Explicit description of the degree function in terms of quantum Lakshmibai-Seshadri paths

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Abstract

We give an explicit and computable description, in terms of the parabolic quantum Bruhat graph, of the degree function defined for quantum Lakshmibai-Seshadri paths, or equivalently, for "projected" (affine) level-zero Lakshmibai-Seshadri paths. This, in turn, gives an explicit and computable description of the global energy function on tensor products of Kirillov-Reshetikhin crystals of one-column type, and also of (classically restricted) one-dimensional sums.

1 Introduction.

Let $\mathfrak{g}$ be an affine Lie algebra with index set I for the simple roots, and let $U_q'(\mathfrak{g})$ be the quantum affine algebra (without the degree operator) associated to $\mathfrak{g}$. Set $I_0 := I \setminus \{0\}$, where

$0 \in I$ corresponds to the "extended" vertex in the Dynkin diagram of $\mathfrak{g}$. In [NS1, NS2, NS3], Naito and Sagaki gave a combinatorial realization of the crystal bases of tensor products of level-zero fundamental representations $W(\varpi_i)$, $i \in I_0$, over $U_q'(\mathfrak{g})$, where the ϖ_i's are the level-zero fundamental weights; the $U_q'(\mathfrak{g})$-modules $W(\varpi_i)$ are often called Kirillov-Reshetikhin (KR for short) modules of one-column type, and accordingly their crystal bases are called KR crystals of one-column type. In the papers above, they realized elements of the crystal bases as projected (affine) level-zero Lakshmibai-Seshadri (LS for short) paths. Here a projected level-zero LS path is obtained from an ordinary LS path of shape λ by factoring out the null root δ of the affine Lie algebra $\mathfrak{g}$, where λ is a level-zero dominant integral weight of the form $\lambda = \sum_{i \in I_0} m_i \varpi_i$, with $m_i \in \mathbb{Z}_{\geq 0}$. However, from the nature of the definition above of projected level-zero LS paths, their description of these objects in [NS1, NS2, NS3] is not as explicit as the one of usual LS paths given by Littelmann in [L1].

By contrast, in our previous paper [LNS^{3}3], we proved that (in the case that $\mathfrak{g}$ is an untwisted affine Lie algebra) a projected level-zero LS path is identical to what we call a quantum LS path, which is described quite explicitly in terms of the parabolic quantum Bruhat graph, instead of (the Hasse diagram of) the usual Bruhat graph.

Also, in [NS5], we defined a certain integer-valued function, called the degree function, on the set $\mathbb{B}(\lambda)_{\mathrm{cl}}$ of projected level-zero LS paths of shape $\lambda = \sum_{i \in I_0} m_i \varpi_i$, and proved that it is identical to the global "energy function" on the tensor product $\bigotimes_{i \in I_0} \mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes m_i}$ under the isomorphism $\mathbb{B}(\lambda)_{\mathrm{cl}} \cong \bigotimes_{i \in I_0} \mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes m_i}$ of $U_q'(\mathfrak{g})$-crystals; recall that for each $i \in I_0$, the crystal $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is isomorphic, as a $U_q'(\mathfrak{g})$-crystal, to a KR crystal of one-column type. However, again from the nature of the definition of projected level-zero LS paths, our description in [NS5] is not very explicit, and hence it is difficult to compute the value of the degree function at a given projected level-zero LS path.

In [LNS^{3}2], we give an explicit and computable description, in terms of the parabolic quantum Bruhat graph, of the degree function defined for quantum LS paths, or equivalently, for projected level-zero LS paths [LNS^{3}3]. This, in turn, gives a new description of the global energy function on tensor products of KR crystals of one-column type, and also of (classically restricted) one-dimensional sums arising from the study of solvable lattice models in statistical mechanics through Baxter's corner transfer matrix method (for details, see [S]).

The purpose of this paper is to give a new proof of the description above, in terms of the parabolic quantum Bruhat graph, of the degree function. We should mention that our proof in this paper is completely different from the one in [LNS^{3}2] in that (at least in appearance) we do not make use of root operators; it is based on a technical lemma (Lemma 2.3.2) about the decomposition of $\mathbb{B}(\lambda)$ into connected components, and also on our results in [LNS^{3}1], where $\mathbb{B}(\lambda)$ denotes the crystal of (not projected) LS paths of shape λ.

This paper is organized as follows. In §2, we fix our basic notation, and review some fundamental facts about level-zero path crystals. Also, we recall the definition of the degree function, and then prove a technical lemma (Lemma 2.3.2), which plays an important rule in the proof of our main result (Theorem 4.1.1). In §3, we recall the notion of parabolic quantum Bruhat graph, and then give the definition of quantum LS paths. In §4, we state and prove our main result about the description of the degree function in terms of the parabolic quantum Bruhat graph.

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2 Lakshmibai-Seshadri paths and the degree function.

2.1 Basic notation. Let $\mathfrak{g}$ be an untwisted affine Lie algebra over $\mathbb{C}$ with Cartan matrix $A = (a_{ij})_{i,\,j \in I}$; throughout this paper, the elements of the index set I are numbered as in [Kac, §4.8, Table Aff 1]. Take a distinguished vertex $0 \in I$ as in [Kac], and set $I_0 := I \setminus \{0\}$. Let $\mathfrak{h} = \left(\bigoplus_{j \in I} \mathbb{C}\alpha_j^{\vee}\right) \oplus \mathbb{C}d$ denote the Cartan subalgebra of $\mathfrak{g}$, where $\Pi^{\vee} := \left\{\alpha_j^{\vee}\right\}_{j \in I} \subset \mathfrak{h}$ is the set of simple coroots, and $d \in \mathfrak{h}$ is the scaling element (or degree operator). We denote by $\langle \cdot\,,\,\cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ the duality pairing between $\mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h},\,\mathbb{C})$ and $\mathfrak{h}$. Denote by $\Pi := \left\{\alpha_j\right\}_{j \in I} \subset \mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ the set of simple roots, and by $\Lambda_j \in \mathfrak{h}^*$, $j \in I$, the fundamental weights; note that $\langle \alpha_j,\, d \rangle = \delta_{j,0}$ and $\langle \Lambda_j,\, d \rangle = 0$ for $j \in I$. Let $\delta = \sum_{j \in I} a_j \alpha_j \in \mathfrak{h}^*$ and $c = \sum_{j \in I} a_j^{\vee} \alpha_j^{\vee} \in \mathfrak{h}$ denote the null root and the canonical central element of $\mathfrak{g}$, respectively. The Weyl group W of $\mathfrak{g}$ is defined as $W := \langle r_j \mid j \in I \rangle \subset \mathrm{GL}(\mathfrak{h}^*)$, where $r_j \in \mathrm{GL}(\mathfrak{h}^*)$ denotes the simple reflection associated to α_j for $j \in I$, with $\ell : W \to \mathbb{Z}_{\geq 0}$ the length function on W. Denote by Δ_{re} the set of real roots, i.e., $\Delta_{\mathrm{re}} := W\Pi$, and by $\Delta_{\mathrm{re}}^+ \subset \Delta_{\mathrm{re}}$ the set of positive real roots; for $\beta \in \Delta_{\mathrm{re}}$, we denote by $\beta^{\vee}$ the dual root of β, and by $r_\beta \in W$ the reflection associated to β. We take a dual weight lattice $P^{\vee}$ and a weight lattice P as follows:

$$P^{\vee} = \left(\bigoplus_{j \in I} \mathbb{Z}\alpha_j^{\vee}\right) \oplus \mathbb{Z}d \subset \mathfrak{h} \quad \text{and} \quad P = \left(\bigoplus_{j \in I} \mathbb{Z}\Lambda_j\right) \oplus \mathbb{Z}\delta \subset \mathfrak{h}^*. \tag{2.1.1}$$

It is clear that P contains the root lattice $Q := \bigoplus_{j \in I} \mathbb{Z}\alpha_j$, and that $P \cong \mathrm{Hom}_{\mathbb{Z}}(P^{\vee}, \mathbb{Z})$.

Let W_0 denote the subgroup of W generated by r_j, $j \in I_0$. Set $Q_0 := \bigoplus_{j \in I_0} \mathbb{Z}\alpha_j$, $Q_0^+ := \sum_{j \in I_0} \mathbb{Z}_{\geq 0}\alpha_j$, $\Delta_0 := \Delta_{\mathrm{re}} \cap Q_0$, $\Delta_0^+ := \Delta_{\mathrm{re}} \cap Q_0^+$, and $\Delta_0^- := -\Delta_0^+$. Note that W_0 (resp., Δ_0, Δ_0^+, Δ_0^-) can be thought of as the (finite) Weyl group (resp., the set of roots, the set of positive roots, the set of negative roots) of the finite-dimensional simple Lie subalgebra of $\mathfrak{g}$ corresponding to the subset I_0 of I. Also, we denote by $\theta \in \Delta_0^+$ the highest root of the (finite) root system Δ_0; note that $\alpha_0 = -\theta + \delta$ and $\alpha_0^{\vee} = -\theta^{\vee} + c$.

