.2. **Modules over the Hecke algebra.** In D_m, we define elements

$$\tau_i = (t - t^{-1})X_i^m + t^{-1}\delta_i^m, \quad i \in [n].$$

Straightforward computations show that τ_i satisfy the braid relations and the quadratic relation $\tau_i^2 = (t^{-1} - t)\tau_i + 1$. So the subalgebra H generated by τ_i for all i is isomorphic to the classical Iwahori-Hecke algebra, and the subalgebra $\mathbf{H} \subset Q_{W,m}$ generated by $R[\Lambda]$ and H is isomorphic to the affine Hecke algebra.

Lemma 4.1. *If $j \in J$, then $\tau_j Y_J^m = t^{-1} Y_J^m$.*

Proof. By definition we know that $\delta_j^m Y_J^m = Y_J^m$, so $X_j^m Y_J^m = 0$. The conclusion then follows. $\square$

Via the embedding $H \subset D_m$ and the $\odot$-action of D_m on $D_{m,J}^\star$, there is a $\odot$-action of H on $D_{m,J}^\star$. Denote $\mathcal{M}^J := H \odot \mathrm{pt}_J^m$, and $\mathcal{M} := \mathcal{M}^\emptyset$. Consider the map of H-modules

$$\varphi^J : H \to \mathcal{M}^J, \quad z \mapsto z \odot \mathrm{pt}_J^m.$$

Lemma 4.2. *The map $\varphi : H \to \mathcal{M}, \quad z \mapsto z \odot \mathrm{pt}^m$ is an isomorphism of H-modules.*

Proof. From [10, Theorem 10.13] we know that $D_m^\star$ is a free D_m-module with basis $\{\mathrm{pt}^m\}$, so φ is an embedding, hence the conclusion follows. $\square$

Lemma 4.3. *We have $\mathcal{M}^J \subset D^\star_{m,J}$, and for any $J' \subset J$, we have the following commutative diagram, where the vertical map in the middle is an H-module homomorphism.*

$$
\begin{array}{ccccc}
H & \xrightarrow{\varphi^{J'}} & \mathcal{M}^{J'} & \hookrightarrow & D^\star_{m,J'} \\
& {\scriptstyle \varphi^J} \searrow & \downarrow{\scriptstyle A^m_{J/J'}} & & \downarrow{\scriptstyle A^m_{J/J'}} \\
& & \mathcal{M}^J & \hookrightarrow & D^\star_{m,J}
\end{array}
$$

Proof. Since the $\odot$-action commutes with the $\bullet$-action, we have

$$\mathcal{M}^J = H \odot \mathrm{pt}^m_J = H \odot Y^m_J \bullet \mathrm{pt}^m = Y^m_J \bullet (H \odot \mathrm{pt}^m) \subsetneq Y^m_J \bullet D^\star_m = D^\star_{m,J}.$$

Since $A^m_{J/J'} = Y^m_{J/J'} \bullet \_$, so the commutativity between $\bullet$ and $\odot$ also implies that $A^m_{J/J'}$ is a map of H-modules. $\square$

Let τ_v be the basis of H. For any $v \in W^J$, we define

$$\mathfrak{m}^J_v = \varphi^J(\tau_v) = \tau_v \odot \mathrm{pt}^m_J \in \mathcal{M}^J, \quad \mathfrak{m}^J_e = \mathrm{pt}^m_J, \quad \mathfrak{m}_v := \mathfrak{m}^\emptyset_v \in \mathcal{M}.$$

Lemma 4.4. *The $\odot$-action of H on the basis $\mathfrak{m}^J_v$ is given by:*

$$
\tau_j \odot \mathfrak{m}^J_v = \begin{cases}
(t^{-1} - t)\mathfrak{m}^J_v + \mathfrak{m}^J_{s_j v}, & \text{if } \ell(s_j v) < \ell(v), \quad \text{in which case } s_j v \in W^J; \\
\mathfrak{m}^J_{s_j v}, & \text{if } \ell(s_j v) > \ell(v) \text{ and } s_j v \in W^J; \\
t^{-1}\mathfrak{m}^J_v, & \text{if } \ell(s_j v) > \ell(v) \text{ and } s_j v = v s_i \text{ for some } i \in J.
\end{cases}
$$

Consequently, $\tau_{vw} \odot \mathfrak{m}^J_e = t^{-\ell(w)}\mathfrak{m}^J_v$ for $v \in W^J, w \in W_J$, and $\{\mathfrak{m}^J_v\}_{v \in W^J}$ is a basis of $\mathcal{M}^J$ as an R-module.

Proof. By definition, we have

$$\tau_j \odot \mathfrak{m}^J_v = \tau_j \odot (\tau_v \odot \mathrm{pt}^m_J) = (\tau_j \tau_v) \odot \mathrm{pt}^m_J.$$

If $\ell(s_j v) < \ell(v)$, then $s_j v \in W^J$ and $\tau_v = \tau_j \tau_{s_j v}$. Therefore,

$$\tau_j \tau_v = (\tau_j)^2 \tau_{s_j v} = ((t^{-1} - t)\tau_j + 1)\tau_{s_j v} = (t^{-1} - t)\tau_v + \tau_{s_j v},$$

which proves the first case.

If $\ell(s_j v) > \ell(v)$ and $s_j v \in W^J$, then $\tau_j \tau_v = \tau_{s_j v}$, hence $\tau_j \odot \mathfrak{m}^J_v = \mathfrak{m}^J_{s_j v}$.

If $\ell(s_j v) > \ell(v)$ and $s_j v = v s_i$ with $i \in J$, then $\tau_j \tau_v = \tau_v \tau_i$. We have

$$\tau_i \odot \mathrm{pt}^m_J = \tau_i \odot Y^m_J \odot \mathrm{pt}^m = (\tau_i Y^m_J) \odot \mathrm{pt}^m \overset{\sharp}{=} t^{-1} Y^m_J \odot \mathrm{pt}^m = t^{-1} \mathrm{pt}^m_J,$$

where $\sharp$ follows from Lemma 4.1, so $\tau_j \odot \mathfrak{m}^J_v = t^{-1}\mathfrak{m}^J_v$.

The last part follows inductively. $\square$

Corollary 4.5. *For $J \supset J'$, we have $A^m_{J/J'}(\mathfrak{m}^{J'}_v) = t^{-\ell(w)}\mathfrak{m}^J_u$ if $v \in W^{J'}$ with $v = uw, u \in W^J, w \in W_J$. In particular, if $v \in W^J$, then*

$$A^m_{J/J'}(\mathfrak{m}^{J'}_v) = \mathfrak{m}^J_v.$$

Proof. We have

$$A^m_{J/J'}(\mathfrak{m}^{J'}_v) = Y^m_{J/J'} \bullet \tau_v \odot Y^m_{J'} \bullet \mathrm{pt}^m \overset{\sharp_1}{=} \tau_v \odot Y^m_{J/J'} \bullet Y^m_{J'} \bullet \mathrm{pt}^m$$

$$\overset{\sharp_2}{=} \tau_v \odot \mathrm{pt}^m_J = \tau_{uw} \odot \mathfrak{m}^J_e \overset{\sharp_3}{=} t^{-\ell(w)}\mathfrak{m}^J_u,$$

where $\sharp_1$ follows from Lemma 3.4, $\sharp_2$ from (9), and $\sharp_3$ from Lemma 4.4. $\square$

Remark 4.6. Beside H, there is another copy of the Iwahori-Hecke algebra inside D_m, denoted H^-. It is generated by

$$\tau_i^- = (t - t^{-1})X_i^m - t\delta_i^m \in D_m.$$

The elements $\mathfrak{m}_v^{J,-} := \tau_v^- \odot \mathrm{pt}_J^m$, for $v \in W^J$ form a basis of $\mathcal{M}^{J,-} := H^- \odot \mathrm{pt}_J^m \subset D_{m,J}^\star$ as an R-module. We will sometimes write H^+ for H, and correspondingly τ_v^+, $\mathcal{M}^{J,+}$, and $\mathfrak{m}_v^{J,+}$ for τ_v, $\mathcal{M}^J$, and $\mathfrak{m}_v^J$.

Let σ be the involution of R defined by $t \mapsto -t^{-1}$. It induces an automorphism of Q_m by $t \mapsto t^{-1}$, $x_\lambda \mapsto x_\lambda$ and, hence, an automorphism of $Q_{W,m}$ with $\sigma(\delta_w^m) = \delta_w^m$. By definition $\sigma(X_i^m) = X_i^m$, so $\sigma(D_m) = D_m$. We have $\sigma(\tau_i^\pm) = \tau_i^\mp$, so σ is an isomorphism between H^+ and H^-. Moreover, it induces an automorphism of $(D_m^\star)^{W_J}$ by $z \odot \mathrm{pt}_J^m \mapsto \sigma(z) \odot \mathrm{pt}_J^m$, $z \in D_m$. Then $\sigma(\mathfrak{m}_v^{J,+}) = \mathfrak{m}_v^{J,-}$, so σ induces an isomorphism between $\mathcal{M}^{J,+}$ and $\mathcal{M}^{J,-}$.

Observe that Lemmas 4.2 and 4.3 hold for the $(\cdot)^-$ version, by means of the isomorphism σ. The t^{-1} in Lemma 4.1 should be replaced by $-t$, the t^{-1} (resp. $t^{-\ell(w)}$) in the third line of Lemma 4.4 should be replaced by $-t$, (resp. $(-t)^{\ell(w)}$), and the $t^{-\ell(w)}$ in Corollary 4.5 should be replaced by $(-t)^{\ell(w)}$.

