A UNIFORM MODEL FOR KIRILLOV-RESHETIKHIN CRYSTALS I: LIFTING THE PARABOLIC QUANTUM BRUHAT GRAPH

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Abstract. We lift the parabolic quantum Bruhat graph into the Bruhat order on the affine Weyl group and into Littelmann's poset on level-zero weights. Also, we establish a quantum analogue of Deodhar's Bruhat-minimum lift from a parabolic quotient of the Weyl group. This result asserts a remarkable compatibility of the quantum Bruhat graph on the Weyl group, with the cosets for every parabolic subgroup.

The results in this paper will be applied in a second paper to establish a uniform construction of tensor products of one-column Kirillov-Reshetikhin (KR) crystals, and the equality, for untwisted affine root systems, between the Macdonald polynomial with t set to zero and the graded character of tensor products of KR modules.

Contents

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1. Introduction

Our goal in this series of papers is to obtain a uniform construction of tensor products of one-column Kirillov-Reshetikhin (KR) crystals. As a consequence we shall prove the equality $P_\lambda(q) = X_\lambda(q)$, where $P_\lambda(q)$ is the Macdonald polynomial $P_\lambda(q,t)$ specialized at $t=0$ and $X_\lambda(q)$ is the graded character of a simple Lie algebra coming from tensor products of KR modules. Both the Macdonald polynomials and KR modules are of arbitrary untwisted affine type. The parameter λ is a dominant weight for the simple Lie subalgebra obtained by removing the affine node. Macdonald polynomials and characters of KR modules have been studied extensively in connection with various fields such as statistical mechanics and integrable systems, representation theory of Coxeter groups and Lie algebras (and their quantized analogues given by Hecke algebras and quantized universal enveloping algebras), geometry of singularities of Schubert varieties, and combinatorics.

Our point of departure is a theorem of Ion [Ion], which asserts that the nonsymmetric Macdonald polynomials at $t=0$ are characters of Demazure submodules of highest weight modules over affine algebras. This applies for the Langlands duals of untwisted affine root systems (and type $A_{2n}^{(2)}$ in the case of nonsymmetric Koornwinder polynomials). Our results apply to the untwisted affine root systems. The overlapping cases are the simply-laced affine root systems $A_n^{(1)}$, $D_n^{(1)}$ and $E_{6,7,8}^{(1)}$.

It is known [FL1, FL2, FSS, KMOU, KMOTU, ST, Na1, Na2] that certain affine Demazure characters (including those for the simply-laced affine root systems) can be expressed in terms of KR crystals, which motivates the relation between P and X. For types $A_n^{(1)}$ and $C_n^{(1)}$, the above mentioned relation between P and X was achieved in [Le, LeS] by establishing a combinatorial formula for the Macdonald polynomials at $t=0$ from the Ram–Yip formula [RY], and by using explicit models for the one-column KR crystals [FOS]. It should be noted that, in types $A_n^{(1)}$ and $C_n^{(1)}$, the one-column KR modules are irreducible when restricted to the canonical simple Lie subalgebra, while in general this is not the case. For the cases considered by Ion [Ion], the corresponding KR crystals are perfect. This is not necessarily true for the untwisted affine root systems considered in this work, especially for the untwisted non-simply-laced affine root systems.

In this work we provide a type-free approach to the connection between P and X for untwisted affine root systems. Lenart's specialization [Le] of the Ram–Yip formula for Macdonald polynomials uses paths in the quantum Bruhat graph, which was defined and studied in [BFP] in relation to the quantum cohomology of the flag variety. On the other hand, Naito and Sagaki [NS1, NS2, NS3, NS4] gave models for tensor products of KR crystals of one-column type in terms of projections of level-zero Lakshmibai–Seshadri (LS) paths to the classical weight lattice. Hence we need to bridge the gap between these two approaches by establishing a bijection between paths in the quantum Bruhat graph and projected level-zero LS paths. For crystal graphs of integrable highest weight modules over quantized universal enveloping algebras of Kac-Moody algebras, Lenart and Postnikov had already established a bijection between LS paths and their alcove model [LP1]. This bijection was refined and reformulated in [LeSh] using Littelmann's direct characterization of LS paths [Li] and Deodhar's lifting construction for Coxeter groups [De].

In this first paper we set the stage for the connection between the level-zero LS paths [NS1, NS2, NS3, NS4] and the quantum alcove model [LeL]. We begin by establishing a *first lift* from the parabolic quantum Bruhat graph (PQBG) to the Bruhat order of the affine Weyl group. This is a parabolic analogue of the fact that the quantum Bruhat graph can be lifted to the affine Bruhat order [LS], which is the combinatorial structure underlying Peterson's theorem [P]; the latter equates the Gromov-Witten invariants of finite-dimensional homogeneous spaces G/P with the Pontryagin homology structure constants of Schubert varieties in the affine Grassmannian. We obtain Diamond Lemmas for the PQBG via projection of the standard Diamond Lemmas for the affine Weyl group. We find a *second lift* of the PQBG into a poset of Littelmann [Li] for level-zero weights and characterize its local structure (such as cover relations) in terms of the PQBG. Littelmann's poset was defined in connection with LS paths for arbitrary (not necessarily dominant) weights, but the local structure was not previously known. Finally, we prove the tilted Bruhat theorem, which is a quantum Bruhat graph analogue of the Deodhar lift [De] for Coxeter groups. This will turn out to be important in our second paper [LNSSS], where we establish the connection between LS paths and the quantum alcove model. Our proof uses the novel notion of quantum length which relies on the fact that the (parabolic) quantum Bruhat graph is strongly connected using only simple transpositions; see [HST]. The theorem ultimately follows from the application of the Diamond Lemmas for the quantum Bruhat graph.