Definition 2.1.1.

(1) An integral weight $\lambda \in P$ is said to be of level zero if $\langle \lambda,\, c \rangle = 0$.

(2) An integral weight $\lambda \in P$ is said to be level-zero dominant if $\langle \lambda,\, c \rangle = 0$, and $\langle \lambda,\, \alpha_j^\vee \rangle \geq 0$ for all $j \in I_0 = I \setminus \{0\}$.

Remark 2.1.2.

(1) If $\lambda \in P$ is of level zero, then $\langle \lambda,\, \alpha_0^\vee \rangle = -\langle \lambda,\, \theta^\vee \rangle$.

(2) For $h \in Q_0^\vee := \bigoplus_{j \in I_0} \mathbb{Z}\alpha_j^\vee$, we denote by $t_h \in W$ the translation with respect to h (see [Kac, §6.5]). If λ is of level-zero, then $t_h \lambda = \lambda - \langle \lambda,\, h \rangle \delta$ for $h \in Q_0^\vee$. Because W is the semidirect product of W_0 and the abelian (normal) subgroup $T = \{t_h \mid h \in Q_0^\vee\}$ of translations by [Kac, Proposition 6.5], we deduce (see also [NS4, Lemma 2.6] for example) that if λ is level-zero dominant, then $W\lambda = W_0 T\lambda \subset W_0\lambda + \mathbb{Z}\delta \subset \lambda - Q_0^+ + \mathbb{Z}\delta$; we define $d_\lambda \in \mathbb{Z}_{>0}$ by: $\{n \in \mathbb{Z} \mid \lambda + n\delta \in T\lambda\} = d_\lambda \mathbb{Z}$.

For each $i \in I_0$, we define a level-zero fundamental weight $\varpi_i \in P$ by

$$\varpi_i := \Lambda_i - a_i^\vee \Lambda_0. \tag{2.1.2}$$

The weights ϖ_i for $i \in I_0$ are actually level-zero dominant integral weights; indeed, $\langle \varpi_i,\, c \rangle = 0$ and $\langle \varpi_i,\, \alpha_j^\vee \rangle = \delta_{i,j}$ for $i,\, j \in I_0$.

Let $\mathrm{cl} : \mathfrak{h}^* \twoheadrightarrow \mathfrak{h}^*/\mathbb{C}\delta$ denote the canonical projection from $\mathfrak{h}^*$ onto $\mathfrak{h}^*/\mathbb{C}\delta$, and define P_{cl} and $P_{\mathrm{cl}}^\vee$ by

$$P_{\mathrm{cl}} := \mathrm{cl}(P) = \bigoplus_{j \in I} \mathbb{Z}\,\mathrm{cl}(\Lambda_j) \quad \text{and} \quad P_{\mathrm{cl}}^\vee := \bigoplus_{j \in I} \mathbb{Z}\alpha_j^\vee \subset P^\vee. \tag{2.1.3}$$

We see that $P_{\mathrm{cl}} \cong P/\mathbb{Z}\delta$, and that P_{cl} can be identified with $\mathrm{Hom}_{\mathbb{Z}}(P_{\mathrm{cl}}^\vee, \mathbb{Z})$ as a $\mathbb{Z}$-module by

$$\langle \mathrm{cl}(\lambda),\, h \rangle = \langle \lambda,\, h \rangle \quad \text{for } \lambda \in P \text{ and } h \in P_{\mathrm{cl}}^\vee. \tag{2.1.4}$$

Also, there exists a natural action of the Weyl group W on $\mathfrak{h}^*/\mathbb{C}\delta$ induced by the one on $\mathfrak{h}^*$, since $W\delta = \delta$; it is obvious that $w \circ \mathrm{cl} = \mathrm{cl} \circ w$ for all $w \in W$.

Remark 2.1.3. Let $\lambda \in P$ be a level-zero integral weight. It is easy to check that $\mathrm{cl}(W\lambda) = W_0\,\mathrm{cl}(\lambda)$ (see the proof of [NS4, Lemma 2.3.3]). In particular, we have $\mathrm{cl}(r_0\lambda) = r_\theta\lambda$ since $\alpha_0 = -\theta + \delta$ and $\alpha_0^\vee = -\theta^\vee + c$.

For simplicity of notation, we often write β instead of $\mathrm{cl}(\beta) \in P_{\mathrm{cl}}$ for $\beta \in \bigoplus_{j \in I} \mathbb{Z}\alpha_j$; note that $\alpha_0 = -\theta$ in P_{cl} since $\alpha_0 = -\theta + \delta$ in P.

2.2 Lakshmibai-Seshadri paths. Here we recall the definition of Lakshmibai-Seshadri (LS for short) paths from [L2, §4]. In this subsection, we fix a level-zero dominant integral weight $\lambda \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \varpi_i$.

Definition 2.2.1. For $\mu, \nu \in W\lambda$, let us write $\mu \geq \nu$ if there exists a sequence $\mu = \mu_0, \mu_1, \ldots, \mu_n = \nu$ of elements in $W\lambda$ and a sequence $\xi_1, \ldots, \xi_n \in \Delta_{\mathrm{re}}^+$ of positive real roots such that $\mu_k = r_{\xi_k}(\mu_{k-1})$ and $\langle \mu_{k-1}, \xi_k^\vee \rangle < 0$ for $k = 1, 2, \ldots, n$. If $\mu \geq \nu$, then we define $\mathrm{dist}(\mu, \nu)$ to be the maximal length n of all possible such sequences $\mu_0, \mu_1, \ldots, \mu_n$ for (μ, ν).

Remark 2.2.2. Keep the notation of Definition 2.2.1. We see that

$$\nu - \mu = \sum_{k=1}^{n} (\mu_k - \mu_{k-1}) = -\sum_{k=1}^{n} \underbrace{\langle \mu_{k-1}, \xi_k^\vee \rangle}_{<0} \xi_k \in \sum_{j \in I} \mathbb{Z}_{\geq 0} \alpha_j.$$

It is obvious that μ covers ν in the poset $W\lambda$ if and only if $\mu > \nu$ with $\mathrm{dist}(\mu, \nu) = 1$. In this case, we write $\mu \gtrdot \nu$.

Remark 2.2.3. Let $\mu, \nu \in W\lambda$ be such that $\mu \gtrdot \nu$, and let $\xi \in \Delta_{\mathrm{re}}^+$ be the positive real root such that $r_\xi \mu = \nu$. We know from [NS4, Lemma 2.11] that $\xi \in \Delta_0^+ \sqcup \{ -\gamma + \delta \mid \gamma \in \Delta_0^+ \}$.

Definition 2.2.4. For $\mu, \nu \in W\lambda$ with $\mu > \nu$ and a rational number $0 < \sigma < 1$, a σ-chain for (μ, ν) is, by definition, a sequence $\mu = \mu_0 \gtrdot \mu_1 \gtrdot \cdots \gtrdot \mu_n = \nu$ of elements in $W\lambda$ such that $\sigma \langle \mu_{k-1}, \xi_k^\vee \rangle \in \mathbb{Z}_{<0}$ for all $k = 1, 2, \ldots, n$, where ξ_k is the positive real root such that $r_{\xi_k} \mu_{k-1} = \mu_k$.

Definition 2.2.5. An LS path of shape λ is, by definition, a pair $(\underline{\nu}\,;\underline{\sigma})$ of a sequence $\underline{\nu} : \nu_1 > \nu_2 > \cdots > \nu_s$ of elements in $W\lambda$ and a sequence $\underline{\sigma} : 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$ of rational numbers satisfying the condition that there exists a σ_k-chain for (ν_k, ν_{k+1}) for each $k = 1, 2, \ldots, s-1$. We denote by $\mathbb{B}(\lambda)$ the set of all LS paths of shape λ.

We identify $\pi = (\nu_1, \nu_2, \ldots, \nu_s\,; \sigma_0, \sigma_1, \ldots, \sigma_s) \in \mathbb{B}(\lambda)$ with the following piecewise-linear, continuous map $\pi : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} P$:

$$\pi(t) = \sum_{l=1}^{k-1} (\sigma_l - \sigma_{l-1}) \nu_l + (t - \sigma_{k-1}) \nu_k \quad \text{for } \sigma_{k-1} \leq t \leq \sigma_k,\ 1 \leq k \leq s. \tag{2.2.1}$$

Remark 2.2.6. It is obvious from the definition that for every $\nu \in W\lambda$, $\pi_\nu := (\nu\,; 0, 1)$ is an LS path of shape λ, which corresponds (under (2.2.1)) to the straight line path $\pi_\nu(t) = t\nu$, $t \in [0,1]$, connecting 0 to ν.