Remark 4.7. The traditional generators of the Iwahori-Hecke algebra H are $T_i^\pm := t^{-1}\tau_i^\pm$. In other words,

$$T_i^+ = (1 - t^{-2})X_i^m + t^{-2}\delta_i^m = (1 - q)X_i^m + q\delta_i^m, \quad T_i^- = (1 - q)X_i^m - \delta_i^m, \quad q := t^{-2},$$

and they satisfy $(T_i^\pm)^2 = (q-1)T_i^\pm + q$. Moreover, we have bases $\{T_w^+ \mid w \in W\}$ and $\{T_w^- \mid w \in W\}$ for H^+ and H^-, respectively.

4.3. Deodhar's modules revisited.
We now explain the correspondence with the setup in Deodhar's work [16]. Consider a generic Iwahori-Hecke algebra with generators T_i. For each $J \subset \Pi$, Deodhar defined two H-modules M^J using explicit $\mathbb{Z}[q, q^{-1}]$-basis $\{m_v^J\}_{v \in W^J}$, and defined the parabolic Kazhdan-Lusztig basis. Let $H_J \subset H$ be the Iwahori-Hecke algebra associated to the sub-system determined by $J \subset \Pi$, which acts on $\mathbb{Z}[q, q^{-1}]$ by the maps $h_\pm : H_J \to \mathbb{Z}[q, q^{-1}]$ sending T_i to q and -1, respectively (here $h_+(T_i) = q$ and $h_-(T_i) = -1$). Then Deodhar's modules are $M^{J,\pm} = Ind_{H_J}^H \mathbb{Z}[q, q^{-1}]$, and the basis elements are $m_v^{J,\pm} = T_v \otimes 1$.

We now observe that the $\odot$-actions of $H^\pm$ on the bases $\mathfrak{m}_v^{J,\pm}$ (see Lemma 4.4) match the classical actions of the Iwahori-Hecke algebra on the parabolic modules $M^{J,\pm}$. So we have the following correspondence.

Lemma 4.8. With $t^{-2} = q$ and $H^\pm \cong H$, there are isomorphisms of H-modules

$$M^{J,+} \overset{\cong}{\to} \mathcal{M}^{J,+} \quad and \quad M^{J,-} \overset{\cong}{\to} \mathcal{M}^{J,-}$$

which map $m_v^{J,+} \mapsto T_v^+ \odot \mathrm{pt}_J^m = t^{-\ell(v)}\mathfrak{m}_v^{J,+}$ and $m_v^{J,-} \mapsto T_v^- \odot \mathrm{pt}_J^m = t^{-\ell(v)}\mathfrak{m}_v^{J,-}$ for any $v \in W^J$, respectively. Moreover, Deodhar's maps φ_J are precisely our φ^J, and Deodhar's $\varphi_{J,L}$ is precisely our $A_{J/J'}^m$ where $L = J' \subset J$.

Remark 4.9. We also have

$$t \leftrightarrow v, \quad \tau_i^+ \leftrightarrow H_i, \quad \mathcal{M}^{J,+} \leftrightarrow \mathcal{M}, \quad \mathcal{M}^{J,-} \leftrightarrow \mathcal{N}$$

where the right-hand side notations are taken from Soergel [34].

4.4. The Kazhdan-Lusztig basis.
We recall several well-known facts about the *Kazhdan-Lusztig basis*. There is an involution on the Iwahori-Hecke algebra H defined as

$$\bar{t} = t^{-1}, \quad \overline{\tau_i} = \tau_i^{-1}, \quad \overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}, \ z_1, z_2 \in H.$$

We use γ_v to denote the Kazhdan-Lusztig basis, that is, γ_v is invariant under the involution and

$$\gamma_v \in \tau_v + \sum_{w<v} t\mathbb{Z}[t]\tau_w.$$

In particular, we have $\gamma_i = \tau_i + t$. We write

$$\gamma_v = \sum_{w\le v} t^{\ell(v)-\ell(w)} P_{w,v}\tau_w, \quad P_{w,v} \in \mathbb{Z}[t^{-1}],$$

where $P_{w,v}$ are the *Kazhdan-Lusztig polynomials*; these are traditionally written as polynomials in q, where $q = t^{-2}$, as mentioned above. It is known that $P_{v,w_J} = 1$ for all $v \le w_J$, so

$$\gamma_J := \gamma_{w_J} = \sum_{w\in W_J} t^{\ell(w_J)-\ell(w)}\tau_w.$$

Defining

$$\gamma_{J/J'} := \sum_{v\in W^{J'}\cap W_J} t^{\ell(w_{J/J'})-\ell(v)}\tau_v \in H,$$

it is not difficult to see that $\gamma_J = \gamma_{J/J'}\gamma_{J'}$.

The following is a classical result restated and reproved in a simple way using our setup; it is needed in the proof of Theorem 5.4 below.

Proposition 4.10. *The following hold.*
 (i) $\delta_i^m(1 + t^{-1}\tau_i) = 1 + t^{-1}\tau_i$.
 (ii) $\gamma_{w_\circ} = t^{-\ell(w_\circ)}\left(\sum_{w\in W} \delta_w^m\right)\prod_{\alpha>0} \frac{t^2-e^\alpha}{1-e^\alpha}$.

Proof. (i) It follows from the identities $(\delta_i^m)^2 = 1$, and

$$\delta_i^m X_i^m = \frac{1}{x_{-\alpha_i}}(1 - \delta_i^m) = \left(\frac{1}{x_{\alpha_i}} - 1\right)(\delta_i^m - 1) = 1 + X_i^m - \delta_i^m.$$

(ii) We can write $\gamma_{w_\circ} = \sum_{w\in W} \delta_w^m a_w$ with $a_w \in Q_m$. We have $\gamma_{w_\circ} \in \tau_{w_\circ} + \sum_{v<w_\circ} t\mathbb{Z}[t]\tau_v$, so $a_{w_\circ}$ is the coefficient of $\delta_{w_\circ}^m$ in the expansion of $\tau_{w_\circ}$ in the basis δ_w^m. Note that $\gamma_i = t^{-1}(\delta_i^m + 1)\frac{t^2-e^{\alpha_i}}{1-e^{\alpha_i}}$. By reasoning as in the proof of [9, Lemma 5.4], we show that

$$a_{w_\circ} = t^{-\ell(w_\circ)}\prod_{\alpha>0} \frac{t^2-e^\alpha}{1-e^\alpha}.$$

To finish the proof, it suffices to show that $a_w = a_{w_\circ}$ for any w. This is equivalent to $\delta_i^m\gamma_{w_\circ} = \gamma_{w_\circ}$ for any s_i. Denote ${}^iW = \{v \in W \mid \ell(s_iv) > \ell(v)\}$. We have

$$\delta_i^m\gamma_{w_\circ} = \delta_i^m \sum_{v\in {}^iW} (t^{\ell(w_\circ)-\ell(v)}\tau_v + t^{\ell(w_\circ)-\ell(v)-1}\tau_i\tau_v)$$

$$= \sum_{v\in {}^iW} t^{\ell(w_\circ)-\ell(v)}\delta_i^m(1 + t^{-1}\tau_i)\tau_v \overset{\sharp}{=} \sum_{v\in {}^iW} t^{\ell(w_\circ)-\ell(v)}(1 + t^{-1}\tau_i)\tau_v = \gamma_{w_\circ}.$$

Here identity $\sharp$ follows from part (i). This concludes the proof. $\qquad\square$

4.5. **Other Kazhdan-Lusztig bases.** There are other ways of defining a Kazhdan-Lusztig basis, as follows:

$$\gamma_v^{+,-} = \sum_{w\le v} (-t)^{\ell(w)-\ell(v)} P_{w,v}^{+,-}\tau_w^+ \in H^+, \quad P_{w,v}^{+,-} \in \mathbb{Z}[t],$$

$$\gamma_v^{-,-} = \sum_{w\le v} (-t)^{\ell(w)-\ell(v)} P_{w,v}^{-,-}\tau_w^- \in H^-, \quad P_{w,v}^{-,-} \in \mathbb{Z}[t],$$

$$\gamma_v^{-,+} = \sum_{w\le v} t^{\ell(v)-\ell(w)} P_{w,v}^{-,+}\tau_w^- \in H^-, \quad P_{w,v}^{-,+} \in \mathbb{Z}[t^{-1}];$$

where the above conditions are in addition to the invariance under the Kazhdan-Lusztig involution. The first sign of γ and P corresponds to the sign of τ, and the second one to the choice of $t^{-1}\mathbb{Z}[t^{-1}]$ or $t\mathbb{Z}[t]$. Under this convention, the Kazhdan-Lusztig basis and polynomials defined above coincide with $\gamma^{+,+} \in H^+$ and $P^{+,+}$.