The paper is organized as follows. In Section 2 we set up the notation for untwisted affine root systems and affine Weyl groups. In Section 3 we give the definitions of stabilizers of orbits of the affine Weyl group and derive properties of J-adjusted elements, where J is the index set of a parablic subgroup. The (parabolic) quantum Bruhat graph is introduced in Section 4 and the lift to the Bruhat order of the affine Weyl group is given in Section 5 (see Proposition 5.2). This gives rise to the Diamond Lemmas in Section 5.5. In Section 6 we state and prove our characterization of Littelmann's level-zero weight poset (see Theorem 6.5) and show that the parabolic quantum Bruhat graph is strongly connected using only simple transpositions (see Lemma 6.12). Finally in Section 7 we prove the tilted Bruhat Theorem (see Theorem 7.1).

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2. Notation: untwisted affine root datum

Let $I_{\mathrm{af}} = I \sqcup \{0\}$ (resp. I) be the Dynkin node set of an untwisted affine algebra $\mathfrak{g}_{\mathrm{af}}$ (resp. its canonical subalgebra $\mathfrak{g}$), $(a_{ij} \mid i,j \in I_{\mathrm{af}})$ the affine Cartan matrix, W_{af} (resp. W) the affine (resp. finite) Weyl group with simple reflections r_i for $i \in I_{\mathrm{af}}$ (resp. $i \in I$), $X_{\mathrm{af}} = \mathbb{Z}\delta \oplus \bigoplus_{i\in I_{\mathrm{af}}} \mathbb{Z}\Lambda_i$ (resp. $X = \bigoplus_{i\in I} \mathbb{Z}\omega_i$) the affine (resp. finite) weight lattice, $X_{\mathrm{af}}^\vee = \mathrm{Hom}_{\mathbb{Z}}(X_{\mathrm{af}}, \mathbb{Z})$ the dual lattice, and $\langle \cdot\,, \cdot\rangle : X_{\mathrm{af}}^\vee \times X_{\mathrm{af}} \to \mathbb{Z}$ the evaluation pairing. Let $X_{\mathrm{af}}^\vee$ have dual basis $\{d\} \cup \{\alpha_i^\vee \mid i \in I_{\mathrm{af}}\}$. The natural projection $\mathrm{cl} : X_{\mathrm{af}} \to X$ has kernel $\mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\delta$ and sends $\Lambda_i \mapsto \omega_i$ for $i \in I$.

Let $\{\alpha_i \mid i \in I_{\mathrm{af}}\} \subset X_{\mathrm{af}}$ be the unique elements such that

$$\langle \alpha_i^\vee\,, \alpha_j\rangle = a_{ij} \qquad \text{for } i,j \in I_{\mathrm{af}} \tag{2.1}$$

$$\langle d\,, \alpha_j\rangle = \delta_{j,0}. \tag{2.2}$$

The affine (resp. finite) root lattice is defined by $Q_{\mathrm{af}} = \bigoplus_{i\in I_{\mathrm{af}}} \mathbb{Z}\alpha_i$ (resp. $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$). The set of affine real roots (resp. roots) of $\mathfrak{g}_{\mathrm{af}}$ (resp. $\mathfrak{g}$) are defined by $\Phi^{\mathrm{af}} = W_{\mathrm{af}}\{\alpha_i \mid i \in I_{\mathrm{af}}\}$ (resp. $\Phi = W\{\alpha_i \mid i \in I\}$). The set of positive affine real (resp. positive) roots are the set $\Phi^{\mathrm{af}+} = \Phi^{\mathrm{af}} \cap \bigoplus_{i\in I_{\mathrm{af}}} \mathbb{Z}_{\geq 0}\alpha_i$ (resp. $\Phi^+ = \Phi \cap \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$). We have $\Phi^{\mathrm{af}} = \Phi^{\mathrm{af}+} \sqcup \Phi^{\mathrm{af}-}$ where $\Phi^{\mathrm{af}-} = -\Phi^{\mathrm{af}+}$ and $\Phi = \Phi^+ \sqcup \Phi^-$ where $\Phi^- = -\Phi^+$.

The null root δ is the unique element such that $\delta \in \bigoplus_{i\in I_{\mathrm{af}}} \mathbb{Z}_{>0}\alpha_i$ which generates the rank 1 sublattice $\{\lambda \in X_{\mathrm{af}} \mid \langle \alpha_i^\vee\,,\,\lambda\rangle = 0 \text{ for all } i \in I_{\mathrm{af}}\}$. Define $a_i \in \mathbb{Z}_{>0}$ by

$$\delta = \sum_{i\in I_{\mathrm{af}}} a_i\alpha_i. \tag{2.3}$$

We have $\delta = \alpha_0 + \theta$, where θ is the highest root for $\mathfrak{g}$, and

$$\Phi^{\mathrm{af}+} = \Phi^+ \sqcup (\Phi + \mathbb{Z}_{>0}\,\delta). \tag{2.4}$$

The canonical central element is the unique element $c \in \bigoplus_{i\in I_{\mathrm{af}}} \mathbb{Z}_{>0}\alpha_i^\vee$ which generates the rank 1 sublattice $\{\mu \in X_{\mathrm{af}}^\vee \mid \langle \mu\,,\,\alpha_i\rangle = 0 \text{ for all } i \in I_{\mathrm{af}}\}$. Define $a_i^\vee \in \mathbb{Z}_{>0}$ by $c = \sum_{i\in I_{\mathrm{af}}} a_i^\vee\alpha_i^\vee$. Then $a_0^\vee = 1$ [Kac]. The level of a weight $\lambda \in X_{\mathrm{af}}$ is defined by $\mathrm{level}(\lambda) = \langle c\,,\,\lambda\rangle$.

