A UNIFORM MODEL FOR KIRILLOV–RESHETIKHIN CRYSTALS III: NONSYMMETRIC MACDONALD POLYNOMIALS AT $t = 0$ AND DEMAZURE CHARACTERS

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Abstract. We establish the equality of the specialization $E_{w\lambda}(x\,;\,q,\,0)$ of the nonsymmetric Macdonald polynomial $E_{w\lambda}(x\,;\,q,\,t)$ at $t=0$ with the graded character $\operatorname{gch} U_w^+(\lambda)$ of a certain Demazure-type submodule $U_w^+(\lambda)$ of a tensor product of "single-column" Kirillov–Reshetikhin modules for an untwisted affine Lie algebra, where λ is a dominant integral weight and w is a (finite) Weyl group element; this generalizes our previous result, that is, the equality between the specialization $P_\lambda(x\,;\,q,\,0)$ of the symmetric Macdonald polynomial $P_\lambda(x\,;\,q,\,t)$ at $t=0$ and the graded character of a tensor product of single-column Kirillov–Reshetikhin modules. We also give two combinatorial formulas for the mentioned specialization of a nonsymmetric Macdonald polynomial: in terms of quantum Lakshmibai-Seshadri paths and the quantum alcove model.

1. Introduction.

In our previous paper [LNS^{3}2], we proved that the specialization $P_\lambda(x\,;\,q,\,0)$ of the symmetric Macdonald polynomial $P_\lambda(x\,;\,q,\,t)$ at $t=0$ is identical to the graded character of a certain tensor product of Kirillov–Reshetikhin (KR for short) modules of one-column type for an untwisted affine Lie algebra $\mathfrak{g}_{\mathrm{af}}$, where λ is a dominant integral weight for the finite-dimensional simple Lie algebra $\mathfrak{g} \subset \mathfrak{g}_{\mathrm{af}}$. The purpose of this paper is to generalize this result to the specialization $E_{w\lambda}(x\,;\,q,\,0)$ of the nonsymmetric Macdonald polynomial $E_{w\lambda}(x\,;\,q,\,t)$ at $t=0$, where w is an element of the (finite) Weyl group W of $\mathfrak{g}$; note that if w is the longest element $w_\circ$ of W, then $E_{w_\circ\lambda}(x\,;\,q,\,0) = P_\lambda(x\,;\,q,\,0)$.

Let us explain our result more precisely. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra (over $\mathbb{C}$), with X its integral weight lattice, and $\mathfrak{g}_{\mathrm{af}}$ the associated untwisted affine Lie algebra. We denote by $\{\alpha_i\}_{i\in I}$ and $\{\alpha_i^\vee\}_{i\in I}$ the simple roots and simple coroots of $\mathfrak{g}$, respectively, and by ϖ_i, $i \in I$, the fundamental weights for $\mathfrak{g}$. For a dominant integral weight $\lambda = \sum_{i\in I} m_i\varpi_i \in X$ with $m_i \in \mathbb{Z}_{\geq 0}$, let $\mathrm{QLS}(\lambda)$ denote the crystal of quantum Lakshmibai-Seshadri (QLS for short) paths of shape λ; for details, see Definition 2.4 below. Then we know from [LNS^{3}2] that the crystal $\mathrm{QLS}(\lambda)$ provides a realization of the crystal basis of the tensor product $\bigotimes_{i\in I} W(\varpi_i)^{\otimes m_i}$ of the level-zero fundamental representations $W(\varpi_i)$, $i \in I$, of the quantum affine algebra $U_q'(\mathfrak{g}_{\mathrm{af}})$ associated to $\mathfrak{g}_{\mathrm{af}}$. The main result of [LNS^{3}2] states that the specialization $P_\lambda(x\,;\,q,\,0)$ of the symmetric Macdonald polynomial at $t=0$ is identical to the graded character of the crystal $\mathrm{QLS}(\lambda)$, where the grading on $\mathrm{QLS}(\lambda)$ is given by the degree function, or equivalently, by the (global) energy function.

Let $W = \langle r_i \mid i \in I\rangle$ denote the (finite) Weyl group of $\mathfrak{g}$, and set $W_J := \langle r_i \mid i \in J\rangle \subset W$, where $J := \{i \in I \mid \langle \alpha_i^\vee,\,\lambda\rangle = 0\}$. Also, let W^J denote the set of minimal(-length) coset representatives for the cosets in W/W_J; for $w \in W$, we denote by $\lfloor w \rfloor = \lfloor w \rfloor^J \in W^J$ the minimal coset representative for the coset wW_J in W/W_J. Now, for $w \in W^J$, we set

$$\mathrm{QLS}_w(\lambda) := \{\eta \in \mathrm{QLS}(\lambda) \mid \iota(\eta) \leq w\},$$

where for a QLS path $\eta = (x_1,\,\ldots,\,x_s\,;\,\sigma_0,\,\sigma_1,\,\ldots,\,\sigma_s) \in \mathrm{QLS}(\lambda)$, we define the initial direction $\iota(\eta)$ of η to be $x_1 \in W^J$; here the symbol $\leq$ is used to denote the Bruhat order on W. Furthermore,

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we define the graded character $\operatorname{gch} \mathrm{QLS}_w(\lambda)$ of $\mathrm{QLS}_w(\lambda) \subset \mathrm{QLS}(\lambda)$ by

$$\operatorname{gch} \mathrm{QLS}_w(\lambda) := \sum_{\eta \in \mathrm{QLS}_w(\lambda)} q^{-\operatorname{Deg}(\eta)} e^{\mathrm{wt}(\eta)},$$

where $\mathrm{wt} : \mathrm{QLS}(\lambda) \to X$ and $\operatorname{Deg} : \mathrm{QLS}(\lambda) \to \mathbb{Z}_{\leq 0}$ denote the weight function and the degree function on $\mathrm{QLS}(\lambda)$, respectively; for the definitions, see (2.7) and (2.9) below. Now, the main result of this paper is as follows.

Theorem 1.1. *For each $w \in W^J$, the equality*

$$\operatorname{gch} \mathrm{QLS}_w(\lambda) = E_{w\lambda}(x\,;\, q,\, 0)$$

holds, where $E_{w\lambda}(x\,;\, q,\, 0)$ denotes the specialization of the nonsymmetric Macdonald polynomial $E_{w\lambda}(x\,;\, q,\, t)$ at $t = 0$.

We should mention that this result generalizes [LNS^{3}2, Proposition 7.8], since it holds that $\mathrm{QLS}_{\lfloor w_\circ \rfloor}(\lambda) = \mathrm{QLS}(\lambda)$ and $E_{\lfloor w_\circ \rfloor \lambda}(x\,;\, q,\, 0) = P_\lambda(x\,;\, q,\, 0)$, where $w_\circ \in W$ denotes the longest element. On the other hand, in Theorems 2.28 and 2.30, we express $E_{w\lambda}(x\,;\, q,\, 0)$ in terms of the so-called quantum alcove model [LL1].

In the following, we explain the representation-theoretic meaning of Theorem 1.1; see §3 for details. Let $V(\lambda)$ denote the extremal weight module of extremal weight λ over the quantum affine algebra $U_q(\mathfrak{g}_{\mathrm{af}})$ associated to $\mathfrak{g}_{\mathrm{af}}$, and set $V_w^+(\lambda) := U_q^+(\mathfrak{g}_{\mathrm{af}}) S_w^{\mathrm{norm}} v_\lambda \subset V(\lambda)$ for $w \in W$, which is the Demazure submodule generated by the extremal weight vector $S_w^{\mathrm{norm}} v_\lambda \in V(\lambda)$ of weight $w\lambda$ over the positive part $U_q^+(\mathfrak{g}_{\mathrm{af}})$ of $U_q(\mathfrak{g}_{\mathrm{af}})$; note that $V_w^+(\lambda) \subset V_{w_\circ}^+(\lambda)$ for all $w \in W$. For $w \in W$, we define $U_w^+(\lambda)$ to be the image of $V_w^+(\lambda)$ under the canonical projection $V_{w_\circ}^+(\lambda) \twoheadrightarrow V_{w_\circ}^+(\lambda)/Z_{w_\circ}^+(\lambda)$; for the definition of $Z_{w_\circ}^+(\lambda)$, see §3.3. Then, $U_{w_\circ}^+(\lambda)$ is isomorphic, as a $U_q(\mathfrak{g})$-module, to the tensor product $\bigotimes_{i \in I} W(\varpi_i)^{\otimes m_i}$ of level-zero fundamental representations $W(\varpi_i)$, $i \in I$; note that this is not an isomorphism of $U_q^+(\mathfrak{g}_{\mathrm{af}})$-modules. Because the module $V_w^+(\lambda)$ is generated by the extremal weight vector $S_w^{\mathrm{norm}} v_\lambda \in V(\lambda)$ over $U_q^+(\mathfrak{g}_{\mathrm{af}})$, it follows that the module $U_w^+(\lambda) \subset U_{w_\circ}^+(\lambda)$ is also generated by the image of $S_w^{\mathrm{norm}} v_\lambda$ over $U_q^+(\mathfrak{g}_{\mathrm{af}})$. Thus, in a sense, we can think of $U_w^+(\lambda) \subset U_{w_\circ}^+(\lambda)$ as a Demazure-type submodule of $U_{w_\circ}^+(\lambda)$, which is isomorphic as a $U_q(\mathfrak{g})$-module to $\bigotimes_{i \in I} W(\varpi_i)^{\otimes m_i}$. Also, if we define the graded character $\operatorname{gch} U_w^+(\lambda)$ of $U_w^+(\lambda)$ by

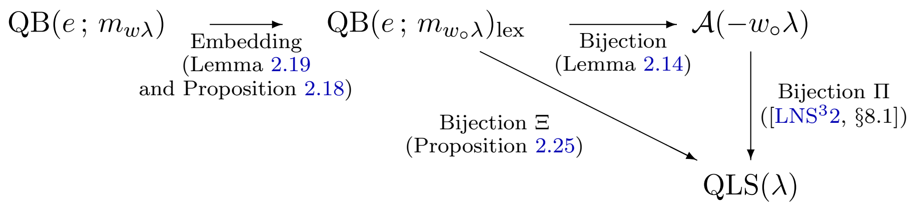
$$\operatorname{gch} U_w^+(\lambda) := \sum_{\gamma \in Q,\, k \in \mathbb{Z}} \dim U_w^+(\lambda)_{\lambda - \gamma + k\delta}\, x^{\lambda - \gamma} q^k,$$

where $Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is the root lattice for $\mathfrak{g}$, δ denotes the null root of $\mathfrak{g}_{\mathrm{af}}$, and $q := x^\delta$, then we have (see Theorem 3.3)

$$\operatorname{gch} U_w^+(\lambda) = \operatorname{gch} \mathrm{QLS}_w(\lambda) \overset{\text{Theorem 1.1}}{=} E_{w\lambda}(x\,;\, q,\, 0).$$

In §2, we give a bijective proof of Theorem 1.1 by making use of the Orr-Shimozono formula for the specialization at $t = 0$ of nonsymmetric Macdonald polynomials [OS]. The outline of our proof is as follows. In §2.3, we briefly review the Orr-Shimozono formula (see Theorem 2.8), which expresses the specialization $E_\mu(x\,;\, q,\, 0)$ of the nonsymmetric Macdonald polynomial $E_\mu(x\,;\, q,\, t)$ at $t = 0$ in terms of the set $\mathrm{QB}(e\,;\, m_\mu)$ of quantum alcove paths from e to m_μ for an integral weight μ, where m_μ denotes the element of the (extended) affine Weyl group that is of minimum length in the coset $t_\mu W$, with t_μ the translation by μ. Next, for a dominant integral weight $\lambda \in X$, we show in Lemma 2.14 that there exists a canonical bijection between the particular set $\mathrm{QB}(e\,;\, m_{w_\circ \lambda})_{\mathrm{lex}}$ and the set $\mathcal{A}(-w_\circ \lambda)$; here, $\mathrm{QB}(e\,;\, m_{w_\circ \lambda})_{\mathrm{lex}}$ is defined by using a specific reduced expression for $m_{w_\circ \lambda} = t_{w_\circ \lambda}$ corresponding to a lexicographic $(-w_\circ \lambda)$-chain of roots. Also, we give an explicit bijection $\Xi : \mathrm{QB}(e\,;\, m_{w_\circ \lambda})_{\mathrm{lex}} \to \mathrm{QLS}(\lambda)$ in such a way that the diagram below is commutative (see

Proposition 2.25). Furthermore, in Lemma 2.19 combined with Proposition 2.18, we show that there exists a natural embedding $\mathrm{QB}(e\,;m_{w\lambda}) \hookrightarrow \mathrm{QB}(e\,;m_{w_\circ\lambda})_{\mathrm{lex}}$ for an arbitrary $w \in W^J$.



Finally, in Proposition 2.21 and Lemma 2.26, we show that the image of $\mathrm{QB}(e\,;m_{w\lambda})$ under the composite of the maps $\mathrm{QB}(e\,;m_{w\lambda}) \hookrightarrow \mathrm{QB}(e\,;m_{w_\circ\lambda})_{\mathrm{lex}} \xrightarrow{\Xi} \mathrm{QLS}(\lambda)$ is identical to $\mathrm{QLS}_w(\lambda)$; we also show in Proposition 2.18, Lemma 2.19, and Proposition 2.25 that both of the embedding $\mathrm{QB}(e\,;m_{w\lambda}) \hookrightarrow \mathrm{QB}(e\,;m_{w_\circ\lambda})_{\mathrm{lex}}$ and the bijection $\Xi : \mathrm{QB}(e\,;m_{w_\circ\lambda}) \to \mathrm{QLS}(\lambda)$ preserve "weights" and "degrees". This implies that the graded character of $\mathrm{QLS}_w(\lambda)$ is identical to that of $\mathrm{QB}(e\,;m_{w\lambda})$. Because we know from the Orr-Shimozono formula that the graded character of $\mathrm{QB}(e\,;m_{w\lambda})$ is identical to $E_{w\lambda}(x\,;q,\,0)$, we conclude from the above that the graded character of $\mathrm{QLS}_w(\lambda)$ is identical to $E_{w\lambda}(x\,;q,\,0)$.

In Appendix A.1, using the crystal structure on the set $\mathrm{QLS}(\lambda)$, we obtain a recursive formula (see Proposition A.1) for the graded characters $\mathrm{gch}\,\mathrm{QLS}_w(\lambda)$, $w \in W^J$, which is described in terms of Demazure operators. Here we note that in view of Theorem 1.1 above, this recursive formula is equivalent to the one (see Proposition A.4) for nonsymmetric Macdonald polynomials $E_{w\lambda}(x\,;q,\,0)$, $w \in W^J$, specialized at $t = 0$; in Appendix A.2, we provide a sketch of how to derive this recursive formula for $E_{w\lambda}(x\,;q,\,0)$ by using the polynomial representation of the double affine Hecke algebra.

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2. Proof of Theorem 1.1.

2.1. **Setting.** Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra (over $\mathbb{C}$). We denote by $\{\alpha_i\}_{i\in I}$ and $\{\alpha_i^\vee\}_{i\in I}$ the simple roots and simple coroots of $\mathfrak{g}$, respectively, and by ϖ_i, $i \in I$, the fundamental weights for $\mathfrak{g}$; we set

$$Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i, \qquad Q^\vee := \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee, \quad \text{and} \quad X := \bigoplus_{i\in I} \mathbb{Z}\varpi_i.$$

Let Φ^+ (resp., $\Phi^{\vee+}$) denote the set of positive roots (resp., coroots), and Φ^- (resp., $\Phi^{\vee-}$) the set of negative roots (resp., coroots). We set $\rho := (1/2)\sum_{\alpha\in\Phi^+}\alpha$. Let $W = \langle r_i \mid i \in I\rangle$ be the (finite) Weyl group of $\mathfrak{g}$, with length function $\ell : W \to \mathbb{Z}_{\ge 0}$. We denote by $w_\circ \in W$ the longest element, and by $e \in W$ the identity element. Also, let us denote by $\omega : I \to I$ the Dynkin diagram automorphism given by: $w_\circ\alpha_i = -\alpha_{\omega(i)}$ for $i \in I$.

For a subset $J \subset I$, we set

$$\Phi_J^+ := \Phi^+ \cap \Big(\bigoplus_{i\in J} \mathbb{Z}\alpha_i\Big), \qquad \rho_J := \frac{1}{2}\sum_{\alpha\in\Phi_J^+} \alpha,$$

$$\Phi_J^{\vee+} := \Phi^{\vee+} \cap \Big(\bigoplus_{i\in J} \mathbb{Z}\alpha_i^\vee\Big), \qquad W_J := \langle r_i \mid i \in J\rangle \subset W;$$

let $w_{J,\circ}$ denote the longest element of W_J. Also, let W^J denote the set of minimal(-length) coset representatives for the cosets in W/W_J; recall that

$$W^J = \{w \in W \mid w\alpha \in \Phi^+ \text{ for all } \alpha \in \Phi_J^+\}, \tag{2.1}$$

$$\ell(wz) = \ell(w) + \ell(z) \qquad \text{for all } w \in W^J \text{ and } z \in W_J. \tag{2.2}$$

For $w \in W$, we denote by $\lfloor w \rfloor = \lfloor w \rfloor^J \in W^J$ the minimal coset representative for the coset wW_J in W/W_J. We use the symbol $\le$ for the Bruhat order on the Weyl group W.

2.2. **Quantum Lakshmibai-Seshadri paths.** In this subsection, we recall the definition of quantum Lakshmibai-Seshadri paths from [LNS[3]2, §3].

Definition 2.1. Let J be a subset of I. The (parabolic) quantum Bruhat graph $\mathrm{QB}(W^J)$ is the $(\Phi^+ \setminus \Phi_J^+)$-labeled, directed graph with vertex set W^J and $(\Phi^+ \setminus \Phi_J^+)$-labeled, directed edges of the following form: $w \xrightarrow{\beta} \lfloor wr_\beta \rfloor$ for $w \in W^J$ and $\beta \in \Phi^+ \setminus \Phi_J^+$, where either
 (i) $\ell(\lfloor wr_\beta \rfloor) = \ell(w) + 1$, or
 (ii) $\ell(\lfloor wr_\beta \rfloor) = \ell(w) - 2\langle \beta^\vee, \rho - \rho_J \rangle + 1$;
if (i) holds (resp., (ii) holds), then the edge is called a Bruhat edge (resp., a quantum edge). If J is the empty set $\emptyset$, then we simply write $\mathrm{QB}(W^J) = \mathrm{QB}(W^\emptyset)$ as $\mathrm{QB}(W)$.

Remark 2.2. (1) We have $\langle \beta^\vee, \rho - \rho_J \rangle > 0$ for all $\beta \in \Phi^+ \setminus \Phi_J^+$. Indeed, since $\langle \alpha_i^\vee, \alpha \rangle \le 0$ for all $i \in I \setminus J$ and $\alpha \in \Phi_J^+$, we see that $\langle \alpha_i^\vee, \rho_J \rangle \le 0$ for all $i \in I \setminus J$, and hence $\langle \alpha_i^\vee, \rho - \rho_J \rangle > 0$ for all $i \in I \setminus J$. Also, we have $\langle \alpha_i^\vee, \rho - \rho_J \rangle = 1 - 1 = 0$ for all $i \in J$. Therefore, $\langle \beta^\vee, \rho - \rho_J \rangle > 0$ for all $\beta \in \Phi^+ \setminus \Phi_J^+$. As a consequence, if $w \xrightarrow{\beta} \lfloor wr_\beta \rfloor$ is a quantum edge, then $\ell(\lfloor wr_\beta \rfloor) < \ell(w)$.

(2) If $w \xrightarrow{\beta} \lfloor wr_\beta \rfloor$ is a Bruhat edge, then $wr_\beta \in W^J$, and hence $\lfloor wr_\beta \rfloor = wr_\beta$ (see [LNS[3]3, Remark 3.1.2]).

(3) Let $x, y \in W^J$ be such that $x \le y$ in the Bruhat order on W. If

$$x = x_0 \xrightarrow{\beta_1} x_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_k} x_k = y \tag{2.3}$$

is a shortest directed path from x to y in $\mathrm{QB}(W^J)$, then all of its edges are Bruhat edges. Indeed, by Definition 2.1 (for Bruhat edges) and part (1) of this remark (for quantum edges), we have

$$\ell(y) - \ell(x) = \sum_{q=1}^{k} (\underbrace{\ell(x_q) - \ell(x_{q-1})}_{=1 \text{ or } <0}) \le \sum_{q=1}^{k} 1 = k; \tag{2.4}$$

note that the equality holds if and only if $\ell(x_q) - \ell(x_{q-1}) = 1$ for all $1 \le q \le k$, or equivalently, all the edges are Bruhat edges. Since $x \le y$ by the assumption, we deduce from the chain property (see [BB, Theorem 2.5.5]) that there exists a directed path from x to y in $\mathrm{QB}(W^J)$ whose edges are all Bruhat edges; the length of this directed path is equal to $\ell(y) - \ell(x)$. Therefore, we obtain $k \le \ell(y) - \ell(x)$ since the directed path (2.3) is a shortest one. Combining this inequality and (2.4), we obtain $k = \ell(y) - \ell(x)$, and hence all the edges in the shortest directed path (2.3) are Bruhat edges.

Now, we fix a dominant integral weight $\lambda \in X$ for $\mathfrak{g}$, and set

$$J = J_\lambda := \{i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0\} \subset I.$$

As above, we simply write $\lfloor w \rfloor^J = \lfloor w \rfloor^{J_\lambda} \in W^J$ for $w \in W$ as: $\lfloor w \rfloor$, unless stated otherwise explicitly.

Definition 2.3. For a given rational number σ, we define $\mathrm{QB}_{\sigma\lambda}(W^J)$ to be the subgraph of the parabolic quantum Bruhat graph $\mathrm{QB}(W^J)$ with the same vertex set but having only the edges:

$$w \xrightarrow{\beta} \lfloor wr_\beta \rfloor \quad \text{with} \quad \langle \beta^\vee, \sigma\lambda \rangle = \sigma\langle \beta^\vee, \lambda \rangle \in \mathbb{Z}.$$

Definition 2.4. A quantum Lakshmibai-Seshadri (QLS for short) path of shape λ is a pair

$$\eta = (x_1, x_2, \ldots, x_s\,;\ \sigma_0, \sigma_1, \ldots, \sigma_s) \tag{2.5}$$

of a sequence $x_1, x_2, \ldots, x_s$ of elements in W^J with $x_u \neq x_{u+1}$ for $1 \le u \le s-1$ and a sequence $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$ of rational numbers satisfying the condition that there exists a directed path from x_{u+1} to x_u in $\mathrm{QB}_{\sigma_u\lambda}(W^J)$ for each $1 \le u \le s-1$; we denote this $x_u \xLeftarrow{\sigma_u\lambda} x_{u+1}$. Let $\mathrm{QLS}(\lambda)$ denote the set of all QLS paths of shape λ.