Since the isomorphism σ defined in Remark 4.6 commutes with the Kazhdan-Lusztig involution, $\sigma(\gamma^{+,+})$ and $\sigma(\gamma^{+,-})$ are also invariant under the involution, and therefore

$$\sigma(\gamma_v^{+,+}) = \gamma_v^{-,-}, \quad \sigma(\gamma_v^{+,-}) = \gamma_v^{-,+}, \quad P_{w,v}^{-,-} = \sigma(P_{w,v}^{+,+}), \quad P_{w,v}^{-,+} = \sigma(P_{w,v}^{+,-}).$$

Applying σ to the identity in Proposition 4.10.(ii), we have

$$\gamma_{w_\circ}^{-,-} = \sum_{w \in W} (-t)^{\ell(w)-\ell(w_\circ)} \tau_w^- = (-t)^{\ell(w_\circ)} \left(\sum_{w \in W} \delta_w^m \right) \prod_{\alpha > 0} \frac{t^{-2} - e^\alpha}{1 - e^\alpha} \, .$$

For $\gamma^{+,-}$ and $\gamma^{-,+}$, we have a similar identity to the first one, but not to the second one.

5. The Kazhdan-Lusztig theory and hyperbolic formal group laws

In this section, we give a functorial treatment of the parabolic Kazhdan-Lusztig basis in the new setup of the oriented cohomology of flag varieties. The main results below are new, and they are direct consequences of this setup.

5.1. From the multiplicative to the generic hyperbolic formal group law.

We first relate the realizations of the Iwahori-Hecke algebra H inside the Demazure algebras associated with the multiplicative formal group law F_m (see §4) and the generic hyperbolic formal group law F_t; this is based on two morphisms from F_t to F_m. Then we identify and study Deodhar's modules inside the dual of the Demazure algebra for F_t, which is isomorphic to the corresponding cohomology of G/B.

We first introduce some notation for the remainder of the paper. Let $\mu := t + t^{-1}$, $\mathbf{R} := \mathbb{Z}[t, t^{-1}, \frac{1}{\mu}]$, and consider F_m as a FGL over $\mathbf{R}$; also recall the notation $u := \mu^{-2}$. Let S_m and S_t be the formal group algebra for F_m and F_t, respectively. For example, $S_m = \mathbf{R}[[\Lambda]]$ if we identify $x_\lambda \in S_m$ with $1 - e^{-\lambda} \in \mathbf{R}[[\Lambda]]$ for any $\lambda \in \Lambda$.

We use X_i^m, Y_i^m and X_i^t, Y_i^t to denote the corresponding divided difference and push-pull elements for F_m and F_t, respectively. Let $D_{F_t} \subset \mathbf{D}_{F_t}$ be the subalgebra generated by X_i^t for all i (so D_{F_t} is the formal Demazure algebra associated to the hyperbolic formal group law). It follows from [27, Example 4.12] that D_{F_t} is actually an $\mathbf{R}$-module with basis $\{X_{I_w}^t \mid w \in W\}$, a fact that does not hold for a general FGL.

Denote

$$\mathrm{pt}_J^t = Y_J^t \bullet \mathrm{pt}^t = Y_J^t \odot \mathrm{pt}^t \in \mathbf{D}_{F_t,J}^\star \subset \mathbf{D}_{F_t}^\star.$$

The automorphism σ of $\mathbf{R}$ with $\sigma(t) = -t^{-1}$, defined in Remark 4.6, extends to an automorphism of S_t with $\sigma(x_\lambda) = x_\lambda$, hence induces an automorphism of D_{F_t} and also of $\mathbf{D}_{F_t}$, satisfying that $\sigma(Y_i^t) = Y_i^t$. We also have $\sigma(Y_J) = Y_J$. Moreover, it induces an automorphism of $\mathbf{D}_{F_t}^\star \cong \mathbf{D}_{F_t} \bullet \mathrm{pt}^t$ by $\sigma(z \bullet \mathrm{pt}^t) = \sigma(z) \bullet \mathrm{pt}^t$. From the definition of $\odot$-action, and the fact that ι commutes with the involution σ, we also have

$$\sigma(z \odot \mathrm{pt}_J^t) = \sigma(Y_J \bullet \iota(z) \bullet \mathrm{pt}^t) = \sigma(Y_J)\iota(\sigma(z)) \bullet \mathrm{pt}^t = \sigma(z) \odot \mathrm{pt}_J^t, \quad z \in \mathbf{D}_{F_t}.$$

Lemma 5.1. *There is a morphism of FGLs $g \colon F_t \to F_m$ over $\mathbf{R}$, defined by*

$$g(x) = \frac{(1-t^2)x}{x-(t^2+1)},$$

so that $F_m(g(x), g(y)) = g(F_t(x,y))$.

It follows from [8, Lemma 2.6] that g induces a W-equivariant embedding of rings

$$\psi = \psi^+ \colon S_m \hookrightarrow S_t \text{ by } f(x_\lambda) \mapsto f(g(x_\lambda)) \text{ for any } \lambda \in \Lambda, f(x) \in \mathbf{R}[[x]].$$

Note that it is not an isomorphism unless one inverts $t^2 - 1$ in $\mathbf{R}$. The map ψ induces a morphism of algebras $\mathbf{Q}_{F_m} \to \mathbf{Q}'_{F_t} := \mathbf{R}[\frac{1}{1-t^2}][[\Lambda]]_{F_t}$, and, hence, a morphism of twisted group algebras

$$\psi \colon \mathbf{Q}_{F_m} \rtimes \mathbf{R}[W] \to \mathbf{Q}'_{F_t} \rtimes \mathbf{R}[W], \quad \text{where } p \in \mathbf{Q}_{F_m} \mapsto \psi(p) \in \mathbf{Q}'_{F_t} \text{ and } \delta_i^m \mapsto \delta_i^t.$$

By definition we have

(18)
$$\psi(\tau_i) = \mu Y_i^t - t = \mu X_i^t + t^{-1}.$$

Since the Hecke algebra H is generated by τ_i and D_{F_t} is generated by Y_i^t, we obtain an isomorphism

$$\psi : H \subset \mathbf{Q}_{W,F_m} \xrightarrow{\sim} D_{F_t} \subset \mathbf{Q}'_{W,F_t}.$$

Indeed, it also induces an embedding of the affine Hecke algebra $\mathbf{H}$ into the affine hyperbolic Demazure algebra $\mathbf{D}_{F_t}$.

As in (10), we have $Y_i^t Y_J^t = Y_J^t$ for any $i \in J$, so we have

(19)
$$\psi(\tau_i) Y_J^t = t^{-1} Y_J^t,$$

which is the analogue of Lemma 4.1.

Remark 5.2. There is another morphism $F_t \to F_m$ defined by $g^-(x) := \frac{(1-t^{-2})x}{-(t^{-2}+1)+x}$. The map $\psi^- \colon \mathbf{Q}_{W,F_m} \to \mathbf{Q}'_{W,F_t}$ determined by g^- satisfies that $\psi^- = \sigma \circ \psi^+ \circ \sigma$, so it induces an isomorphism $H^- \xrightarrow{\sim} D_{F_t}$. Indeed, it is precisely the other isomorphism mentioned in [10, Proposition 9.2], i.e., it satisfies $\tau_i^- \mapsto -\mu Y_i^t + t^{-1}$. In summary, we have the following commutative diagram (ψ^+ is the ψ defined before).

We define the H-module $\mathcal{N}^J := \psi(H) \odot \mathrm{pt}_J^t \subset \mathbf{D}_{F_t}^\star$ and define $\mathfrak{n}_v^J := \psi(\tau_v) \odot \mathrm{pt}_J^t$. In particular, $\mathfrak{n}_e^J = \mathrm{pt}_J^t$. Because of (19), all results of §4 hold if one replaces $\mathcal{M}^J$ by $\mathcal{N}^J$, pt_J^m by pt_J^t and $\mathfrak{m}_v^J$ by $\mathfrak{n}_v^J$, respectively. Indeed, $\mathcal{M}^J \cong \mathcal{N}^J$ as H-modules by identifying $\mathfrak{m}_v^J$ with $\mathfrak{n}_v^J$. So $\{\mathfrak{n}_v^J\}_{v \in W^J}$ is a basis of $\mathcal{N}^J$. Furthermore, for $J' \subset J$, we have the following commutative diagram.

Remark 5.3. By using the isomorphisms $\psi : H \cong D_{F_t}$, it is not difficult to see that for the generic hyperbolic FGL, the $\ker \partial_0$ of the complex (17) is a free $\mathbf{S}$-module of rank 1, generated by $\sum_{v \in W} t^{-\ell(v)} \mathfrak{n}_v$.

Theorem 5.4. We have

$$Y_\Pi^t = \mu^{-\ell(w_\circ)} \psi(\gamma_{w_\circ}).$$

Proof. Using the identification $e^\alpha = 1 - x_{-\alpha}$ in S_m, straightforward computations show that

$$\psi\left(t^{-1}\tfrac{t^2-e^\alpha}{1-e^\alpha}\right) = \psi\left(\tfrac{t-t^{-1}}{x_{-\alpha}} + t^{-1}\right) = \tfrac{\mu}{x_{-\alpha}} \in S_t\,.$$

Therefore, using Proposition 4.10, we obtain

$$\psi(\gamma_{w_\circ}) = \sum_{w \in W} \delta_w^t \prod_{\alpha>0} \psi\left(t^{-1}\tfrac{t^2-e^\alpha}{1-e^\alpha}\right) = \sum_{w \in W} \delta_w^t \prod_{\alpha>0} \tfrac{\mu}{x_{-\alpha}} = \mu^{\ell(w_\circ)}Y_\Pi^t. \qquad \square$$

Corollary 5.5. *Let $J' \subset J$. Then we have*

(1) $\mu^{-\ell(w_J)}\psi(\gamma_J) = Y_J^t$. *In particular, $Y_J^t \in D_{F_t}$.*

(2) $\mu^{-\ell(w_{J/J'})}\psi(\gamma_{J/J'})Y_{J'}^t = Y_J^t$.