W_{af} acts on X_{af} and $X_{\mathrm{af}}^\vee$ by

$$\begin{aligned} r_i\lambda &= \lambda - \langle \alpha_i^\vee\,,\,\lambda\rangle\alpha_i \\ r_i\mu &= \mu - \langle \mu\,,\,\alpha_i\rangle\alpha_i^\vee \end{aligned}$$

for $i \in I_{\mathrm{af}}$, $\lambda \in X_{\mathrm{af}}$, and $\mu \in X_{\mathrm{af}}^\vee$. The pairing is W_{af}-invariant:

$$\langle w\mu\,,\,w\lambda\rangle = \langle \mu\,,\,\lambda\rangle \qquad \text{for } \lambda \in X_{\mathrm{af}} \text{ and } \mu \in X_{\mathrm{af}}^\vee.$$

Since the action of W_{af} on X_{af} is level-preserving, the sublattice $X_{\mathrm{af}}^0 \subset X_{\mathrm{af}}$ of level-zero elements is W_{af}-stable. There is a section $X \to X_{\mathrm{af}}^0$ given by $\omega_i \mapsto \Lambda_i - \mathrm{level}(\Lambda_i)\Lambda_0$ for $i \in I$.

For $\beta \in \Phi^{\mathrm{af}}$ let $w \in W_{\mathrm{af}}$ and $i \in I_{\mathrm{af}}$ be such that $\beta = w\alpha_i$. Define the associated reflection $r_\beta \in W_{\mathrm{af}}$ and associated coroot $\beta^\vee \in X_{\mathrm{af}}^\vee$ by

$$r_\beta = wr_iw^{-1} \tag{2.5}$$

$$\beta^\vee = w\alpha_i^\vee. \tag{2.6}$$

Both are independent of w and i. Of course $r_{-\beta} = r_\beta$. We have

$$\begin{aligned} r_\beta\lambda &= \lambda - \langle \beta^\vee\,,\,\lambda\rangle\beta && \text{for } \lambda \in X_{\mathrm{af}} \\ r_\beta\mu &= \mu - \langle \mu\,,\,\beta\rangle\beta^\vee && \text{for } \mu \in X_{\mathrm{af}}^\vee. \end{aligned}$$

There is an isomorphism

$$W_{\mathrm{af}} \cong W \ltimes Q^\vee. \tag{2.7}$$

Consider the injective group homomorphism $Q^\vee := \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee \to W_{\mathrm{af}}$ from the finite coroot lattice into W_{af}, denoted by $\mu \mapsto t_\mu$. Then $wt_\mu w^{-1} = t_{w\mu}$ for $w \in W$. Under the map (2.7), for $\alpha \in \Phi$ and $n \in \mathbb{Z}$, we have

$$\begin{aligned} r_{\alpha+n\delta} &\mapsto r_\alpha t_{n\alpha^\vee} \\ r_0 &\mapsto r_\theta t_{-\theta^\vee} \end{aligned}$$

the latter holding since $\alpha_0 = \delta - \theta$.

Let $W_e = W \ltimes X^\vee$ be the extended affine Weyl group where $X^\vee = \bigoplus_{i\in I} \mathbb{Z}\omega_i^\vee$ is the coweight lattice of $\mathfrak{g}$. Let $I^s \subset I_{\mathrm{af}}$ be the subset of special or cominuscule nodes, the set of nodes $i \in I^{\mathrm{af}}$ which are the image of 0 under some automorphism of the affine Dynkin diagram. There is a bijection from I^s to $X^\vee/Q^\vee$ given by $i \mapsto \omega_i^\vee + Q^\vee$ where $\omega_0^\vee := 0$ and $Q^\vee = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$ is the finite coroot lattice. For each $i \in I^s$ there is a permutation τ_i of $X^\vee/Q^\vee$ (and therefore a permutation of I^s) defined by adding $-\omega_i + Q^\vee$. The induced permutation of I^s extends uniquely to an automorphism τ_i of the affine Dynkin diagram. The group $\mathrm{Aut}^s(I^{\mathrm{af}})$ of special automorphisms is defined to be the group of τ_i for $i \in I^s$. It acts on X_{af}, $X_{\mathrm{af}}^\vee$, Q_{af}, $Q_{\mathrm{af}}^\vee = \bigoplus_{i\in I_{\mathrm{af}}} \mathbb{Z}\alpha_i^\vee$, and W_{af} by permuting I_{af} on basis elements and for W_{af}, indices of simple reflections.