Remark 2.5. We identify $\eta \in \mathrm{QLS}(\lambda)$ of the form (2.5) with the following piecewise-linear, continuous map $\eta : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} X$:

$$\eta(t) = \sum_{p=1}^{u-1} (\sigma_p - \sigma_{p-1}) x_p \lambda + (t - \sigma_{u-1}) x_u \lambda \quad \text{for } \sigma_{u-1} \le t \le \sigma_u,\ 1 \le u \le s. \tag{2.6}$$

In [LNS³2, Theorem 3.3], we proved that $\mathrm{QLS}(\lambda)$ is identical (as a set of piecewise-linear, continuous maps from $[0,1]$ to $\mathbb{R} \otimes_{\mathbb{Z}} X$) to the set $\mathbb{B}(\lambda)_{\mathrm{cl}}$ of "projected" Lakshmibai-Seshadri paths of shape λ; for the definition of $\mathbb{B}(\lambda)_{\mathrm{cl}}$, see [LNS³2, §2.2].

Let $\eta = (x_1, \ldots, x_s\,;\, \sigma_0, \sigma_1, \ldots, \sigma_s) \in \mathrm{QLS}(\lambda)$. We define the weight $\mathrm{wt}(\eta)$ of $\eta \in \mathrm{QLS}(\lambda)$ by

$$\mathrm{wt}(\eta) := \eta(1) = \sum_{u=1}^{s} (\sigma_u - \sigma_{u-1}) x_u \lambda; \tag{2.7}$$

we can show in exactly the same way as [L2, Lemma 4.5 a)] that $\mathrm{wt}(\eta) \in X$. Also, we define the degree $\mathrm{Deg}(\eta)$ as follows (see [LNS³2, §4.2 and Theorem 4.5]). First, let $x, y \in W^J$, and let

$$x = y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_k} y_k = y$$

be a shortest directed path from x to y in $\mathrm{QB}(W^J)$. Then we set

$$\mathrm{wt}_\lambda(x \Rightarrow y) := \sum_{\substack{1 \le p \le k \\ y_{p-1} \xrightarrow{\beta_p} y_p \text{ is a quantum edge}}} \langle \beta_p^\vee, \lambda \rangle \in \mathbb{Z}_{\ge 0}; \tag{2.8}$$

we see from [LNS³2, Proposition 4.1] that this value does not depend on the choice of a shortest directed path from x to y in $\mathrm{QB}(W^J)$. For $\eta = (x_1, \ldots, x_s\,;\, \sigma_0, \sigma_1, \ldots, \sigma_s) \in \mathrm{QLS}(\lambda)$, we define

$$\mathrm{Deg}(\eta) := -\sum_{u=1}^{s-1} (1 - \sigma_u)\, \mathrm{wt}_\lambda(x_{u+1} \Rightarrow x_u) \in \mathbb{Z}_{\le 0}. \tag{2.9}$$

For $\eta = (x_1, \ldots, x_s\,;\, \sigma_0, \sigma_1, \ldots, \sigma_s) \in \mathrm{QLS}(\lambda)$, we set $\iota(\eta) := x_1 \in W^J$, and call it the initial direction of η. Now, for each $w \in W^J$, we set

$$\mathrm{QLS}_w(\lambda) := \bigl\{ \eta \in \mathrm{QLS}(\lambda) \mid \iota(\eta) \le w \bigr\}, \tag{2.10}$$

and define the graded character $\mathrm{gch}\,\mathrm{QLS}_w(\lambda)$ of $\mathrm{QLS}_w(\lambda) \subset \mathrm{QLS}(\lambda)$ by

$$\mathrm{gch}\,\mathrm{QLS}_w(\lambda) := \sum_{\eta \in \mathrm{QLS}_w(\lambda)} q^{-\mathrm{Deg}(\eta)} e^{\mathrm{wt}(\eta)}.$$

We will prove that for each $w \in W^J$, the equality

$$\mathrm{gch}\,\mathrm{QLS}_w(\lambda) = E_{w\lambda}(x\,;\, q, 0) \tag{2.11}$$

holds, where $E_{w\lambda}(x\,;\, q, 0)$ denotes the specialization of the nonsymmetric Macdonald polynomial $E_{w\lambda}(x\,;\, q, t)$ at $t = 0$.

2.3. **Orr-Shimozono formula.** In this subsection, we review a formula ([OS, Corollary 4.4]) for the specialization at $t = 0$ of nonsymmetric Macdonald polynomials.

Let $\widetilde{\mathfrak{g}}$ denote the dual Lie algebra of $\mathfrak{g}$, and let $\{\widetilde{\alpha}_i\}_{i\in I}$ and $\{\widetilde{\alpha}_i^\vee\}_{i\in I}$ be the simple roots and the simple coroots of $\widetilde{\mathfrak{g}}$, respectively. We denote by $\widetilde{W}$ the Weyl group of $\widetilde{\mathfrak{g}}$; note that $W \cong \widetilde{W}$. As is well-known, for $w \in W \cong \widetilde{W}$ and $i \in I$,

$$w\widetilde{\alpha}_i = \sum_{j\in I} c_j \widetilde{\alpha}_j \quad \text{if and only if} \quad w\alpha_i^\vee = \sum_{j\in I} c_j \alpha_j^\vee. \tag{2.12}$$

Hence we identify $w\widetilde{\alpha}_i$ with $w\alpha_i^\vee$ for $w \in W \cong \widetilde{W}$ and $i \in I$:

$$w\widetilde{\alpha}_i \overset{\text{identify}}{\longleftrightarrow} w\alpha_i^\vee. \tag{2.13}$$

Let $\widetilde{\Phi}^+$ denote the set of positive roots of $\widetilde{\mathfrak{g}}$, which we identify with the set $\Phi^{\vee+}$ of positive coroots of $\mathfrak{g}$ by (2.13).

Now, let $\widetilde{\mathfrak{g}}_{\mathrm{af}}$ denote the untwisted affine Lie algebra associated to $\widetilde{\mathfrak{g}}$. Let $\{\widetilde{\alpha}_i\}_{i\in I_{\mathrm{af}}}$ be the simple roots of $\widetilde{\mathfrak{g}}_{\mathrm{af}}$, where $I_{\mathrm{af}} = I \sqcup \{0\}$, and $\widetilde{\delta}$ the null root of $\widetilde{\mathfrak{g}}_{\mathrm{af}}$. We denote by $\widetilde{\Phi}^{\mathrm{af}+}$ (resp., $\widetilde{\Phi}^{\mathrm{af}-}$) the set of positive (resp., negative) real roots of $\widetilde{\mathfrak{g}}_{\mathrm{af}}$; note that

$$\widetilde{\Phi}^{\mathrm{af}+} = \big(\underbrace{\mathbb{Z}_{\geq 0}\widetilde{\delta} + \widetilde{\Phi}^+}_{\substack{\text{identified with} \\ \mathbb{Z}_{\geq 0}\widetilde{\delta}+\Phi^{\vee+}}}\big) \sqcup \big(\underbrace{\mathbb{Z}_{>0}\widetilde{\delta} - \widetilde{\Phi}^+}_{\substack{\text{identified with} \\ \mathbb{Z}_{>0}\widetilde{\delta}-\Phi^{\vee+}}}\big).$$

Denote by $\widetilde{W}_{\mathrm{af}}$ the Weyl group of $\widetilde{\mathfrak{g}}_{\mathrm{af}}$; note that $\widetilde{W}_{\mathrm{af}} \cong Q \rtimes \widetilde{W} \cong Q \rtimes W$. Also, we denote by $\widetilde{W}_{\mathrm{ext}} := X \rtimes \widetilde{W} \cong X \rtimes W$ the extended affine Weyl group of $\widetilde{\mathfrak{g}}_{\mathrm{af}}$, and by $t_\mu \in \widetilde{W}_{\mathrm{ext}}$ the translation by $\mu \in X$. For $x \in \widetilde{W}_{\mathrm{ext}}$, define $\mathrm{wt}(x) \in X$ and $\mathrm{dir}(x) \in W$ by:

$$x = t_{\mathrm{wt}(x)} \, \mathrm{dir}(x).$$

For an integral weight $\mu \in X$ for $\mathfrak{g}$, we set

$$m_\mu := t_\mu v(\mu)^{-1} \in X \rtimes W \cong \widetilde{W}_{\mathrm{ext}},$$

where $v(\mu)$ denotes the shortest element in W such that $v(\mu)\mu$ is an antidominant integral weight (see [OS, (2.45)]). The following lemma will be used later.

Lemma 2.6. *Let $\lambda \in X$ be a dominant integral weight, and let $w \in W^J$, where $J = J_\lambda = \{i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0\}$. Then, $v(w\lambda) = \lfloor w_\circ \rfloor w^{-1}$, and hence*

$$m_{w\lambda} = t_{w\lambda}(\lfloor w_\circ \rfloor w^{-1})^{-1} = w(\lfloor w_\circ \rfloor)^{-1} t_{w_\circ\lambda}.$$

In particular,

$$\begin{cases} v(w_\circ\lambda) = v(\lfloor w_\circ \rfloor \lambda) = e, & m_{w_\circ\lambda} = t_{w_\circ\lambda}, \\ v(\lambda) = \lfloor w_\circ \rfloor, & m_\lambda = (\lfloor w_\circ \rfloor)^{-1} t_{w_\circ\lambda}, \end{cases}$$

and $m_{w\lambda} = w m_\lambda$.

Proof. It is obvious that $(\lfloor w_\circ \rfloor w^{-1}) w\lambda = w_\circ\lambda$ is antidominant. Hence it suffices to show that $\ell(x) \geq \ell(\lfloor w_\circ \rfloor w^{-1})$ for all $x \in W$ such that $xw\lambda = w_\circ\lambda$. If $xw\lambda = w_\circ\lambda$, then $w_\circ xw \in W_J$, and hence $x = w_\circ z w^{-1}$ for some $z \in W_J$; note that $\ell(zw^{-1}) = \ell(wz^{-1}) = \ell(w) + \ell(z^{-1})$ since $w \in W^J$ and $z \in W_J$. Therefore,

$$\ell(x) = \ell(w_\circ) - \ell(zw^{-1}) = \ell(w_\circ) - \ell(w) - \ell(z^{-1}).$$

Here we remark that $\lfloor w_\circ \rfloor = w_\circ w_{J,\circ}$, where $w_{J,\circ} \in W_J$ is the longest element. Hence it follows from the computation above (with z replaced by $w_{J,\circ}$) that

$$\ell(\lfloor w_\circ \rfloor w^{-1}) = \ell(w_\circ w_{J,\circ} w^{-1}) = \ell(w_\circ) - \ell(w) - \ell(w_{J,\circ}^{-1}).$$

Since $\ell(z^{-1}) \le \ell(w_{J,\circ}^{-1})$, we obtain $\ell(x) \ge \ell(\lfloor w_\circ \rfloor w^{-1})$, as desired. □

We fix an arbitrary $\mu \in X$, and apply the argument in [OS, §3.3] to the case that $u = e$ (the identity element) and $w = m_\mu$; we generally follow the notation thereof. Let

$$m_\mu = \pi \underbrace{r_{i_1} r_{i_2} \cdots r_{i_\ell}}_{\in \widetilde{W}_{\mathrm{af}}} \tag{2.14}$$

be a reduced expression for m_μ, where π is an (affine) Dynkin diagram automorphism of $\widetilde{\mathfrak{g}}_{\mathrm{af}}$, and set

$$\beta_k^{\mathsf{OS}} := r_{i_\ell} \cdots r_{i_{k+1}} \widetilde{\alpha}_{i_k} \quad \text{for } 1 \le k \le \ell, \tag{2.15}$$

which is a positive real root of $\widetilde{\mathfrak{g}}_{\mathrm{af}}$ contained in $\mathbb{Z}_{>0}\widetilde{\delta} - \widetilde{\Phi}^+$ (see [OS, Remark 3.17]). Then we can write β_k^{OS} as:

$$\beta_k^{\mathsf{OS}} = a_k \widetilde{\delta} + \overline{\beta_k^{\mathsf{OS}}} \quad \text{for } a_k \in \mathbb{Z}_{>0} \text{ and } \overline{\beta_k^{\mathsf{OS}}} \in \widetilde{\Phi}^-, \quad 1 \le k \le \ell; \tag{2.16}$$

we think of $\overline{\beta_k^{\mathsf{OS}}}$ as an element of $\Phi^{\vee -}$ under the identification (2.13) of $\widetilde{\Phi}^+$ and $\Phi^{\vee +}$, and set $\gamma_k^{\mathsf{OS}} := -(\overline{\beta_k^{\mathsf{OS}}})^\vee \in \Phi^+$.

Let $A = \{j_1 < j_2 < \cdots < j_r\}$ be a subset of $\{1, 2, \ldots, \ell\}$. Following [OS, (3.16) and (3.17)] (recall that $u = e$ and $w = m_\mu$), we set

$$z_0 := m_\mu, \qquad z_k := z_{k-1} r_{\beta_{j_k}^{\mathsf{OS}}} \quad \text{for } 1 \le k \le r;$$

or equivalently, $z_0 = m_\mu$, and z_k is obtained from the reduced expression (2.14) by removing the j_1-th reflection, the j_2-th reflection, …, and the j_k-th reflection. We express these data as:

$$p_A = \Big(z_0 \xrightarrow{\beta_{j_1}^{\mathsf{OS}}} z_1 \xrightarrow{\beta_{j_2}^{\mathsf{OS}}} \cdots \xrightarrow{\beta_{j_r}^{\mathsf{OS}}} z_r \Big). \tag{2.17}$$

Definition 2.7 ([OS, §4.2])**.** Keep the notation and setting above. We say that p_A is an element of $\mathrm{QB}(e\,;\,m_\mu)$ if

$$\mathrm{dir}(z_0) \xrightarrow{\gamma_{j_1}^{\mathsf{OS}}} \mathrm{dir}(z_1) \xrightarrow{\gamma_{j_2}^{\mathsf{OS}}} \cdots \xrightarrow{\gamma_{j_r}^{\mathsf{OS}}} \mathrm{dir}(z_r)$$

is a directed path in the quantum Bruhat graph $\mathrm{QB}(W) = \mathrm{QB}(W^{\emptyset})$ for W.

For an element $p_A \in \mathrm{QB}(e\,;\,m_\mu)$, we set (see [OS, (3.19)])

$$A^- := \Big\{ j_k \in A \mid \mathrm{dir}(z_{k-1}) \xrightarrow{\gamma_{j_k}^{\mathsf{OS}}} \mathrm{dir}(z_k) \text{ is a quantum edge} \Big\} \subset A, \tag{2.18}$$

and then set (see [OS, (4.1)])

$$\mathrm{qwt}(p_A) := \sum_{j \in A^-} \beta_j^{\mathsf{OS}}, \tag{2.19}$$

which is contained in $\mathbb{Z}_{>0}\widetilde{\delta} - \widetilde{Q}^+$ if $A^- \neq \emptyset$, where $\widetilde{Q}^+ := \sum_{i \in I} \mathbb{Z}_{\ge 0} \widetilde{\alpha}_i$. Furthermore, in view of equation (2.16), we set (in the notation of [OS, (2.4)])

$$\deg(\mathrm{qwt}(p_A)) := \sum_{j \in A^-} a_j \in \mathbb{Z}_{\ge 0}. \tag{2.20}$$

Also, if $p_A \in \mathrm{QB}(e\,;\,m_\mu)$ is of the form (2.17), then we set

$$\mathrm{end}(p_A) := z_r \in \widetilde{W}_{\mathrm{ext}} = X \rtimes W \quad \text{and} \quad \mathrm{wt}(p_A) := \mathrm{wt}(\mathrm{end}(p_A)). \tag{2.21}$$

Theorem 2.8 ([OS, Corollary 4.4])**.** *Keep the notation and setting above. We have*

$$E_\mu(x\,;\,q,\,0) = \sum_{p \in \mathrm{QB}(e\,;\,m_\mu)} e^{\mathrm{wt}(p)} q^{\deg(\mathrm{qwt}(p))}.$$

2.4. **Bijective correspondence between** $\mathrm{QB}(e\,;\,m_{w_\circ\lambda})$ **and** $\mathcal{A}(-w_\circ\lambda)$**.** First, we recall the quantum alcove model from [LL1] (see also [LNS^{3}2, §5.1]). We set $H_{\alpha,n} := \{\zeta \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \alpha^\vee, \zeta \rangle = n\}$ for $\alpha \in \Phi$ and $n \in \mathbb{Z}$, where $\mathfrak{h}^*_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} X = \bigoplus_{i\in I} \mathbb{R}\alpha_i$. An alcove is, by definition, a connected component (with respect to the usual topology on $\mathfrak{h}^*_{\mathbb{R}}$) of

$$\mathfrak{h}^*_{\mathbb{R}} \setminus \bigcup_{\alpha\in\Phi^+,\, n\in\mathbb{Z}} H_{\alpha,n}.$$

We say that two alcoves are adjacent if they are distinct and have a common wall. For adjacent alcoves A and B, we write $A \xrightarrow{\alpha} B$, with $\alpha \in \Phi$, if their common wall is contained in the hyperplane $H_{\alpha,n}$ for some $n \in \mathbb{Z}$, and if α points in the direction from A to B. An alcove path is a sequence of alcoves $(A_0, A_1, \dots, A_s)$ such that A_{u-1} and A_u are adjacent for each $u = 1, 2, \dots, s$. We say that $(A_0, A_1, \dots, A_s)$ is reduced if it has minimal length among all alcove paths from A_0 to A_s.

Recall that $\widetilde{W}_{\mathrm{ext}} \cong X \rtimes W$ acts (as affine transformations) on $\mathfrak{h}^*_{\mathbb{R}}$ by

$$(t_\xi w)\cdot\zeta = w\zeta + \xi \qquad \text{for } \xi \in X,\ w \in W, \text{ and } \zeta \in \mathfrak{h}^*_{\mathbb{R}}.$$

Remark 2.9. For $\beta = \alpha^\vee + n\widetilde{\delta} \in \widetilde{\Phi}^{\mathrm{af}+}$ with $\alpha \in \Phi^+$ and $n \in \mathbb{Z}_{\geq 0}$ (here we identify $\widetilde{\Phi}^+$ with $\Phi^{\vee+}$ under (2.13)), we have $r_{\alpha^\vee+n\widetilde{\delta}}\cdot\zeta = (t_{-n\alpha}r_{\alpha^\vee})\cdot\zeta = r_{\alpha^\vee}\zeta - n\alpha = r_\alpha\zeta - n\alpha$ for $\zeta \in \mathfrak{h}^*_{\mathbb{R}}$. Hence $r_{\alpha^\vee+n\widetilde{\delta}} \in \widetilde{W}_{\mathrm{ext}}$ acts on $\mathfrak{h}^*_{\mathbb{R}}$ as the affine reflection with respect to the hyperplane $H_{\alpha,-n} = H_{-\alpha,n}$.

Now, let $\lambda \in X$ be a dominant integral weight; note that $w_\circ\lambda \in X$ is antidominant, where $w_\circ \in W$ denotes the longest element. We set

$$A_\circ := \{\zeta \in \mathfrak{h}^*_{\mathbb{R}} \mid 0 < \langle \alpha^\vee, \zeta \rangle < 1 \text{ for all } \alpha \in \Phi^+\},$$

and $A_{w_\circ\lambda} := A_\circ + w_\circ\lambda$.

Definition 2.10. The sequence of roots $(\gamma_1, \gamma_2, \dots, \gamma_\ell)$ is called a $(-w_\circ\lambda)$-chain of roots if

$$A_\circ = A_0 \xrightarrow{-\gamma_1} A_1 \xrightarrow{-\gamma_2} \cdots \xrightarrow{-\gamma_\ell} A_\ell = A_{w_\circ\lambda}$$

is a reduced alcove path.

Here we note that $m_{w_\circ\lambda} = t_{w_\circ\lambda}$ by Lemma 2.6. It follows from [LP1, Lemma 5.3] that there exists a bijection:

$$\{\text{reduced expressions for } m_{w_\circ\lambda} = t_{w_\circ\lambda}\} \quad \stackrel{1:1}{\longleftrightarrow} \quad \{(-w_\circ\lambda)\text{-chains of roots}\}. \tag{2.22}$$

More precisely, let $m_{w_\circ\lambda} = t_{w_\circ\lambda} = \pi r_{i_1} r_{i_2} \cdots r_{i_\ell}$ be a reduced expression for $m_{w_\circ\lambda} = t_{w_\circ\lambda} \in \widetilde{W}_{\mathrm{ext}}$. We set $A_k := (\pi r_{i_1} r_{i_2} \cdots r_{i_k})\cdot A_\circ$ for $0 \leq k \leq \ell$, and

$$\beta^{\mathsf{L}}_k := \pi r_{i_1} \cdots r_{i_{k-1}}(\widetilde{\alpha}_{i_k}) = r_{\pi(i_1)} \cdots r_{\pi(i_{k-1})}(\widetilde{\alpha}_{\pi(i_k)}) \quad \text{for } 1 \leq k \leq \ell; \tag{2.23}$$

note that β^{L}_k is a positive real root of $\widetilde{\mathfrak{g}}_{\mathrm{af}}$ contained in $\mathbb{Z}_{\geq 0}\widetilde{\delta} + \widetilde{\Phi}^+$. In fact, by [M, (2.4.7)], we have

$$\begin{aligned}\{\beta^{\mathsf{L}}_k \mid 1 \leq k \leq \ell\} &= \widetilde{\Phi}^{\mathrm{af}+} \cap m_{w_\circ\lambda}\widetilde{\Phi}^{\mathrm{af}-} = \widetilde{\Phi}^{\mathrm{af}+} \cap t_{w_\circ\lambda}\widetilde{\Phi}^{\mathrm{af}-} \\ &= \{b\widetilde{\delta} + \beta^\vee \mid \beta \in \Phi^+,\ 0 \leq b < -\langle \beta^\vee, w_\circ\lambda \rangle\}\end{aligned} \tag{2.24}$$

under the identification (2.13) of $\widetilde{\Phi}^+$ and $\Phi^{\vee+}$. Therefore, we can write β^{L}_k in the form

$$\beta^{\mathsf{L}}_k = b_k\widetilde{\delta} + \overline{\beta^{\mathsf{L}}_k}, \quad \text{with } b_k \in \mathbb{Z}_{\geq 0} \text{ and } \overline{\beta^{\mathsf{L}}_k} \in -w_\circ(\Phi^{\vee+} \setminus \Phi^{\vee+}_J), \tag{2.25}$$

for each $1 \leq k \leq \ell$. If we set $\gamma^{\mathsf{L}}_k := (\overline{\beta^{\mathsf{L}}_k})^\vee \in -w_\circ(\Phi^+ \setminus \Phi^+_J)$, then

$$A_\circ = A_0 \xrightarrow{-\gamma^{\mathsf{L}}_1} A_1 \xrightarrow{-\gamma^{\mathsf{L}}_2} \cdots \xrightarrow{-\gamma^{\mathsf{L}}_\ell} A_\ell = A_{w_\circ\lambda} \tag{2.26}$$

is a $(-w_\circ\lambda)$-chain of roots.