(3) $\mathcal{N}^J = \psi(H\gamma_J) \odot \mathrm{pt}^t \subset \mathbf{D}_{F_t}^\star$.

(4) *The embedding $i_{J/J'} \colon \mathbf{D}_{F_t,J}^\star \to \mathbf{D}_{F_t,J'}^\star$ restricts to a map $i_{J/J'} \colon \mathcal{N}^J \subset \mathcal{N}^{J'}$.*

(5) $\psi(z) \odot \mathrm{pt}_J^t = \mu^{-\ell(w_{J/J'})}\psi(z\gamma_{J/J'}) \odot \mathrm{pt}_{J'}^t$. *In particular, $\psi(z) \odot \mathrm{pt}_J^t = \mu^{-\ell(w_J)}\psi(z\gamma_J) \odot \mathrm{pt}^t$ and $\mathfrak{n}_v^J = \mu^{-\ell(w_J)}\psi(\tau_v\gamma_J) \odot \mathrm{pt}^t$.*

Proof. (1) Follows similar as in Theorem 5.4 replacing $w_\circ$ by w_J. The second property follows since $\psi(H) \subset D_{F_t}$.

(2) Follows from Theorem 5.4 and the identities $Y_{J/J'}^t Y_{J'}^t = Y_J^t$ and $\gamma_{J/J'}\gamma_{J'} = \gamma_J$.

(3) Follows from the definition of $\mathcal{N}^J$.

(4) Follows from part (2) and the definition of $\mathcal{N}^J$.

(5) We have

$$\psi(z) \odot \mathrm{pt}_J^t \overset{\sharp_1}{=} \psi(z) \odot \mu^{-\ell(w_J)}\psi(\gamma_J) \odot \mathrm{pt}^t$$

$$\overset{\sharp_2}{=} \mu^{-\ell(w_J)}\psi(z\gamma_{J/J'}) \odot \psi(\gamma_{J'}) \odot \mathrm{pt}^t$$

$$\overset{\sharp_3}{=} \mu^{-\ell(w_{J/J'})}\psi(z\gamma_{J/J'}) \odot \mu^{-\ell(w_{J'})}\psi(\gamma_{J'}) \odot \mathrm{pt}^t$$

$$\overset{\sharp_4}{=} \mu^{-\ell(w_{J/J'})}\psi(z\gamma_{J/J'}) \odot \mathrm{pt}_{J'}^t,$$

where $\sharp_1$ and $\sharp_4$ follow from part (2), $\sharp_2$ follows from the identity $\gamma_J = \gamma_{J/J'}\gamma_{J'}$, and $\sharp_3$ follows from the identity $\ell(w_J) = \ell(w_{J/J'}) + \ell(w_{J'})$. $\qquad \square$

Remark 5.6. Corollary 5.5 (1) shows that $\mu^{-\ell(w_{J/J'})}\psi(\gamma_{J/J'}) \in \mathbf{D}_{F_t}$ behaves similarly to $Y_{J/J'} \in \mathbf{Q}_{F_t,W}$, while $Y_{J/J'}$ may not even belong to $\mathbf{D}_{F_t}$ when $J \supsetneq J' \supsetneq \emptyset$. This is an important advantage. For example, it can be used to study the functoriality of the module $\mathcal{N}^J$ as below.

From Corollary 5.5 (4) we have the following commutative diagram.

$$
\begin{array}{ccccc}
\mathcal{N}^J & \hookrightarrow & \mathbf{D}_{F_t,J}^\star & \hookrightarrow & \mathbf{D}_{F_t}^\star \\
{\scriptstyle i_{J/J'}}\downarrow & & {\scriptstyle i_{J/J'}}\uparrow & & \parallel \\
\mathcal{N}^{J'} & \hookrightarrow & \mathbf{D}_{F_t,J'}^\star & \hookrightarrow & \mathbf{D}_{F_t}^\star
\end{array}
$$

Remark 5.7. Note that $\mathcal{M}^J$ does not satisfy the property in the previous diagram, that is, in general the embedding $i_{J/J'} \colon D_{m,J}^\star \to D_{m,J'}^\star$ does not map $\mathcal{M}^J$ into $\mathcal{M}^{J'}$.

5.2. Parabolic KL-Schubert classes. We now recall Deodhar's definition of the *parabolic Kazhdan-Lusztig (KL) basis* and define the parabolic KL-Schubert classes for the generic hyperbolic FGL in a canonical way, independent of choices of reduced words, unlike the parabolic Bott-Samelson classes.

We define the Kazhdan-Lusztig involution on $\mathcal{N}^J$

$$\overline{\mathfrak{n}_e^J} = \mathfrak{n}_e^J, \quad \overline{\psi(z) \odot \mathfrak{n}_e^J} = \psi(\bar{z}) \odot \mathfrak{n}_e^J, \quad z \in H.$$

Then $\overline{\psi(z) \odot f} = \psi(\bar{z}) \odot \overline{f}, z \in H, f \in \mathcal{N}^J$. In particular, if z is invariant, then so is $\psi(z) \odot \mathfrak{n}_e^J$. Since γ_J is invariant, so $i_{J/J'}$ commutes with the involution.

Definition 5.8. For $v \in W^J$, let $C_v^J \in \mathfrak{n}_v^J + \sum_{w<v} t\mathbb{Z}[t]\mathfrak{n}_w^J \in \mathcal{N}^J$ which is invariant under the involution. By [34, Theorem 3.1] C_v^J are uniquely determined by these conditions. Denote $C_v = C_v^\emptyset$.

Write

(20) $$C_v^J = \sum_{w \leq v, w \in W^J} t^{\ell(v)-\ell(w)} P_{w,v}^J \mathfrak{n}_w^J, \quad P_{w,v}^J \in \mathbb{Z}[t^{-1}],$$

where $P_{w,v}^J$ are the *parabolic Kazhdan-Lusztig polynomials*. By the uniqueness of the Kazhdan-Lusztig basis, we see that

(21) $$C_v = \psi(\gamma_v) \odot \mathrm{pt}^t \in \mathcal{N}.$$

For $J \neq \emptyset$, this is not true. However, see Corollary 5.13 below.

Definition 5.9. For a $v \in W^J$, we view $\mu^{-\ell(v)} C_v^J$ under the canonical embedding of $\mathcal{N}^J$ into $\mathbf{D}_{F_t,J}^\star \simeq \mathrm{h}_T(G/P_J)$ and call it the *parabolic KL-Schubert class*.

As a short summary, we have

$$\mathfrak{n}_v^J = \mu^{-\ell(w_J)}\psi(\tau_v \gamma_J) \odot \mathrm{pt}^t, \quad C_v^J = \mu^{-\ell(w_J)}\psi(\gamma_{vw_J}) \odot \mathrm{pt}^t.$$

Remark 5.10. (i) By definition, the parabolic KL-Schubert classes form an **R**-basis of $\mathcal{N}^J$, and a S_t-basis of $\mathbf{D}_{F_t,J}^\star$.

(ii) In the G/B case, in [29] we used $\psi(\gamma_{v^{-1}}) \bullet \mathrm{pt}^t$ instead of (21) to define the KL-Schubert classes. The two definitions turn out to be equivalent, as seen below:

$$C_v = \psi(\gamma_v) \odot \mathrm{pt}^t = \left(\sum_{w \leq v} t^{\ell(v)-\ell(w)} P_{w,v}\, \psi(\tau_w)\right) \odot \mathrm{pt}^t$$

$$\overset{\sharp_1}{=} \left(\sum_{w \leq v} t^{\ell(v)-\ell(w)} P_{w,v}\, \psi(\tau_{w^{-1}})\right) \bullet \mathrm{pt}^t$$

$$\overset{\sharp_2}{=} \left(\sum_{w \leq v} t^{\ell(v)-\ell(w)} P_{w^{-1},v^{-1}}\, \psi(\tau_{w^{-1}})\right) \bullet \mathrm{pt}^t = \psi(\gamma_{v^{-1}}) \bullet \mathrm{pt}^t.$$

Here $\sharp_1$ follows from (18) and the identity $X_i^t \bullet \mathrm{pt}^t = X_i^t \odot \mathrm{pt}^t$, while $\sharp_2$ is based on a classical fact which can be found, for instance, in [4, Chapter 5, Exercise 12].

Example 5.11. We calculate the parabolic KL-Schubert classes corresponding to the example in §3.6.