For $\pi \in \mathbb{B}(\lambda)$, we define $\mathrm{cl}(\pi) : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} P_{\mathrm{cl}}$ by

$$\mathrm{cl}(\pi)(t) := \mathrm{cl}(\pi(t)) \quad \text{for } t \in [0,1].$$

Also, we set

$$\mathbb{B}(\lambda)_{\mathrm{cl}} := \mathrm{cl}(\mathbb{B}(\lambda)) = \{ \mathrm{cl}(\pi) \mid \pi \in \mathbb{B}(\lambda) \}.$$

Remark 2.2.7. For $\mu \in P_{\mathrm{cl}}$, we define $\eta_\mu(t) := t\mu$ for $t \in [0,1]$. It is easily seen from Remark 2.2.6 that η_μ is contained in $\mathbb{B}(\lambda)_{\mathrm{cl}}$ for all $\mu \in \mathrm{cl}(W\lambda) = W_0\,\mathrm{cl}(\lambda)$.

We can endow the set $\mathbb{B}(\lambda)$ of LS paths of shape λ (resp., the set $\mathbb{B}(\lambda)_{\mathrm{cl}}$ of "cl-projected" LS paths of shape λ) with a crystal structure with weights in P (resp., in P_{cl}) by defining root operators on $\mathbb{B}(\lambda)$ (resp., $\mathbb{B}(\lambda)_{\mathrm{cl}}$); since we do not use root operators in this paper, we omit the details (see [L2], and also [NS5, §2.2], [LNS^{3}2, §2.3]).

2.3 Degree function. As in the previous subsection, we fix a level-zero dominant integral weight $\lambda \in \sum_{i\in I_0} \mathbb{Z}_{\geq 0}\varpi_i$. Let us recall the definition of the degree function $\mathrm{Deg} = \mathrm{Deg}_\lambda : \mathbb{B}(\lambda)_{\mathrm{cl}} \to \mathbb{Z}_{\leq 0}$ from [NS5, §3.1]. We know the following proposition from [NS5, Proposition 3.1.3].

Proposition 2.3.1. *Let $\lambda \in \sum_{i\in I_0} \mathbb{Z}_{\geq 0}\varpi_i$ be a level-zero dominant integral weight. For each $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, there exists a unique element $\pi_\eta \in \mathbb{B}(\lambda)$ satisfying the following conditions:*

(1) $\mathrm{cl}(\pi_\eta) = \eta$*;*

(2) *the element π_η is contained in the connected component $\mathbb{B}_0(\lambda)$ of $\mathbb{B}(\lambda)$ containing the straight line path $\pi_\lambda = (\lambda\,;\,0,\,1) \in \mathbb{B}(\lambda)$;*

(3) *if we write π_η in the form $(\nu_1, \nu_2, \dots, \nu_s\,;\,\underline{\sigma})$ as in Definition 2.2.5, then ν_1 is contained in the set $\lambda - Q_0^+$ (see Remark 2.1.2 (2)).*

Let $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$. It follows from [NS5, Lemma 3.1.1] that $\pi_\eta(1) \in P$ is of the form $\pi_\eta(1) = \lambda - \beta + K\delta$ for some $\beta \in Q_0^+$ and $K \in \mathbb{Z}_{\geq 0}$. We define

$$\mathrm{Deg}(\eta) = \mathrm{Deg}_\lambda(\eta) := -K \in \mathbb{Z}_{\leq 0}.$$

The following lemma plays an important role in the proof of Theorem 4.1.1.

Lemma 2.3.2. *Let C be a connected component of $\mathbb{B}(\lambda)$.*

(1) *For each $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$, there exists a unique element $\pi_\eta^C \in C$ satisfying the same conditions as (1) and (3) of Proposition 2.3.1.*

(2) *If $\pi_\eta(1) = \lambda - \beta - \mathrm{Deg}(\eta)\delta$ with $\beta \in Q_0^+$, then $\pi_\eta^C(1) = \lambda - \beta + \big(-\mathrm{Deg}(\eta) + L\big)\delta$ for some $L \in \mathbb{Z}_{\geq 0}$.*

(3) *In part (2) above, $C = \mathbb{B}_0(\lambda)$ if and only if $L = 0$.*

Proof. If $C = \mathbb{B}_0(\lambda)$, then we have $\pi_\eta^C = \pi_\eta$. In this case, part (1) follows from Proposition 2.3.1; part (2) and the "only if" part of part (3) are obvious.

Assume that $C \neq \mathbb{B}_0(\lambda)$. We see from [NS4, Theorem 3.1 and Remark 2.15] that the connected component C contains a unique element π_λ^C of the form

$$\pi_\lambda^C = (\lambda - N_1\delta, \ldots, \lambda - N_{s-1}\delta, \lambda; \tau_0, \tau_1, \ldots, \tau_{s-1}, \tau_s) \tag{2.3.1}$$

for some integers $N_1 > N_2 > \cdots > N_{s-1} > N_s = 0$ and rational numbers $0 = \tau_0 < \tau_1 < \cdots < \tau_s = 1$; since $C \neq \mathbb{B}_0(\lambda)$ (and hence $\pi_\lambda^C \neq \pi_\lambda$), we have $s > 1$. From (2.3.1), by using (2.2.1), we deduce that

$$\pi_\lambda^C(1) = \lambda - \underbrace{\left(\sum_{u=1}^{s}(\tau_u - \tau_{u-1})N_u\right)}_{=:N}\delta;$$

note that $N \in \mathbb{Z}$ since $\pi_\lambda^C(1) \in P$, which in turn follows from the integrality condition on LS paths (see Definitions 2.2.4 and 2.2.5). Also, since $N_1 > N_2 > \cdots > N_{s-1} > N_s = 0$ with $s > 1$, it follows that

$$N = \sum_{u=1}^{s}(\tau_u - \tau_{u-1})N_u < \sum_{u=1}^{s}(\tau_u - \tau_{u-1})N_1 = N_1.$$

Therefore, we have $\pi_\lambda^C(1) = \lambda - N_1\delta + L\delta$, with $L := N_1 - N \in \mathbb{Z}_{>0}$.

Let us denote by $F : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} P$ the piecewise-linear, continuous function such that $\pi_\lambda^C(t) = \pi_\lambda(t) + F(t)\delta$ for all $t \in [0,1]$; note that $F(0) = 0$,

$$\lim_{\substack{t\to 0\\ t>0}} \frac{F(t) - F(0)}{t - 0} = \lim_{\substack{t\to 0\\ t>0}} \frac{F(t)}{t} = -N_1, \tag{2.3.2}$$

and $F(1) = -N_1 + L$. Then, by using [NS4, Lemma 2.26], we deduce that $C = \bigl\{\pi(t) + F(t)\delta \mid \pi \in \mathbb{B}_0(\lambda)\bigr\}$. Hence it follows from [NS5, Lemma 3.1.2] that

$$\bigl\{\pi \in C \mid \mathrm{cl}(\pi) = \eta\bigr\} = \bigl\{\pi_\eta(t) + t(M\delta) + F(t)\delta \mid M \in d_\lambda\mathbb{Z}\bigr\};$$

recall the notation $d_\lambda \in \mathbb{Z}_{\geq 0}$ from Remark 2.1.2 (2). Therefore, we conclude by Proposition 2.3.1 and (2.3.2) that $\pi_\eta^C(t) := \pi_\eta(t) + F(t)\delta + t(N_1\delta)$ is a unique element in C satisfying the same conditions as (1) and (3) of Proposition 2.3.1. This proves part (1) for $C \neq \mathbb{B}_0(\lambda)$. Moreover, part (2) for $C \neq \mathbb{B}_0(\lambda)$ and the "if" part of part (3) follow immediately since

$$\pi_\eta^C(1) = \pi_\eta(1) + F(1)\delta + N_1\delta = \pi_\eta(1) + L\delta$$

with $L > 0$. This completes the proof of the lemma. $\square$

2.4 Global energy function. We know from [NS1, Proposition 5.8] and [NS3, Theorem 2.1.1 and Proposition 3.4.2] that for each $i \in I_0$, the crystal $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is isomorphic, as a crystal with weights in P_{cl}, to the crystal basis of the level-zero fundamental representation $W(\varpi_i)$ introduced in [Kas2, Theorem 5.17]; the level-zero fundamental modules $W(\varpi_i)$,

$i \in I_0$, are often called Kirillov-Reshetikhin (KR for short) modules of one-column type, and accordingly their crystal bases are called KR crystals of one-column type. Also, we know the following from [NS2, Theorem 3.2]. Let $\mathbf{i} = (i_1, i_2, \dots, i_p)$ be an arbitrary sequence of elements of I_0 (with repetitions allowed), and set $\lambda := \varpi_{i_1} + \varpi_{i_2} + \cdots + \varpi_{i_p}$. Then the crystal $\mathbb{B}(\lambda)_{\mathrm{cl}}$ is isomorphic, as a crystal with weights in P_{cl}, to the tensor product $\mathbb{B}_{\mathbf{i}} := \mathbb{B}(\varpi_{i_1})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_2})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_p})_{\mathrm{cl}}$ of KR crystals of one-column type. Moreover, in [NS5, Theorem 4.1], we proved that the degree function $\mathrm{Deg} = \mathrm{Deg}_\lambda : \mathbb{B}(\lambda)_{\mathrm{cl}} \to \mathbb{Z}_{\le 0}$ in §2.3 is identical, up to a constant, to the global energy function $D_{\mathbf{i}}$ (which is called the energy function in [LNS^{3}2], and the right energy function in [LS]; note that the order of tensor factors in tensor products of crystals in [LS] is "opposite" to the one in this paper and [LNS^{3}2]) on $\mathbb{B}_{\mathbf{i}} = \mathbb{B}(\varpi_{i_1})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_2})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_p})_{\mathrm{cl}}$ under the isomorphism $\Psi : \mathbb{B}(\lambda)_{\mathrm{cl}} \xrightarrow{\sim} \mathbb{B}_{\mathbf{i}}$ of crystals above.