Define $v_i \in W$ by the length-additive product

$$w_0 = v_i w_0^J \qquad \text{for } i \in I^s \tag{2.8}$$

where $w_0 \in W$ and $w_0^J \in W_J$ are the longest elements in W and the subgroup W_J of W generated by r_j for $j \in J = I \setminus \{i\}$ respectively. In particular $v_0 = \mathrm{id}$. Then there is an injective group homomorphism

$$\begin{aligned} \mathrm{Aut}^s(I^{\mathrm{af}}) &\to W_e \\ \tau_i &= v_i t_{-\omega_i^\vee} \qquad \text{for } i \in I^s. \end{aligned}$$

$\mathrm{Aut}^s(I^{\mathrm{af}})$ acts on W_{af} by conjugation. This action may be defined by relabeling indices of simple reflections: $\tau r_i \tau^{-1} = r_{\tau(i)}$ for all $\tau \in \mathrm{Aut}^s(I^{\mathrm{af}})$ and $i \in I^{\mathrm{af}}$. As such we have $W_e \cong \mathrm{Aut}^s(I^{\mathrm{af}}) \ltimes W_{\mathrm{af}}$. There is an injective group homomorphism

$$\begin{aligned} \mathrm{Aut}^s(I^{\mathrm{af}}) &\to W \\ \tau_i &\mapsto v_i. \end{aligned} \tag{2.9}$$

Lemma 2.1. *For every $i \in I^s$, $a_i = 1$ and α_i occurs in θ with coefficient 1 for $i \in I^s \setminus \{0\}$.*

Proof. For untwisted affine algebras $a_0 = 1$ [Kac]. The lemma follows since $\mathrm{Aut}^s(I^{\mathrm{af}})$ acts transitively on I^s and fixes δ. □

Lemma 2.2. *For every $i \in I^s$*

$$\ell(v_i) = \langle \omega_i^\vee \,,\, 2\rho \rangle. \tag{2.10}$$

Proof. Fix $i \in I^s$. Since θ is the highest root, it follows from Lemma 2.1 that if α_i occurs in a positive root then its coefficient is 1. Consequently the right hand side of (2.10) equals the number of positive roots that contain α_i. This is the complement of the number of positive roots in the parabolic subsystem for $J = I \setminus \{i\}$. But this is equal to $\ell(w_0) - \ell(w_0^J) = \ell(v_i)$. □

3. Orbits of level-zero weights

3.1. **W_{af}-orbit and W-orbit.** The action of W_{af} on X_{af}^0 is given by

$$w t_\mu \lambda = w\lambda - \langle \mu \,,\, \lambda \rangle \delta \tag{3.1}$$

for $w \in W$, $\mu \in Q^\vee$, and $\lambda \in X_{\mathrm{af}}^0$.

Lemma 3.1. *For a dominant weight $\lambda \in X \cong X_{\mathrm{af}}^0/\mathbb{Z}\delta$ we have $W_{\mathrm{af}}\lambda = W\lambda$ in $X_{\mathrm{af}}^0/\mathbb{Z}\delta$.*

Proof. This follows immediately from (3.1). □

3.2. **Stabilizers.** Let $\lambda \in X$ be a dominant weight, which will be used several times in this paper, so the notation below applies throughout. Let W_J be the stabilizer of λ in W. It is a parabolic subgroup, being generated by r_i for $i \in J$ where

$$J = \{ i \in I \mid \langle \alpha_i^\vee \,,\, \lambda \rangle = 0 \}. \tag{3.2}$$

Let $Q_J^\vee = \bigoplus_{i \in J} \mathbb{Z}\alpha_i^\vee$ be the associated coroot lattice, W^J the set of minimum-length coset representatives in W/W_J, $\Phi_J = \Phi_J^+ \sqcup \Phi_J^-$ the set of roots and positive/negative roots respectively, and $\rho_J = (1/2)\sum_{\alpha \in \Phi_J^+} \alpha$.

Lemma 3.2. *The stabilizer of λ in W_{af} under its level-zero action on $X \cong X_{\mathrm{af}}^0/\mathbb{Z}\delta$, is given by the subgroup of elements of the form wt_μ where $w \in W_J$ and $\mu \in Q^\vee$ satisfies $\langle \mu \,,\, \lambda \rangle = 0$.*

Proof. This follows immediately from the definitions and Lemma 3.1. □

3.3. **Affinization of stabilizer.** Let $J = \bigsqcup_{m=1}^k I_m$ have connected components with vertex sets $I_1, I_2, \dots, I_k$. The coweight lattice $X_J^\vee$ is the direct sum $\bigoplus_{m=1}^k X_{I_m}^\vee$ where $X_{I_m}^\vee$ is the coweight lattice for the root system defined by the component I_m. Define $J^{\mathrm{af}} = \bigsqcup_m I_m^{\mathrm{af}}$, where $I_m^{\mathrm{af}} = I_m \sqcup \{0_m\}$ and 0_m is a separate additional affine node attached to I_m. Define $(W_J)_{\mathrm{af}} = \prod_{m=1}^k (W_{I_m})_{\mathrm{af}}$, where W_{I_m} and $(W_{I_m})_{\mathrm{af}}$ are the finite and affine Weyl groups for the root subsystem with Dynkin node set I_m. Under this isomorphism $r_{0_m} = r_{\theta_m} t_{-\theta_m^\vee}$ where θ_m is the highest root for I_m.