All the relevant Kazhdan-Lusztig polynomials are equal to 1, except for

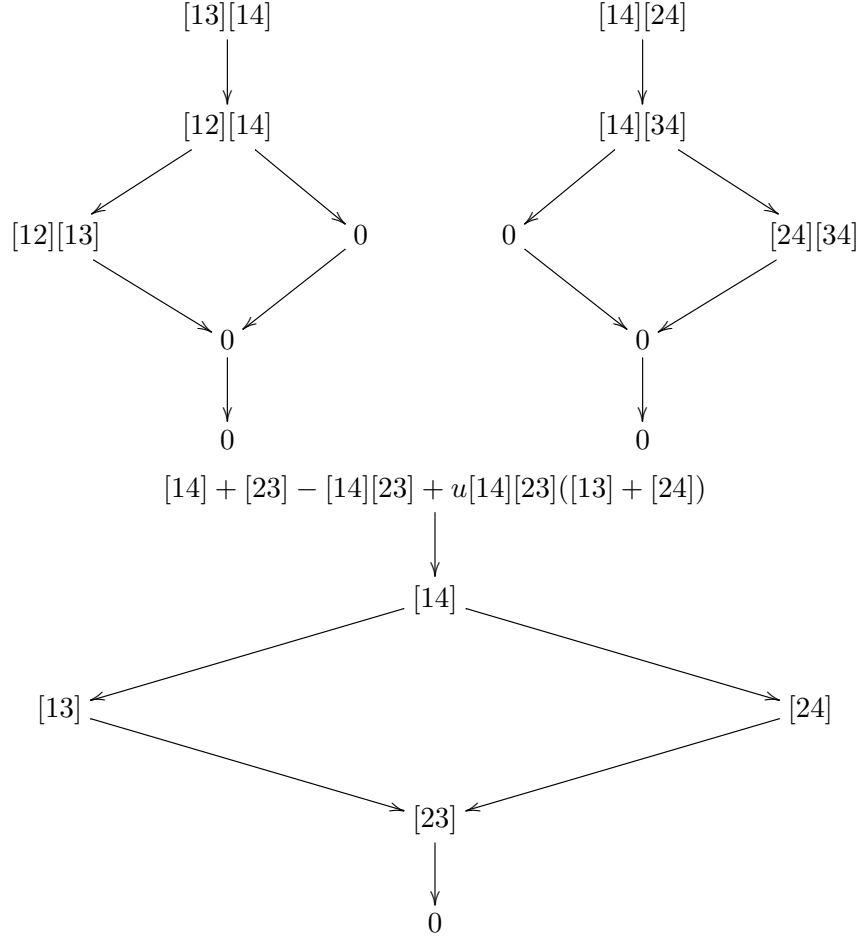$$P_{e,s_1s_3s_2}^J = P_{s_1s_3,s_1s_3s_2s_1s_3} = 1 + q = 1 + t^{-2};$$

see Proposition 5.19 below. By plugging (18) into the definition (20) of the parabolic KL-Schubert classes and cancelling terms, we obtain:

$$C_e^J = \mathrm{pt}_J^t = \zeta_e^J, \quad \mu^{-1} C_{s_2}^J = Y_2^t \odot \mathrm{pt}_J^t = \zeta_{s_2}^J,$$

$$\mu^{-2} C_{s_i s_2}^J = \left(Y_i^t Y_2^t - \mu^{-1} t Y_i^t + \mu^{-2} t^2\right) \odot \mathrm{pt}_J^t = \zeta_{s_i s_2}^J - u\zeta_e^J,$$

$$\mu^{-3} C_{s_1 s_3 s_2}^J = \left(Y_1^t Y_3^t Y_2^t - \mu^{-1} t Y_1^t Y_3^t + \mu^{-3}(t^3 + t)\right) \odot \mathrm{pt}_J^t = \zeta_{s_1 s_3 s_2}^J - u\zeta_e^J,$$

where in the third formula $i \in \{1,3\}$.

Thus, the parabolic Schubert classes $\mu^{-2}C_{s_3 s_2}^J$, $\mu^{-2}C_{s_1 s_2}^J$, and $\mu^{-3}C_{s_1 s_3 s_2}^J$ are as follows, respectively:

$$[13][14] \qquad\qquad [14][24]$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$[12][14] \qquad\qquad [14][34]$$

$$[12][13] \qquad\qquad 0 \qquad\qquad 0 \qquad\qquad [24][34]$$

$$0 \qquad\qquad\qquad 0$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$0 \qquad\qquad\qquad 0$$

$$[14] + [23] - [14][23] + u[14][23]([13] + [24])$$

$$\downarrow$$

$$[14]$$

$$[13] \qquad\qquad\qquad\qquad\qquad [24]$$

$$[23]$$

$$\downarrow$$

$$0$$

Furthermore, by Corollary 5.13 (2) below, we have $\mu^{-4}C_{s_2 s_1 s_3 s_2}^J = \sum_{w \in S_4} f_w = \mathbf{1}$. Thus, we verified Conjecture 5.14 below for this Grassmannian. Furthermore, note that even for the singular Schubert variety (indexed by $s_1 s_3 s_2$), the parabolic KL-Schubert class is given by a simpler formula than the corresponding parabolic Bott-Samelson class computed in §3.6.

5.3. **Functorial properties.** Let $J \supset J'$. Since $i_{J/J'}$ commutes with the Kazhdan-Lusztig involution, by the uniqueness of the parabolic Kazhdan-Lusztig basis, we recover the following classical result.

Theorem 5.12. *For any $J \supset J'$ and $v \in W^J$, via the embedding $i_{J/J'} : \mathcal{N}^J \to \mathcal{N}^{J'}$, we have*

$$C_v^J = \mu^{-\ell(w_{J/J'})} C_{vw_{J/J'}}^{J'} .$$

Corollary 5.13.

(1) *If $v \in W^J$, we have $C_v^J = \mu^{-\ell(w_J)} C_{vw_J} = \mu^{-\ell(w_J)} \psi(\gamma_{vw_J}) \odot \mathrm{pt}^t \in \mathcal{N}$.*

(2) *As an element of $\mathbf{D}_{F_t}^\star$, the expansion of the parabolic KL-Schubert class $\mu^{-\ell(v)} C_v^J$ in the $\{f_w\}$ basis has coefficients in the subring of S_t corresponding to the base ring $\mathbb{Z}[u]$ (rather than $\mathbf{R}$).*

(3) *For $J \supset J'$, we have*

$$(22) \qquad C_{w_{J/J'}}^{J'} = \mu^{\ell(w_{J/J'})} C_e^J = \mu^{\ell(w_{J/J'})} \mathrm{pt}_J^t = \mu^{\ell(w_{J/J'})} \frac{x_\Pi}{x_J} \sum_{w \in W_J} f_w.$$

In particular, $\mu^{-\ell(w_{\Pi/J})} C_{w_{\Pi/J}}^J = \sum_{w \in W} f_w = \mathbf{1} \in \mathbf{D}_{F_t}^\star$.

(4) *In the limit $t \to 0$ (which implies $u \to 0$), the parabolic KL-Schubert class $\mu^{-\ell(v)} C_v^J$ becomes the Bott-Samelson class ζ_v^J in K-theory (which coincides with the corresponding Schubert class, defined topologically).*

Proof. (1) The first identity follows from Theorem 5.12 by letting $J' = \emptyset$, and the second one follows from (21).

(2) This result follows from the first part of the corollary and [29, Proposition 3.4], which it generalizes.

(3) In (22), the first identity follows from Theorem 5.12 by letting $v = e$, the second one follows from the fact that $C_e^J = \mathfrak{n}_e^J = \mathrm{pt}_J^t$, and the last one follows from [10, Lemma 6.6].

The last part follows from (22).

(4) This result follows from the first part of the corollary and [29, Corollary 3.6], which it generalizes. Recalling the notation in §3.3, we also use the fact that $\zeta_v^J = A_J(\zeta_v)$ coincides with ζ_{vw_J} in K-theory, for $v \in W^J$. For example, see Remark 3.13 (ii).   □

5.4. **Smoothness.** In the G/B case, it was conjectured in [29] that if the Schubert variety corresponding to v is smooth, then C_v is equal to the corresponding topologically defined Schubert class. We now generalize this conjecture to the parabolic case, and then prove some special cases; other results are also derived along the way.

We first recall from [12, 13] the localization formula for the class $[X(v)]$ of a smooth Schubert variety in G/P, where $v, w \in W^J$:

$$
(23) \qquad [X(v)]_w = \frac{\displaystyle\prod_{\alpha \in w\left(\Sigma_{\Pi/J}^-\right)} x_\alpha}{\displaystyle\prod_{\substack{\alpha \in w\left(\Sigma_{\Pi/J}^-\right) \\ s_\alpha w \le v w_J}} x_\alpha},
$$

if $w \le v$, and otherwise $[X(v)]_w = 0$.

Conjecture 5.14. If the Schubert variety $X(v)$ in G/P is smooth, then the topologically defined Schubert class $[X(v)]$ given by (23) coincides with the parabolic KL-Schubert class $\mu^{-\ell(v)} C_v^J$.

Recall that the Weyl group for the root system of type C_n is the group of signed permutations, represented (in the window notation) as words of length n with letters $\{1, \ldots, n, \overline{n}, \ldots, \overline{1}\}$; we use the indexing of the corresponding Dynkin diagram $\Pi := \{0, \ldots, n-1\}$, where the 0-node is the sign change in position 1.

Corollary 5.15. *Conjecture 5.14 is true in the following cases (which all correspond to non-singular Schubert varieties):*

(1) *in all types for Schubert varieties indexed by $w_{J/J'}$ in $G/P_{J'}$;*

(2) *in type A_{n-1} in the maximal parabolic case $J = \Pi \setminus \{n-k\}$ (with $1 < k \le n-1$), for $v \in W^J$ of the form $[k, k+1, \ldots, n-1, 1, 2, \ldots, k-1, n]$ (in one-line notation);*

(3) *in type C_n in the maximal parabolic case $J = \Pi \setminus \{k\}$ (with $1 \le k \le n-1$), for $v \in W^J$ of the form $[1, 2, \ldots, k-1, n, \overline{n-1}, \overline{n-2}, \ldots, \overline{k}]$ (in the window notation);*

(4) *for the complex projective spaces.*

In addition, the conjecture is true for a Schubert variety in G/P_J indexed by $v \in W^J$ if and only if it is true for the one indexed by $vw_{J/J'}$ in $G/P_{J'}$, for $J' \subset J$.