Now we explain the relation between the degree function and the global energy function more precisely. Following [HKOTY, §3] and [HKOTT, §3.3] (see also [NS5, §4.1]), we define the global energy function $D_{\mathbf{i}} : \mathbb{B}_{\mathbf{i}} = \mathbb{B}(\varpi_{i_1})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_2})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_p})_{\mathrm{cl}} \to \mathbb{Z}$ as follows. First we recall that there exists a unique isomorphism

$$\begin{aligned}\mathbb{B}(\varpi_{i_k})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_{k+1}})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_{l-1}})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_l})_{\mathrm{cl}} \\ \xrightarrow{\sim} \mathbb{B}(\varpi_{i_l})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_k})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_{l-2}})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_{l-1}})_{\mathrm{cl}}\end{aligned}$$

of crystals, which is given as the composite of combinatorial R-matrices (see [NS5, §2.4]). For an element $\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_p \in \mathbb{B}_{\mathbf{i}}$, we define $\eta_l^{(k)} \in \mathbb{B}(\varpi_{i_l})_{\mathrm{cl}}$, $1 \le k < l \le p$, to be the first factor (which lies in $\mathbb{B}(\varpi_{i_l})_{\mathrm{cl}}$) of the image of $\eta_k \otimes \eta_{k+1} \otimes \cdots \otimes \eta_l \in \mathbb{B}(\varpi_{i_k})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_{k+1}})_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_l})_{\mathrm{cl}}$ under the above isomorphism of crystals. For convenience, we set $\eta_l^{(l)} := \eta_l$ for $1 \le l \le p$. Furthermore, for each $1 \le k \le p$, take (and fix) an arbitrary element $\eta_k^\flat \in \mathbb{B}(\varpi_{i_k})_{\mathrm{cl}}$ such that $f_j \eta_k^\flat = \mathbf{0}$ for all $j \in I_0$. Then we set

$$\begin{aligned}&D_{\mathbf{i}}(\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_p) = \\ &\qquad \sum_{1 \le k < l \le p} H_{\varpi_{i_k}, \varpi_{i_l}}(\eta_k \otimes \eta_l^{(k+1)}) + \sum_{k=1}^{p} H_{\varpi_{i_k}, \varpi_{i_k}}(\eta_k^\flat \otimes \eta_k^{(1)}).\end{aligned}$$

Here, $H_{\varpi_{i_k}, \varpi_{i_l}} : \mathbb{B}(\varpi_{i_k})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_l})_{\mathrm{cl}} \to \mathbb{Z}$ is the local energy function, which is a unique $\mathbb{Z}$-valued function on $\mathbb{B}(\varpi_{i_k})_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_{i_l})_{\mathrm{cl}}$ satisfying the conditions [NS5, (H1) and (H2) in Theorem 2.5.1]. Also, we define a constant $D_{\mathbf{i}}^{\mathrm{ext}} \in \mathbb{Z}$ by

$$D_{\mathbf{i}}^{\mathrm{ext}} := \sum_{k=1}^{p} H_{\varpi_{i_k}, \varpi_{i_k}}(\eta_k^\flat \otimes \mathrm{cl}(\pi_{\varpi_{i_k}})).$$

In [NS5, Theorem 4.1], we proved that for every $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$,

$$\mathrm{Deg}(\eta) = D_{\mathbf{i}}(\Psi(\eta)) - D_{\mathbf{i}}^{\mathrm{ext}},$$

where $\Psi : \mathbb{B}(\lambda)_{\mathrm{cl}} \xrightarrow{\sim} \mathbb{B}_{\mathbf{i}}$ is the isomorphism of crystals above.

Remark 2.4.1. We can verify that the function $D_{\mathbf{i}} \circ \Psi : \mathbb{B}(\lambda)_{\mathrm{cl}} \to \mathbb{Z}$ is a unique function on $\mathbb{B}(\lambda)_{\mathrm{cl}}$ satisfying [NS5, (3.2.1)] (with Deg replaced by $D_{\mathbf{i}} \circ \Psi$) and the condition that $D_{\mathbf{i}} \circ \Psi(\mathrm{cl}(\pi_\lambda)) = D_{\mathbf{i}}^{\mathrm{ext}}$ (see [NS5, Lemma 3.2.1 (1)]).

3 Quantum Lakshmibai-Seshadri paths.

3.1 Parabolic quantum Bruhat graph. In this subsection, we fix a subset J of I_0. Set

$$W_{0,J} := \langle r_j \mid j \in J \rangle \subset W_0.$$

It is well-known that each coset in $W_0/W_{0,J}$ has a unique element of minimal length, called the minimal coset representative for the coset; we denote by $W_0^J \subset W_0$ the set of minimal coset representatives for the cosets in $W_0/W_{0,J}$, and by $\lfloor \cdot \rfloor = \lfloor \cdot \rfloor_J : W_0 \twoheadrightarrow W_0^J \cong W_0/W_{0,J}$ the canonical projection. Also, we set $\Delta_{0,J} := \Delta_0 \cap \left(\bigoplus_{j \in J} \mathbb{Z}\alpha_j\right)$, $\Delta_{0,J}^{\pm} := \Delta_0^{\pm} \cap \left(\bigoplus_{j \in J} \mathbb{Z}\alpha_j\right)$, and $\rho := (1/2) \sum_{\alpha \in \Delta_0^+} \alpha$, $\rho_J := (1/2) \sum_{\alpha \in \Delta_{0,J}^+} \alpha$.

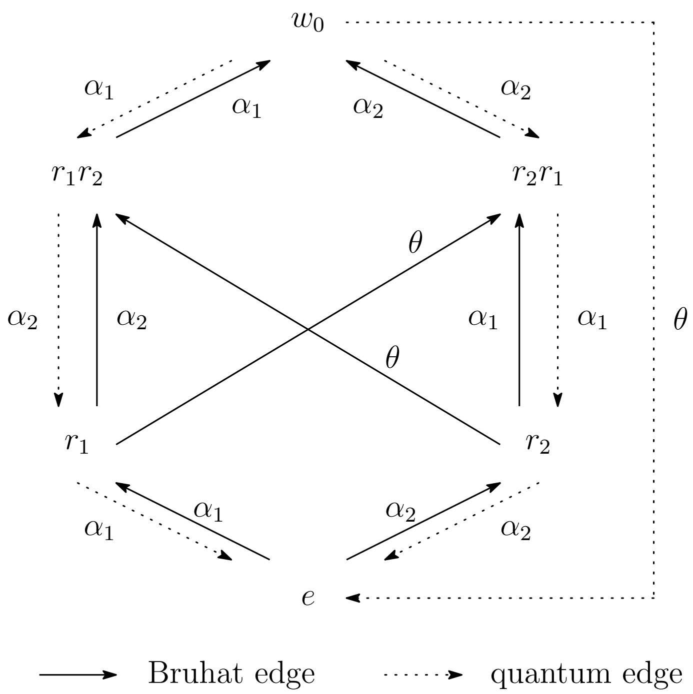
Definition 3.1.1. The parabolic quantum Bruhat graph is a $(\Delta_0^+ \setminus \Delta_{0,J}^+)$-labeled, directed graph with vertex set W_0^J and $(\Delta_0^+ \setminus \Delta_{0,J}^+)$-labeled, directed edges of the following form: $w \xrightarrow{\beta} \lfloor w r_\beta \rfloor$ for $w \in W_0^J$ and $\beta \in \Delta_0^+ \setminus \Delta_{0,J}^+$ such that either

(i) $\ell(\lfloor w r_\beta \rfloor) = \ell(w) + 1$, or

(ii) $\ell(\lfloor w r_\beta \rfloor) = \ell(w) - 2\langle \rho - \rho_J, \beta^\vee \rangle + 1$;

if (i) holds (resp., (ii) holds), then the edge is called a Bruhat edge (resp., a quantum edge).