Remark 2.11 (see [LNS^{3}2, §6.1]). Let $1 \le k \le \ell$. We see from Remark 2.9 that the action of $r_{\beta_k^{\mathsf{L}}} \in \widetilde{W}_{\mathrm{af}}$ on $\mathfrak{h}_{\mathbb{R}}^*$ is the affine reflection with respect to the hyperplane $H_{\gamma_k^{\mathsf{L}}, -b_k}$. Also, we know that

$$0 \le b_k = \#\{1 \le p < k \mid \gamma_p^{\mathsf{L}} = \gamma_k^{\mathsf{L}}\} < \langle \overline{\beta_k^{\mathsf{L}}}, -w_\circ\lambda \rangle; \tag{2.27}$$

the sequence $(b_1, \dots, b_\ell)$ is called the height sequence for the $(-w_\circ\lambda)$-chain (2.26).

Remark 2.12. Keep the notation and setting above. If we define β_k^{OS}, $1 \le k \le \ell$, by (2.15) for the reduced expression $m_{w_\circ\lambda} = t_{w_\circ\lambda} = \pi r_{i_1} r_{i_2} \cdots r_{i_\ell}$, then we have $\beta_k^{\mathsf{L}} = -t_{w_\circ\lambda}(\beta_k^{\mathsf{OS}})$ for all $1 \le k \le \ell$. In particular, $\overline{\beta_k^{\mathsf{L}}} = -\overline{\beta_k^{\mathsf{OS}}}$ (see (2.16) and (2.25)), and hence $\gamma_k^{\mathsf{L}} = \gamma_k^{\mathsf{OS}} =: \gamma_k$. Also, we have $b_k = \langle \gamma_k^\vee, -w_\circ\lambda \rangle - a_k$.

Now, let

$$A_\circ = A_0 \xrightarrow{-\gamma_1} A_1 \xrightarrow{-\gamma_2} \cdots \xrightarrow{-\gamma_\ell} A_\ell = A_{w_\circ\lambda} \tag{2.28}$$

be a $(-w_\circ\lambda)$-chain of roots.

Definition 2.13. Let $\mathcal{A}(-w_\circ\lambda)$ denote the set of all subsets $A = \{j_1 < \cdots < j_r\}$ of $\{1, 2, \dots, \ell\}$ such that

$$e \xrightarrow{\gamma_{j_1}} r_{\gamma_{j_1}} \xrightarrow{\gamma_{j_2}} r_{\gamma_{j_1}} r_{\gamma_{j_2}} \xrightarrow{\gamma_{j_3}} \cdots \xrightarrow{\gamma_{j_r}} r_{\gamma_{j_1}} r_{\gamma_{j_2}} \cdots r_{\gamma_{j_r}} =: \phi(A) \tag{2.29}$$

is a directed path in the quantum Bruhat graph $\mathrm{QB}(W)$ for W. The subsets A are called admissible subsets, and $\phi(A)$ is called the final direction of A.

For $A = \{j_1 < \cdots < j_r\} \in \mathcal{A}(-w_\circ\lambda)$, we define $\mathrm{wt}(A) \in X$, $\mathrm{height}(A) \in \mathbb{Z}_{\ge 0}$ (see [LNS^{3}2, Definition 5.1 and (7.1)]), and $\mathrm{coheight}(A) \in \mathbb{Z}_{\ge 0}$ as follows:

$$\begin{aligned} \mathrm{wt}(A) &:= -r_{\beta_{j_1}^{\mathsf{L}}} r_{\beta_{j_2}^{\mathsf{L}}} \cdots r_{\beta_{j_r}^{\mathsf{L}}} \cdot (w_\circ\lambda) \\ &= -r_{\gamma_{j_1}^{\mathsf{L}}, -b_{j_1}} r_{\gamma_{j_2}^{\mathsf{L}}, -b_{j_2}} \cdots r_{\gamma_{j_r}^{\mathsf{L}}, -b_{j_r}} \cdot (w_\circ\lambda), \end{aligned} \tag{2.30}$$

$$\mathrm{height}(A) := \sum_{j \in A_-} \Big(\langle (\gamma_j^{\mathsf{L}})^\vee, -w_\circ\lambda \rangle - b_j \Big), \tag{2.31}$$

$$\mathrm{coheight}(A) := \sum_{j \in A_-} b_j, \tag{2.32}$$

where

$$A_- := \{ j_k \in A \mid \ r_{\gamma_{j_1}} \cdots r_{\gamma_{j_{k-1}}} \xrightarrow{\gamma_{j_k}^{\mathsf{L}}} r_{\gamma_{j_1}} \cdots r_{\gamma_{j_{k-1}}} r_{\gamma_{j_k}} \text{ is a quantum edge} \}. \tag{2.33}$$

Let $m_{w_\circ\lambda} = t_{w_\circ\lambda} = \pi r_{i_1} r_{i_2} \cdots r_{i_\ell}$ be the reduced expression for $m_{w_\circ\lambda} = t_{w_\circ\lambda}$ corresponding to the $(-w_\circ\lambda)$-chain of roots (2.28) under the correspondence (2.22). We define $\mathrm{QB}(e\,; m_{w_\circ\lambda})$ by using this reduced expression for $m_{w_\circ\lambda} = t_{w_\circ\lambda}$. Note that

$$\gamma_k = \gamma_k^{\mathsf{L}} = \gamma_k^{\mathsf{OS}} \qquad \text{for } 1 \le k \le \ell. \tag{2.34}$$

Lemma 2.14. *Keep the notation and setting above. Then,*

$$A \in \mathcal{A}(-w_\circ\lambda) \quad \text{if and only if} \quad p_A \in \mathrm{QB}(e\,; m_{w_\circ\lambda}).$$

Hence we have a bijection from $\mathcal{A}(-w_\circ\lambda)$ *onto* $\mathrm{QB}(e\,; m_{w_\circ\lambda})$ *that maps* $A \in \mathcal{A}(-w_\circ\lambda)$ *to* $p_A \in \mathrm{QB}(e\,; m_{w_\circ\lambda})$. *Moreover, we have*

$$\mathrm{wt}(A) = -\,\mathrm{wt}(p_A) \quad \text{and} \quad \mathrm{height}(A) = \deg(\mathrm{qwt}(p_A)) \quad \text{for all } A \in \mathcal{A}(-w_\circ\lambda). \tag{2.35}$$

Proof. Let $A = \{j_1 < \cdots < j_r\}$. Then, we have

$$p_A \in \mathrm{QB}(e\,;\, m_{w_\circ\lambda}) \iff \underbrace{\mathrm{dir}(z_0)}_{=e} \xrightarrow{\gamma_{j_1}^{\mathsf{OS}}} \mathrm{dir}(z_1) \xrightarrow{\gamma_{j_2}^{\mathsf{OS}}} \cdots \xrightarrow{\gamma_{j_r}^{\mathsf{OS}}} \mathrm{dir}(z_r) \quad \text{in } \mathrm{QB}(W)$$

$$\iff e \xrightarrow{\gamma_{j_1}} r_{\gamma_{j_1}} \xrightarrow{\gamma_{j_2}} \cdots \xrightarrow{\gamma_{j_r}} r_{\gamma_{j_1}} r_{\gamma_{j_2}} \cdots r_{\gamma_{j_r}} \quad \text{in } \mathrm{QB}(W) \text{ by (2.34)}$$
$$\iff A \in \mathcal{A}(-w_\circ\lambda).$$

Next, we prove that $\mathrm{height}(A) = \deg(\mathrm{qwt}(p_A))$ for all $A \in \mathcal{A}(-w_\circ\lambda)$. Let $A = \{j_1 < \cdots < j_r\} \in \mathcal{A}(-w_\circ\lambda)$; we see from the argument above that the set A^- in (2.18) is identical to the set A_- in (2.33). Then, we see that

$$\mathrm{height}(A) = \sum_{j\in A_-} \Big(\langle(\gamma_j^{\mathsf{L}})^\vee,\, -w_\circ\lambda\rangle - b_j\Big) \quad \text{by definition (2.31)}$$

$$= \sum_{j\in A^-} \Big(\underbrace{\langle\gamma_j^\vee,\, -w_\circ\lambda\rangle - b_j}_{=a_j}\Big) \quad \text{by Remark 2.12}$$

$$= \sum_{j\in A^-} a_j = \deg(\mathrm{qwt}(p_A)) \quad \text{by (2.20)}.$$

Finally, we show that $\mathrm{wt}(A) = -\mathrm{wt}(p_A)$ for all $A \in \mathcal{A}(-w_\circ\lambda)$; we proceed by induction on the cardinality of $A \in \mathcal{A}(-w_\circ\lambda)$. First, observe that this equality is obvious if $A = \emptyset$. Now, let us take $A = \{j_1 < \cdots < j_{r-1} < j_r\} \in \mathcal{A}(-w_\circ\lambda)$, and set $A' := \{j_1 < \cdots < j_{r-1}\}$, which is also an element of $\mathcal{A}(-w_\circ\lambda)$. By direct computation, together with definition (2.30), we can show that

$$\mathrm{wt}(A) = \mathrm{wt}(A') - \big(\langle\gamma_{j_r}^\vee,\, -w_\circ\lambda\rangle - b_{j_r}\big) r_{\gamma_{j_1}} \cdots r_{\gamma_{j_{r-1}}}(\gamma_{j_r}); \tag{2.36}$$

or, we may refer the reader to the proof of [LNS[3]2, Proposition 6.7]. Also, we have

$$z_r = z_{r-1} r_{\beta_{j_r}^{\mathsf{OS}}} = z_{r-1} r_{a_{j_r}\widetilde{\delta} + \overline{\beta_{j_r}^{\mathsf{OS}}}} = z_{r-1}\big(t_{-a_{j_r}(\overline{\beta_{j_r}^{\mathsf{OS}}})^\vee} r_{\overline{\beta_{j_r}^{\mathsf{OS}}}}\big) = z_{r-1} t_{a_{j_r}\gamma_{j_r}} r_{\gamma_{j_r}}.$$

Therefore, if we write $z_r = t_{\mathrm{wt}(z_r)}\,\mathrm{dir}(z_r)$ and $z_{r-1} = t_{\mathrm{wt}(z_{r-1})}\,\mathrm{dir}(z_{r-1})$, then we deduce that

$$\begin{aligned} t_{\mathrm{wt}(z_r)}\,\mathrm{dir}(z_r) &= t_{\mathrm{wt}(z_{r-1})}\,\mathrm{dir}(z_{r-1}) t_{a_{j_r}\gamma_{j_r}} r_{\gamma_{j_r}} = t_{\mathrm{wt}(z_{r-1})} t_{a_{j_r}\,\mathrm{dir}(z_{r-1})\gamma_{j_r}}\,\mathrm{dir}(z_{r-1}) r_{\gamma_{j_r}} \\ &= t_{\mathrm{wt}(z_{r-1}) + a_{j_r}\,\mathrm{dir}(z_{r-1})\gamma_{j_r}}\big(\mathrm{dir}(z_{r-1}) r_{\gamma_{j_r}}\big), \end{aligned}$$

and hence

$$\mathrm{wt}(p_A) = \mathrm{wt}(z_r) = \mathrm{wt}(z_{r-1}) + a_{j_r}\,\mathrm{dir}(z_{r-1})\gamma_{j_r}.$$

Here, since $a_{j_r} = \langle\gamma_{j_r}^\vee,\, -w_\circ\lambda\rangle - b_{j_r}$ by Remark 2.12, we obtain

$$\begin{aligned} \mathrm{wt}(p_A) &= \mathrm{wt}(z_{r-1}) + \big(\langle\gamma_{j_r}^\vee,\, -w_\circ\lambda\rangle - b_{j_r}\big)\,\mathrm{dir}(z_{r-1})\gamma_{j_r} \\ &= \mathrm{wt}(p_{A'}) + \big(\langle\gamma_{j_r}^\vee,\, -w_\circ\lambda\rangle - b_{j_r}\big)\,\mathrm{dir}(z_{r-1})\gamma_{j_r}; \end{aligned}$$

note that $\mathrm{dir}(z_{r-1}) = r_{\gamma_{j_1}} \cdots r_{\gamma_{j_{r-1}}}$ since $\mathrm{dir}(z_0) = \mathrm{dir}(m_{w_\circ\lambda}) = e$. Hence it follows that

$$\begin{aligned} \mathrm{wt}(p_A) &= \mathrm{wt}(p_{A'}) + \big(\langle\gamma_{j_r}^\vee,\, -w_\circ\lambda\rangle - b_{j_r}\big) r_{\gamma_{j_1}} \cdots r_{\gamma_{j_{r-1}}}(\gamma_{j_r}) \\ &= -\mathrm{wt}(A') + \big(\langle\gamma_{j_r}^\vee,\, -w_\circ\lambda\rangle - b_{j_r}\big) r_{\gamma_{j_1}} \cdots r_{\gamma_{j_{r-1}}}(\gamma_{j_r}) \\ &\qquad\qquad \text{by our induction hypothesis} \\ &= -\mathrm{wt}(A) \quad \text{by (2.36)}, \end{aligned}$$

as desired. This completes the proof of the lemma. □

2.5. **Lexicographic (lex) $(-w_\circ\lambda)$-chains of roots.** We keep the notation and setting of the previous subsection; we fix a dominant integral weight $\lambda \in X$, and set $J = J_\lambda = \{i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0\}$. For $w \in W$, we simply write $\lfloor w \rfloor^J = \lfloor w \rfloor^{J_\lambda} \in W^J$ as $\lfloor w \rfloor$ unless stated otherwise explicitly.

In [LP2, §4] (see also [LNS³2, Proposition 5.4]), the authors introduced a specific $(-w_\circ\lambda)$-chain of roots, called a lexicographic (lex for short) $(-w_\circ\lambda)$-chain of roots. We will frequently make use of the following property of a lex $(-w_\circ\lambda)$-chain of roots: If $m_{w_\circ\lambda} = \pi r_{i_1} r_{i_2} \cdots r_{i_\ell}$ is the reduced expression for $m_{w_\circ\lambda}$ corresponding to a lex $(-w_\circ\lambda)$-chain of roots (recall from (2.22) the one-to-one correspondence between the reduced expressions for $m_{w_\circ\lambda} = t_{w_\circ\lambda}$ and the $(-w_\circ\lambda)$-chains of roots), then we have

$$0 \le \frac{b_1}{\left\langle \overline{\beta_1^{\mathsf{L}}},\, -w_\circ\lambda \right\rangle} \le \frac{b_2}{\left\langle \overline{\beta_2^{\mathsf{L}}},\, -w_\circ\lambda \right\rangle} \le \cdots \le \frac{b_\ell}{\left\langle \overline{\beta_\ell^{\mathsf{L}}},\, -w_\circ\lambda \right\rangle} < 1, \tag{2.37}$$

where $\beta_k^{\mathsf{L}} = b_k\widetilde{\delta} + \overline{\beta_k^{\mathsf{L}}}$ for $1 \le k \le \ell$ is given as in (2.23) and (2.25) (see also Remark 2.11).

We know from Lemma 2.6 that $m_{w_\circ\lambda} = t_{w_\circ\lambda} = \lfloor w_\circ \rfloor (t_\lambda \lfloor w_\circ \rfloor^{-1})$ and $m_\lambda = t_\lambda \lfloor w_\circ \rfloor^{-1}$. It follows that $m_{w_\circ\lambda} = \lfloor w_\circ \rfloor m_\lambda$. Also, since $\ell(t_\lambda) = \ell(m_\lambda \lfloor w_\circ \rfloor) = \ell(m_\lambda) + \ell(\lfloor w_\circ \rfloor)$ by [M, (2.4.5)], we have

$$\ell(m_{w_\circ\lambda}) = \ell(t_{w_\circ\lambda}) = \ell(t_\lambda) = \ell(m_\lambda) + \ell(\lfloor w_\circ \rfloor); \tag{2.38}$$

note that $\ell(t_{w_\circ\lambda}) = \ell(t_\lambda)$ by [M, (2.4.1)]. This implies that the product of a reduced expression for $\lfloor w_\circ \rfloor$ and a reduced expression for m_λ is a reduced expression for $m_{w_\circ\lambda} = t_{w_\circ\lambda}$. We set $M := \ell(\lfloor w_\circ \rfloor)$.

Lemma 2.15. *Let $m_{w_\circ\lambda} = \pi r_{i_1} r_{i_2} \cdots r_{i_\ell}$ be the reduced expression for $m_{w_\circ\lambda}$ corresponding to a lex $(-w_\circ\lambda)$-chain of roots under the correspondence (2.22). Then,*

$$\lfloor w_\circ \rfloor = r_{\pi(i_1)} \cdots r_{\pi(i_M)} \quad \text{and} \quad m_\lambda = \pi r_{i_{M+1}} \cdots r_{i_\ell}.$$

Namely,

$$m_{w_\circ\lambda} = \pi r_{i_1} r_{i_2} \cdots r_{i_\ell} = (\underbrace{r_{\pi(i_1)} \cdots r_{\pi(i_M)}}_{=\lfloor w_\circ \rfloor \ \text{(reduced)}})(\underbrace{\pi r_{i_{M+1}} \cdots r_{i_\ell}}_{=m_\lambda \ \text{(reduced)}}).$$

Proof. We make use of (2.37). Let K be the maximal index such that $b_K / \langle \overline{\beta_K^{\mathsf{L}}}, -w_\circ\lambda \rangle = 0$. Then we see that

$$\left\{\beta_k^{\mathsf{L}} \mid 1 \le k \le \ell\right\} \cap \Big(-w_\circ(\Phi^{\vee+} \setminus \Phi_J^{\vee+})\Big) = \left\{\beta_k^{\mathsf{L}} \mid 1 \le k \le K\right\}. \tag{2.39}$$

Also, we see from (2.24) that $-w_\circ(\Phi^{\vee+} \setminus \Phi_J^{\vee+}) \subset \{\beta_k^{\mathsf{L}} \mid 1 \le k \le \ell\}$. Hence the left-hand side of (2.39) is identical to $-w_\circ(\Phi^{\vee+} \setminus \Phi_J^{\vee+})$. From these, by noting that $\#(\Phi^{\vee+} \setminus \Phi_J^{\vee+}) = \ell(w_\circ) - \ell(w_{J,\circ}) = \ell(\lfloor w_\circ \rfloor) = M$ (recall that $w_{J,\circ}$ is the longest element of W_J), we conclude that $K = M$, and hence that $\{\beta_k^{\mathsf{L}} \mid 1 \le k \le M\} = -w_\circ(\Phi^{\vee+} \setminus \Phi_J^{\vee+})$. In addition, since

$$\beta_k^{\mathsf{L}} = \pi r_{i_1} \cdots r_{i_{k-1}}(\widetilde{\alpha}_{i_k}) = r_{\pi(i_1)} \cdots r_{\pi(i_{k-1})}(\widetilde{\alpha}_{\pi(i_k)}) \in -w_\circ(\Phi^{\vee+} \setminus \Phi_J^{\vee+}) \tag{2.40}$$

for all $1 \le k \le M$, we see easily that $\pi(i_1), \ldots, \pi(i_M) \in I$.