Define

$$\Phi_J^{\mathrm{af}+} = \{\beta \in \Phi^{\mathrm{af}+} \mid \mathrm{cl}(\beta) \in \Phi_J\} = \Phi_J^+ \cup (\mathbb{Z}_{>0}\delta + \Phi_J)\,, \quad \Phi_J^{\mathrm{af}-} = -\Phi_J^{\mathrm{af}+}\,, \tag{3.3}$$

$$(W^J)_{\mathrm{af}} = \{x \in W_{\mathrm{af}} \mid x\beta > 0 \text{ for all } \beta \in \Phi_J^{\mathrm{af}+}\}. \tag{3.4}$$

Lemma 3.3. [LS, Lemma 10.1] *$wt_\mu \in (W^J)_{\mathrm{af}}$ if and only if, for all $\alpha \in \Phi_J^+$, $w\alpha > 0$ implies that $\langle \mu\,,\,\alpha\rangle = 0$ and $w\alpha < 0$ implies that $\langle \mu\,,\,\alpha\rangle = -1$.*

Proposition 3.4. [LS, Lemma 10.5] [P] *Given $w \in W_{\mathrm{af}}$ there exist unique $w_1 \in (W^J)_{\mathrm{af}}$ and $w_2 \in (W_J)_{\mathrm{af}}$ such that $w = w_1 w_2$. If $w \in W$, then $w_1 \in W^J$ is the minimum-length representative of the coset wW_J.*

Define $\pi_J : W_{\mathrm{af}} \to (W^J)_{\mathrm{af}}$ by

$$w \mapsto w_1\,, \tag{3.5}$$

with w_1 as in Proposition 3.4. Note that for $x \in W_{\mathrm{af}}$, $x \in (W^J)_{\mathrm{af}}$ if and only if $\pi_J(x) = x$.

Let W_{af}^- be the set of minimum-length coset representatives in W_{af}/W.

Proposition 3.5. [LS, Proposition 10.8] [P] *Let $x \in W_{\mathrm{af}}$ and $\mu \in Q^\vee$. Then*

(1) $\pi_J(xv) = \pi_J(x)$ *if* $v \in (W_J)_{\mathrm{af}}$.
(2) $\pi_J(W) \subset W^J \subset (W^J)_{\mathrm{af}}$.
(3) $\pi_J(W_{\mathrm{af}}^-) \subset W_{\mathrm{af}}^-$.
(4) $\pi_J(xt_\mu) = \pi_J(x)\pi_J(t_\mu)$.

We shall employ the explicit description of π_J in [LS, Lemma 10.7]. The element $\mu \in Q^\vee$ can be written uniquely in the form

$$\mu = \sum_{i \in I\setminus J} c_i \omega_i^\vee - \phi_J(\mu) - \sum_{m=1}^k \omega_{j_m}^\vee\,, \tag{3.6}$$

where $\phi_J(\mu) \in Q_J^\vee$ and each $j_m \in I_m$ is a cominuscule node. The element μ is first separated into the part in $X_J^\vee$ and the part not in it, and then one considers the projection of the part in $X_J^\vee$ to $X_J^\vee/Q_J^\vee$, takes a canonical lift (the last sum). Then $\phi_J(\mu) \in Q_J^\vee$ is the correction term. We write $z_\mu = \prod_{m=1}^k v_{j_m}^{I_m}$ where $v_{j_m} \in W_{I_m} \subset W_J$ is defined in (2.8). Then for $w \in W$ and $\mu \in Q^\vee$ we have

$$\pi_J(wt_\mu) = \pi_J(w)\pi_J(t_\mu) = \pi_J(w) z_\mu t_{\mu+\phi_J(\mu)}. \tag{3.7}$$

Remark 3.6. By Proposition 3.5 the map

$$\begin{aligned} Q^\vee &\to \mathrm{Aut}^s(J^{\mathrm{af}}) \subset W_J \\ \mu &\mapsto z_\mu \end{aligned} \tag{3.8}$$

is a group homomorphism.

Denote by $\Sigma_J \subset \mathrm{Aut}^s(J^{\mathrm{af}}) \subset W_J$ the image of the homomorphism (3.8):

$$\Sigma_J = \{z \in W_J \mid z = z_\mu \text{ for some } \mu \in Q^\vee\}. \tag{3.9}$$

3.4. **J-adjusted elements.** We say that $\mu \in Q^\vee$ is *J-adjusted* if $\phi_J(\mu) = 0$ or equivalently

$$\pi_J(t_\mu) = z_\mu t_\mu. \tag{3.10}$$

This notion gives a nice parametrization of the set $(W^J)_{\mathrm{af}}$.

Lemma 3.7. *Let $w \in W^J$, $z \in W_J$, and $\mu \in Q^\vee$. Then $wzt_\mu \in (W^J)_{\mathrm{af}}$ if and only if μ is J-adjusted and $z = z_\mu$. In particular every element of $(W^J)_{\mathrm{af}}$ can be uniquely written as $w\pi_J(t_\mu) = wz_\mu t_\mu$ where $w \in W^J$ and $\mu \in Q^\vee$ is J-adjusted.*

Proof. $wzt_\mu \in (W^J)_{\mathrm{af}}$ if and only if $wzt_\mu = \pi_J(wzt_\mu) = \pi_J(wz)\pi_J(t_\mu) = w\pi_J(t_\mu)$ from which the result follows. □

Lemma 3.8. *Let $\mu \in Q^\vee$ and consider (3.6). The following are equivalent:*

(1) *μ is J-adjusted.*
(2) *For every component I_m of J, either*
  (a) *$\langle \mu\,,\,\alpha_i\rangle = 0$ for all $i \in I_m$ (that is, $j_m = 0_m \in I_m^{\mathrm{af}}$), or*
  (b) *there is a unique $j \in I_m$ such that $\langle \mu\,,\,\alpha_j\rangle \neq 0$, and in this case $j = j_m$ and $\langle \mu\,,\,\alpha_{j_m}\rangle = -1$.*
(3) *$\langle \mu\,,\,\alpha\rangle \in \{0, -1\}$ for all $\alpha \in \Phi_J^+$.*

Proof. Given [LS, Lemma 10.7], (1) and (2) are equivalent. Suppose (2) holds. Let $\alpha \in \Phi_J^+$. Then α is a positive root in the subrootsystem Φ_m^+ of Φ for some component I_m of J. Let $\alpha = \sum_{i \in I_m} b_i \alpha_i$. Since $j_m \in I_m$ is cominuscule, $\langle \omega_{j_m}^\vee\,,\,\theta_m\rangle = 1$ where $\theta_m \in \Phi_m^+$ is the highest root. Therefore $b_{j_m} \in \{0, 1\}$. Since $b_i = 0$ for $i \in I_m \setminus \{j_m\}$, (3) follows.