Example 3.1.2. Assume that $\mathfrak{g}$ is of type $A_2^{(1)}$ (and hence Δ_0 and W_0 are of type A_2), and $J = \emptyset$. Then the quantum Bruhat graph is as follows, where $\theta = \alpha_1 + \alpha_2 \in \Delta_0^+$, the highest root of A_2:



Let $x,\ y \in W_0^J$. A directed path $\mathbf{d}$ from y to x in the parabolic quantum Bruhat graph is, by definition, a pair of a sequence $w_0,\ w_1,\ \ldots,\ w_n$ of elements in W_0^J and a sequence $\beta_1,\ \beta_2,\ \ldots,\ \beta_n$ of elements in $\Delta_0^+ \setminus \Delta_{0,J}^+$ such that in the parabolic quantum Bruhat graph,

$$\mathbf{d} : x = w_0 \stackrel{\beta_1}{\leftarrow} w_1 \stackrel{\beta_2}{\leftarrow} \cdots \stackrel{\beta_n}{\leftarrow} w_n = y. \tag{3.1.1}$$

A directed path $\mathbf{d}$ from y to x is said to be shortest if its length n is minimal among all possible directed paths from y to x; let $\ell(y,\ x)$ denote the length of a shortest directed path from y to x in the parabolic quantum Bruhat graph. Also, we define the weight $\mathrm{wt}(\mathbf{d}) \in Q^\vee = \bigoplus_{j \in I_0} \mathbb{Z}\alpha_j^\vee$ of a directed path of the form (3.1.1) by

$$\mathrm{wt}(\mathbf{d}) := \sum_{\substack{1 \le k \le n\,; \\ w_{k-1} \stackrel{\beta_k}{\leftarrow} w_k \text{ is} \\ \text{a quantum edge}}} \beta_k^\vee. \tag{3.1.2}$$

We recall the following proposition from [LNS^{3}1, Theorem 6.5].

Proposition 3.1.3. *Set* $\Lambda := \mathrm{cl}(\lambda) \in P_{\mathrm{cl}}$.

(1) *Let* $w \in W_0^J$ *and* $\beta \in \Delta_0^+ \setminus \Delta_{0,J}^+$ *be such that* $\lfloor w r_\beta \rfloor \xleftarrow{\beta} w$ *in the parabolic quantum Bruhat graph. We set*

$$\xi := \begin{cases} w\beta & \text{if } \lfloor w r_\beta \rfloor \xleftarrow{\beta} w \text{ is a Bruhat edge}, \\ w\beta + \delta & \text{if } \lfloor w r_\beta \rfloor \xleftarrow{\beta} w \text{ is a quantum edge}. \end{cases}$$

Then, $\xi \in \Delta_{\mathrm{re}}^+$, *and* $r_\xi \nu > \nu$ *for all* $\nu \in W\lambda$ *such that* $\mathrm{cl}(\nu) = w\Lambda$.

(2) *Let* $\mu,\ \nu \in W\lambda$ *be such that* $\mu > \nu$, *and let* $\xi \in \Delta_{\mathrm{re}}^+$ *be the positive real root such that* $r_\xi \mu = \nu$; *recall from Remark 2.2.3 that* $\xi \in \Delta_0^+ \sqcup \{-\gamma + \delta \mid \gamma \in \Delta_0^+\}$. *Let* $w \in W_0^J$ *be a unique element in* W_0^J *such that* $\mathrm{cl}(\nu) = w\Lambda$, *and set*

$$\beta := \begin{cases} w^{-1}\xi & \text{if } \xi \in \Delta_0^+, \\ w^{-1}(\xi - \delta) & \text{if } \xi \in \{-\gamma + \delta \mid \gamma \in \Delta_0^+\}. \end{cases}$$

Then, $\beta \in \Delta_0^+ \setminus \Delta_J^+$, *and* $\lfloor w r_\beta \rfloor \xleftarrow{\beta} w$ *in the parabolic quantum Bruhat graph; note that* $\mathrm{cl}(\mu) = \lfloor w r_\beta \rfloor \Lambda$. *Moreover, the edge* $\lfloor w r_\beta \rfloor \xleftarrow{\beta} w$ *is a Bruhat (resp., quantum) edge if* $\xi \in \Delta_0^+$ *(resp.,* $\xi \in \{-\gamma + \delta \mid \gamma \in \Delta_0^+\}$*).*

3.2 Definition of quantum Lakshmibai-Seshadri paths. In this subsection, we fix a level-zero dominant integral weight $\lambda \in \sum_{i \in I_0} \mathbb{Z}_{\ge 0}\varpi_i$, and set $\Lambda := \mathrm{cl}(\lambda)$ for simplicity of notation. Also, we set

$$J := \{ j \in I_0 \mid \langle \Lambda,\ \alpha_j^\vee \rangle = 0 \} \subset I_0.$$

Definition 3.2.1. Let $x,\, y \in W_0^J$, and let $\sigma \in \mathbb{Q}$ be such that $0 < \sigma < 1$. A directed σ-path from y to x is, by definition, a directed path

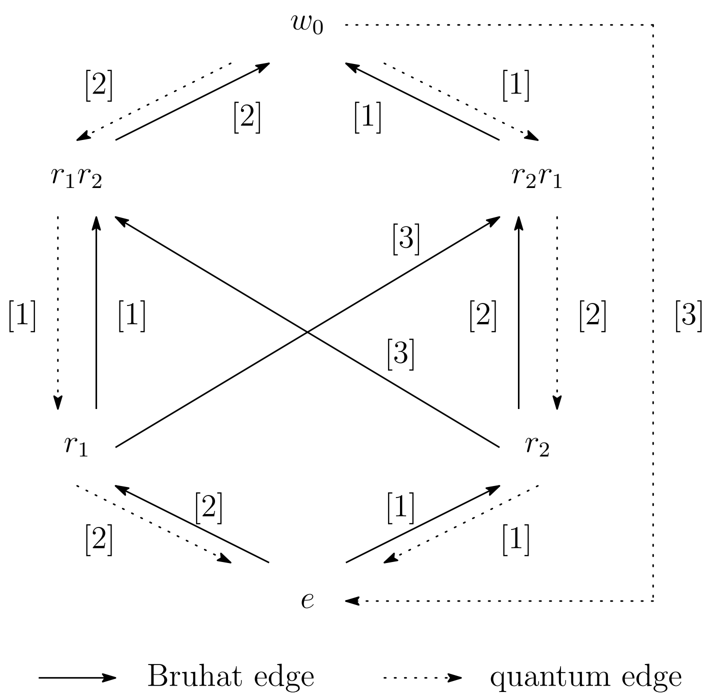
$$x = w_0 \xleftarrow{\beta_1} w_1 \xleftarrow{\beta_2} w_2 \xleftarrow{\beta_3} \cdots \xleftarrow{\beta_n} w_n = y$$

from y to x in the parabolic quantum Bruhat graph satisfying the condition that

$$\sigma\langle \Lambda,\, \beta_k^\vee \rangle \in \mathbb{Z} \quad \text{for all } 1 \le k \le n.$$

Remark 3.2.2. Keep the notation and setting of Proposition 3.1.3 (1). Let $0 < \sigma < 1$ be a rational number. If an edge $\lfloor w r_\beta \rfloor \xleftarrow{\beta} w$ satisfies $\sigma\langle \Lambda,\, \beta^\vee \rangle \in \mathbb{Z}$, then $r_\xi \nu > \nu$ is a σ-chain for $(r_\xi \nu,\, \nu)$. Indeed, we have $\sigma\langle \nu,\, \xi^\vee \rangle = \sigma\langle w\Lambda,\, w\beta^\vee \rangle = \sigma\langle \Lambda,\, \beta^\vee \rangle \in \mathbb{Z}$.

Example 3.2.3. Assume that $\mathfrak{g}$ is of type $A_2^{(1)}$, and $\lambda = 2\varpi_1 + \varpi_2$. Then, J is the empty set, and hence the corresponding (parabolic) quantum Bruhat graph is the one in Example 3.1.2. In the figure below, the symbol $[a]$ on an edge indicates that the value of $\Lambda = \mathrm{cl}(\lambda)$ at the coroot of the label of the edge is equal to a:



From this, we see that the directed edges $r_1 \xrightarrow{\theta} r_2r_1$, $w_0 \xrightarrow{\theta} e$, and $r_2 \xrightarrow{\theta} r_1r_2$ are (1/3)-paths, and hence (2/3)-paths. Also, we see that the directed edges $e \xrightarrow{\alpha_1} r_1$, $r_1r_2 \xrightarrow{\alpha_1} w_0$, and $r_2r_1 \xrightarrow{\alpha_1} r_2$ are (1/2)-paths.

Definition 3.2.4. Let us denote by $\widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$ (resp., $\widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}$) the set of all pairs $\eta = (\underline{x}\,;\, \underline{\sigma})$ of a sequence $\underline{x} \,:\, x_1,\, x_2,\, \dots,\, x_s$ of elements in W_0^J, with $x_k \ne x_{k+1}$ for $1 \le k \le s-1$, and a sequence $\underline{\sigma} \,:\, 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$ of rational numbers satisfying the condition that there exists a directed σ_k-path (resp., directed σ_k-path of length $\ell(x_{k+1},\, x_k)$) from x_{k+1} to x_k for each $1 \le k \le s-1$. We call an element of $\widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}}$ a quantum Lakshmibai-Seshadri path of shape λ.