We will show that $v := r_{\pi(i_1)} \cdots r_{\pi(i_M)} \in W$ is identical to $\lfloor w_\circ \rfloor$. By the argument above, we have

$$\left\{\alpha \in \Phi^{\vee+} \mid v^{-1}\alpha \in \Phi^{\vee-}\right\} = \left\{\beta_k^{\mathsf{L}} \mid 1 \le k \le M\right\} = -w_\circ(\Phi^{\vee+} \setminus \Phi_J^{\vee+}).$$

From this, we see that

$$-v^{-1}w_\circ(\Phi^{\vee+} \setminus \Phi_J^{\vee+}) \subset \Phi^{\vee-}, \quad \text{so that} \quad v^{-1}w_\circ(\Phi^{\vee+} \setminus \Phi_J^{\vee+}) \subset \Phi^{\vee+}. \tag{2.41}$$

Hence it follows that $\{\alpha \in \Phi^{\vee+} \mid v^{-1}w_\circ\alpha \in \Phi^{\vee-}\} \subset \Phi_J^{\vee+}$. Since $v = r_{\pi(i_1)} \cdots r_{\pi(i_M)}$ is a reduced expression, we have $\ell(v) = M$, and hence

$$\#\left\{\alpha \in \Phi^{\vee+} \mid v^{-1}w_\circ\alpha \in \Phi^{\vee-}\right\} = \ell(v^{-1}w_\circ) = N - M.$$

Also, we have $\#\Phi_J^{\vee+} = \ell(w_{J,\circ}) = \ell(w_\circ) - \ell(\lfloor w_\circ \rfloor) = N - M$. Therefore, we deduce that

$$\{\alpha \in \Phi^{\vee+} \mid v^{-1}w_\circ\alpha \in \Phi^{\vee-}\} = \Phi_J^{\vee+}. \tag{2.42}$$

Since $w_{J,\circ}(\Phi^{\vee+} \setminus \Phi_J^{\vee+}) \subset \Phi^{\vee+} \setminus \Phi_J^{\vee+}$ and $w_{J,\circ}(\Phi_J^{\vee+}) \subset \Phi_J^{\vee-}$, we have

$$\begin{aligned} &v^{-1}w_\circ w_{J,\circ}(\Phi^{\vee+} \setminus \Phi_J^{\vee+}) \subset v^{-1}w_\circ(\Phi^{\vee+} \setminus \Phi_J^{\vee+}) \subset \Phi^{\vee+} \qquad \text{by (2.41)},\\ &v^{-1}w_\circ w_{J,\circ}(\Phi_J^{\vee+}) \subset v^{-1}w_\circ(\Phi_J^{\vee-}) \subset \Phi_J^{\vee+} \qquad \text{by (2.42)}.\end{aligned}$$

From these, we obtain $v^{-1}w_\circ w_{J,\circ}(\Phi^{\vee+}) \subset \Phi^{\vee+}$, which implies that $v^{-1}w_\circ w_{J,\circ} = e$, and hence that $v = w_\circ w_{J,\circ} = \lfloor w_\circ \rfloor$, as desired. Finally, because $\ell(m_\lambda) = \ell(m_{w_\circ\lambda}) - \ell(\lfloor w_\circ \rfloor) = \ell - M$ and $m_{w_\circ\lambda} = \lfloor w_\circ \rfloor m_\lambda$, it follows that $m_\lambda = \pi r_{i_{M+1}} \cdots r_{i_\ell}$ is a reduced expression for m_λ. This proves Lemma 2.15. □

Fix a lex $(-w_\circ\lambda)$-chain of roots. We construct $\mathrm{QB}(e\,;\, m_{w_\circ\lambda})$ from the reduced expression $m_{w_\circ\lambda} = \pi r_{i_1} r_{i_2} \cdots r_{i_\ell}$ corresponding to the lex $(-w_\circ\lambda)$-chain of roots under (2.22), which we denote by $\mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{\mathrm{lex}}$; recall from (2.15), (2.16), and Remark 2.12 that for $1 \le k \le \ell$,

$$\begin{cases} \beta_k^{\mathsf{OS}} = r_{i_\ell} \cdots r_{i_{k+1}} \widetilde{\alpha}_{i_k} = a_k \widetilde{\delta} + \overline{\beta_k^{\mathsf{OS}}}, \quad \text{with } a_k \in \mathbb{Z}_{>0} \text{ and } \overline{\beta_k^{\mathsf{OS}}} \in \widetilde{\Phi}^-,\\ \gamma_k = \gamma_k^{\mathsf{OS}} = -(\overline{\beta_k^{\mathsf{OS}}})^\vee \in \Phi^+,\\ b_k = \langle \gamma_k^\vee, -w_\circ\lambda \rangle - a_k; \end{cases}$$

We see from (2.40) that

$$\gamma_k = \gamma_k^{\mathsf{L}} = (\beta_k^{\mathsf{L}})^\vee = r_{\pi(i_1)} \cdots r_{\pi(i_{k-1})}(\alpha_{\pi(i_k)}) \qquad \text{for } 1 \le k \le M = \ell(\lfloor w_\circ \rfloor), \tag{2.43}$$

and hence

$$\{\gamma_1, \dots, \gamma_M\} = \{(\beta_1^{\mathsf{L}})^\vee, \dots, (\beta_M^{\mathsf{L}})^\vee\} = -w_\circ(\Phi^+ \setminus \Phi_J^+) = \Phi^+ \setminus \Phi_{\omega(J)}^+, \tag{2.44}$$

where $\omega : I \to I$ is the Dynkin diagram automorphism given by: $w_\circ\alpha_i = -\alpha_{\omega(i)}$ for $i \in I$. Also, it follows from the equality "$K = M$" (shown in the proof of Lemma 2.15), together with (2.37), that

$$0 = \frac{b_1}{\langle \gamma_1^\vee, -w_\circ\lambda \rangle} = \cdots = \frac{b_M}{\langle \gamma_M^\vee, -w_\circ\lambda \rangle} < \frac{b_{M+1}}{\langle \gamma_{M+1}^\vee, -w_\circ\lambda \rangle} \le \cdots \le \frac{b_\ell}{\langle \gamma_\ell^\vee, -w_\circ\lambda \rangle} < 1. \tag{2.45}$$

Now, let $\lfloor w_\circ \rfloor = r_{p_1} r_{p_2} \cdots r_{p_M}$ be an (arbitrary) reduced expression for $\lfloor w_\circ \rfloor$, and set $i'_k := \pi^{-1}(p_k)$ for $1 \le k \le M$. We see from Lemma 2.15 that

$$m_{w_\circ\lambda} = \underbrace{(r_{p_1} \cdots r_{p_M})}_{=\lfloor w_\circ \rfloor} \underbrace{(\pi r_{i_{M+1}} \cdots r_{i_\ell})}_{=m_\lambda} = \pi r_{i'_1} \cdots r_{i'_M} r_{i_{M+1}} \cdots r_{i_\ell} \tag{2.46}$$

is a reduced expression for $m_{w_\circ\lambda}$, which we denote by R. We construct $\mathrm{QB}(e\,;\, m_{w_\circ\lambda})$ from this reduced expression R of $m_{w_\circ\lambda}$, and denote it by $\mathrm{QB}(e\,;\, m_{w_\circ\lambda})_R$. Then,

$$\beta_k^{\mathsf{OS},R} := \begin{cases} \underbrace{r_{i_\ell} \cdots r_{i_{M+1}}}_{=m_\lambda^{-1}\pi} r_{i'_M} \cdots r_{i'_{k+1}} \widetilde{\alpha}_{i'_k} = m_\lambda^{-1} r_{p_M} \cdots r_{p_{k+1}} \widetilde{\alpha}_{p_k} & \text{for } 1 \le k \le M,\\ r_{i_\ell} \cdots r_{i_{k+1}} \widetilde{\alpha}_{i_k} = \beta_k^{\mathsf{OS}} & \text{for } M+1 \le k \le \ell, \end{cases}$$

$$= a_k^R \widetilde{\delta} + \overline{\beta_k^{\mathsf{OS},R}} \quad \text{for some } a_k^R \in \mathbb{Z}_{>0} \text{ and } \overline{\beta_k^{\mathsf{OS},R}} \in \widetilde{\Phi}^-,$$

$$\gamma_k^{\mathsf{OS},R} := -(\overline{\beta_k^{\mathsf{OS},R}})^\vee \in \Phi^+.$$

Also, for the reduced expression R of $m_{w_\circ\lambda}$ in (2.46), we define $\beta_k^{\mathsf{L},R}$, $1 \le k \le \ell$, as in (2.23), and write it as: $\beta_k^{\mathsf{L},R} = b_k^R\widetilde{\delta} + \overline{\beta_k^{\mathsf{L},R}}$, with some $b_k^R \in \mathbb{Z}_{\ge 0}$ and $\overline{\beta_k^{\mathsf{L},R}} \in -w_\circ(\Phi^{\vee+} \setminus \Phi_J^{\vee+})$ (see (2.25)). Then we set $\gamma_k^{\mathsf{L},R} := (\overline{\beta_k^{\mathsf{L},R}})^\vee$ for $1 \le k \le \ell$. By Remark 2.12, we have

$$\gamma_k^{\mathsf{L},R} = \gamma_k^{\mathsf{OS},R} =: \gamma_k^R \quad \text{and} \quad b_k^R = \langle (\gamma_k^R)^\vee, -w_\circ\lambda \rangle - a_k^R \quad \text{for } 1 \le k \le \ell.$$

Notice that $\beta_k^{\mathsf{L},R} = r_{p_1} \cdots r_{p_{k-1}}(\widetilde{\alpha}_{p_k})$ for $1 \le k \le M$. Since $p_1, \ldots, p_M \in I$, we see that $\beta_k^{\mathsf{L},R} \in \Phi^{\vee+}$ for all $1 \le k \le M$, which implies that $b_k^R = 0$ and

$$\gamma_k^R = \gamma_k^{\mathsf{L},R} = (\overline{\beta_k^{\mathsf{L},R}})^\vee = (\beta_k^{\mathsf{L},R})^\vee = r_{p_1} \cdots r_{p_{k-1}}(\alpha_{p_k}) \tag{2.47}$$

for all $1 \le k \le M$.

Lemma 2.16. *Keep the notation and setting above. We have*

$$\{\beta_k^{\mathsf{OS},R} \mid 1 \le k \le M\} = \{\beta_k^{\mathsf{OS}} \mid 1 \le k \le M\}, \tag{2.48}$$

$$\beta_k^{\mathsf{OS},R} = \beta_k^{\mathsf{OS}} \qquad \text{for all } M+1 \le k \le \ell. \tag{2.49}$$

Hence

$$\{\gamma_k^R \mid 1 \le k \le M\} = \{\gamma_k \mid 1 \le k \le M\} = \Phi^+ \setminus \Phi^+_{\omega(J)}, \tag{2.50}$$

$$\gamma_k^R = \gamma_k \quad \text{for all } M+1 \le k \le \ell, \tag{2.51}$$

$$b_k^R = \begin{cases} 0 & \text{for } 1 \le k \le M, \\ b_k > 0 & \text{for } M+1 \le k \le \ell. \end{cases} \tag{2.52}$$

Proof. It is obvious from the definitions that $\beta_k^{\mathsf{OS},R} = \beta_k^{\mathsf{OS}}$ for all $M+1 \le k \le \ell$. We see from this equality and (2.45) that

$$\gamma_k^R = \gamma_k \quad \text{and} \quad b_k^R = b_k > 0 \quad \text{for all } M+1 \le k \le \ell.$$

Also, we have shown that $b_k^R = 0$ for all $1 \le k \le M$ (see the comment preceding this lemma).

It remains to show (2.48) and (2.50). Since $\lfloor w_\circ \rfloor = r_{p_1} \cdots r_{p_M} = r_{\pi(i_1)} \cdots r_{\pi(i_M)}$ are reduced expressions, it follows that

$$\begin{aligned} \{r_{p_M} \cdots r_{p_{k+1}}(\widetilde{\alpha}_{p_k}) \mid 1 \le k \le M\} &= \{\widetilde{\alpha} \in \Phi^{\vee+} \mid \lfloor w_\circ \rfloor \widetilde{\alpha} \in -\Phi^{\vee+}\} \\ &= \{r_{\pi(i_M)} \cdots r_{\pi(i_{k+1})}(\widetilde{\alpha}_{\pi(i_k)}) \mid 1 \le k \le M\}; \end{aligned}$$

notice that

$$\{\widetilde{\alpha} \in \Phi^{\vee+} \mid \lfloor w_\circ \rfloor \widetilde{\alpha} \in -\Phi^{\vee+}\} = \Phi^{\vee+} \setminus \Phi_J^{\vee+}. \tag{2.53}$$

Indeed, we see from (2.1) that $\{\widetilde{\alpha} \in \Phi^{\vee+} \mid \lfloor w_\circ \rfloor \widetilde{\alpha} \in -\Phi^{\vee+}\} \subset \Phi^{\vee+} \setminus \Phi_J^{\vee+}$. Conversely, if $\widetilde{\alpha} \in \Phi^{\vee+} \setminus \Phi_J^{\vee+}$, then $w_{J,\circ}\widetilde{\alpha} \in \Phi^{\vee+}$, and hence $\lfloor w_\circ \rfloor \widetilde{\alpha} = w_\circ w_{J,\circ}\widetilde{\alpha} \in -\Phi^{\vee+}$, as desired. Therefore, we deduce from the definitions that

$$\{\beta_k^{\mathsf{OS},R} \mid 1 \le k \le M\} = m_\lambda^{-1}(\Phi^{\vee+} \setminus \Phi_J^{\vee+}) = \{\beta_k^{\mathsf{OS}} \mid 1 \le k \le M\},$$

and hence that

$$\{\gamma_k^R \mid 1 \le k \le M\} = \{\gamma_k \mid 1 \le k \le M\} \overset{(2.44)}{=} \Phi^+ \setminus \Phi^+_{\omega(J)}.$$

This proves the lemma. ☐

We set $\ell(w_\circ) := N$; since $w_\circ = \lfloor w_\circ \rfloor w_{J,\circ}$, it follows that $\ell(w_{J,\circ}) = N - M$. Fix a reduced expression $w_{J,\circ} = r_{t_{M+1}} r_{t_{M+2}} \cdots r_{t_N}$ for $w_{J,\circ}$. Then,

$$w_\circ = \underbrace{r_{\pi(i_1)} \cdots r_{\pi(i_M)}}_{=\lfloor w_\circ \rfloor} \underbrace{r_{t_{M+1}} r_{t_{M+2}} \cdots r_{t_N}}_{=w_{J,\circ}} = \underbrace{r_{p_1} \cdots r_{p_M}}_{=\lfloor w_\circ \rfloor} \underbrace{r_{t_{M+1}} r_{t_{M+2}} \cdots r_{t_N}}_{=w_{J,\circ}}$$

are reduced expressions for $w_\circ$. Now we set

$$\xi_k = \xi_k^R := \lfloor w_\circ \rfloor r_{t_{M+1}} \cdots r_{t_{k-1}} \alpha_{t_k} \in \Phi^+ \qquad \text{for } M+1 \le k \le N. \tag{2.54}$$

Then, by (2.43) and (2.47), both of the sets $\{\gamma_k \mid 1 \le k \le M\} \cup \{\xi_k \mid M+1 \le k \le N\}$ and $\{\gamma_k^R \mid 1 \le k \le M\} \cup \{\xi_k^R \mid M+1 \le k \le N\}$ are identical to Φ^+. Hence it follows from (2.50) that

$$\{\xi_k^R \mid M+1 \le k \le N\} = \{\xi_k \mid M+1 \le k \le N\} = \Phi^+_{\omega(J)}. \tag{2.55}$$

If we define total orders $\prec$ and $\prec_R$ on Φ^+ by:

$$\underbrace{\gamma_1 \prec \cdots \prec \gamma_M}_{\in \Phi^+ \setminus \Phi^+_{\omega(J)}} \prec \underbrace{\xi_{M+1} \prec \cdots \prec \xi_N}_{\in \Phi^+_{\omega(J)}}, \tag{2.56}$$

$$\underbrace{\gamma_1^R \prec_R \cdots \prec_R \gamma_M^R}_{\in \Phi^+ \setminus \Phi^+_{\omega(J)}} \prec \underbrace{\xi_{M+1}^R \prec_R \cdots \prec_R \xi_N^R}_{\in \Phi^+_{\omega(J)}}, \tag{2.57}$$

respectively, then these total orders are reflection orders (see, for example, [BB, Chap. 5, Exerc. 20]).

Let $A = \{j_1, j_2, \dots, j_r\} \subset \{1, 2, \dots, \ell\}$ be such that

$$p_A = \left(m_{w_\circ\lambda} = t_{w_\circ\lambda} = z_0 \xrightarrow{\beta_{j_1}^{\mathsf{OS}}} \cdots \xrightarrow{\beta_{j_r}^{\mathsf{OS}}} z_r \right) \in \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{\mathrm{lex}},$$

$$(\text{resp., } p_A = \left(m_{w_\circ\lambda} = t_{w_\circ\lambda} = z_0^R \xrightarrow{\beta_{j_1}^{\mathsf{OS},R}} \cdots \xrightarrow{\beta_{j_r}^{\mathsf{OS},R}} z_r^R \right) \in \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_R\ );$$

we set $j_0 := 0$ by convention. By the definition (see Definition 2.7), we have a directed path

$$e = \mathrm{dir}(z_0) \xrightarrow{\gamma_{j_1}} \cdots \xrightarrow{\gamma_{j_r}} \mathrm{dir}(z_r)$$

$$(\text{resp., } e = \mathrm{dir}(z_0^R) \xrightarrow{\gamma_{j_1}^R} \cdots \xrightarrow{\gamma_{j_r}^R} \mathrm{dir}(z_r^R))$$

in the quantum Bruhat graph $\mathrm{QB}(W)$. Let us take $0 \le s \le r$ such that $j_s \le M$ and $j_{s+1} \ge M+1$, and set

$$\begin{aligned} &\widetilde{\iota}(p_A) := \mathrm{dir}(z_s) = r_{\gamma_{j_1}} \cdots r_{\gamma_{j_s}} \in W \\ &(\text{resp., } \ \widetilde{\iota}(p_A) := \mathrm{dir}(z_s^R) = r_{\gamma_{j_1}^R} \cdots r_{\gamma_{j_s}^R} \in W). \end{aligned} \tag{2.58}$$

Remark 2.17. Because $\gamma_{j_1} \prec \gamma_{j_2} \prec \cdots \prec \gamma_{j_s}$ with respect to the reflection order $\prec$ on Φ^+ (see (2.56)), we deduce from [LNS³2, Theorem 6.3] that $e = \mathrm{dir}(z_0) \xrightarrow{\gamma_{j_1}} \cdots \xrightarrow{\gamma_{j_s}} \mathrm{dir}(z_s) = \widetilde{\iota}(p_A)$ is a shortest directed path from e to $\widetilde{\iota}(p_A)$ in the quantum Bruhat graph $\mathrm{QB}(W)$. Therefore, all the edges in this directed path are Bruhat edges by Remark 2.2 (3). We show by induction on u that $\mathrm{dir}(z_u) \in W^{\omega(J)}$ for all $0 \le u \le s$. If $u = 0$, then it is obvious that $\mathrm{dir}(z_0) = e \in W^{\omega(J)}$. Assume that $0 < u \le s$. Since $\mathrm{dir}(z_{u-1}) \in W^{\omega(J)}$ by our induction hypothesis, and since $\mathrm{dir}(z_{u-1}) \xrightarrow{\gamma_{j_u}} \mathrm{dir}(z_u) = \mathrm{dir}(z_{u-1}) r_{\gamma_{j_u}}$ is a Bruhat edge in $\mathrm{QB}(W)$, we see by [BB, Corollary 2.5.2] that $\mathrm{dir}(z_u) \in W^{\omega(J)}$ or $\mathrm{dir}(z_u) = \mathrm{dir}(z_{u-1}) r_i$ for some $i \in \omega(J)$. Suppose that $\mathrm{dir}(z_u) = \mathrm{dir}(z_{u-1}) r_i$ for some $i \in \omega(J)$. Since $\mathrm{dir}(z_{u-1}) r_{\gamma_{j_u}} = \mathrm{dir}(z_u) = \mathrm{dir}(z_{u-1}) r_i$, we have $r_{\gamma_{j_u}} = r_i$, and hence $\gamma_{j_u} = \alpha_i \in \Phi^+_{\omega(J)}$, which contradicts $\gamma_{j_u} \in \Phi^+ \setminus \Phi^+_{\omega(J)}$. Thus we obtain $\mathrm{dir}(z_u) \in W^{\omega(J)}$, as desired. In particular, $\widetilde{\iota}(p_A) = \mathrm{dir}(z_s) \in W^{\omega(J)}$. The same argument works also for the reduced expression R.

Here we define a map $\Theta_R^{\mathrm{lex}} : \mathrm{QB}(e\,;\,m_{w_\circ\lambda})_R \to \mathrm{QB}(e\,;\,m_{w_\circ\lambda})_{\mathrm{lex}}$. Let $A = \{j_1, j_2, \dots, j_r\} \subset \{1, 2, \dots, \ell\}$ be such that

$$p_A = \Bigl(m_{w_\circ\lambda} = t_{w_\circ\lambda} = z_0^R \xrightarrow{\beta_{j_1}^{\mathsf{OS},R}} \cdots \xrightarrow{\beta_{j_r}^{\mathsf{OS},R}} z_r^R\Bigr) \in \mathrm{QB}(e\,;\,m_{w_\circ\lambda})_R, \tag{2.59}$$

that is,

$$e = \mathrm{dir}(z_0^R) \xrightarrow{\gamma_{j_1}^R} \mathrm{dir}(z_1^R) \xrightarrow{\gamma_{j_2}^R} \cdots \xrightarrow{\gamma_{j_r}^R} \mathrm{dir}(z_r^R) \tag{2.60}$$

is a directed path in the quantum Bruhat graph $\mathrm{QB}(W)$. If we take $0 \le s \le r$ such that $j_s \le M$ and $j_{s+1} \ge M+1$, then we have a shortest directed path

$$e = \mathrm{dir}(z_0^R) \xrightarrow{\gamma_{j_1}^R} \mathrm{dir}(z_1^R) \xrightarrow{\gamma_{j_2}^R} \cdots \xrightarrow{\gamma_{j_s}^R} \mathrm{dir}(z_s^R) = \widetilde{\iota}(p_A) \in W^{\omega(J)}$$

in the quantum Bruhat graph $\mathrm{QB}(W)$; note that $\gamma_{j_1}^R \prec_R \cdots \prec_R \gamma_{j_s}^R$ with respect to the reflection order $\prec_R$ on Φ^+ (see (2.57)). We know from [LNS^{3}2, Theorem 6.3] that there exists a unique shortest directed path

$$e = x_0 \xrightarrow{\gamma_{q_1}} \cdots \xrightarrow{\gamma_{q_u}} x_u \xrightarrow{\xi_{q_{u+1}}} \cdots \xrightarrow{\xi_{q_s}} x_s = \widetilde{\iota}(p_A)$$

from e to $\widetilde{\iota}(p_A)$ in $\mathrm{QB}(W)$ such that $1 \le q_1 < \cdots < q_u \le M < q_{u+1} \le \cdots \le q_s \le N = \ell(w_\circ)$ (see (2.43) and (2.54)) for some $0 \le u \le s$; note that all the edges in this directed path are Bruhat edges by Remark 2.2 (3). We claim that $u = s$. Indeed, suppose for a contradiction that $u < s$; in this case, $\xi_{q_s} \in \Phi^+_{\omega(J)}$ by (2.55), and hence $r_{\xi_{q_s}} \in W_{\omega(J)}$. We write $x_{s-1} = \lfloor x_{s-1} \rfloor^{\omega(J)} z$ for some $z \in W_{\omega(J)}$; note that $\ell(x_{s-1}) = \ell(\lfloor x_{s-1} \rfloor^{\omega(J)}) + \ell(z)$. We see that

$$\widetilde{\iota}(p_A) = x_s = x_{s-1} r_{\xi_{q_s}} = \underbrace{\lfloor x_{s-1} \rfloor^{\omega(J)}}_{\in W^{\omega(J)}} \underbrace{z r_{\xi_{q_s}}}_{\in W_{\omega(J)}},$$

and hence $\ell(x_s) = \ell(\lfloor x_{s-1} \rfloor^{\omega(J)}) + \ell(z r_{\xi_{q_s}})$. Because $x_{s-1} \xrightarrow{\xi_{q_s}} x_s$ is a Bruhat edge in $\mathrm{QB}(W)$ as seen above, we have $\ell(x_s) = \ell(x_{s-1}) + 1$. Combining these equalities, we obtain

$$\ell(\lfloor x_{s-1} \rfloor^{\omega(J)}) + \ell(z r_{\xi_{q_s}}) = \ell(x_s) = \ell(x_{s-1}) + 1 = \ell(\lfloor x_{s-1} \rfloor^{\omega(J)}) + \ell(z) + 1,$$

and hence $\ell(z r_{\xi_{q_s}}) = \ell(z) + 1 \ge 1$. Hence it follows that $z r_{\xi_{q_s}} \ne e$, which implies that $\widetilde{\iota}(p_A) = x_s = \lfloor x_{s-1} \rfloor^{\omega(J)} z r_{\xi_{q_s}} \notin W^{\omega(J)}$. However, this contradicts the fact that $\widetilde{\iota}(p_A) \in W^{\omega(J)}$ (see Remark 2.17). Thus, we obtain $u = s$, and hence a directed path

$$e = x_0 \xrightarrow{\gamma_{q_1}} \cdots \xrightarrow{\gamma_{q_s}} x_s = \widetilde{\iota}(p_A) \tag{2.61}$$

such that $1 \le q_1 < \cdots < q_s \le M$.