Conversely, suppose (3) holds. Let I_m be a component of J. Applying (3) to θ_m and to each of the α_i for $i \in I_m$, we see that (2) must hold. □

Lemma 3.9. *For $\mu \in Q^\vee$, μ is W_J-invariant if and only if μ is J-adjusted and $z_\mu = \mathrm{id}$.*

Proof. The first condition holds if and only if no fundamental coweight $\omega_i^\vee$ occurs in μ for $i \in J$, which for the expression (3.6) means that $\phi_J(\mu) = 0$ and $j_m = 0_m$ for all m. But this holds if and only if $\pi_J(t_\mu) = t_\mu$ by (3.7). □

Lemma 3.10. *Let $\mu \in Q^\vee$ be J-adjusted. Then*

$$\ell(z_\mu) = -\langle \mu\,,\,2\rho_J\rangle. \tag{3.11}$$

Proof. The proof reduces to considering each component I_m of J. Note that $-\mu$ pairs with roots of I_m like a fundamental cominuscule coweight by Lemma 3.8 and the result follows by Lemma 2.2. □

Lemma 3.11. *For every J-adjusted element $\mu \in Q^\vee$ and $v \in W_J$, $z_\mu = z_{v\mu}$.*

Proof. We have $z_\mu t_\mu = \pi_J(t_\mu) = \pi_J(vt_\mu v^{-1}) = \pi_J(t_{v\mu})$, which implies the result. □

Lemma 3.12. *Given $\alpha \in \Phi^+$ and $x = wt_\mu \in W_{\mathrm{af}}$ with $w \in W$ and $\mu \in Q^\vee$, let $\ell_\alpha(x)$ be the number of roots $\pm\alpha + n\delta \in \Phi^{\mathrm{af}+}$ with $n \in \mathbb{Z}$, which x sends to $\Phi^{\mathrm{af}-}$. Then*

$$\ell_\alpha(x) = |\chi(w\alpha \in \Phi^-) + \langle \mu\,,\,\alpha\rangle|. \tag{3.12}$$

Here $\chi(S) = 1$ if S is true and $\chi(S) = 0$ if S is false.

Proof. This follows from $x(\pm\alpha + n\delta) = \pm w\alpha + (n - \langle \mu\,,\,\pm\alpha\rangle)\delta$. □

Lemma 3.13. *Let $w \in W^J$, $z \in W_J$, and $\mu \in Q^\vee$ be such that $\langle \mu\,,\,\alpha\rangle < 0$ for all $\alpha \in \Phi^+ \setminus \Phi_J^+$ and $x = wzt_\mu \in (W^J)_{\mathrm{af}}$. Then μ is J-adjusted, $z = z_\mu$, and*

$$\ell(x) = -\langle \mu\,,\,2\rho - 2\rho_J\rangle - \ell(w). \tag{3.13}$$

Proof. By Lemma 3.7 we need only prove the length condition. We have $\ell(x) = \sum_{\alpha\in\Phi^+} \ell_\alpha(x)$. Fix $\alpha \in \Phi^+$. Since $x \in (W^J)_{\mathrm{af}}$, if $\alpha \in \Phi_J^+$ then $\ell_\alpha(x) = 0$. Let $\alpha \in \Phi^+ \setminus \Phi_J^+$. By Lemma 3.12 we have $\ell_\alpha(wzt_\mu) = -\chi(wz\alpha \in \Phi^-) - \langle \mu\,,\, \alpha\rangle$. Summing this over $\alpha \in \Phi^+ \setminus \Phi_J^+$ we have

$$\ell(x) = -\langle \mu\,,\, 2\rho - 2\rho_J\rangle + \sum_{\alpha\in\Phi^+\setminus\Phi_J^+} -\chi(wz\alpha \in \Phi^-).$$

But $z \in W_J$ so it permutes the set $\Phi^+ \setminus \Phi_J^+$. Moreover $w \in W^J$ so $w\Phi_J^+ \subset \Phi^+$. The lemma follows. □

Let $\mu \in Q^\vee$. We say that μ is antidominant if

$$\langle \mu\,,\, \alpha\rangle \le 0 \qquad \text{for all } \alpha \in \Phi^+. \tag{3.14}$$

Say that μ is *strictly J-antidominant* if it is antidominant and

$$\langle \mu\,,\, \alpha\rangle < 0 \qquad \text{for } \alpha \in \Phi^+ \setminus \Phi_J^+. \tag{3.15}$$

Say that μ is *J-superantidominant* if μ is antidominant and

$$\langle \mu\,,\, \alpha\rangle \ll 0 \qquad \text{for } \alpha \in \Phi^+ \setminus \Phi_J^+. \tag{3.16}$$

In the notation of (3.6), the condition (3.16) means that $c_i \ll 0$ for all $i \in I \setminus J$.