Example 3.2.5. Keep the notation and setting of Example 3.2.3. We can check that

$$\begin{aligned}\eta_1 &= (r_2,\, r_2r_1,\, r_1\,;\, 0,\, 1/2,\, 2/3,\, 1),\\ \eta_2 &= (r_1,\, e,\, w_0\,;\, 0,\, 1/2,\, 2/3,\, 1),\\ \eta_3 &= (e,\, w_0,\, r_1r_2\,;\, 0,\, 1/3,\, 1/2,\, 1)\end{aligned}$$

are quantum LS paths of shape λ.

Let $\eta = (x_1,\, x_2,\, \ldots,\, x_s\,;\, \sigma_0,\, \sigma_1,\, \ldots,\, \sigma_s)$ be a rational path, that is, a pair of a sequence $x_1,\, x_2,\, \ldots,\, x_s$ of elements in W_0^J, with $x_k \neq x_{k+1}$ for $1 \leq k \leq s-1$, and a sequence $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$ of rational numbers. We identify η with the following piecewise-linear, continuous map $\eta : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} P_{\mathrm{cl}}$ (cf. (2.2.1)):

$$\eta(t) = \sum_{l=1}^{k-1}(\sigma_l - \sigma_{l-1})x_l\Lambda + (t - \sigma_{k-1})x_k\Lambda \quad \text{for } \sigma_{k-1} \leq t \leq \sigma_k,\ 1 \leq k \leq s; \tag{3.2.1}$$

here we note that the map $W_0^J \to W_0\Lambda$, $w \mapsto w\Lambda$, is bijective.

We know the following from [LNS^{3}3, Theorem 4.1.1] (see also [LNS^{3}2]).

Theorem 3.2.6. *With the notation and setting above, we have*

$$\widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}} = \widetilde{\mathbb{B}}(\lambda)_{\mathrm{cl}} = \mathbb{B}(\lambda)_{\mathrm{cl}}.$$

4 Main result.

4.1 Description of the degree function in terms of the parabolic quantum Bruhat graph. As in §3.2, we fix a level-zero dominant integral weight $\lambda \in \sum_{j\in I_0} \mathbb{Z}_{\geq 0}\varpi_j$, and set $J = \bigl\{j \in I_0 \mid \langle \Lambda,\, \alpha_j^\vee\rangle = 0\bigr\}$, where $\Lambda := \mathrm{cl}(\lambda)$.

Let $\eta \in \mathbb{B}(\lambda)_{\mathrm{cl}}$. By Theorem 3.2.6, we can write η in the form:

$$\eta = (x_1,\, x_2,\, \ldots,\, x_s\,;\, \sigma_0,\, \sigma_1,\, \ldots,\, \sigma_s) \in \widehat{\mathbb{B}}(\lambda)_{\mathrm{cl}}.$$

For each $1 \leq p \leq s-1$, let $\mathbf{d}_p$ denote a directed σ_p-path from x_{p+1} to x_p of length $\ell(x_{p+1},\, x_p)$; observe that the value $\langle \Lambda,\, \mathrm{wt}(\mathbf{d}_p)\rangle$ does not depend on the choice of such a directed σ_p-path $\mathbf{d}_p$. Indeed, if $\mathbf{d}_p'$ is another directed σ_p-path from x_{p+1} to x_p of length $\ell(x_{p+1},\, x_p)$, then it follows from [LNS^{3}1, Proposition 8.1] that $\mathrm{wt}(\mathbf{d}_p) - \mathrm{wt}(\mathbf{d}_p') \in Q_J^\vee := \bigoplus_{j\in J}\mathbb{Z}\alpha_j^\vee$. Since $J = \bigl\{j \in I_0 \mid \langle \Lambda,\, \alpha_j^\vee\rangle = 0\bigr\}$ by the definition, we have

$$\langle \Lambda,\, \mathrm{wt}(\mathbf{d}_p) - \mathrm{wt}(\mathbf{d}_p')\rangle = 0, \quad \text{and hence} \quad \langle \Lambda,\, \mathrm{wt}(\mathbf{d}_p)\rangle = \langle \Lambda,\, \mathrm{wt}(\mathbf{d}_p')\rangle.$$

Now, we define

$$\widetilde{\nu}_1 := x_1\lambda, \qquad \widetilde{\nu}_p := x_p\lambda + \left(\sum_{u=1}^{p-1}\bigl\langle \Lambda,\, \mathrm{wt}(\mathbf{d}_u)\bigr\rangle\right)\delta \quad \text{for } 2 \leq p \leq s, \tag{4.1.1}$$

and set

$$\widetilde{\pi}_\eta := (\widetilde{\nu}_1,\, \widetilde{\nu}_2,\, \dots,\, \widetilde{\nu}_s\,;\, \sigma_0,\, \sigma_1,\, \dots,\, \sigma_s).$$

The following is the main result of this paper; its proof will be given in the next subsection.

Theorem 4.1.1. *Keep the notation above. Then, the element $\widetilde{\pi}_\eta$ defined above is identical to the element $\pi_\eta \in \mathbb{B}_0(\lambda) \subset \mathbb{B}(\lambda)$ in Proposition 2.3.1. Moreover, we have*

$$\mathrm{Deg}(\eta) = -\sum_{p=1}^{s-1} (1-\sigma_p)\big\langle \Lambda,\, \mathrm{wt}(\mathbf{d}_p)\big\rangle. \tag{4.1.2}$$

Remark 4.1.2. The formula (4.1.2) is identical to the one obtained in [LNS^{3}2, Theorem 4.5], but the proof given there is completely different from the proof given in the next subsection.

Example 4.1.3. Keep the notation and setting of Examples 3.2.3 and 3.2.5. Let us compute $\mathrm{Deg}(\eta_1)$. It is obvious that $r_2 \xleftarrow{\alpha_1} r_2r_1$ (resp., $r_2r_1 \xleftarrow{\theta} r_1$) is a shortest directed path from r_2r_1 to r_2 (resp., from r_1 to r_2r_1). Because $r_2 \xleftarrow{\alpha_1} r_2r_1$ (resp., $r_2r_1 \xleftarrow{\theta} r_1$) is a quantum edge (resp., Bruhat edge), it follows from the definition (3.1.2) of the weight of a directed path that

$$\mathrm{wt}(r_2 \xleftarrow{\alpha_1} r_2r_1) = \alpha_1^\vee \quad \text{and} \quad \mathrm{wt}(r_2r_1 \xleftarrow{\theta} r_1) = 0.$$

Hence, by Theorem 4.1.1, we have

$$\begin{aligned}
\mathrm{Deg}(\eta_1) &= -\left(1-\frac{1}{2}\right)\big\langle \Lambda,\, \mathrm{wt}(r_2 \xleftarrow{\alpha_1} r_2r_1)\big\rangle - \left(1-\frac{2}{3}\right)\big\langle \Lambda,\, \mathrm{wt}(r_2r_1 \xleftarrow{\theta} r_1)\big\rangle \\
&= -\left(1-\frac{1}{2}\right)\underbrace{\big\langle \Lambda,\, \alpha_1^\vee\big\rangle}_{=2} - \left(1-\frac{2}{3}\right)\langle \Lambda,\, 0\rangle = -1.
\end{aligned}$$

Similarly, we have

$$\mathrm{Deg}(\eta_2) = -\left(1-\frac{1}{2}\right)\big\langle \Lambda,\, \underbrace{\mathrm{wt}(r_1 \xleftarrow{\alpha_1} e)}_{=0}\big\rangle - \left(1-\frac{2}{3}\right)\big\langle \Lambda,\, \underbrace{\mathrm{wt}(e \xleftarrow{\theta} w_0)}_{=\theta^\vee}\big\rangle = -1,$$

$$\mathrm{Deg}(\eta_3) = -\left(1-\frac{1}{3}\right)\big\langle \Lambda,\, \underbrace{\mathrm{wt}(e \xleftarrow{\theta} w_0)}_{=\theta^\vee}\big\rangle - \left(1-\frac{1}{2}\right)\big\langle \Lambda,\, \underbrace{\mathrm{wt}(w_0 \xleftarrow{\alpha_1} r_1r_2)}_{=0}\big\rangle = -2.$$

4.2 Proof of Theorem 4.1.1. Keep the notation of the previous subsection. First we claim that $\widetilde{\pi}_\eta \in \mathbb{B}(\lambda)$. We will show by induction on p that $\widetilde{\nu}_p \in W\lambda$ for all $1 \le p \le s$. If $p = 1$, then the assertion is obvious from the definition: $\widetilde{\nu}_1 = x_1\lambda$. Assume now that $s-1 \ge p \ge 1$, and $\mathbf{d}_p$ is of the form

$$\mathbf{d}_p : x_p = w_0 \xleftarrow{\beta_1} w_1 \xleftarrow{\beta_2} w_2 \xleftarrow{\beta_3} \cdots \xleftarrow{\beta_n} w_n = x_{p+1}.$$