Proof. We need [29, Lemma 4.1 (2)]; this says that, given a non-singular Schubert variety, if the coefficients in the expansion of a KL-Schubert class (as an element of $\mathbf{D}_{F_t}^\star$) are products $\prod_\alpha x_\alpha$ over

some subsets of the negative roots, then the class coincides with the topologically defined Schubert class. This lemma was proved in the G/B case, but it extends in a straightforward way to the parabolic case, using Corollary 5.13 (4). This lemma combined with Theorem 5.12 imply the last statement of the corollary. Moreover, based on the lemma, part (1) follows from Corollary 5.13 (3), while parts (2) and (3) follow from [29, Theorem 3.14 (2)] and Corollary 5.13 (1). Finally, part (4) is the content of Theorem 5.25. $\qquad \square$

5.5. **Positivity.** A positivity property in the G/B case was conjectured [28, Conjecture 6.4] for the (hyperbolic) Bott-Samelson classes, and in [29, Conjecture 3.9] for the corresponding KL-Schubert classes. Here we conjecture the same property for the parabolic KL-Schubert classes. If the conjecture is true, it would be interesting to find the geometric reason behind it.

Conjecture 5.16. The coefficient of f_w in the expansion of the parabolic KL-Schubert class $\mu^{-\ell(v)} C_v^J$ (as an element of $\mathbf{D}_{F_t}^\star$, where $w \leq v$) can be expressed as a (possibly infinite) sum of monomials in x_α, where α are negative roots, such that the coefficient of each monomial is of the form
$$(-1)^{k-(N-\ell(v))} \, c \, u^{(m-k)/2} \, ;$$
here c is a positive integer, m is the degree of the monomial, $N - \ell(v) \leq k \leq m$, $m - k$ is even, and N is the cardinality of $\Sigma_{\Pi/J}^+$.

Remark 5.17. (i) The above positivity property is a generalization of the one in K-theory which is made explicit in Graham's formula [19] for the localization of Schubert classes at torus fixed points, cf. also [28].

(ii) The conjecture does not hold for the parabolic Bott-Samelson classes, cf. the example in §3.6. However, when passing to the corresponding KL-Schubert classes, the terms violating the positivity condition disappear, so the conjecture holds; see Example 5.11.

5.6. **Other results in Kazhdan-Lusztig theory reinterpreted.** The following corollary was known to experts, since it is equivalent to Proposition 5.19, which is a classical result in [16]. Here we give a new proof using the embedding i_J.

Corollary 5.18. (i) *For any $z \in H$, we have*
$$\psi(z) \odot \mathfrak{n}_e^J = C_v^J \iff z \gamma_J = \gamma_{v w_J}.$$

(ii) *For any $v \in W^J$, there exists $z \in \sum_{w \in W^J} \mathbf{R} \tau_w$ such that $\gamma_{v w_J} = z \gamma_{w_J}$.*

Proof. (i) Inside $\mathcal{N}$, by Corollary 5.5 (5), we know that $z \odot \mathfrak{n}_e^J = \mu^{-\ell(w_J)} \psi(z \gamma_J) \odot \mathrm{pt}^t$, and by Theorem 5.12 we have $C_v^J = \mu^{-\ell(w_J)} \psi(\gamma_{v w_J}) \odot \mathrm{pt}^t$. By Lemma 4.2 the map $H \to \mathcal{N}, z \mapsto \psi(z) \odot \mathrm{pt}^t$ is an isomorphism, we get that $\psi(z) \odot \mathfrak{n}_e^J = C_v^J$ if and only if $z \gamma_J = \gamma_{v w_J}$.

(ii) The set $\{\mathfrak{n}_w^J\}_{w \in W^J}$ is a $\mathbf{R}$-basis of $\mathcal{N}^J$, and we know that $\mathfrak{n}_w^J = \psi(\tau_w) \odot \mathfrak{n}_e^J$, so the map $\sum_{w \in W^J} \mathbf{R} \tau_w \to \mathcal{N}^J, z \mapsto \psi(z) \odot \mathfrak{n}_e^J$ is surjective. Therefore, for any $v \in W^J$, there exists $z \in \sum_{w \in W^J} \mathbf{R} \tau_w$ such that $\psi(z) \odot \mathfrak{n}_e^J = C_v^J$. Then part (i) implies that $z \gamma_{w_J} = \gamma_{v w_J}$. $\qquad \square$

The following result appears as [16, Proposition 3.4] and [34, Proposition 3.4]; it follows from Theorem 5.12 and the uniqueness of the parabolic Kazhdan-Lusztig basis.

Proposition 5.19. [16, Proposition 3.4] $P_{wu,vw_J} = P_{w,v}^J$ *for* $w, v \in W^J, u \in W_J$.

5.7. **Other parabolic KL-Schubert bases.** Continuing Remark 4.6, §4.5, and Remark 5.2, one can define $\mathcal{N}^{J,-} = \psi^-(H^-) \odot \mathrm{pt}_J^t$ and $\mathfrak{n}_v^{J,-} = \psi^-(\tau_v^-) \odot \mathrm{pt}_J^t, v \in W^J$, then the $\mathcal{N}^J, \mathfrak{n}_v^J$ defined in §5.1 can be denoted by $\mathcal{N}^{J,+}$ and $\mathfrak{n}_v^{J,+}$, respectively. One can similarly define $C_v^{J,+,-} \in \mathcal{N}^{J,+}$ and $C_v^{J,-,-}, C_v^{J,-,+}$ in $\mathcal{N}^{J,-}$, and then Definition 5.8 above can be denoted by $C_v^{J,+,+} \in \mathcal{N}^{J,+}$. All properties concerning $\gamma^{+,+}$ and $C^{J,+,+}$ hold for the $(-,-)$ version, i.e., after replacing ψ^+ by

ψ^-, $\gamma^{+,+}$ by $\gamma^{-,-}$, $\mathcal{N}^{J,+}$ by $\mathcal{N}^{J,-}$, $\mathfrak{n}^{J,+}$ by $\mathfrak{n}^{J,-}$, $C^{J,+,+}$ by $C^{J,-,-}$, and μ by $-\mu$. For example, $C_v^{-,-} = \psi^-(\gamma_v^{-,-}) \odot \mathrm{pt}^t$, and the analogue to Theorem 5.4 is

$$Y_\Pi = (-\mu)^{-\ell(w_\circ)}\psi^-(\gamma_{w_\circ}^{-,-}).$$

Considering the involution σ on $\mathbf{D}_{F_t}^\star$ defined at the beginning of §5.1, we have

$$\sigma(\mathfrak{n}_v^{J,+}) = \mathfrak{n}_v^{J,-}, \quad \sigma(C_v^{J,+,+}) = C_v^{J,-,-}, \quad \sigma(C_v^{J,+,-}) = C_v^{J,-,+}.$$

By applying σ to the identity in Theorem 5.12, we obtain the following version:

$$(24) \qquad C_v^{J,-,-} = \sigma(C_v^{J,+,+}) = (-\mu)^{-\ell(w_{J/J'})}C_{vw_{J/J'}}^{J',-,-}.$$

In fact, we can get the following more precise relationship between the above bases:

$$C_v^{J,-,-} = \sigma(C_v^{J,+,+}) = \sigma(\mu^{\ell(v)})\sigma(\mu^{-\ell(v)}C_v^{J,+,+}) \overset{\sharp}{=} (-1)^{\ell(v)}C_v^{J,+,+}.$$

Here $\sharp$ is based on Corollary 5.13 (2), which implies that $\mu^{-\ell(v)}C_v^{J,+,+}$ is fixed by σ. Hence, by considering $(-\mu)^{-\ell(v)}C_v^{J,-,-}$, we obtain the same class as in Definition 5.9.

It is not difficult to check that $A_J : \mathcal{N}^\pm \to \mathcal{N}^{J,\pm}$ commutes with the Kazhdan-Lusztig involution, so we have

$$A_J(C_v^{+,-}) = C_v^{J,+,-}, \quad A_J(C_v^{-,+}) = C_v^{J,-,+}, \quad v \in W^J,$$

which imply

$$(25) \qquad P_{u,v}^{J,+,-} = \sum_{w \in W_J}(-1)^{\ell(w)}P_{uw,v}^{+,-}, \quad P_{u,v}^{J,-,+} = \sum_{w \in W_J}(-1)^{\ell(w)}P_{uw,v}^{-,+}.$$

Classically, in [16], with $q = t^{-2}$, Deodhar considered $\gamma^{+,-} \in H^+, \gamma^{-,-} \in H^-$ and $C^{J,+,-} \in \mathcal{N}^{J,+}, C^{J,-,-} \in \mathcal{N}^{J,-}$. For example, in *loc.cit.*, Proposition 3.4 coincides with our Proposition 5.19 for $P^{-,-}$ which can be derived from (24), and Remark 3.8 is the first identity of (25). In [34], Proposition 3.4.1 and 3.4.2 correspond to our Proposition 3.4 and the second identity of (25).