Remark 3.14. If $J = \emptyset$, then the J-superantidominant property becomes the superantidominant one in [LS]. If μ is superantidominant, then (3.6) and (3.7) show that, in the projection $\pi_J(t_\mu) = z_\mu t_\nu$, the element ν is J-superantidominant.

Lemma 3.15. *Let $z \in \Sigma_J$ (see (3.9)). Then there is a J-superantidominant, J-adjusted element $\mu \in Q^\vee$ such that $z = z_\mu$.*

Proof. By assumption there is a $\nu \in Q^\vee$ such that $\pi_J(t_\nu) = zt_{\nu+\phi_J(\nu)}$. Since $\gamma = \phi_J(\nu) \in Q_J^\vee$, by (3.7) we have $\pi_J(t_\gamma) = \mathrm{id}$. We have $\pi_J(t_{\nu+\gamma}) = \pi_J(t_\nu)\pi_J(t_\gamma) = zt_{\nu+\gamma}$ so that $\nu+\gamma$ is a J-adjusted element of $Q^\vee$ with $z_{\nu+\gamma} = z$. Let $\eta \in Q^\vee$ be J-superantidominant and W_J-invariant, so that $z_\eta = \mathrm{id}$. Then $\nu + \gamma + \eta$ is the required element. □

Lemma 3.16. *Let $w \in W^J$ and let $\mu \in Q^\vee$ be J-adjusted and strictly J-antidominant. Then $wz_\mu t_\mu \in W_{\mathrm{af}}^-$.*

Proof. By [LS, Lemma 3.3] $wt_\mu \in W_{\mathrm{af}}^-$. We have $\pi_J(wt_\mu) = \pi_J(w)\pi_J(t_\mu) = wz_\mu t_\mu \in W_{\mathrm{af}}^-$ by Proposition 3.5. □

4. Quantum Bruhat graph

The quantum Bruhat graph was first introduced in a paper by Brenti, Fomin and Postnikov [BFP] and later appeared in connection with the quantum cohomology of flag varieties in a paper by Fulton and Woodward [FW]. In this section we define the quantum Bruhat graph and its parabolic analogue, and prove some properties we need.

Say that $\alpha \in \Phi^+$ is a *quantum root* if $\ell(r_\alpha) = \langle \alpha^\vee\,,\, 2\rho\rangle - 1$.

Lemma 4.1. [BFP, Lemma 4.3] [M, Lemma 3.2] *For any positive root $\alpha \in \Phi^+$, we have $\ell(r_\alpha) \le -1 + \langle \alpha^\vee\,,\, 2\rho\rangle$. In simply-laced type all roots are quantum roots.*

Lemma 4.2. [BMO] *$\alpha \in \Phi^+$ is a quantum root if and only if*

(1) *α is a long root, or*
(2) *α is a short root, and writing $\alpha = \sum_i c_i\alpha_i^\vee$, we have $c_i = 0$ for all i such that α_i is long.*

Here for simply-laced root systems we consider all roots to be long.

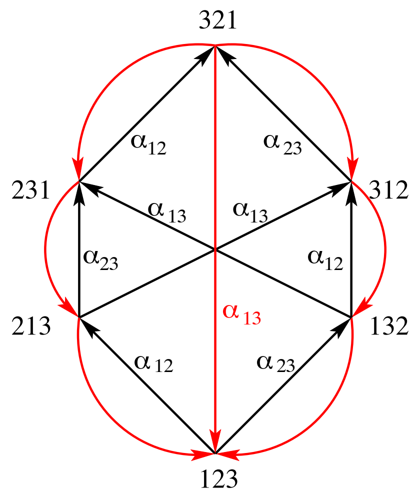

FIGURE 1. Quantum Bruhat graph for S_3

4.1. **Regular case.** The *quantum Bruhat graph* $\mathrm{QB}(W)$ is a directed graph structure on W that contains two kinds of directed edges. For $w \in W$ there is a directed edge $w \xrightarrow{\alpha} wr_\alpha$ if $\alpha \in \Phi^+$ and one of the following holds.

(1) (Bruhat edge) $w \lessdot wr_\alpha$ is a covering relation in Bruhat order, that is, $\ell(wr_\alpha) = \ell(w) + 1$.
(2) (Quantum edge) $\ell(wr_\alpha) = \ell(w) - \ell(r_\alpha)$ and α is a quantum root.

Condition (2) is equivalent to

(2′) $\ell(wr_\alpha) = \ell(w) + 1 - \langle \alpha^\vee , 2\rho \rangle$.

An example is given in Figure 1, where the quantum edges are drawn in red and $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$.

4.2. **Parabolic case.** Let $\mathrm{QB}(W^J)$ be the *parabolic quantum Bruhat graph*. Its vertex set is W^J. There are two kinds of directed edges. Both are labeled by some $\alpha \in \Phi^+ \setminus \Phi_J^+$. We use the notation $\lfloor w \rfloor$ to indicate the minimum-length coset representative in the coset wW_J.

(1) (Bruhat edge) $w \xrightarrow{\alpha} \lfloor wr_\alpha \rfloor$ where $w \lessdot wr_\alpha$. (One may deduce that $wr_\alpha \in W^J$.)
(2) (Quantum edge)

$$\ell(\lfloor wr_\alpha \rfloor) = \ell(w) + 1 - \langle \alpha^\vee , 2\rho - 2\rho_J \rangle. \tag{4.1}$$

Condition (2) is equivalent to

(2′) $wr_\alpha \xleftarrow{\alpha} w$ is a quantum edge in $\mathrm{QB}(W)$ and $wr_\alpha t_{\alpha^\vee} \in (W^J)_{\mathrm{af}}$.