For each $1 \le k \le n$, we define $\xi_k \in \Delta_{\mathrm{re}}^+$ as follows (see Proposition 3.1.3):

$$\xi_k = \begin{cases} w_k\beta_k & \text{if } w_{k-1} = \lfloor w_k r_{\beta_k} \rfloor \xleftarrow{\beta_k} w_k \text{ is a Bruhat edge,} \\ w_k\beta_k + \delta & \text{if } w_{k-1} = \lfloor w_k r_{\beta_k} \rfloor \xleftarrow{\beta_k} w_k \text{ is a quantum edge.} \end{cases} \tag{4.2.1}$$

Then, for $0 \le k \le n$, we obtain

$$\widetilde{\mu}_k := r_{\xi_k} \cdots r_{\xi_2} r_{\xi_1} \widetilde{\nu}_p = w_k\lambda + \left(\sum_{u=1}^{p-1} \langle \Lambda,\, \mathrm{wt}(\mathbf{d}_u) \rangle \right) \delta + \left(\sum_{l \in [1,\, k]_{\mathsf{q}}} \langle \Lambda,\, \beta_l^{\vee} \rangle \right) \delta, \tag{4.2.2}$$

where $[1,\, k]_{\mathsf{q}} := \bigl\{ 1 \le l \le k \mid w_{l-1} = \lfloor w_l r_{\beta_l} \rfloor \xleftarrow{\beta_l} w_l \text{ is a quantum edge} \bigr\}$. Indeed, this equation follows by induction on k. If $k = 0$, then equation (4.2.2) is obvious by (4.1.1). Assume that $k \ge 1$; by the induction hypothesis,

$$\widetilde{\mu}_k = r_{\xi_k} \widetilde{\mu}_{k-1} = r_{\xi_k} w_{k-1}\lambda + \left(\sum_{u=1}^{p-1} \langle \Lambda,\, \mathrm{wt}(\mathbf{d}_u) \rangle \right) \delta + \left(\sum_{l \in [1,\, k-1]_{\mathsf{q}}} \langle \Lambda,\, \beta_l^{\vee} \rangle \right) \delta. \tag{4.2.3}$$

If $w_{k-1} \xleftarrow{\beta_k} w_k$ is a Bruhat edge, then we have $[1,\, k]_{\mathsf{q}} = [1,\, k-1]_{\mathsf{q}}$. Also, since $\xi_k = w_k\beta_k$, it follows that

$$r_{\xi_k} w_{k-1}\lambda = w_k r_{\beta_k} w_k^{-1} w_{k-1}\lambda = w_k r_{\beta_k} w_k^{-1} \lfloor w_k r_{\beta_k} \rfloor \lambda = w_k r_{\beta_k} w_k^{-1} w_k r_{\beta_k} \lambda = w_k\lambda.$$

Therefore, the right-hand side of equation (4.2.3) is identical to

$$w_k\lambda + \left(\sum_{u=1}^{p-1} \langle \Lambda,\, \mathrm{wt}(\mathbf{d}_u) \rangle \right) \delta + \left(\sum_{l \in [1,\, k]_{\mathsf{q}}} \langle \Lambda,\, \beta_l^{\vee} \rangle \right) \delta.$$

If $w_{k-1} \xleftarrow{\beta_k} w_k$ is a quantum edge, then we have $[1,\, k]_{\mathsf{q}} = [1,\, k-1]_{\mathsf{q}} \cup \bigl\{k\bigr\}$. Also, since $\xi_k = w_k\beta_k + \delta$, it follows that

$$\begin{aligned} r_{\xi_k} w_{k-1}\lambda &= r_{w_k\beta_k + \delta} w_{k-1}\lambda = r_{w_k\beta_k} t_{w_k\beta_k^{\vee}} w_{k-1}\lambda = \underbrace{r_{w_k\beta_k} w_{k-1}\lambda}_{= w_k\lambda \text{ as above}} - \langle w_{k-1}\Lambda,\, w_k\beta_k^{\vee} \rangle \delta \\ &= w_k\lambda - \langle \lfloor w_k r_{\beta_k} \rfloor \Lambda,\, w_k\beta_k^{\vee} \rangle \delta = w_k\lambda - \langle w_k r_{\beta_k} \Lambda,\, w_k\beta_k^{\vee} \rangle \delta \\ &= w_k\lambda + \langle \Lambda,\, \beta_k^{\vee} \rangle \delta. \end{aligned}$$

Therefore, the right-hand side of equation (4.2.3) is identical to

$$\begin{aligned} & w_k\lambda + \langle \Lambda,\, \beta_k^{\vee} \rangle \delta + \left(\sum_{u=1}^{p-1} \langle \Lambda,\, \mathrm{wt}(\mathbf{d}_u) \rangle \right) \delta + \left(\sum_{l \in [1,\, k-1]_{\mathsf{q}}} \langle \Lambda,\, \beta_l^{\vee} \rangle \right) \delta \\ &\qquad = w_k\lambda + \left(\sum_{u=1}^{p-1} \langle \Lambda,\, \mathrm{wt}(\mathbf{d}_u) \rangle \right) \delta + \left(\sum_{l \in [1,\, k]_{\mathsf{q}}} \langle \Lambda,\, \beta_l^{\vee} \rangle \right) \delta. \end{aligned}$$

This proves equation (4.2.2). In particular, for $k = n$, we obtain

$$\widetilde{\mu}_n = r_{\xi_n} \cdots r_{\xi_2} r_{\xi_1} \widetilde{\nu}_p = w_n \lambda + \left(\sum_{u=1}^{p-1} \langle \Lambda,\ \mathrm{wt}(\mathbf{d}_u) \rangle \right) \delta + \left(\sum_{l \in [1,\,n]_{\mathfrak{q}}} \langle \Lambda,\ \beta_l^{\vee} \rangle \right) \delta$$

$$= x_{p+1}\lambda + \left(\sum_{u=1}^{p-1} \langle \Lambda,\ \mathrm{wt}(\mathbf{d}_u) \rangle \right) \delta + \langle \Lambda,\ \mathrm{wt}(\mathbf{d}_p) \rangle \delta$$

$$= x_{p+1}\lambda + \left(\sum_{u=1}^{p} \langle \Lambda,\ \mathrm{wt}(\mathbf{d}_u) \rangle \right) \delta = \widetilde{\nu}_{p+1}$$

by the definition (4.1.1) of $\widetilde{\nu}_{p+1}$. Since $\widetilde{\nu}_p \in W\lambda$ by our induction hypothesis, we deduce that $\widetilde{\nu}_{p+1} \in W\lambda$, as desired. Also, by Proposition 3.1.3 (1), we see that for $1 \le p \le s-1$,

$$\widetilde{\nu}_p = \widetilde{\mu}_0 \gtrdot \widetilde{\mu}_1 \gtrdot \widetilde{\mu}_2 \gtrdot \cdots \gtrdot \widetilde{\mu}_n = \widetilde{\nu}_{p+1},$$

where $\widetilde{\mu}_k = r_{\xi_k} \widetilde{\mu}_{k-1}$ for $1 \le k \le n$ by the definitions. Moreover, since $\mathbf{d}_p$ is a directed σ_p-path, it follows from Remark 3.2.2 that the sequence above is a σ_p-chain for $(\widetilde{\nu}_p,\ \widetilde{\nu}_{p+1})$. Thus we conclude that $\widetilde{\pi}_\eta \in \mathbb{B}(\lambda)$.

Because $\widetilde{\pi}_\eta \in \mathbb{B}(\lambda)$ as shown above, and because $\mathrm{cl}(\widetilde{\pi}_\eta) = \eta$ and $\widetilde{\nu}_1 = x_1\lambda \in W_0\lambda \subset \lambda - Q_0^+$ by the definitions, the element $\widetilde{\pi}_\eta$ satisfies conditions (1) and (3) of Proposition 2.3.1. Therefore, we deduce from Lemma 2.3.2 that $\widetilde{\pi}_\eta(1)$ is of the form:

$$\widetilde{\pi}_\eta(1) = \lambda - \beta + \bigl(-\mathrm{Deg}(\eta) + L \bigr) \delta$$

for some $\beta \in Q_0^+$ and $L \in \mathbb{Z}_{\ge 0}$. By Lemma 2.3.2, in order to prove that $\widetilde{\pi}_\eta = \pi_\eta$, it suffices to show that $L = 0$, or equivalently, $-\mathrm{Deg}(\eta) + L \le -\mathrm{Deg}(\eta)$ since $L \in \mathbb{Z}_{\ge 0}$. By using (2.2.1), we see from the definition of $\widetilde{\pi}_\eta$ that

$$-\mathrm{Deg}(\eta) + L = \sum_{p=0}^{s-1} (\sigma_{p+1} - \sigma_p) \left(\sum_{u=1}^{p} \langle \Lambda,\ \mathrm{wt}(\mathbf{d}_u) \rangle \right). \tag{4.2.4}$$