Now, we set $B := \{q_1, \dots, q_s, j_{s+1}, \dots, j_r\}$, and consider

$$p_B = \Bigl(m_{w_\circ\lambda} = t_{w_\circ\lambda} = z_0 \xrightarrow{\beta_{q_1}^{\mathsf{OS}}} \cdots \xrightarrow{\beta_{q_s}^{\mathsf{OS}}} z_s \xrightarrow{\beta_{j_{s+1}}^{\mathsf{OS}}} \cdots \xrightarrow{\beta_{j_r}^{\mathsf{OS}}} z_r\Bigr). \tag{2.62}$$

Since $M+1 \le j_{s+1} < \cdots < j_r \le \ell$, we see from (2.51) that $\gamma_{j_u} = \gamma_{j_u}^R$ for all $s+1 \le u \le r$. Therefore, by replacing the first s edges in (2.60) with (2.61), we obtain a directed path

$$e = \mathrm{dir}(z_0) \xrightarrow{\gamma_{q_1}} \cdots \xrightarrow{\gamma_{q_s}} \mathrm{dir}(z_s) = \widetilde{\iota}(p_A) \xrightarrow{\gamma_{j_{s+1}}} \cdots \xrightarrow{\gamma_{j_r}} \mathrm{dir}(z_r)$$

in the quantum Bruhat graph $\mathrm{QB}(W)$. Hence we conclude that $p_B \in \mathrm{QB}(e\,;\,m_{w_\circ\lambda})_{\mathrm{lex}}$; we set $\Theta_R^{\mathrm{lex}}(p_A) := p_B$.

Proposition 2.18. *The map* $\Theta_R^{\mathrm{lex}} : \mathrm{QB}(e\,;\,m_{w_\circ\lambda})_R \to \mathrm{QB}(e\,;\,m_{w_\circ\lambda})_{\mathrm{lex}}$ *is bijective. Moreover, for every* $p \in \mathrm{QB}(e\,;\,m_{w_\circ\lambda})_R$,

$$\mathrm{wt}(\Theta_R^{\mathrm{lex}}(p)) = \mathrm{wt}(p), \qquad \mathrm{qwt}(\Theta_R^{\mathrm{lex}}(p)) = \mathrm{qwt}(p), \qquad \widetilde{\iota}(\Theta_R^{\mathrm{lex}}(p)) = \widetilde{\iota}(p).$$

Proof. Let $A = \{j_1, j_2, \ldots, j_r\}$ and $B = \{q_1, \ldots, q_s, j_{s+1}, \ldots, j_r\}$ be as in the definition above of the map Θ_R^{lex}. Recall that

$$p_A = \Big(m_{w_\circ\lambda} = z_0^R \xrightarrow{\beta_{j_1}^{\mathsf{OS},R}} \cdots \xrightarrow{\beta_{j_s}^{\mathsf{OS},R}} z_s^R \xrightarrow{\beta_{j_{s+1}}^{\mathsf{OS},R}} \cdots \xrightarrow{\beta_{j_r}^{\mathsf{OS},R}} z_r^R = \mathrm{end}(p_A)\Big);$$

$$e = \mathrm{dir}(z_0^R) \xrightarrow{\gamma_{j_1}^R} \cdots \xrightarrow{\gamma_{j_s}^R} \mathrm{dir}(z_s^R) = \widetilde{\iota}(p_A) \xrightarrow{\gamma_{j_{s+1}}^R} \cdots \xrightarrow{\gamma_{j_r}^R} \mathrm{dir}(z_r^R),$$

and that

$$p_B = \Big(m_{w_\circ\lambda} = z_0 \xrightarrow{\beta_{q_1}^{\mathsf{OS}}} \cdots \xrightarrow{\beta_{q_s}^{\mathsf{OS}}} z_s \xrightarrow{\beta_{j_{s+1}}^{\mathsf{OS}}} \cdots \xrightarrow{\beta_{j_r}^{\mathsf{OS}}} z_r^R = \mathrm{end}(p_B)\Big);$$

$$e = \mathrm{dir}(z_0) \xrightarrow{\gamma_{q_1}} \cdots \xrightarrow{\gamma_{q_s}} \mathrm{dir}(z_s) = \widetilde{\iota}(p_A) \xrightarrow{\gamma_{j_{s+1}}} \cdots \xrightarrow{\gamma_{j_r}} \mathrm{dir}(z_r).$$

Since $q_s \le M$ and $j_{s+1} \ge M+1$, it follows that $\widetilde{\iota}(p_B) = \mathrm{dir}(z_s) = \widetilde{\iota}(p_A)$.

Next, we prove that

$$\mathrm{wt}(p_B) = \mathrm{wt}(p_A) \quad \text{and} \quad \mathrm{qwt}(p_B) = \mathrm{qwt}(p_A). \tag{2.63}$$

Recall from (2.18) and (2.19) that

$$\mathrm{qwt}(p_B) = \sum_{j \in B^-} \beta_j^{\mathsf{OS}} \quad \text{and} \quad \mathrm{qwt}(p_A) = \sum_{j \in A^-} \beta_j^{\mathsf{OS},R}.$$

We know from Remark 2.17 that all the edges in $e = \mathrm{dir}(z_0^R) \xrightarrow{\gamma_{j_1}^R} \cdots \xrightarrow{\gamma_{j_s}^R} \mathrm{dir}(z_s^R) = \widetilde{\iota}(p_A)$ and $e = \mathrm{dir}(z_0) \xrightarrow{\gamma_{q_1}} \cdots \xrightarrow{\gamma_{q_s}} \mathrm{dir}(z_s) = \widetilde{\iota}(p_A)$ are Bruhat edges, which implies that $A^-, B^- \subset \{j_{s+1}, \ldots, j_r\}$. Since $M+1 \le j_{s+1} < \cdots < j_r \le \ell$, we see from (2.51) that $\gamma_{j_u}^R = \gamma_{j_u}$ for all $s+1 \le u \le r$. Therefore, the directed paths $\mathrm{dir}(z_s^R) = \widetilde{\iota}(p_A) \xrightarrow{\gamma_{j_{s+1}}^R} \cdots \xrightarrow{\gamma_{j_r}^R} \mathrm{dir}(z_r^R)$ and $\mathrm{dir}(z_s) = \widetilde{\iota}(p_A) \xrightarrow{\gamma_{j_{s+1}}} \cdots \xrightarrow{\gamma_{j_r}} \mathrm{dir}(z_r)$ are identical, which implies that $A^- = B^-$. Since $\beta_{j_u}^{\mathsf{OS},R} = \beta_{j_u}^{\mathsf{OS}}$ for all $s+1 \le u \le r$ by (2.49), we obtain $\mathrm{qwt}(p_B) = \mathrm{qwt}(p_A)$.

Finally, we prove that $\mathrm{wt}(p_B) = \mathrm{wt}(p_A)$; it suffices to show that $\mathrm{end}(p_B) = \mathrm{end}(p_A)$ (see (2.21)). Since $b_k^R = b_k = 0$ for all $1 \le k \le M$ by (2.45) and (2.52), we see that

$$\beta_k^{\mathsf{OS},R} = \langle (\gamma_k^R)^\vee, -w_\circ\lambda\rangle\widetilde{\delta} - (\gamma_k^R)^\vee, \qquad \beta_k^{\mathsf{OS}} = \langle \gamma_k^\vee, -w_\circ\lambda\rangle\widetilde{\delta} - \gamma_k^\vee$$

for $1 \le k \le M$, which implies that

$$r_{\beta_k^{\mathsf{OS},R}} = \big(t_{\langle(\gamma_k^R)^\vee, -w_\circ\lambda\rangle\gamma_k^R}\big) r_{\gamma_k^R}, \qquad r_{\beta_k^{\mathsf{OS}}} = \big(t_{\langle\gamma_k^\vee, -w_\circ\lambda\rangle\gamma_k}\big) r_{\gamma_k}.$$

Using these equalities together with $z_0 = z_0^R = m_{w_\circ\lambda} = t_{w_\circ\lambda}$, we can show by induction on $0 \le u \le s$ that

$$z_u^R = t_{\mathrm{dir}(z_u^R)w_\circ\lambda}\,\mathrm{dir}(z_u^R), \qquad z_u = t_{\mathrm{dir}(z_u)w_\circ\lambda}\,\mathrm{dir}(z_u).$$

Since $\mathrm{dir}(z_s^R) = \widetilde{\iota}(p_A) = \mathrm{dir}(z_s)$, we deduce that

$$z_s^R = t_{\mathrm{dir}(z_s^R)w_\circ\lambda}\,\mathrm{dir}(z_s^R) = t_{\mathrm{dir}(z_s)w_\circ\lambda}\,\mathrm{dir}(z_s) = z_s.$$

Since $\beta_{j_u}^{\mathsf{OS},R} = \beta_{j_u}^{\mathsf{OS}}$ for all $s+1 \le u \le r$ as seen above, we obtain

$$\mathrm{end}(p_A) = z_s^R r_{\beta_{j_{s+1}}^{\mathsf{OS},R}} \cdots r_{\beta_{j_r}^{\mathsf{OS},R}} = z_s r_{\beta_{j_{s+1}}^{\mathsf{OS}}} \cdots r_{\beta_{j_r}^{\mathsf{OS}}} = \mathrm{end}(p_B).$$

This proves (2.63).

If we define a map $\Theta_{\mathrm{lex}}^{R} : \mathrm{QB}(e\,;m_{w_\circ\lambda})_{\mathrm{lex}} \to \mathrm{QB}(e\,;m_{w_\circ\lambda})_R$ in exactly the same manner as for the map $\Theta_R^{\mathrm{lex}} : \mathrm{QB}(e\,;m_{w_\circ\lambda})_R \to \mathrm{QB}(e\,;m_{w_\circ\lambda})_{\mathrm{lex}}$, then from the uniqueness of a directed path in $\mathrm{QB}(W)$ whose labels are increasing in a reflection order (see [LNS^{3}2, Theorem 6.3]), we deduce that both of the composites $\Theta_{\mathrm{lex}}^{R} \circ \Theta_R^{\mathrm{lex}}$ and $\Theta_R^{\mathrm{lex}} \circ \Theta_{\mathrm{lex}}^{R}$ are the identity maps. This proves the bijectivity of the map Θ_R^{lex}, and hence completes the proof of the proposition. □

2.6. **Embedding of** $\mathrm{QB}(e\,;m_{w\lambda})$ **into** $\mathrm{QB}(e\,;m_{w_\circ\lambda})_{\mathrm{lex}}$**.** We keep the notation and setting of the previous subsection. Recall that $m_{w_\circ\lambda} = t_{w_\circ\lambda} = \pi r_{i_1} r_{i_2} \cdots r_{i_\ell}$ is the reduced expression for $m_{w_\circ\lambda} = t_{w_\circ\lambda}$ corresponding to the (fixed) lex $(-w_\circ\lambda)$-chain of roots; we know from Lemma 2.15 that

$$m_{w_\circ\lambda} = t_{w_\circ\lambda} = \pi r_{i_1} r_{i_2} \cdots r_{i_\ell} = \underbrace{\bigl(r_{\pi(i_1)} \cdots r_{\pi(i_M)}\bigr)}_{=\lfloor w_\circ \rfloor} \underbrace{\bigl(\pi r_{i_{M+1}} \cdots r_{i_\ell}\bigr)}_{=m_\lambda}, \tag{2.64}$$

where $M = \ell(\lfloor w_\circ \rfloor)$.

Let $w \in W^J$, and set $L := \ell(w) \le M$. We can take a reduced expression $\lfloor w_\circ \rfloor = r_{p_1} r_{p_2} \cdots r_{p_M}$ of $\lfloor w_\circ \rfloor$ such that $w = r_{p_{M-L+1}} \cdots r_{p_M}$;

$$\lfloor w_\circ \rfloor = \underbrace{r_{p_1} \cdots r_{p_{M-L}}}_{=\lfloor w_\circ \rfloor w^{-1}} \underbrace{r_{p_{M-L+1}} \cdots r_{p_M}}_{=w}. \tag{2.65}$$

Indeed, recall that $w_\circ = \lfloor w_\circ \rfloor w_{J,\circ}$, with $\ell(w_\circ) = \ell(\lfloor w_\circ \rfloor) + \ell(w_{J,\circ})$. Since $w \in W^J$, we have $\ell(ww_{J,\circ}) = \ell(w) + \ell(w_{J,\circ})$. Hence it follows that

$$\begin{aligned} \ell(\lfloor w_\circ \rfloor) + \ell(w_{J,\circ}) = \ell(w_\circ) &= \ell(w_\circ(ww_{J,\circ})^{-1}) + \ell(ww_{J,\circ}) \\ &= \ell(\lfloor w_\circ \rfloor w^{-1}) + \ell(w) + \ell(w_{J,\circ}), \end{aligned}$$

so that $\ell(\lfloor w_\circ \rfloor) = \ell(\lfloor w_\circ \rfloor w^{-1}) + \ell(w)$, which implies that $\ell(\lfloor w_\circ \rfloor w^{-1}) = M - L$. Therefore, if $\lfloor w_\circ \rfloor w^{-1} = r_{p_1} \cdots r_{p_{M-L}}$ is a reduced expression for $\lfloor w_\circ \rfloor w^{-1}$, and $w = r_{p_{M-L+1}} \cdots r_{p_M}$ is a reduced expression for w, then $\lfloor w_\circ \rfloor = r_{p_1} \cdots r_{p_{M-L}} r_{p_{M-L+1}} \cdots r_{p_M}$ is a reduced expression for $\lfloor w_\circ \rfloor$. Now, we set $i'_k := \pi^{-1}(p_k)$ for $1 \le k \le M$; we see that

$$m_{w_\circ\lambda} = \underbrace{\bigl(r_{p_1} r_{p_2} \cdots r_{p_M}\bigr)}_{=\lfloor w_\circ \rfloor} \underbrace{\bigl(\pi r_{i_{M+1}} \cdots r_{i_\ell}\bigr)}_{=m_\lambda} = \pi r_{i'_1} \cdots r_{i'_M} r_{i_{M+1}} \cdots r_{i_\ell} \tag{2.66}$$

is a reduced expression for $m_{w_\circ\lambda}$. As in §2.5, we construct $\mathrm{QB}(e\,;m_{w_\circ\lambda})$ from this reduced expression R for $m_{w_\circ\lambda}$, and denote it by $\mathrm{QB}(e\,;m_{w_\circ\lambda})_R$; recall from Proposition 2.18 the bijection $\Theta_R^{\mathrm{lex}} : \mathrm{QB}(e\,;m_{w_\circ\lambda})_R \to \mathrm{QB}(e\,;m_{w_\circ\lambda})_{\mathrm{lex}}$. We set $A_0 := \{1, 2, \ldots, M-L\} \subset \{1, 2, \ldots, \ell\}$, and consider p_{A_0}. Using Lemma 2.6, we see by direct computation that

$$\begin{aligned} z_0^R &= m_{w_\circ\lambda} = \pi r_{i'_1} \cdots r_{i'_M} r_{i_{M+1}} \cdots r_{i_\ell} = t_{w_\circ\lambda}, \\ z_1^R &= \pi r_{i'_2} \cdots r_{i'_M} r_{i_{M+1}} \cdots r_{i_\ell} = r_{p_1} t_{w_\circ\lambda}, \\ z_2^R &= \pi r_{i'_3} \cdots r_{i'_M} r_{i_{M+1}} \cdots r_{i_\ell} = r_{p_2} r_{p_1} t_{w_\circ\lambda}, \\ &\cdots\cdots\cdots, \\ z_{M-L-1}^R &= \pi r_{i'_{M-L}} \cdots r_{i'_M} r_{i_{M+1}} \cdots r_{i_\ell} = r_{p_{M-L-1}} \cdots r_{p_2} r_{p_1} t_{w_\circ\lambda}, \\ z_{M-L}^R &= \pi r_{i'_{M-L+1}} \cdots r_{i'_M} r_{i_{M+1}} \cdots r_{i_\ell} = \underbrace{r_{p_{M-L}} \cdots r_{p_2} r_{p_1}}_{=w\lfloor w_\circ \rfloor^{-1}} t_{w_\circ\lambda} = m_{w\lambda}. \end{aligned}$$

From these, we deduce that $\mathrm{dir}(z_K^R) = r_{p_K} \cdots r_{p_2} r_{p_1}$ for $0 \le K \le M-L$, and that

$$e \xrightarrow{\gamma_1^R} r_{p_1} \xrightarrow{\gamma_2^R} \cdots \xrightarrow{\gamma_{M-L-1}^R} r_{p_{M-L-1}} \cdots r_{p_2} r_{p_1} \xrightarrow{\gamma_{M-L}^R} w\lfloor w_\circ \rfloor^{-1} \tag{2.67}$$

is a directed path from e to $w\lfloor w_\circ \rfloor^{-1}$ in the quantum Bruhat graph $\mathrm{QB}(W)$; since $\ell(\mathrm{dir}(z_K)) = \ell(\mathrm{dir}(z_{K-1})) + 1$ for $1 \le K \le M - L$, all the edges in this directed path are Bruhat edges. Hence we obtain

$$p_{A_0} = \Big(m_{w_\circ\lambda} = z_0^R \xrightarrow{\beta_1^{\mathsf{OS},R}} z_1^R \xrightarrow{\beta_2^{\mathsf{OS},R}} \cdots \xrightarrow{\beta_{M-L}^{\mathsf{OS},R}} z_{M-L}^R = m_{w\lambda} \Big) \in \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_R. \tag{2.68}$$

Since $m_{w\lambda} = w\lfloor w_\circ \rfloor^{-1} t_{w_\circ\lambda} = w m_\lambda$ by Lemma 2.6, we have

$$m_{w\lambda} = \underbrace{\big(r_{p_{M-L+1}} \cdots r_{p_M} \big)}_{=w} \underbrace{\big(\pi r_{i_{M+1}} \cdots r_{i_\ell} \big)}_{=m_\lambda} = \pi r_{i'_{M-L+1}} \cdots r_{i'_M} r_{i_{M+1}} \cdots r_{i_\ell}; \tag{2.69}$$

since (2.66) is a reduced expression (for $m_{w_\circ\lambda}$), we see that (2.69) is also a reduced expression (for $m_{w\lambda}$). Let us construct $\mathrm{QB}(e\,;\, m_{w\lambda})$ from this reduced expression. Namely, for a subset $B = \{ j_1 < j_2 < \cdots < j_r \} \subset \{ M-L+1,\, M-L+2,\, \dots,\, \ell \}$, we define

$$y_0^R = m_{w\lambda}, \qquad y_k^R = y_0 r_{\beta_{j_1}^{\mathsf{OS},R}} \cdots r_{\beta_{j_k}^{\mathsf{OS},R}} \quad \text{for } 1 \le k \le r,$$

where $\beta_k^{\mathsf{OS},R}$, $M-L+1 \le k \le \ell$, are those used in the definition of $\mathrm{QB}(e\,;\, m_{w_\circ\lambda})_R$, and set

$$p_B := \Big(m_{w\lambda} = y_0^R \xrightarrow{\beta_{j_1}^{\mathsf{OS},R}} y_1^R \xrightarrow{\beta_{j_2}^{\mathsf{OS},R}} y_2^R \xrightarrow{\beta_{j_3}^{\mathsf{OS},R}} \cdots \xrightarrow{\beta_{j_r}^{\mathsf{OS},R}} y_r^R \Big).$$

Then, $p_B \in \mathrm{QB}(e\,;\, m_{w\lambda})$ if

$$w\lfloor w_\circ \rfloor^{-1} = \mathrm{dir}(y_0^R) \xrightarrow{\gamma_{j_1}^R} \mathrm{dir}(y_1^R) \xrightarrow{\gamma_{j_2}^R} \cdots \xrightarrow{\gamma_{j_r}^R} \mathrm{dir}(y_r^R)$$

is a directed path in the quantum Bruhat graph $\mathrm{QB}(W)$.

Since $\mathrm{end}(p_{A_0}) = m_{w\lambda}$, we can "concatenate" p_{A_0} with an arbitrary $p_B \in \mathrm{QB}(e\,;\, m_{w\lambda})$, which is just $p_{A_0 \sqcup B}$; we see easily that $p_{A_0 \sqcup B} \in \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_R$.

Lemma 2.19. *There exists an embedding* $\mathrm{QB}(e\,;\, m_{w\lambda}) \hookrightarrow \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_R$, *which maps* $p_B \in \mathrm{QB}(e\,;\, m_{w\lambda})$ *to* $p_{A_0 \sqcup B} \in \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_R$. *Moreover,* $\mathrm{wt}(p_{A_0 \sqcup B}) = \mathrm{wt}(p_B)$, *and* $\mathrm{qwt}(p_{A_0 \sqcup B}) = \mathrm{qwt}(p_B)$ (*and hence* $\deg(\mathrm{qwt}(p_{A_0 \sqcup B})) = \deg(\mathrm{qwt}(p_B))$).

Proof. The injectivity of the map is obvious. Since $\mathrm{end}(p_{A_0 \sqcup B}) = \mathrm{end}(p_B)$ by the definition, we have $\mathrm{wt}(p_{A_0 \sqcup B}) = \mathrm{wt}(p_B)$. Because all the edges in the directed path (2.67) are Bruhat edges, we see from the definition (2.18) that $(A_0 \sqcup B)^- = B^-$. Hence we obtain $\mathrm{qwt}(p_{A_0 \sqcup B}) = \mathrm{qwt}(p_B)$ by the definition (2.19) of qwt. This proves the lemma. □

We set

$$\mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{R,w} := \big\{ p_A \in \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_R \mid \{1,\, 2,\, \dots,\, M-L\} \subset A \big\}.$$

We see from the argument above that $\mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{R,w}$ is identical to the image of the embedding $\mathrm{QB}(e\,;\, m_{w\lambda}) \hookrightarrow \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_R$ of Lemma 2.19.

Lemma 2.20. *Let* $p_A \in \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_R$. *Then,* $p_A \in \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{R,w}$ *if and only if* $\widetilde{\iota}(p_A) \ge w\lfloor w_\circ \rfloor^{-1}$ *with respect to the Bruhat order* $\ge$ *on* W.

Proof. First, we prove the "only if" part. Since $p_A \in \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{R,w}$, it follows that A is of the form: $A = \{1,\, 2,\, \dots,\, M-L,\, j_1,\, \dots,\, j_r\}$ for some $M-L+1 \le j_1 < \cdots < j_r \le \ell$; we set $j_0 = 0$ by convention. Take $0 \le s \le r$ such that $j_s \le M$ and $j_{s+1} \ge M+1$. Then, by (2.67) and the definition of $\widetilde{\iota}(p_A)$, we have a directed path

$$e \xrightarrow{\gamma_1^R} r_{p_1} \xrightarrow{\gamma_2^R} \cdots \xrightarrow{\gamma_{M-L}^R} w\lfloor w_\circ \rfloor^{-1} \xrightarrow{\gamma_{j_1}^R} \cdots \xrightarrow{\gamma_{j_s}^R} \widetilde{\iota}(p_A)$$

in the quantum Bruhat graph. Since all the edges in this directed path are Bruhat edges (see Remark 2.17), we obtain $\widetilde{\iota}(p_A) \ge w\lfloor w_\circ \rfloor^{-1}$, as desired.