Remark 5.20. It seems that in order to generalize the Kazhdan-Lusztig polynomials to the parabolic case, H^+ (resp. H^-) and $P^{J,+,+}$ (resp. $P^{J,-,-}$) are the correct objects to generalize $P^+ := P^{+,+}$ (resp. $P^- := P^{-,-}$). Moreover, we believe that the parabolic KL-Schubert class $\mu^{-\ell(v)}C_v^{J,+,+}$ will have geometric significance, cf. Conjecture 5.14, Corollary 5.13, and Corollary 5.15.

5.8. **Other hyperbolic formal group laws.** So far we focused entirely on the hyperbolic formal group law (1) with the special choice of the parameters μ_1, μ_2 in (2). So it is natural to ask what happens in general; for instance, the case of the Lorentz formal group law, given by $\mu_1 = 0$, is also an important one.

We will show that the general case is related to the *generic Iwahori-Hecke algebra*, which depends on two parameters q_1, q_2. This is defined in the same way as the usual one, except that we use the more general relation $(T_i - q_1)(T_i - q_2) = 0$. The classical case is recovered by setting $q_1 = q$ and $q_2 = -1$. It is well-known that one can define a Kazhdan-Lusztig basis for the generic Iwahori-Hecke algebra (via reduction to the classical case) whenever $-q_1q_2$ is a square in the base ring [22].

We have seen that the hyperbolic formal group law corresponds to the 2-parameter Todd genus $T_{\alpha,\beta}$, where $\mu_1 = \alpha + \beta$ and $\mu_2 = -\alpha\beta$. So we assume that we have such elements α, β in the base ring. Imposing the type A braid relations on elements $T_i = aY_i + b$ in the hyperbolic Demazure algebra amounts to a quadratic equation for b. The solutions are $b = -a\alpha$ and $b = -a\beta$. Correspondingly, using $Y_i^2 = \mu_1 Y_i$, we easily obtain

$$(T_i + \alpha)(T_i - \beta) = 0, \quad \text{and} \quad (T_i - \alpha)(T_i + \beta) = 0.$$

Thus, the Kazhdan-Lusztig basis can be defined as long as $-\mu_2$ is a square in the base ring. In this case, all of the above constructions apply with minor changes.

Recall that our previous choice (2) for μ_1, μ_2 corresponds to $\alpha = \frac{t}{t+t^{-1}}$ and $\beta = \frac{t^{-1}}{t+t^{-1}}$. In this case, the two choices of T_i above are precisely the T_i^+ and T_i^-, cf. Remark 4.7.

5.9. Complex projective spaces.

We compute the Bott-Samelson classes and KL-Schubert classes corresponding to the complex projective space $\mathbb{P}^{n-1}$, so $W = S_n$, $J = \{2, \ldots, n-1\}$, and

$$W^J = \{e, s_1, s_2 s_1, \ldots, s_{n-1} \ldots s_1\}.$$

For simplicity, we let $w_i := s_i \ldots s_1$, where $w_0 := e$. We use the same notation as in §3.6 and Example 5.11. Note that $\mathrm{pt}_J^t = [12] \ldots [1n] \sum_{w \in W_J} f_w$.

We start with the Bott-Samelson classes $\zeta_{w_i}^J = Y_i^t \ldots Y_1^t \odot \mathrm{pt}_J^t$. We let

$$\zeta_{w_i}^J = \sum_{j=1}^{i+1} q_{ij} \sum_{w \in W_J} f_{w_{j-1}w}.$$

Throughout, we make the convention that sums are 0 and products are 1 if the initial and final values of the corresponding index are in contradiction with the variation of this index (for instance, the initial value is strictly greater than the final value when the index is increasing).

Proposition 5.21. *We have*

$$q_{ij} = [j, i+2] \ldots [jn] + u \sum_{k=j+1}^{i} [jk] \ldots [jn].$$

Proof. We proceed by induction on i, which starts at $i = 0$. Assume that the formula holds for $i - 1 \geq 0$ and all $j = 1, \ldots, i$. We compute $\zeta_{w_i}^J = Y_i^t \odot \zeta_{w_{i-1}}^J$. We have the following two cases.

Case 1: $j = i$ or $j = i+1$. We have

$$q_{ii} = \frac{[i,i+1][i,i+2]\ldots[in]}{[i,i+1]} = [i, i+2] \ldots [in],$$

$$q_{i,i+1} = \frac{[i+1,i][i+1,i+2]\ldots[i+1,n]}{[i+1,i]} = [i+1, i+2] \ldots [i+1, n].$$

Case 2: $j < i$. We have

$$q_{ij} = \frac{1}{[i,i+1]} \left([j, i+1][j, i+2] \ldots [jn] + u \sum_{k=j+1}^{i-1} [jk] \ldots [jn] \right)$$

$$+ \frac{1}{[i+1,i]} \left([ji][j, i+2] \ldots [jn] + u \sum_{k=j+1}^{i-1} [jk] \ldots [jn] \right)$$

$$= [j, i+2] \ldots [jn] + u[ji][j, i+1][j, i+2] \ldots [jn] + u \sum_{k=j+1}^{i-1} [jk] \ldots [jn],$$

as needed; here we used the first identity in (16) and $k_\alpha = 1$. $\square$

For the calculation of the KL-Schubert classes, we need several lemmas.

Lemma 5.22. *We have*

$$\tau_k \ldots \tau_1 \odot \mathrm{pt}_J^t = \left(\mu^k Y_k^t \ldots Y_1^t - \sum_{l=0}^{k-1} t^{2-k+l} \mu^l Y_l^t \ldots Y_1^t \right) \odot \mathrm{pt}_J^t.$$

Proof. Since $k_\alpha = 1$, we note first that $Y_l^t \odot \mathrm{pt}_J^t = \mathrm{pt}_J^t$ for $l > 1$. This implies that, for $l > j + 1$, we have

$$\tau_l Y_j^t \ldots Y_1^t \odot \mathrm{pt}_J^t = (\mu Y_l^t - t) Y_j^t \ldots Y_1^t \odot \mathrm{pt}_J^t = t^{-1} Y_j^t \ldots Y_1^t \odot \mathrm{pt}_J^t .$$

The result follows easily by using this fact in order to expand

$$\tau_k \ldots \tau_1 \odot \mathrm{pt}_J^t = (\mu Y_k^t - t) \ldots (\mu Y_1^t - t) \odot \mathrm{pt}_J^t .$$

$\square$

Let

$$\mu_k := \frac{t^{k+1} - t^{-(k+1)}}{t - t^{-1}} = t^k + t^{k-2} + \ldots + t^{-(k-2)} + t^{-k} .$$

This notation is local to this section, so it does not interfere with similar notation used earlier. In particular, $\mu_1 = \mu$ and $\mu_0 = 1$.

The following lemma expresses a KL-Schubert class in terms of Bott-Samelson classes.

Lemma 5.23. *We have*

$$(26) \qquad C_{w_i}^J = \mu^i \, \zeta_{w_i}^J - \sum_{k=0}^{i-2} \mu_{i-2-k} \, \mu^k \, \zeta_{w_k}^J .$$

Proof. Since the complex projective spaces are smooth, all the corresponding parabolic Kazhdan-Lusztig polynomials are equal to 1. So we have

$$C_{w_i}^J = \left(\sum_{k=0}^{i} t^{i-k} \tau_k \ldots \tau_1 \right) \odot \mathrm{pt}_J^t .$$

By plugging the formula in Lemma 5.22 into the previous one, we obtain:

$$C_{w_i}^J = \left(\sum_{k=0}^{i} t^{i-k} \left(\mu^k \, Y_k^t \ldots Y_1^t - \sum_{l=0}^{k-1} t^{2-k+l} \mu^l \, Y_l^t \ldots Y_1^t \right) \right) \odot \mathrm{pt}_J^t .$$

For each k between 0 and $i - 1$, we now cancel the term $t^{i-k} \mu^k \, Y_k^t \ldots Y_1^t$ in the outer sum with the term corresponding to $l = k$ in the inner sum, for the index $k + 1$ in the outer sum; indeed, the latter is

$$t^{i-(k+1)} t^{2-(k+1)+k} \mu^k \, Y_k^t \ldots Y_1^t .$$

The only positive term surviving is $\mu^i \, Y_i^t \ldots Y_1^t$ (throughout, we have to take into account the $\odot$-action on pt_J^t). By collecting the remaining terms, we obtain the desired formula. $\square$

Lemma 5.24. *We have*

$$\mu^k - \sum_{l=0}^{k-2} \mu_{k-2-l} \, \mu^l - \mu_k = 0 .$$

Proof. We use induction on k which starts at $k = 1$ (in which case the summation in the formula is void). Assuming that the above formula holds for $k - 1 > 0$, and multiplying it through by μ, we obtain:

$$\mu^k - \sum_{l=1}^{k-2} \mu_{k-2-l} \, \mu^l = \mu_{k-1} \, \mu .$$

Plugging the right-hand side into the formula for k to be proved, it remains to show that

$$\mu_{k-1} \, \mu = \mu_{k-2} + \mu_k , \quad \text{i.e.,} \ (t^k - t^{-k})(t + t^{-1}) = (t^{k-1} - t^{-(k-1)}) + (t^{k+1} - t^{-(k+1)}) .$$

But this is immediate. $\square$

It now remains to plug the formula for Bott-Samelson classes in Proposition 5.21 into the formula for a KL-Schubert class in Lemma 5.23. We will show that $\mu^{-i} C_{w_i}^J$ is obtained by simply setting $u = 0$ in the formula for $\zeta_{w_i}^J$.