Now, if we write π_η as

$$\pi_\eta = (\nu_1,\ \nu_2,\ \ldots,\ \nu_b\,;\ \tau_0,\ \tau_1,\ \ldots,\ \tau_b),$$

then we have $\nu_q \in \lambda - Q_0^+ + K_q\delta$ for some $K_q \in \mathbb{Z}$, $1 \le q \le b$ (see Remark 2.1.2 (2)); observe that $K_1 = 0$ by the definition of π_η (see Proposition 2.3.1 (3)), and that $0 = K_1 \le K_2 \le \cdots \le K_b$ by Remark 2.2.2. Since $\mathrm{cl}(\pi_\eta) = \eta$, we deduce that there exist $0 = c_0 < c_1 < c_2 < \cdots < c_s = b$ such that $\tau_{c_p} = \sigma_p$ for $0 \le p \le s$, and hence π_η can be written as:

$$\Bigl(\underbrace{\nu_1,\ \ldots,\ \nu_{c_1}}_{\substack{\text{mapped to}\\ x_1\lambda \text{ by cl}}},\ \underbrace{\nu_{c_1+1},\ \ldots,\ \nu_{c_2}}_{\substack{\text{mapped to}\\ x_2\lambda \text{ by cl}}},\ \ldots,\ \underbrace{\nu_{c_{s-1}+1},\ \ldots,\ \nu_{c_s} = \nu_b}_{\text{mapped to } x_s\lambda \text{ by cl}}\,;$$

$$\underbrace{0 = \tau_0}_{=\sigma_0},\ \tau_1,\ \ldots,\ \underbrace{\tau_{c_1}}_{=\sigma_1},\ \tau_{c_1+1},\ \ldots,\ \underbrace{\tau_{c_2}}_{=\sigma_2},\ \ldots,\ \tau_{c_{s-1}+1},\ \ldots,\ \underbrace{\tau_{c_s} = \tau_b = 1}_{=\sigma_s} \Bigr).$$

From this, we compute

$$
\begin{aligned}
-\operatorname{Deg}(\eta) &= \sum_{q=1}^{b}(\tau_q - \tau_{q-1})K_q = \sum_{p=0}^{s-1}\sum_{q=c_p+1}^{c_{p+1}}(\tau_q - \tau_{q-1})K_q \\
&\geq \sum_{p=0}^{s-1}\sum_{q=c_p+1}^{c_{p+1}}(\tau_q - \tau_{q-1})K_{c_p+1} \quad \text{since } K_q \geq K_{c_p+1} \text{ for all } c_p+1 \leq q \leq c_{p+1} \\
&= \sum_{p=0}^{s-1}(\tau_{c_{p+1}} - \tau_{c_p})K_{c_p+1} = \sum_{p=0}^{s-1}(\sigma_{p+1} - \sigma_p)K_{c_p+1}.
\end{aligned} \tag{4.2.5}
$$

Therefore, by (4.2.4) and (4.2.5), in order to show the inequality $-\operatorname{Deg}(\eta) + L \leq -\operatorname{Deg}(\eta)$, it suffices to show that

$$
K_{c_p+1} \geq \sum_{u=1}^{p}\langle \Lambda,\, \mathrm{wt}(\mathbf{d}_u)\rangle \quad \text{for all } 0 \leq p \leq s-1. \tag{4.2.6}
$$

We show this inequality by induction on p. If $p = 0$, then the assertion is obvious since $K_{c_p+1} = K_1 = 0$ as seen above. Assume that $s-1 \geq p > 0$. Take $\mu_0, \mu_1, \ldots, \mu_m \in W\lambda$ such that

$$
\nu_{c_p} = \mu_0 \gtrdot \mu_1 \gtrdot \cdots \gtrdot \mu_m = \nu_{c_p+1}
$$

(for example, take a τ_{c_p}-chain for $(\nu_{c_p},\, \nu_{c_p+1})$), and let $\zeta_k \in \Delta_{\mathrm{re}}^{+}$ be the positive real root such that $\mu_k = r_{\zeta_k}\mu_{k-1}$, $1 \leq k \leq m$. For each $0 \leq k \leq m$, let $v_k \in W_0^J$ be a unique element in W_0^J such that $\mathrm{cl}(\mu_k) = v_k\Lambda$; remark that $v_0 = x_p$ and $v_m = x_{p+1}$. By repeated application of Proposition 3.1.3 (2), we obtain a directed path (not shortest in general)

$$
\mathbf{d} : x_p = v_0 \xleftarrow{\gamma_1} v_1 \xleftarrow{\gamma_2} v_2 \xleftarrow{\gamma_3} \cdots \xleftarrow{\gamma_m} v_m = x_{p+1}
$$

from x_{p+1} to x_p in the parabolic quantum Bruhat graph, where $\gamma_k \in \Delta_0^{+} \setminus \Delta_{0,J}^{+}$ for $1 \leq k \leq m$ are defined by

$$
\gamma_k := \begin{cases} v_k^{-1}\zeta_k & \text{if } \zeta_k \in \Delta_0^{+}, \\ v_k^{-1}(\zeta_k - \delta) & \text{if } \zeta_k \in \{-\gamma + \delta \mid \gamma \in \Delta_0^{+}\}; \end{cases}
$$

recall that $v_{k-1} \xleftarrow{\gamma_k} v_k$ is a Bruhat edge if and only if $\zeta_k \in \Delta_0^{+}$. By the same argument as for equation (4.2.2), we can show that for $0 \leq k \leq m$,

$$
\mu_k = r_{\zeta_k}\cdots r_{\zeta_2}r_{\zeta_1}\nu_{c_p} = v_k\lambda + K_{c_p}\delta + \left(\sum_{l}\langle \Lambda,\, \gamma_l^{\vee}\rangle\right)\delta,
$$

where the summation above is over all $1 \leq l \leq k$ for which $v_{l-1} = \lfloor v_l r_{\gamma_l}\rfloor \xleftarrow{\gamma_l} v_l$ is a quantum edge. In particular, for $k = m$, we obtain

$$
\nu_{c_p+1} = \mu_m = x_{p+1}\lambda + K_{c_p}\delta + \langle \Lambda,\, \mathrm{wt}(\mathbf{d})\rangle\delta,
$$

and hence $K_{c_p+1} = K_{c_p} + \langle \Lambda, \mathrm{wt}(\mathbf{d}) \rangle$. Here we see from [LNS^{3}1, Proposition 8.1] that $\langle \Lambda, \mathrm{wt}(\mathbf{d}) \rangle \geq \langle \Lambda, \mathrm{wt}(\mathbf{d}_p) \rangle$. Also, by the induction hypothesis (note that $c_{p-1} < c_p$),

$$K_{c_p} \geq K_{c_{p-1}+1} \geq \sum_{u=1}^{p-1} \bigl\langle \Lambda, \mathrm{wt}(\mathbf{d}_u) \bigr\rangle.$$

Combining these, we obtain

$$K_{c_p+1} = K_{c_p} + \langle \Lambda, \mathrm{wt}(\mathbf{d}) \rangle \geq \sum_{u=1}^{p-1} \bigl\langle \Lambda, \mathrm{wt}(\mathbf{d}_u) \bigr\rangle + \langle \Lambda, \mathrm{wt}(\mathbf{d}_p) \rangle = \sum_{u=1}^{p} \bigl\langle \Lambda, \mathrm{wt}(\mathbf{d}_u) \bigr\rangle.$$

Thus, we have proved the inequality $-\mathrm{Deg}(\eta) + L \leq -\mathrm{Deg}(\eta)$, and hence the equality $\widetilde{\pi}_\eta = \pi_\eta$, as desired.

Finally, from equation (4.2.4) together with $L = 0$ shown above, we deduce that

$$\begin{aligned} -\mathrm{Deg}(\eta) &= \sum_{p=0}^{s-1} (\sigma_{p+1} - \sigma_p) \left(\sum_{u=1}^{p} \bigl\langle \Lambda, \mathrm{wt}(\mathbf{d}_u) \bigr\rangle \right) = \sum_{p=1}^{s-1} \sum_{u=1}^{p} (\sigma_{p+1} - \sigma_p) \bigl\langle \Lambda, \mathrm{wt}(\mathbf{d}_u) \bigr\rangle \\ &= \sum_{p=1}^{s-1} \left\{ \sum_{q=p}^{s-1} (\sigma_{q+1} - \sigma_q) \right\} \bigl\langle \Lambda, \mathrm{wt}(\mathbf{d}_p) \bigr\rangle = \sum_{p=1}^{s-1} (\sigma_s - \sigma_p) \bigl\langle \Lambda, \mathrm{wt}(\mathbf{d}_p) \bigr\rangle \\ &= \sum_{p=1}^{s-1} (1 - \sigma_p) \bigl\langle \Lambda, \mathrm{wt}(\mathbf{d}_p) \bigr\rangle. \end{aligned}$$

Thus we have proved formula (4.1.2). This completes the proof of Theorem 4.1.1. □

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