Next, we prove the "if" part. Assume that $\widetilde{\iota}(p_A) \geq w\lfloor w_\circ \rfloor^{-1}$, with $A = \{j_1, \ldots, j_r\} \subset \{1, 2, \ldots, \ell\}$. If we take $0 \leq s \leq \ell$ such that $j_s \leq M$ and $j_{s+1} \geq M+1$, then we have a shortest directed path

$$e = \mathrm{dir}(z_0^R) \xrightarrow{\gamma_{j_1}^R} \mathrm{dir}(z_1^R) \xrightarrow{\gamma_{j_2}^R} \cdots \xrightarrow{\gamma_{j_s}^R} \mathrm{dir}(z_s^R) = \widetilde{\iota}(p_A) \tag{2.70}$$

in the quantum Bruhat graph $\mathrm{QB}(W)$ whose edges are all Bruhat edges (see Remark 2.17); note that $s = \ell(\widetilde{\iota}(p_A))$. Here, because $\widetilde{\iota}(p_A) \geq w\lfloor w_\circ \rfloor^{-1}$ with respect to the Bruhat order on W, we deduce by the chain property of the Bruhat order (see [BB, Theorem 2.2.6]) that there exists a directed path $w\lfloor w_\circ \rfloor^{-1} = x_0 \xrightarrow{\xi_1} \cdots \xrightarrow{\xi_{s-M+L}} x_{s-M+L} = \widetilde{\iota}(p_A)$ of length $\ell(\widetilde{\iota}(p_A)) - \ell(w\lfloor w_\circ \rfloor^{-1}) = s - (M - L)$ from $w\lfloor w_\circ \rfloor^{-1}$ to $\widetilde{\iota}(p_A)$ in the quantum Bruhat graph $\mathrm{QB}(W)$ whose edges are all Bruhat edges. Concatenating this directed path with the directed path (2.67), we get the directed path

$$e \xrightarrow{\gamma_1^R} r_{p_1} \xrightarrow{\gamma_2^R} \cdots \xrightarrow{\gamma_{M-L}^R} w\lfloor w_\circ \rfloor^{-1} = x_0 \xrightarrow{\xi_1} \cdots \xrightarrow{\xi_{s-M+L}} x_{s-M+L} = \widetilde{\iota}(p_A). \tag{2.71}$$

Since the length of this directed path is equal to $s = \ell(\widetilde{\iota}(p_A)) - \ell(e)$, this directed path is also a shortest directed path from e to $\widetilde{\iota}(p_A)$ in the quantum Bruhat graph $\mathrm{QB}(W)$. Because the labels in the directed path (2.70) are strictly increasing with respect to the reflection order $\prec_R$ (see (2.57)), that is, $\gamma_{j_1}^R \prec_R \cdots \prec_R \gamma_{j_s}^R$, it follows from [LNS^{3}2, Theorem 6.3] that the directed path (2.70) is lexicographically minimal among all shortest directed paths from e to $\widetilde{\iota}(p_A)$; in particular, the directed path (2.70) is less than or equal to the directed path (2.71), which implies that $j_1 = 1, j_2 = 2, \ldots, j_{M-L} = M - L$. Thus, we obtain $\{1, 2, \ldots, M-L\} \subset A$, and hence $p_A \in \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{R,w}$. This completes the proof of the lemma. □

From Lemma 2.20 (together with the comment preceding it), Lemma 2.19, and Proposition 2.18, we obtain the following proposition.

Proposition 2.21. *The image of* $\mathrm{QB}(e\,;\, m_{w\lambda})$ *under the composite*

$$\mathrm{QB}(e\,;\, m_{w\lambda}) \overset{\text{Lemma 2.19}}{\hookrightarrow} \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_R \xrightarrow{\Theta_R^{\mathrm{lex}}} \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{\mathrm{lex}}$$

is identical to

$$\mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{\mathrm{lex},w} := \{p \in \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{\mathrm{lex}} \mid \widetilde{\iota}(p) \geq w\lfloor w_\circ \rfloor^{-1}\}. \tag{2.72}$$

Hence we have

$$\sum_{p \in \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{\mathrm{lex},w}} e^{\mathrm{wt}(p)} q^{\deg(\mathrm{qwt}(p))} = \sum_{p \in \mathrm{QB}(e\,;\, m_{w\lambda})} e^{\mathrm{wt}(p)} q^{\deg(\mathrm{qwt}(p))} = E_{w\lambda}(x\,;\, q,\, 0). \tag{2.73}$$

2.7. **Bijection between** $\mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{\mathrm{lex}}$ **and** $\mathrm{QLS}(\lambda)$**.** As in the previous subsection, we fix a lex $(-w_\circ\lambda)$-chain of roots

$$A_\circ = A_0 \xrightarrow{-\gamma_1} A_1 \xrightarrow{-\gamma_2} \cdots \xrightarrow{-\gamma_\ell} A_\ell = A_{w_\circ\lambda}, \tag{2.74}$$

and let $m_{w_\circ\lambda} = \pi r_{i_1} r_{i_2} \cdots r_{i_\ell}$ be the corresponding reduced expression for $m_{w_\circ\lambda}$ under (2.22). We construct $\mathcal{A}(-w_\circ\lambda)$ from this reduced expression $m_{w_\circ\lambda} = \pi r_{i_1} r_{i_2} \cdots r_{i_\ell}$, which we denote by $\mathcal{A}(-w_\circ\lambda)_{\mathrm{lex}}$; recall from Remark 2.12 and (2.25) that $\gamma_k = \gamma_k^{\mathsf{OS}} = \gamma_k^{\mathsf{L}} = (\overline{\beta_k^{\mathsf{L}}})^\vee \in -w_\circ(\Phi^+ \setminus \Phi_J^+)$ for all $1 \leq k \leq \ell$. We set (see Remark 2.12)

$$d_k := \frac{b_k}{\left\langle \overline{\beta_k^{\mathsf{L}}},\, -w_\circ\lambda \right\rangle} = 1 - \frac{a_k}{\left\langle \overline{\beta_k^{\mathsf{OS}}},\, w_\circ\lambda \right\rangle} \qquad \text{for } 1 \leq k \leq \ell. \tag{2.75}$$

Because $m_{w_\circ\lambda} = \pi r_{i_1} r_{i_2} \cdots r_{i_\ell}$ is the reduced expression corresponding to the lex $(-w_\circ\lambda)$-chain of roots, it follows from (2.37) that

$$0 \leq d_1 \leq d_2 \leq \cdots \leq d_\ell < 1.$$

Remark 2.22. Let $1 \le k < p \le \ell$ be such that $d_k = d_p$. Then we know from [LNS[3]2, Remark 6.4] that $\gamma_k \prec \gamma_p$ in the reflection order $\prec$ (see (2.56)).

In the following, we define a map $\Xi : \mathrm{QB}(e\,;m_{w_\circ\lambda})_{\mathrm{lex}} \to \mathrm{QLS}(\lambda)$ (resp., $\Pi : \mathcal{A}(-w_\circ\lambda)_{\mathrm{lex}} \to \mathrm{QLS}(\lambda)$; see [LNS[3]2, §6.1]). Let $A = \{j_1 < \cdots < j_r\} \subset \{1,\ldots,\ell\}$ be such that

$$p_A = \left(m_{w_\circ\lambda} = z_0 \xrightarrow{\beta^{\mathsf{OS}}_{j_1}} z_1 \xrightarrow{\beta^{\mathsf{OS}}_{j_2}} \cdots \xrightarrow{\beta^{\mathsf{OS}}_{j_r}} z_r\right)$$

is an element of $\mathrm{QB}(e\,;m_{w_\circ\lambda})_{\mathrm{lex}}$ (resp., $A \in \mathcal{A}(-w_\circ\lambda)_{\mathrm{lex}}$); if we set $x_k := r_{\gamma_{j_1}} \cdots r_{\gamma_{j_k}} = \mathrm{dir}(z_k) \in W$ for $0 \le k \le r$, then

$$e = x_0 \xrightarrow{\gamma_{j_1}} x_1 \xrightarrow{\gamma_{j_2}} \cdots \xrightarrow{\gamma_{j_r}} x_r$$

is a directed path in the quantum Bruhat graph $\mathrm{QB}(W)$. Take $0 = u_0 \le u_1 < u_2 < \cdots < u_{s-1} < u_s = r$ (with $s \ge 1$) in such a way that

$$(2.76)\quad \underbrace{0 = d_{j_1} = \cdots = d_{j_{u_1}}}_{=:\sigma_0} < \underbrace{d_{j_{u_1+1}} = \cdots = d_{j_{u_2}}}_{=:\sigma_1} < \underbrace{d_{j_{u_2+1}} = \cdots = d_{j_{u_3}}}_{=:\sigma_2} < \cdots\cdots\cdots < \underbrace{d_{j_{u_{s-1}+1}} = \cdots = d_{j_r}}_{=:\sigma_{s-1}} < 1 =: \sigma_s;$$

note that $u_1 = 0$ if $d_{j_1} > 0$. We set $w'_p := x_{u_p}$ for $1 \le p \le s-1$, and $w'_s := x_r$. For each $1 \le p \le s-1$, we have a directed path

$$w'_p = x_{u_p} \xrightarrow{\gamma_{j_{u_p+1}}} x_{u_p+1} \xrightarrow{\gamma_{j_{u_p+2}}} \cdots \xrightarrow{\gamma_{j_{u_{p+1}}}} x_{u_{p+1}} = w'_{p+1}$$

in the quantum Bruhat graph $\mathrm{QB}(W)$. We claim that this directed path is a shortest directed path from w'_p to w'_{p+1}. Indeed, since $d_{j_{u_p+1}} = \cdots = d_{j_{u_{p+1}}}$ by (2.76), it follows from Remark 2.22 that $\gamma_{j_{u_p+1}} \prec \cdots \prec \gamma_{j_{u_{p+1}}}$ in the reflection order $\prec$ (see (2.56)). Therefore, we deduce from [LNS[3]2, Theorem 6.3] that the directed path above is a shortest directed path from w'_p to w'_{p+1}, as desired. Hence it follows that

$$(2.77)\quad w_p := w'_p w_\circ = x_{u_p} w_\circ \xleftarrow{-w_\circ\gamma_{j_{u_p+1}}} x_{u_p+1} w_\circ \xleftarrow{-w_\circ\gamma_{j_{u_p+2}}} \cdots\cdots \xleftarrow{-w_\circ\gamma_{j_{u_{p+1}}}} x_{u_{p+1}} w_\circ = w'_{p+1} w_\circ =: w_{p+1}$$

is also a shortest directed path in the quantum Bruhat graph $\mathrm{QB}(W)$, where $-w_\circ\gamma_{j_u} \in \Phi^+ \setminus \Phi^+_J$ for all $u_p + 1 \le u \le u_{p+1}$ since $\gamma_{j_u} \in -w_\circ(\Phi^+ \setminus \Phi^+_J)$ as mentioned at the beginning of this subsection. Moreover, for $u_p + 1 \le u \le u_{p+1}$, we have

$$\sigma_p \langle -w_\circ\gamma^\vee_{j_u}, \lambda\rangle = d_{j_u} \langle \gamma^\vee_{j_u}, -w_\circ\lambda\rangle = \frac{b_{j_u}}{\left\langle \overline{\beta^{\mathsf{L}}_{j_u}}, -w_\circ\lambda \right\rangle} \times \left\langle \overline{\beta^{\mathsf{L}}_{j_u}}, -w_\circ\lambda \right\rangle = b_{j_u} \in \mathbb{Z}.$$

Hence the directed path (2.77) is a directed path in $\mathrm{QB}_{\sigma_p\lambda}(W)$. We deduce from [LNS[3]1, Lemma 6.1] that there exists a directed path from $\lfloor w_{p+1} \rfloor = \lfloor w_{p+1} \rfloor^J$ to $\lfloor w_p \rfloor = \lfloor w_p \rfloor^J$ in $\mathrm{QB}_{\sigma_p\lambda}(W^J)$. Therefore, we conclude that

$$\eta := (\lfloor w_1 \rfloor, \lfloor w_2 \rfloor, \ldots, \lfloor w_s \rfloor\,; \sigma_0, \sigma_1, \ldots, \sigma_s) \in \mathrm{QLS}(\lambda);$$

we set $\Xi(p_A) := \eta$.

Remark 2.23. Keep the setting above. Because $0 = d_{j_1} = \cdots = d_{j_{u_1}} < d_{j_{u_1+1}}$ by the definition of u_1, we see from (2.45) that $j_{u_1} \le M$ and $j_{u_1+1} \ge M+1$, where $\ell(\lfloor w_\circ \rfloor) = M$. Therefore, by definition (2.58), $\widetilde{\iota}(p_A)$ is just $\mathrm{dir}(z_{u_1}) = x_{u_1} = w'_1$. Hence we obtain

$$(2.78)\quad \iota(\Xi(p_A)) = \iota(\eta) = \lfloor w_1 \rfloor = \lfloor w'_1 w_\circ \rfloor = \lfloor \widetilde{\iota}(p_A) w_\circ \rfloor.$$

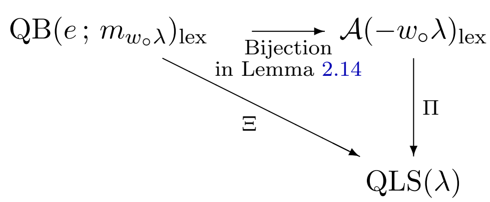
Proposition 2.24 ([LNS[3]2, Proposition 6.7 and Theorem 7.3])**.** *The map* $\Pi : \mathcal{A}(-w_\circ\lambda)_{\mathrm{lex}} \to \mathrm{QLS}(\lambda)$ *is bijective. Moreover, for every* $A \in \mathcal{A}(-w_\circ\lambda)_{\mathrm{lex}}$*,*

$$\mathrm{wt}(\Pi(A)) = -\,\mathrm{wt}(A) \qquad \text{and} \qquad \mathrm{Deg}(\Pi(A)) = -\,\mathrm{height}(A). \tag{2.79}$$

Proposition 2.25. *The map* $\Xi : \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{\mathrm{lex}} \to \mathrm{QLS}(\lambda)$ *is bijective. Moreover, for every* $p_A \in \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{\mathrm{lex}}$*,*

$$\mathrm{wt}(\Xi(p_A)) = \mathrm{wt}(p_A) \qquad \text{and} \qquad \mathrm{Deg}(\Xi(p_A)) = -\deg(\mathrm{qwt}(p_A)).$$

Proof. From the constructions, we see that the map $\Xi : \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{\mathrm{lex}} \to \mathrm{QLS}(\lambda)$ above is identical to the composite of the bijection $\mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{\mathrm{lex}} \xrightarrow{\sim} \mathcal{A}(-w_\circ\lambda)_{\mathrm{lex}}$ of Lemma 2.14 and the bijection $\Pi : \mathcal{A}(-w_\circ\lambda)_{\mathrm{lex}} \xrightarrow{\sim} \mathrm{QLS}(\lambda)$ in Proposition 2.24. Hence the map $\Xi : \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{\mathrm{lex}} \to \mathrm{QLS}(\lambda)$ is also bijective.



We know from (2.35) that $\mathrm{wt}(A) = -\,\mathrm{wt}(p_A)$ and $\mathrm{height}(A) = \deg(\mathrm{qwt}(p_A))$ for all $A \in \mathcal{A}(-w_\circ\lambda)$. Combining this equality and (2.79), we obtain the equalities $\mathrm{wt}(\Xi(p_A)) = \mathrm{wt}(p_A)$ and $\mathrm{Deg}(\Xi(p_A)) = -\deg(\mathrm{qwt}(p_A))$ for all $p_A \in \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{\mathrm{lex}}$, as desired. □

Lemma 2.26. *The image of* $\mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{\mathrm{lex},w}$ *(see Proposition 2.21) under the bijection* $\Xi : \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{\mathrm{lex}} \to \mathrm{QLS}(\lambda)$ *of Proposition 2.25 is identical to* $\mathrm{QLS}_w(\lambda)$*.*

Proof. Let $p \in \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{\mathrm{lex}}$. Then,

$$p \in \mathrm{QB}(e\,;\, m_{w_\circ\lambda})_{\mathrm{lex},w} \overset{(2.72)}{\iff} \widetilde{\iota}(p) \ge w\lfloor w_\circ \rfloor^{-1} \iff \widetilde{\iota}(p)w_\circ \le w\lfloor w_\circ \rfloor^{-1} w_\circ.$$

Since $\widetilde{\iota}(p) \in W^{\omega(J)}$ (see Remark 2.17), it follows by (2.1) that

$$\widetilde{\iota}(p)w_\circ w_{J,\circ}(\Phi_J^+) = \widetilde{\iota}(p)w_\circ(-\Phi_J^+) = \widetilde{\iota}(p)(\Phi_{\omega(J)}^+) \subset \Phi^+.$$

From this, we deduce that $\widetilde{\iota}(p)w_\circ w_{J,\circ} \in W^J$ again by (2.1), which implies that $\lfloor \widetilde{\iota}(p)w_\circ \rfloor w_{J,\circ} = \lfloor \widetilde{\iota}(p)w_\circ w_{J,\circ} \rfloor w_{J,\circ} = (\widetilde{\iota}(p)w_\circ w_{J,\circ})w_{J,\circ} = \widetilde{\iota}(p)w_\circ$. Therefore,

$$\widetilde{\iota}(p)w_\circ \le w\lfloor w_\circ \rfloor^{-1} w_\circ \iff \lfloor \widetilde{\iota}(p)w_\circ \rfloor w_{J,\circ} \le ww_{J,\circ}.$$

Here we have

$$\lfloor \widetilde{\iota}(p)w_\circ \rfloor w_{J,\circ} \le ww_{J,\circ} \qquad \iff \qquad \lfloor \widetilde{\iota}(p)w_\circ \rfloor \le w.$$

Indeed, the "only if" part ($\Rightarrow$) follows immediately from [BB, Proposition 2.5.1]. Let us show the "if" part ($\Leftarrow$). Fix reduced expressions for $w_{J,\circ} \in W_J$ and $w \in W^J$, and then take a reduced expression of $\lfloor \widetilde{\iota}(p)w_\circ \rfloor \in W^J$ that is a "subword" of the fixed reduced expression of w (see [BB, Theorem 2.2.2]). By [BB, Proposition 2.4.4], the product of this reduced expression for $\lfloor \widetilde{\iota}(p)w_\circ \rfloor$ (resp., $w \in W^J$) and a reduced expression for $w_{J,\circ}$ is a reduced expression for $\lfloor \widetilde{\iota}(p)w_\circ \rfloor w_{J,\circ}$ (resp., $ww_{J,\circ}$); observe that the obtained reduced expression for $\lfloor \widetilde{\iota}(p)w_\circ \rfloor w_{J,\circ}$ is a subword of the obtained reduced expression for $ww_{J,\circ}$. Therefore, by [BB, Theorem 2.2.2], we see that $\lfloor \widetilde{\iota}(p)w_\circ \rfloor w_{J,\circ} \le ww_{J,\circ}$, as desired. Finally, we have

$$\begin{aligned} \lfloor \widetilde{\iota}(p)w_\circ \rfloor \le w &\iff \iota(\Xi(p)) \le w \qquad \text{by (2.78)} \\ &\iff \Xi(p) \in \mathrm{QLS}_w(\lambda). \end{aligned}$$

This proves the lemma. □

Proof of Theorem 1.1. We compute:

$$\sum_{\eta \in \mathrm{QLS}_w(\lambda)} e^{\mathrm{wt}(\eta)} q^{-\operatorname{Deg}(\eta)} = \sum_{p \in \mathrm{QB}(e\,;\, m_{w_\circ \lambda})_{\mathrm{lex},w}} e^{\mathrm{wt}(p)} q^{\deg(\mathrm{qwt}(p))}$$

by Lemma 2.26 and Proposition 2.25

$$= E_{w\lambda}(x\,;\, q,\, 0) \quad \text{by (2.73).}$$

This completes the proof of Theorem 1.1. □

2.8. **The formula in terms of the quantum alcove model.** We start with some review from [LNS[3]2]. Recall the Dynkin diagram automorphism $\omega : I \to I$ induced by $w_\circ \alpha_j = -\alpha_{\omega(j)}$ for $j \in I$. Note that ω acts as $-w_\circ$ on the integral weight lattice X. There exists a group automorphism, denoted also by ω, of the Weyl group W such that $\omega(r_j) = r_{\omega(j)}$ for all $j \in I$.

Now, fix $\lambda \in X$ be a dominant integral weight with $J = \{ i \in I \mid \langle \alpha_i^\vee,\, \lambda \rangle = 0 \}$, and let

$$\eta = (x_1,\, \ldots,\, x_s\,;\, \sigma_0,\, \sigma_1,\, \ldots,\, \sigma_s) \in \mathrm{QLS}(\lambda), \tag{2.80}$$

with $x_1, \ldots, x_s \in W^J$ and rational numbers $0 = \sigma_0 < \cdots < \sigma_s = 1$. Then we define

$$\eta^* := \left(\lfloor x_s w_\circ \rfloor^{\omega(J)},\, \ldots,\, \lfloor x_1 w_\circ \rfloor^{\omega(J)}\,;\, 1 - \sigma_s,\, 1 - \sigma_{s-1},\, \ldots,\, 1 - \sigma_0 \right). \tag{2.81}$$

We also define $\omega(\eta)$ by:

$$\omega(\eta) = (\omega(x_1),\, \ldots,\, \omega(x_s)\,;\, \sigma_0,\, \sigma_1,\, \ldots,\, \sigma_s). \tag{2.82}$$

Both maps, $*$ and ω, are bijections between $\mathrm{QLS}(\lambda)$ and $\mathrm{QLS}(-w_\circ\lambda)$, and they change the weight of a path by a negative sign and ω, respectively. Finally, we set $S(\eta) := \omega(\eta^*) = (\omega(\eta))^*$, which turns out to be the Lusztig involution on $\mathrm{QLS}(\lambda)$.

Replacing λ by $-w_\circ\lambda$ in §2.4 and §2.5, let us consider a lex λ-chain of roots, and the quantum alcove model $\mathcal{A}(\lambda)_{\mathrm{lex}}$ associated to it. Recall the map Π (in Proposition 2.24 with λ replaced by $-w_\circ\lambda$) and the corresponding commutative diagram:

$$\begin{array}{ccc} \mathcal{A}(\lambda)_{\mathrm{lex}} & \xrightarrow{\ \Pi\ } & \mathrm{QLS}(-w_\circ\lambda) \\ & \searrow^{\Pi^*} & \downarrow * \\ & & \mathrm{QLS}(\lambda). \end{array} \tag{2.83}$$

We need an analogue of [LNS[3]2, Theorem 7.3] for the coheight statistic, which was defined in (2.32). This is stated as follows, and is proved in a completely similar way, based on the results in [LNS[3]2].