Theorem 5.25. *We have*

$$\mu^{-i} C_{w_i}^J = \sum_{j=1}^{i+1} [j, i+2] \ldots [jn] \sum_{w \in W_J} f_{w_{j-1}w} \,.$$

This expression agrees with the localization formula (23) *for the corresponding Schubert class.*

Proof. Let us focus on the coefficient of $f_{w_{j-1}}$ in $C_{w_i}^J$, where $1 \le j \le i+1$ is fixed. For simplicity, we write $[[k]] := [jk] \ldots [jn]$. We calculate the mentioned coefficient as mentioned above. We write (26) by replacing the summation index k by p, and note that $\zeta_{w_p}^J$ contains $f_{w_{j-1}}$ precisely when $j-1 \le p \le i-2$. Thus, the cases $j=i$ and $j=i+1$ are immediate, so we assume $j < i$.

The desired coefficient is expressed as follows:

$$\mu^i [[i+2]] + u\mu^i \sum_{k=j+1}^{i} [[k]] - \sum_{p=j-1}^{i-2} \mu_{i-2-p}\, \mu^p \left([[p+2]] + u \sum_{l=j+1}^{p} [[l]] \right)$$

(27)
$$= \mu^i [[i+2]] + u\mu^i \sum_{k=j+1}^{i} [[k]] - \sum_{p=j+1}^{i} \mu_{i-p}\, \mu^{p-2} \left([[p]] + u \sum_{l=j+1}^{p-2} [[l]] \right) \,.$$

The only products appearing in (27) beside the first term are $[[k]]$ for $j+1 \le k \le i$. Fixing such a value k, it suffices to show that the coefficient of $[[k]]$ is 0. Let us first note that $[[k]]$ appears in the inner sum in (27) for the index p of the outer sum precisely when $j+1 \le k \le p-2$, so p needs to be in the range $k+2 \le p \le i$. It follows that the coefficient of $[[k]]$ in (27) is given by

$$u\mu^i - \mu_{i-k}\, \mu^{k-2} - u \sum_{p=k+2}^{i} \mu_{i-p}\, \mu^{p-2} = \mu^{i-2} - \mu_{i-k}\, \mu^{k-2} - \sum_{p=k}^{i-2} \mu_{i-p-2}\, \mu^{p-2}$$

$$= \mu^{k-2} \left(\mu^{i-k} - \mu_{i-k} - \sum_{p=k}^{i-2} \mu_{i-p-2}\, \mu^{p-k} \right) \,.$$

The latter expression is 0 by Lemma 5.24.

We conclude the proof by easily checking that the obtained formula agrees with the localization formula (23). □

References

[1] A. Arabia, Cycles de Schubert et cohomologie équivariante de K/T, *Invent. Math.* 85 (1986), 39–52.

[2] _____, *Cohomologie T-équivariante de la variété de drapeaux d'un groupe de Kac-Moody*, *Bull. Soc. Math. France* 117 (1989), 129–165.

[3] I. N. Bernstein, I. M. Gelfand, S. I. Gelfand, Schubert cells and cohomology of the spaces G/P, *Russian Math. Surveys* 28 (1973), 1–26.

[4] A. Björner, F. Brenti, *Combinatorics of Coxeter Groups*, Graduate Texts in Mathematics 231, Springer, 2005, ix+363 pp.

[5] P. Bressler, S. Evens, The Schubert calculus, braid relations and generalized cohomology, *Trans. Amer. Math. Soc.* 317 (1990), 799–811.

[6] M. Brion, Equivariant Chow groups for torus actions, *Transform. Groups* 2 (1997), 225–267.

[7] V. Buchstaber, A. Kholodov, Formal groups, functional equations and generalized cohomology theories. *Mat. Sbornik* 181 (1990), 75–94.

[8] B. Calmès, V. Petrov, K. Zainoulline, Invariants, torsion indices and cohomology of complete flags, *Ann. Sci. École Norm. Sup.* (4) 46 (2013), 405–448.

[9] B. Calmès, K. Zainoulline, C. Zhong, A coproduct structure on the formal affine Demazure algebra, *Math. Zeitschrift* 282 (2016), 1191–1218.

[10] B. Calmès, K. Zainoulline, C. Zhong, Push-pull operators on the formal affine Demazure algebra and its dual, `arXiv:1312.0019`.

[11] B. Calmès, K. Zainoulline, C. Zhong, Equivariant oriented cohomology of flag varieties, *Documenta Math.*, Extra Volume: Alexander S. Merkurjev's Sixtieth Birthday(2015), 113-144.

[12] J. Carrell, The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties, *Algebraic groups and their generalizations: classical methods (University Park, PA, 1991)*, Proc. Sympos. Pure Math., 56, Part 1, Amer. Math. Soc., Providence, RI, 1994, 53-61.

[13] J. Carrell, J. Kuttler, Smooth points of T-stable varieties in G/B and the Peterson map. *Invent. Math.* 151 (2003), 353-379.

[14] M. Demazure, Invariants symétriques entiers des groupes de Weyl et torsion, *Invent. Math.* 21 (1973), 287–301.

[15] M. Demazure, Désingularisation des variétés de Schubert généralisées, *Ann. Sci. École Norm. Sup.* (4) 7 (1974), 53–88.

[16] V. Deodhar, On some geometric aspects of Bruhat orderings II. The parabolic analogue of Kazhdan-Lusztig polynomials, *J. Algebra* 111 (1987), 483–506.

[17] D. Edidin, W. Graham, Equivariant intersection theory, *Invent. Math.* 131 (1998), 595–634.

[18] W. Fulton, *Young Tableaux*, Cambridge University Press, 1997.

[19] W. Graham, Equivariant K-theory and Schubert varieties, Preprint, 2002.

[20] J. Heller, J. Malagon-Lopez, Equivariant algebraic cobordism, *J. Reine Angew. Math.* 684 (2013), 87–112.

[21] A. Hoffnung, J. Malagón-López, A. Savage, K. Zainoulline, Formal Hecke algebras and algebraic oriented cohomology theories, *Selecta Math.* 20 (2014), 1213–1245.

[22] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* 53 (1979), 165-184.

[23] A. Knutson, A Schubert calculus recurrence from the noncomplex W-action on G/B, `arXiv:math.CO/0306304`.

[24] B. Kostant, S. Kumar, T-Equivariant K-theory of generalized flag varieties, *J. Differential Geometry* 32 (1990), 549–603.

[25] B. Kostant, S. Kumar, The nil Hecke ring and cohomology of G/P for a Kac-Moody group G, *Adv. Math.* 62 (1986), 187–237.

[26] I. Krichever, Formal groups and the Atiyah-Hirzebruch formula, *Izv. Akad. Nauk SSSR Ser. Math.* 38 (1974), 1289–1304; English transl. in *Math. USSR Izv.* 8 (1974), 1271–1285.

[27] M.-A. Leclerc, The hyperbolic formal affine Demazure algebra, to appear in *Algebras and Representation theory*, (`arXiv:1505.05097`).

[28] C. Lenart, K. Zainoulline, Towards generalized cohomology Schubert calculus via formal root polynomials, to appear in *Math. Res. Lett.* (`arXiv:1408.5952`).

[29] C. Lenart, K. Zainoulline, A Schubert basis in equivariant elliptic cohomology, `arXiv:1508.03134`.

[30] M. Levine, F. Morel, *Algebraic Cobordism*, Springer Monographs in Math., Springer, Berlin, 2007, xii+244pp.

[31] D. Maulik, A. Okounkov, Quantum groups and quantum cohomology, `arXiv:1211.1287`.

[32] D. Peterson, *Lecture Notes on Schubert Calculus*, MIT, Spring 1997.

[33] J. H. Silverman, *The Arithmetic of Elliptic Curves*, Second edition, Graduate Texts in Mathematics 106, Springer, Dordrecht, 2009, xx+513 pp.

[34] W. Soergel, Kazhdan-Lusztig polynomials and a combinatorics for tilting modules, *Represent. Theory* 1 (1997), 83–114.

[35] Schémas en groupes III: Structure des schémas en groupes réductifs. (SGA 3, Vol. 3), *Lecture Notes in Math.* 153, Springer-Verlag, Berlin-New York, 1970, viii+529 pp.

[36] J. Stembridge, On the fully commutative elements of Coxeter groups, *J. Algebraic Combin.* 5 (1996), 353–385.

[37] C. Su, G. Zhao, C. Zhong, On the K-theory stable bases of the Springer resolution, in preparation.

[38] B. Totaro, The Chow ring of a classifying space. Algebraic K-theory (Seattle, WA, 1997), 249–281, Proc. Sympos. Pure Math., 67, Amer. Math. Soc., Providence, RI, 1999.

[39] J. Tymoczko, Permutation representations on Schubert varieties, *Amer. J. Math.* 130 (2008), 1171–1194.

[40] J. Tymoczko, Divided difference operators for partial flag varieties, `arXiv:0912.2545`.

[41] G. Zhao, C. Zhong, Elliptic affine Hecke algebras and their representations, `arXiv:1507.01245`

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