Theorem 2.27. *Consider an admissible subset $A \in \mathcal{A}(\lambda)_{\mathrm{lex}}$, and the corresponding QLS path $\Pi(A) \in \mathrm{QLS}(-w_\circ\lambda)$. Write $\Pi(A)$ as follows (cf. Definition 2.4):*

$$x_1 \xLeftarrow{-\sigma_1 w_\circ \lambda} x_2 \xLeftarrow{-\sigma_2 w_\circ \lambda} \cdots \xLeftarrow{-\sigma_{s-1} w_\circ \lambda} x_s, \tag{2.84}$$

with $x_i \in W^{\omega(J)}$ and $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$. Then, we have

$$\operatorname{coheight}(A) = \sum_{i=1}^{s-1} \sigma_i \, \mathrm{wt}_{-w_\circ\lambda}(x_{i+1} \Rightarrow x_i), \tag{2.85}$$

where $\mathrm{wt}_{-w_\circ\lambda}(x_{i+1} \Rightarrow x_i)$ was defined in (2.8).

We will now express the nonsymmetric Macdonald polynomial in terms of the quantum alcove model. Recall that the final direction $\phi(A)$ of an admissible subset A was defined in (2.29).

Theorem 2.28. *We have*

$$E_{w\lambda}(x;q,0) = \sum_{\substack{A\in\mathcal{A}(\lambda)\\ \lfloor\phi(A)\rfloor^J\le w}} q^{\mathrm{coheight}(A)}x^{\mathrm{wt}(A)}. \tag{2.86}$$

Proof. We derive this formula directly from Theorem 1.1, based on the map Π^*, which is known to be a weight-preserving bijection, by [LNS^{3}2, Proposition 6.7]. Using the very explicit description of the map Π^* in [LNS^{3}2, §6.1], we can see that it switches initial and final directions, i.e., for $A\in\mathcal{A}(\lambda)$ we have

$$\iota(\Pi^*(A)) = \lfloor\phi(A)\rfloor^J.$$

Finally, by using the notation (2.84) for $\Pi(A)$, we deduce:

$$\begin{aligned}\mathrm{coheight}(A) &= \sum_{u=1}^{s-1}\sigma_u\,\mathrm{wt}_{-w_\circ\lambda}(x_{u+1}\Rightarrow x_u) = -\operatorname{Deg}(S(\Pi(A)))\\ &= -\operatorname{Deg}(\omega(\Pi^*(A))) = -\operatorname{Deg}(\Pi^*(A)).\end{aligned}$$

Here the first equality is based on Theorem 2.27, the second one on [LNS^{3}2, Corollary 4.7], the third one on the above definition of the Lusztig involution S, and the last one on [LNS^{3}2, Corollary 7.4]. □

Remark 2.29. In [LNS^{3}2], we realized an appropriate tensor product of Kirillov–Reshetikhin crystals $\mathbb{B}$ in terms of $\mathrm{QLS}(\lambda)$. Based on this, we expressed the so-called "right" energy function on $\mathbb{B}$ as $\operatorname{Deg}(\eta)$ for $\eta\in\mathrm{QLS}(\lambda)$. In these terms, $\operatorname{Deg}(S(\eta))$ expresses the corresponding "left" energy function, see [LNS^{3}2, Remark 4.9]. We also realized $\mathbb{B}$ in terms of the quantum alcove model, and in this setup the two energy functions are expressed by the height and coheight statistics.

When Γ is an (arbitrary) λ-chain of roots, we denote by $\mathcal{A}(\lambda)_\Gamma$ the quantum alcove model associated to Γ. In [LL2], we defined certain combinatorial moves (called quantum Yang-Baxter moves) in the quantum alcove model, namely on the collection of $\mathcal{A}(\lambda)_\Gamma$, where Γ is any λ-chain (of roots). We showed that these define an affine crystal isomorphism between $\mathcal{A}(\lambda)_\Gamma$ and $\mathcal{A}(\lambda)_{\Gamma'}$ for any two λ-chains Γ and Γ'. We also showed that the moves preserve the weight, the height and coheight, as well as the final direction of (the path in $\mathrm{QB}(W)$ associated with) an admissible subset. Based on these facts, we can generalize Theorem 2.28.

Theorem 2.30. *Theorem 2.28 still holds if we replace the admissible subsets $\mathcal{A}(\lambda)_{\mathrm{lex}}$ for a lex λ-chain with the ones for an arbitrary λ-chain Γ, namely $\mathcal{A}(\lambda)_\Gamma$.*

Remark 2.31. The formulas in Theorems 1.1 and 2.28 (in fact, the latter can be replaced with the mentioned generalization) specialize, upon setting $q=0$, to the formulas for Demazure characters in terms of LS paths [L1, Theorem 5.2] and the alcove model [L, Theorem 6.3].

3. Graded characters of quotients of Demazure modules.

3.1. **Additional setting.** The untwisted affine Lie algebra $\mathfrak{g}_{\mathrm{af}}$ is written as: $\mathfrak{g}_{\mathrm{af}} = \mathfrak{g}\otimes\mathbb{C}[t,\,t^{-1}]\oplus\mathbb{C}c\oplus\mathbb{C}D$, where $c=\sum_{j\in I_{\mathrm{af}}}a_j^\vee\alpha_j^\vee$ is the canonical central element, and D is the scaling element (or the degree operator); note that the Cartan subalgebra $\mathfrak{h}_{\mathrm{af}}$ of $\mathfrak{g}_{\mathrm{af}}$ is $\mathfrak{h}\oplus\mathbb{C}c\oplus\mathbb{C}D$, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$.

Let us denote by $\{\alpha_i\}_{i\in I_{\mathrm{af}}}$ and $\{\alpha_i^\vee\}_{i\in I_{\mathrm{af}}}$ the simple roots and simple coroots of $\mathfrak{g}_{\mathrm{af}}$, respectively, and by $\Lambda_j\in\mathfrak{h}_{\mathrm{af}}^*$, $j\in I_{\mathrm{af}}$, the fundamental weights for $\mathfrak{g}_{\mathrm{af}}$; note that $\langle D,\,\alpha_j\rangle=\delta_{j,0}$ and $\langle D,\,\Lambda_j\rangle=0$ for $j\in I_{\mathrm{af}}$. We take a weight lattice X_{af} for $\mathfrak{g}_{\mathrm{af}}$ as follows:

$$X_{\mathrm{af}} = \left(\bigoplus_{j\in I_{\mathrm{af}}}\mathbb{Z}\Lambda_j\right)\oplus\mathbb{Z}\delta\subset\mathfrak{h}_{\mathrm{af}}^*, \tag{3.1}$$

where $\delta \in \mathfrak{h}_{\mathrm{af}}^*$ denotes the null root of $\mathfrak{g}_{\mathrm{af}}$. We think of a weight $\mu \in \mathfrak{h}^*$ for $\mathfrak{g}$ as a weight $(\in \mathfrak{h}_{\mathrm{af}}^*)$ for $\mathfrak{g}_{\mathrm{af}}$ by: $\langle c, \mu \rangle = \langle D, \mu \rangle = 0$. Then, for each $i \in I$, the fundamental weight ϖ_i for $\mathfrak{g}$ is identical to $\Lambda_i - a_i^\vee \Lambda_0 \in \mathfrak{h}_{\mathrm{af}}^*$; we call the weights $\varpi_i = \Lambda_i - a_i^\vee \Lambda_0 \in \mathfrak{h}_{\mathrm{af}}^*$, $i \in I$, the level-zero fundamental weights.

The (affine) Weyl group W_{af} of $\mathfrak{g}_{\mathrm{af}}$ is the subgroup $\langle r_j \mid j \in I_{\mathrm{af}} \rangle \subset \mathrm{GL}(\mathfrak{h}_{\mathrm{af}}^*)$ generated by the simple reflections r_j associated to α_j for $j \in I_{\mathrm{af}}$, with length function $\ell : W_{\mathrm{af}} \to \mathbb{Z}_{\geq 0}$ and unit element $e \in W_{\mathrm{af}}$; recall that $W_{\mathrm{af}} \cong W \ltimes Q^\vee$. We denote by Φ_{af} the set of real roots, and by $\Phi_{\mathrm{af}}^+ \subset \Phi_{\mathrm{af}}$ the set of positive real roots.

Definition 3.1 ([P]). Let $x \in W_{\mathrm{af}} \cong W \ltimes Q^\vee$, and write it as $x = wt_\xi$ for $w \in W$ and $\xi \in Q^\vee$. Then we define the semi-infinite length $\ell^{\frac{\infty}{2}}(x)$ of x by $\ell^{\frac{\infty}{2}}(x) := \ell(w) + 2\langle \xi, \rho \rangle$.

Now, let J be a subset of I. Following [P] (see also [LS, §10]), we define

$$(\Phi_J)_{\mathrm{af}}^+ := \left(\bigoplus_{i \in J} \mathbb{Z}\alpha_i + \mathbb{Z}\delta \right) \cap \Phi_{\mathrm{af}}^+, \tag{3.2}$$

$$(W^J)_{\mathrm{af}} := \left\{ x \in W_{\mathrm{af}} \mid x\beta \in \Phi_{\mathrm{af}}^+ \text{ for all } \beta \in (\Phi_J)_{\mathrm{af}}^+ \right\}. \tag{3.3}$$

Definition 3.2. (1) The (parabolic) semi-infinite Bruhat graph SiB^J is the Φ_{af}^+-labeled, directed graph with vertex set $(W^J)_{\mathrm{af}}$ and Φ_{af}^+-labeled, directed edges of the following form: $x \xrightarrow{\beta} r_\beta x$ for $x \in (W^J)_{\mathrm{af}}$ and $\beta \in \Phi_{\mathrm{af}}^+$, where $r_\beta x \in (W^J)_{\mathrm{af}}$ and $\ell^{\frac{\infty}{2}}(r_\beta x) = \ell^{\frac{\infty}{2}}(x) + 1$.

(2) The semi-infinite Bruhat order is a partial order $\preceq$ on $(W^J)_{\mathrm{af}}$ defined as follows: for $x, y \in (W^J)_{\mathrm{af}}$, we write $x \preceq y$ if there exists a directed path from x to y in SiB^J; also, we write $x \prec y$ if $x \preceq y$ and $x \neq y$.

Finally, let $U_q(\mathfrak{g}_{\mathrm{af}})$ denote the quantum affine algebra associated to $\mathfrak{g}_{\mathrm{af}}$ with integral weight lattice X_{af}, and E_j, F_j, $j \in I_{\mathrm{af}}$, the Chevalley generators of $U_q(\mathfrak{g}_{\mathrm{af}})$. Also, let $U_q^+(\mathfrak{g}_{\mathrm{af}})$ and $U_q^-(\mathfrak{g}_{\mathrm{af}})$ denote the subalgebras of $U_q(\mathfrak{g}_{\mathrm{af}})$ generated by E_j, $j \in I_{\mathrm{af}}$, and F_j, $j \in I_{\mathrm{af}}$, respectively.

3.2. **Extremal weight modules and Demazure modules.** For an arbitrary integral weight $\lambda \in X_{\mathrm{af}}$ of $\mathfrak{g}_{\mathrm{af}}$, let $V(\lambda)$ denote the extremal weight module of extremal weight λ over $U_q(\mathfrak{g}_{\mathrm{af}})$, which is an integrable $U_q(\mathfrak{g}_{\mathrm{af}})$-module generated by a single element v_λ with the defining relation that v_λ is an "extremal weight vector" of weight λ (for details, see [Kas1, §8] and [Kas2, §3]). We know from [Kas1, Proposition 8.2.2] that $V(\lambda)$ has a crystal basis $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ with corresponding global basis $\{G(b) \mid b \in \mathcal{B}(\lambda)\}$; we denote by u_λ the element of $\mathcal{B}(\lambda)$ such that $G(u_\lambda) = v_\lambda \in V(\lambda)$.

Now, let λ be a dominant integral weight for $\mathfrak{g}$, and set $J = J_\lambda = \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \}$; note that λ is regarded as an element of X_{af} by $\langle c, \lambda \rangle = \langle D, \lambda \rangle = 0$. For each $x \in W_{\mathrm{af}}$, we set

$$V_x^\pm(\lambda) := U_q^\pm(\mathfrak{g}_{\mathrm{af}}) S_x^{\mathrm{norm}} v_\lambda \subset V(\lambda), \tag{3.4}$$

where S_x^{norm} denotes the action of the (affine) Weyl group W_{af} on the set of extremal weight vectors (see [NS2, (3.2.1)]). We know from [Kas3, §2.8] (see also [NS2, §4.1]) that there exists a subset $\mathcal{B}_x^\pm(\lambda)$ of the crystal basis $\mathcal{B}(\lambda)$ such that $\{G(b) \mid b \in \mathcal{B}_x^\pm(\lambda)\}$ is the global basis of $V_x^\pm(\lambda)$.

3.3. **Quotients of Demazure modules and their graded characters.** We fix a dominant integral weight λ for $\mathfrak{g}$. As in [NS2, §7.2], we set

$$Z_{w_\circ}^+(\lambda) := \sum_{\substack{\mathbf{c}_0 \in \overline{\mathrm{Par}(\lambda)} \\ \mathbf{c}_0 \neq (\emptyset)_{i \in I}}} U_q^+(\mathfrak{g}_{\mathrm{af}}) S_{\mathbf{c}_0} S_{w_\circ}^{\mathrm{norm}} v_\lambda.$$

Here, $\overline{\mathrm{Par}(\lambda)}$ denotes a certain set of multi-partitions indexed by I (see [NS2, (2.5.1)]), and $S_{\mathbf{c}_0} \in U_q^+(\mathfrak{g}_{\mathrm{af}})$ denotes the PBW-type basis element of weight $|\mathbf{c}_0|\delta$ corresponding to the "purely imaginary part" (see [BN, page 352]), where $|\mathbf{c}_0|$ is the sum of all parts in the multi-partition $\mathbf{c}_0$. Notice that $Z^+_{w_\circ}(\lambda) \subset V^+_{w_\circ}(\lambda) = U_q^+(\mathfrak{g}_{\mathrm{af}})S^{\mathrm{norm}}_{w_\circ}v_\lambda$ since $S_{\mathbf{c}_0} \in U_q^+(\mathfrak{g}_{\mathrm{af}})$ for all $\mathbf{c}_0 \in \overline{\mathrm{Par}(\lambda)}$.

Now, let $w \in W$; in what follows, we may assume that $w \in W^J \subset (W^J)_{\mathrm{af}}$ since $V^+_w(\lambda) = V^+_{\lfloor w \rfloor}(\lambda)$ for $w \in W$ by [NS2, Lemma 4.1.2]. Then, noting that $V^+_w(\lambda) = V^+_{\lfloor w \rfloor}(\lambda) \subset V^+_{\lfloor w_\circ \rfloor}(\lambda) = V^+_{w_\circ}(\lambda)$ by [NS2, Corollary 5.2.5] since $\lfloor w \rfloor \preceq \lfloor w_\circ \rfloor$, we define $U^+_w(\lambda)$ to be the image of $V^+_w(\lambda)$ under the canonical projection $V^+_{w_\circ}(\lambda) \twoheadrightarrow V^+_{w_\circ}(\lambda)/Z^+_{w_\circ}(\lambda)$. We write the weight space decomposition of $U^+_w(\lambda)$ with respect to $\mathfrak{h}_{\mathrm{af}}$ as:

$$U^+_w(\lambda) = \bigoplus_{\gamma \in Q,\, k \in \mathbb{Z}} U^+_w(\lambda)_{\lambda-\gamma+k\delta},$$

and define the graded character $\mathrm{gch}\, U^+_w(\lambda)$ of $U^+_w(\lambda)$ to be

$$\mathrm{gch}\, U^+_w(\lambda) := \sum_{\gamma \in Q,\, k \in \mathbb{Z}} \dim U^+_w(\lambda)_{\lambda-\gamma+k\delta}\, x^{\lambda-\gamma} q^k, \qquad \text{where } q = x^\delta.$$

The following is the main result of this section.

Theorem 3.3. *Keep the notation and setting above. We have*

$$\mathrm{gch}\, U^+_w(\lambda) = E_{w\lambda}(x\,;\, q,\, 0).$$

3.4. **Semi-infinite Lakshmibai-Seshadri paths.** We keep the notation and setting of §3.3; recall that $\lambda = \sum_{i \in I} m_i \varpi_i$ is a dominant integral weight for $\mathfrak{g}$, and $J = J_\lambda = \{ i \in I \mid \langle \alpha_i^\vee,\, \lambda \rangle = 0 \} \subset I$.

Definition 3.4. For a rational number $0 < \tau < 1$, define $\mathrm{SiB}(\lambda\,;\,\tau)$ to be the subgraph of SiB^J with the same vertex set but having only the edges of the form: $x \xrightarrow{\beta} y$ with $\tau\langle \beta^\vee,\, x\lambda \rangle \in \mathbb{Z}$.

Definition 3.5. A semi-infinite Lakshmibai-Seshadri path (SiLS path for short) of shape λ is, by definition, a pair $(\boldsymbol{y}\,;\,\boldsymbol{\tau})$ of a (strictly) decreasing sequence $\boldsymbol{y} : y_1 \succ \cdots \succ y_s$ of elements in $(W^J)_{\mathrm{af}}$ and an increasing sequence $\boldsymbol{\tau} : 0 = \tau_0 < \tau_1 < \cdots < \tau_s = 1$ of rational numbers satisfying the condition that there exists a directed path from y_{u+1} to y_u in $\mathrm{SiB}(\lambda\,;\,\tau_u)$ for each $u = 1, 2, \ldots, s-1$. We denote by $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ the set of all SiLS paths of shape λ.

In [INS, §3.1], we defined root operators e_j and f_j, $j \in I_{\mathrm{af}}$, on $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$, and proved that the set $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$, equipped with these root operators, is a crystal with weights in X_{af}.

Theorem 3.6 ([INS, Theorem 3.2.1])**.** *Keep the notation and setting above. There exists an isomorphism of crystals between the crystal basis $\mathcal{B}(\lambda)$ of the extremal weight module $V(\lambda)$ of extremal weight λ and the crystal $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ of SiLS paths of shape λ.*

Remark 3.7. For $\pi = (y_1, \ldots, y_s\,;\, \tau_0, \tau_1, \ldots, \tau_s) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$, we define the piecewise-linear continuous map $\overline{\pi} : [0,1] \to \mathbb{R} \otimes_{\mathbb{Z}} X_{\mathrm{af}}$ by

$$\overline{\pi}(t) = \sum_{p=1}^{u-1} (\tau_p - \tau_{p-1}) y_p \lambda + (t - \tau_{u-1}) y_u \lambda \quad \text{for } \tau_{u-1} \le t \le \tau_u,\ 1 \le u \le s. \tag{3.5}$$

Then we know from [INS, Proposition 3.1.3] that $\overline{\pi}$ is a Lakshmibai-Seshadri (LS for short) path of shape λ; for the definition of an LS path of shape λ, see [L2] and [LNS^{3}2, §2.2 and 2.3]. We denote by $\mathbb{B}(\lambda)$ the set of LS paths of shape λ. In fact, the map $\overline{} : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \to \mathbb{B}(\lambda)$, $\pi \mapsto \overline{\pi}$, is a surjective, strict crystal morphism.

Define a surjective map $\mathrm{cl} : (W^J)_{\mathrm{af}} \twoheadrightarrow W^J$ by

$$\mathrm{cl}(x) := w \quad \text{if } x = wzt_\xi \text{ for } w \in W^J,\ z \in W_J, \text{ and } \xi \in Q^\vee.$$

Then, for $\pi = (y_1, \ldots, y_s\,;\, \tau_0, \tau_1, \ldots, \tau_s) \in \mathbb{B}^{\frac{\infty}{2}}(\lambda)$, we set (see [NS2, Remark 6.2.1])

$$\mathrm{cl}(\pi) := (\mathrm{cl}(y_1), \ldots, \mathrm{cl}(y_s)\,;\, \tau_0, \tau_1, \ldots, \tau_s);$$

for each $1 \le p < q \le s$ such that $\mathrm{cl}(y_p) = \cdots = \mathrm{cl}(y_q)$, we drop $\mathrm{cl}(y_p), \ldots, \mathrm{cl}(y_{q-1})$ and $\tau_p, \ldots, \tau_{q-1}$ from this expression of $\mathrm{cl}(\pi)$.

Let $\mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$ denote the connected component of $\mathbb{B}^{\frac{\infty}{2}}(\lambda)$ containing $\pi_e := (e\,;\,0,\,1)$. We know from [NS2, Lemma 7.1.2] that for each $\eta \in \mathrm{QLS}(\lambda) = \mathbb{B}(\lambda)_{\mathrm{cl}}$ (see Remark 2.5), there exists a unique element $\pi_\eta = (y_1, \ldots, y_s\,;\,\boldsymbol{\tau}) \in \mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$ such that $\iota(\pi_\eta) := y_1 \in W^J$ and $\mathrm{cl}(\pi_\eta) = \eta$. We claim that

$$\mathrm{wt}(\pi_\eta) = \lambda - \beta - \mathrm{Deg}(\eta)\delta, \qquad \text{where } \beta \in Q^+ := \sum_{j \in I} \mathbb{Z}_{\ge 0}\alpha_j. \tag{3.6}$$

Indeed, since $\mathrm{wt}(\pi_\eta) = \mathrm{wt}(\overline{\pi_\eta})$ by their definitions, it suffices to show that $\overline{\pi_\eta} \in \mathbb{B}(\lambda)$ satisfies the following conditions (see [NS1, Proposition 3.1.3] and [LNS[3]2, §4.2 and Theorem 4.5]):

(a) $\mathrm{cl}(\overline{\pi_\eta}(t)) = \eta(t)$ for all $t \in [0,1]$, where $\mathrm{cl} : \mathbb{R} \otimes_{\mathbb{Z}} X_{\mathrm{af}} \twoheadrightarrow (\mathbb{R} \otimes_{\mathbb{Z}} X_{\mathrm{af}})/\mathbb{R}\delta$ denotes the canonical projection;

(b) $\overline{\pi_\eta}$ is contained in the connected component $\mathbb{B}_0(\lambda)$ of $\mathbb{B}(\lambda)$ containing π_λ, where $\pi_\lambda(t) := t\lambda$ for $t \in [0,1]$;

(c) $\iota(\overline{\pi_\eta}) = y_1\lambda \in \lambda - Q^+$.

If $x \in W_{\mathrm{af}}$ is of the form: $x = wzt_\xi$ with $w \in W^J$, $z \in W_J$, and $\xi \in Q^\vee$, then $x\lambda = w\lambda - \langle \xi, \lambda\rangle\delta$ (recall that $\langle c, \lambda\rangle = 0$), and hence $x\lambda \equiv w\lambda$ modulo $\mathbb{R}\delta$. Therefore, the assertion (a) is obvious from the definitions of $\overline{} : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \to \mathbb{B}(\lambda)$ and the maps cl. Also, since $\pi_\eta \in \mathbb{B}_0^{\frac{\infty}{2}}(\lambda)$, there exists a monomial Y in root operators such that $\pi_\eta = Y\pi_e$. Because $\overline{} : \mathbb{B}^{\frac{\infty}{2}}(\lambda) \to \mathbb{B}(\lambda)$ commutes with the action of root operators, we have $\overline{\pi_\eta} = \overline{Y\pi_e} = Y\overline{\pi_e} = Y\pi_\lambda$. Hence we obtain $\overline{\pi_\eta} \in \mathbb{B}_0(\lambda)$. Finally, since $\iota(\pi_\eta) = y_1 \in W^J$ and λ is a dominant integral weight for $\mathfrak{g}$, it follows that $\iota(\overline{\pi_\eta}) = y_1\lambda$ is contained in $\lambda - Q^+$. This proves (3.6).

3.5. **Proof of Theorem 3.3.** We know from [NS2, Theorem 7.2.2 (1)] that there exists a subset $\mathcal{B}(Z_{w_\circ}^+(\lambda))$ of $\mathcal{B}(\lambda)$ such that $\{G(b) \mid b \in \mathcal{B}(Z_{w_\circ}^+(\lambda))\}$ is the global basis of $Z_{w_\circ}^+(\lambda)$. Also, recall that $\{G(b) \mid b \in \mathcal{B}_w^+(\lambda)\}$ is the global basis of $V_w^+(\lambda)$. Therefore,

$$\{G(b) \bmod Z_{w_\circ}^+(\lambda) \mid b \in \mathcal{B}(U_w^+(\lambda)) := \mathcal{B}_w^+(\lambda) \setminus \mathcal{B}(Z_{w_\circ}^+(\lambda))\}$$

is the global basis of $U_w^+(\lambda)$, which is the image of $V_w^+(\lambda)$ under the canonical projection $V_{w_\circ}^+(\lambda) \twoheadrightarrow V_{w_\circ}^+(\lambda)/Z_{w_\circ}^+(\lambda)$.

We know from [NS2, Theorem 7.2.2 (2)] that there exists an isomorphism $\Psi_\lambda^\vee : \mathcal{B}(\lambda) \xrightarrow{\sim} \mathbb{B}^{\frac{\infty}{2}}(\lambda)$ of crystals, which maps $\mathcal{B}(U_w^+(\lambda)) \subset \mathcal{B}(\lambda)$ to

$$\{\pi_\eta \mid \eta \in \mathrm{QLS}(\lambda) \text{ such that } w \ge \iota(\pi_\eta)\} \subset \mathbb{B}^{\frac{\infty}{2}}(\lambda).$$

Since $\iota(\pi_\eta) \in W^J$, we see that $\iota(\pi_\eta) = \iota(\eta)$. Therefore, the subset above is identical to the set $\{\pi_\eta \mid \eta \in \mathrm{QLS}_w(\lambda)\}$. Hence we compute:

$$\begin{aligned}
\mathrm{gch}\, U_w^+(\lambda) &= \left(\sum_{b \in \mathcal{B}(U_w^+(\lambda))} x^{\mathrm{wt}(b)} \right)\Bigg|_{x^\delta = q} = \left(\sum_{\eta \in \mathrm{QLS}_w(\lambda)} x^{\mathrm{wt}(\pi_\eta)} \right)\Bigg|_{x^\delta = q} \\
&= \left(\sum_{\eta \in \mathrm{QLS}_w(\lambda)} x^{\mathrm{wt}(\eta) - \mathrm{Deg}(\eta)\delta} \right)\Bigg|_{x^\delta = q} \qquad \text{by (3.6)} \\
&= \sum_{\eta \in \mathrm{QLS}_w(\lambda)} q^{-\mathrm{Deg}(\eta)} x^{\mathrm{wt}(\eta)} = E_{w\lambda}(x\,;\,q,\,0) \qquad \text{by Theorem 1.1.}
\end{aligned}$$

This completes the proof of Theorem 3.3.

Appendix.

Appendix A. Recursive formulas in terms of Demazure operators.

We use the notation of §2.1 and §3.1. Fix a dominant integral weight $\lambda \in X$, and set $J = J^{\lambda} = \{i \in I \mid \langle \alpha_i^{\vee}, \lambda \rangle = 0\}$. For each $i \in I$, we define a $\mathbb{Z}[q]$-linear operator D_i (called a Demazure operator) on $(\mathbb{Z}[q])[e^{\xi} ; \xi \in X]$ by:

$$D_i(e^{\xi}) := \frac{e^{\xi+\rho} - e^{r_i(\xi+\rho)}}{1 - e^{-\alpha_i}} e^{-\rho}$$

$$= \begin{cases} e^{\xi}\bigl(1 + e^{-\alpha_i} + \cdots + e^{-n\alpha_i}\bigr) & \text{if } n = \langle \alpha_i^{\vee}, \xi \rangle \geq 0, \\ 0 & \text{if } n = \langle \alpha_i^{\vee}, \xi \rangle = -1, \\ -e^{\xi}\bigl(e^{\alpha_i} + \cdots + e^{(-n-1)\alpha_i}\bigr) & \text{if } n = \langle \alpha_i^{\vee}, \xi \rangle \leq -2. \end{cases} \tag{A.1}$$

In this appendix, we give a recursive formula for $\operatorname{gch} \mathrm{QLS}_w(\lambda)$ (Proposition A.1) and one for $E_{w\lambda}(x\,;\,q,\,0)$ (Proposition A.4), both of which are in terms of Demazure operators.

A.1. Recursive formula for $\operatorname{gch} \mathrm{QLS}_w(\lambda)$.

Proposition A.1. *Let $w \in W^J$ and $i \in I$ be such that $w > r_i w$; note that $r_i w \in W^J$ by* [LNS[3]1, Lemma 5.8]. *Then we have*

$$\operatorname{gch} \mathrm{QLS}_w(\lambda) = D_i \operatorname{gch} \mathrm{QLS}_{r_i w}(\lambda).$$

Let $U_q'(\mathfrak{g}_{\mathrm{af}})$ denote the quantum affine algebra without the degree operator associated to $\mathfrak{g}_{\mathrm{af}}$. We know that the set $\mathrm{QLS}(\lambda) = \mathbb{B}(\lambda)_{\mathrm{cl}}$ (see Remark 2.5), equipped with root operators e_j, f_j, $j \in I_{\mathrm{af}}$, is a $U_q'(\mathfrak{g}_{\mathrm{af}})$-crystal; for the definition of root operators, see [LNS[3]2, §2.3] and [NS1, §2.2]. We prove Proposition A.1 by using this $U_q'(\mathfrak{g}_{\mathrm{af}})$-crystal structure on $\mathrm{QLS}(\lambda) = \mathbb{B}(\lambda)_{\mathrm{cl}}$ (cf. [L1, Theorem in §5.2]).

Lemma A.2 (recursive relation)**.** *Let $w \in W^J$ and $i \in I$ be such that $w > r_i w$. We have*

$$\mathrm{QLS}_w(\lambda) = \bigcup_{p \geq 0} f_i^p \mathrm{QLS}_{r_i w}(\lambda) \setminus \{\mathbf{0}\}.$$

Proof. First we prove the inclusion $\subset$. Let $\eta \in \mathrm{QLS}_w(\lambda)$, and set $\eta' := e_i^{\max}\eta$. It suffices to show that $\eta' \in \mathrm{QLS}_{r_i w}(\lambda)$; for simplicity of notation, we set $x := \iota(\eta) \in W^J$. If $\iota(\eta') = \iota(\eta) = x$, then it follows from the definition of the root operator e_i that

$$\langle \alpha_i^{\vee}, x\lambda \rangle = \langle \alpha_i^{\vee}, \iota(\eta)\lambda \rangle = \langle \alpha_i^{\vee}, \iota(\eta')\lambda \rangle \geq 0,$$

since $e_i\eta' = \mathbf{0}$. Because $\eta \in \mathrm{QLS}_w(\lambda)$ by the assumption, we have $x = \iota(\eta) \leq w$. Also, from the assumption that $w > r_i w$ and $w \in W^J$, it follows that $r_i w \in W^J$ and $\langle \alpha_i^{\vee}, w\lambda \rangle < 0$ by [LNS[3]1, Lemmas 5.8 and 5.9]. Therefore, we deduce from [L2, Lemma 4.1 a)] applied to $x \leq w$ that $x \leq r_i w$, and hence $\iota(\eta') = \iota(\eta) = x \leq r_i w$. Thus we obtain $\eta' \in \mathrm{QLS}_{r_i w}(\lambda)$, as desired. If $\iota(\eta') \neq \iota(\eta)$, then it follows from the definition of the root operator e_i that $\iota(\eta') = r_i\iota(\eta) = r_i x$ and

$$\langle \alpha_i^{\vee}, x\lambda \rangle = -\langle \alpha_i^{\vee}, r_i x\lambda \rangle = -\langle \alpha_i^{\vee}, r_i\iota(\eta)\lambda \rangle = -\langle \alpha_i^{\vee}, r_i\iota(\eta')\lambda \rangle < 0.$$

Since $x = \iota(\eta) \leq w$ by the assumption and $\langle \alpha_i^{\vee}, w\lambda \rangle < 0$ as seen above, we deduce from [L2, Lemma 4.1 c)] applied to $x \leq w$ that $r_i x \leq r_i w$, and hence $\iota(\eta') = r_i x \leq r_i w$. Therefore, we obtain $\eta' \in \mathrm{QLS}_{r_i w}(\lambda)$, as desired. This proves the inclusion $\subset$.

Next we prove the opposite inclusion $\supset$. Let $\eta' \in \mathrm{QLS}_{r_i w}(\lambda)$, and assume that $\eta := f_i^p\eta' \neq \mathbf{0}$ for some $p \geq 0$. If $\iota(\eta) = \iota(\eta')$, then

$$\iota(\eta) = \iota(\eta') \leq r_i w < w,$$

and hence $\eta \in \mathrm{QLS}_w(\lambda)$. Assume now that $\iota(\eta) \neq r_i\iota(\eta')$, and hence $\iota(\eta) = r_i\iota(\eta')$; for simplicity of notation, we set $x' := \iota(\eta') \in W^J$. Then we see from the definition of the root operator f_i that $\langle \alpha_i^\vee, x'\lambda\rangle = \langle \alpha_i^\vee, \iota(\eta')\lambda\rangle > 0$. Also, we have $\langle \alpha_i^\vee, r_iw\lambda\rangle = -\langle \alpha_i^\vee, w\lambda\rangle > 0$ as seen above and $x' = \iota(\eta') \leq r_iw$ by the assumption. It follows from [L2, Lemma 4.1 c)] applied to $x' \leq r_iw$ that $r_ix' \leq w$, and hence $\iota(\eta) = r_ix' \leq w$. Therefore, we obtain $\eta \in \mathrm{QLS}_w(\lambda)$, as desired. This proves the opposite inclusion $\supset$. This completes the proof of the lemma. $\square$

For $i \in I$ and $\eta \in \mathrm{QLS}(\lambda)$, let $S_i(\eta)$ denote the α_i-string through η, that is,

$$S_i(\eta) := \left\{e_i^p\eta,\ f_i^q\eta \mid p,\ q \geq 0\right\} \setminus \{\mathbf{0}\}.$$

Lemma A.3 (string property). *Let $\eta \in \mathrm{QLS}(\lambda)$, and $i \in I$. For $z \in W^J$,*

$$\mathrm{QLS}_z(\lambda) \cap S_i(\eta) = \emptyset,\ \left\{e_i^{\max}\eta\right\},\ \text{or } S_i(\eta).$$

Proof. For simplicity of notation, we set $\eta' := e_i^{\max}\eta$. We will prove that if $\mathrm{QLS}_z(\lambda) \cap S_i(\eta)$ is neither $\emptyset$ nor $\left\{e_i^{\max}\eta\right\}$, then $\mathrm{QLS}_z(\lambda) \cap S_i(\eta) = S_i(\eta)$, or equivalently, $S_i(\eta) \subset \mathrm{QLS}_z(\lambda)$. By our assumption, $\mathrm{QLS}_z(\lambda) \cap S_i(\eta)$ contains an element η'' that is not η'. We can write the element η'' as $\eta'' = f_i^p\eta'$ for some $p \geq 1$. Here, from the definition of the root operator f_i, we can deduce that

$$\iota(f_i\eta') = \iota(f_i^2\eta') = \cdots = \iota(f_i^p\eta') = \cdots = \iota(f_i^{\max}\eta').$$

Since $\iota(f_i^p\eta') = \iota(\eta'') \leq z$ by the assumption that $\eta'' \in \mathrm{QLS}_z(\lambda)$, we see that the elements $f_i\eta',\ f_i^2\eta',\ \ldots,\ f_i^{\max}\eta'$ are all contained in $\mathrm{QLS}_z(\lambda)$. Namely,

$$S_i(\eta) \setminus \left\{\eta'\right\} \subset \mathrm{QLS}_z(\lambda). \tag{A.2}$$

Hence it remains to show that $\eta' \in \mathrm{QLS}_z(\lambda)$. If $\iota(\eta') = \iota(\eta'')$, then we have $\iota(\eta') \leq z$ since $\iota(\eta'') \leq z$ by the assumption that $\eta'' \in \mathrm{QLS}_z(\lambda)$. This implies that $\eta' \in \mathrm{QLS}_z(\lambda)$. Assume now that $\iota(\eta'') \neq r_i\iota(\eta')$, and hence that $\iota(\eta'') = r_i\iota(\eta')$. Then, by the definition of the root operator f_i, we see that $\langle \alpha_i^\vee, \iota(\eta')\lambda\rangle > 0$. Therefore, we deduce that $\langle \alpha_i^\vee, \iota(\eta'')\lambda\rangle = \langle \alpha_i^\vee, r_i\iota(\eta')\lambda\rangle < 0$, and hence that $\iota(\eta') = r_i\iota(\eta'') < \iota(\eta'') \leq z$. Thus we obtain $\eta' \in \mathrm{QLS}_z(\lambda)$. Combining this with (A.2), we conclude that $S_i(\eta) \subset \mathrm{QLS}_z(\lambda)$, as desired. This completes the proof of the lemma. $\square$

Proof of Proposition A.1. First, we show that for each $\eta \in \mathrm{QLS}(\lambda)$,

$$\mathrm{QLS}_w(\lambda) \cap S_i(\eta) = \emptyset \text{ or } S_i(\eta). \tag{A.3}$$

Now, assume that $\mathrm{QLS}_w(\lambda) \cap S_i(\eta) \neq \emptyset$. Then, we see from Lemma A.3 that $\mathrm{QLS}_w(\lambda) \cap S_i(\eta) = \left\{e_i^{\max}\eta\right\}$ or $S_i(\eta)$; in both cases, we have $e_i^{\max}\eta \in \mathrm{QLS}_w(\lambda)$. Here we recall from the proof of Lemma A.2 that if $\psi \in \mathrm{QLS}_w(\lambda)$, then $e_i^{\max}\psi \in \mathrm{QLS}_{r_iw}(\lambda)$. Hence it follows that $e_i^{\max}(e_i^{\max}\eta) = e_i^{\max}\eta$ is contained in $\mathrm{QLS}_{r_iw}(\lambda)$. Therefore, we see from Lemma A.2 that $f_i^pe_i^{\max}\eta \in \mathrm{QLS}_w(\lambda)$ for all $p \geq 0$ unless $f_i^pe_i^{\max}\eta = \mathbf{0}$. From this, we conclude that $S_i(\eta) \subset \mathrm{QLS}_w(\lambda)$, as desired.

From (A.3), we deduce that $\mathrm{QLS}_w(\lambda)$ decomposes into a disjoint union of α_i-strings:

$$\mathrm{QLS}_w(\lambda) = S^{(1)} \sqcup S^{(2)} \sqcup \cdots \sqcup S^{(n)}, \quad \text{where } S^{(m)} \text{ is an } \alpha_i\text{-string for each } 1 \leq m \leq n.$$

Since $i \in I$, the degree function Deg is constant on $S^{(m)}$ for each $1 \leq m \leq n$ (see [LNS³2, (4.2)]); we set $d_m := \mathrm{Deg}\,|_{S^{(m)}}$ for $1 \leq m \leq n$. Then we have

$$\mathrm{gch}\,\mathrm{QLS}_w(\lambda) = \sum_{m=1}^{n} q^{-d_m} \sum_{\eta \in S^{(m)}} e^{\mathrm{wt}(\eta)}.$$

Next, let us consider the intersection $\mathrm{QLS}_{r_iw}(\lambda) \cap S^{(m)}$ for each $1 \leq m \leq n$. Recall that if $\psi \in \mathrm{QLS}_w(\lambda)$, then $e_i^{\max}\psi \in \mathrm{QLS}_{r_iw}(\lambda)$. Since $S^{(m)} \subset \mathrm{QLS}_w(\lambda)$, it follows from the above that $\mathrm{QLS}_{r_iw}(\lambda)$ contains a unique element $\eta_m \in S^{(m)}$ such that $e_i\eta_m = \mathbf{0}$; in particular, $\mathrm{QLS}_{r_iw}(\lambda) \cap S^{(m)} \neq \emptyset$. Therefore, from Lemma A.3, we deduce that

$$\mathrm{QLS}_{r_iw}(\lambda) \cap S^{(m)} = \left\{\eta_m\right\} \text{ or } S^{(m)} \quad \text{for each } 1 \leq m \leq n;$$

here we assume that

$$\mathrm{QLS}_{r_i w}(\lambda) \cap S^{(m)} = \begin{cases} \{\eta_m\} & \text{for } 1 \le m \le p, \\ S^{(m)} & \text{for } p+1 \le m \le n, \end{cases}$$

for some $0 \le p \le n$ for simplicity of notation. Then, we have

$$\operatorname{gch} \mathrm{QLS}_{r_i w}(\lambda) = \sum_{m=1}^{p} q^{-d_m} e^{\mathrm{wt}(\eta_m)} + \sum_{m=p+1}^{n} q^{-d_m} \sum_{\eta \in S^{(m)}} e^{\mathrm{wt}(\eta)}.$$

Combining all the above, we compute:

$$\begin{aligned} D_i \operatorname{gch} \mathrm{QLS}_{r_i w}(\lambda) &= \sum_{m=1}^{p} q^{-d_m} D_i e^{\mathrm{wt}(\eta_m)} + \sum_{m=p+1}^{n} q^{-d_m} D_i \underbrace{\Bigg(\sum_{\eta \in S^{(m)}} e^{\mathrm{wt}(\eta)} \Bigg)}_{\substack{= D_i e^{\mathrm{wt}(\eta_m)} \\ \text{by (A.1)}}} \\ &= \sum_{m=1}^{p} q^{-d_m} D_i e^{\mathrm{wt}(\eta_m)} + \sum_{m=p+1}^{n} q^{-d_m} \underbrace{D_i D_i e^{\mathrm{wt}(\eta_m)}}_{= D_i e^{\mathrm{wt}(\eta_m)}} \\ &= \sum_{m=1}^{n} q^{-d_m} D_i e^{\mathrm{wt}(\eta_m)} = \sum_{m=1}^{n} q^{-d_m} \sum_{\eta \in S^{(m)}} e^{\mathrm{wt}(\eta)} \quad \text{by (A.1)} \\ &= \operatorname{gch} \mathrm{QLS}_{w}(\lambda). \end{aligned}$$

This completes the proof of the proposition. □

A.2. **Recursive formula for $E_{w\lambda}(x\,;\,q,\,0)$.** In view of Theorem 1.1, Proposition A.1 is equivalent to the following proposition.

Proposition A.4. *Let $w \in W^J$ and $i \in I$ be such that $w > r_i w$; note that $r_i w \in W^J$ by* [LNS[3]1, Lemma 5.8]. *Then we have*

$$E_{w\lambda}(x\,;\,q,\,0) = D_i E_{r_i w \lambda}(x\,;\,q,\,0).$$

We can also show this proposition by using the polynomial representation of the double affine Hecke algebra as follows.

Proof. Note that $r_i w \in W^J$ and $\langle \alpha_i^\vee,\, r_i w \lambda \rangle > 0$ by [LNS[3]1, Lemmas 5.8 and 5.9]. We set $\mu := r_i w \lambda$. Since $\langle \alpha_i^\vee,\, \mu \rangle = \langle \alpha_i^\vee,\, r_i w\lambda \rangle > 0$ as seen above, it follows from [M, (5.10.7)] that

$$\left(t^2 r_i + (t^2 - 1) \frac{1 - r_i}{1 - e^{\alpha_i}} - (t^2 - 1) \frac{1}{1 - Y^{-\alpha_i}} \right) \cdot E_\mu(x\,;\,q,\,t) = E_{r_i \mu}(x\,;\,q,\,t). \tag{A.4}$$

Also, we know from [M, (5.2.2')] (in the notation thereof) that

$$Y^{-\alpha_i} E_\mu(x\,;\,q,\,t) = q^{\langle \alpha_i^\vee,\, \mu \rangle} t^{-2\langle v(\mu)\alpha_i^\vee,\, \rho \rangle} E_\mu(x\,;\,q,\,t).$$

Since $\langle \alpha_i^\vee,\, \mu \rangle > 0$ as seen above, it follows that $\langle v(\mu)\alpha_i^\vee,\, v(\mu)\mu \rangle = \langle \alpha_i^\vee,\, \mu \rangle > 0$. Since $v(\mu)\mu$ is antidominant by the definition of $v(\mu)$, we see that $v(\mu)\alpha_i^\vee$ is a negative coroot, and hence $-2\langle v(\mu)\alpha_i^\vee,\, \rho \rangle > 0$. Therefore, by taking the limit $t \to 0$, we deduce from (A.4) that

$$\left(\frac{r_i - 1}{1 - e^{\alpha_i}} + 1 \right) E_\mu(x\,;\,q,\,0) = E_{r_i \mu}(x\,;\,q,\,0).$$

We see by direct computation that

$$\frac{r_i - 1}{1 - e^{\alpha_i}} + 1 = D_i.$$

Thus, we obtain $D_i E_\mu(x\,;\,q,\,0) = E_{r_i\mu}(x\,;\,q,\,0)$, and hence $E_{w\lambda}(x\,;\,q,\,0) = D_i E_{r_i w\lambda}(x\,;\,q,\,0)$, as desired. This proves Proposition A.4. □

Remark A.5. If we employ the special case $w = e$ of Theorem 1.1 as the start of the induction and use Proposition A.1 and Proposition A.4 (proved as above) in the induction step, then we can give an inductive proof of Theorem 1.1.

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