This equivalence may be deduced from [LS, Lemma 10.14] and the proof of [LS, Theorem 10.18]. The arguments there rely on geometry, namely, the quantum Chevalley rule and the Peterson-Woodward comparison theorem. An example of a parabolic quantum Bruhat graph is given in Figure 2.

4.3. **Duality antiautomorphism of** $\mathrm{QB}(W^J)$**.** Let $w_0 \in W$ be the longest element. There is an involution on W defined by $w \mapsto w_0 w$. It reverses length in that $\ell(w_0 w) = \ell(w_0) - \ell(w)$. It also reverses Bruhat order in W: $v \lessdot w$ if and only if $w_0 v \gtrdot w_0 w$. The map $w \mapsto ww_0$ also has the same properties. In particular $w \mapsto w^* = w_0 w w_0$ is a group automorphism of W which preserves length. Define the involution $*$ on the Dynkin diagram I by $w_0 r_i w_0 = r_{i^*}$ or equivalently $w_0 \alpha_i = -\alpha_{i^*}$.

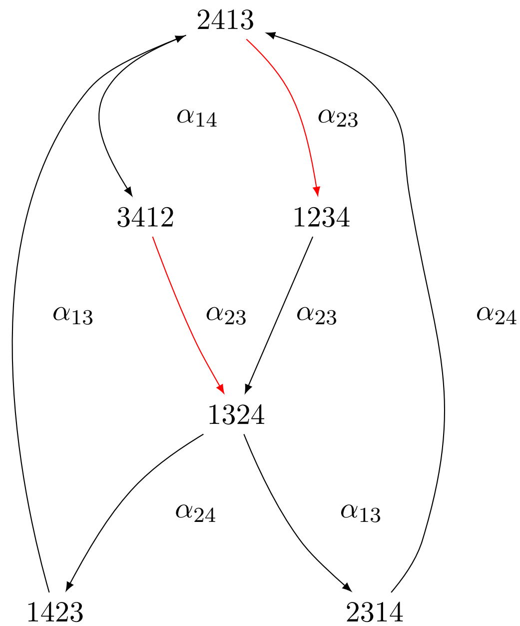

FIGURE 2. Parabolic quantum Bruhat graph for S_4 with $J = \{1, 3\}$

Then $*$ is an automorphism of I. The map $w \mapsto w^*$ can be computed on reduced words by replacing each r_i by r_{i^*}.

Define the map $w \mapsto w^\circ$ on W^J by $w^\circ = \lfloor w_0 w \rfloor$. Let $w_0^J \in W_J$ be the longest element.

Proposition 4.3. *The map $w \mapsto w^\circ$ is an involution on W^J such that*

(1) $w^\circ = w_0 w w_0^J$.
(2) $\ell(w^\circ) = \ell(w_0) - \ell(w_0^J) - \ell(w) = |\Phi^+ \setminus \Phi_J^+| - \ell(w)$.
(3) $v \xleftarrow{\beta} w$ *is an edge in* $\mathrm{QB}(W^J)$ *if and only if* $w^\circ \xleftarrow{w_0^J \beta} v^\circ$ *is an edge in* $\mathrm{QB}(W^J)$*. Moreover both edges are Bruhat or both are quantum.*

In particular this involution reverses arrows in $\mathrm{QB}(W^J)$ *and preserves whether an arrow is quantum or not.*

Proof. For $\alpha \in \Phi_J^+$ we have $w_0^J \alpha \in \Phi_J^-$. Since $w \in W^J$, $w w_0^J \alpha \in \Phi^-$. Then $w_0 w w_0^J \alpha \in \Phi^+$. Therefore $w_0 w w_0^J \in W^J$ and $w^\circ = w_0 w w_0^J$. This implies (1).

Since elements of W^J permute Φ_J^+, $\mathrm{Inv}(w)$ and $\mathrm{Inv}(w^\circ)$ are subsets of $\Phi^+ \setminus \Phi_J^+$. Note that for $\alpha \in \Phi^+ \setminus \Phi_J^+$, $\alpha \in \mathrm{Inv}(w)$ if and only if $w\alpha \in \Phi^-$, if and only if $w_0 w w_0^J w_0^J \alpha \in \Phi^+$, if and only if $w_0^J \alpha \in \Phi^+ \setminus \Phi_J^+ \setminus \mathrm{Inv}(w^\circ)$. Therefore the map $\alpha \mapsto w_0^J \alpha$ defines a bijection from $\mathrm{Inv}(w)$ to $(\Phi^+ \setminus \Phi_J^+) \setminus \mathrm{Inv}(w^\circ)$. This implies (2).

Let $v = w r_\beta$. Then $v^\circ = w_0 v w_0^J = w_0 w w_0^J w_0^J r_\beta w_0^J = w^\circ r_{w_0^J \beta}$, that is, $w^\circ = v^\circ r_{w_0^J \beta}$. Since $\beta \in \Phi^+ \setminus \Phi_J^+$, $w_0^J \beta \in \Phi^+ \setminus \Phi_J^+$. Let χ be 0 or 1 according as the edge $v \xleftarrow{\beta} w$ is Bruhat or quantum. By (2) we have

$$\begin{aligned}
\ell(w^\circ) &= |\Phi^+ \setminus \Phi_J^+| - \ell(w) \\
&= |\Phi^+ \setminus \Phi_J^+| - (\ell(v) - 1 + \chi \langle \beta^\vee \, , \, 2\rho - 2\rho_J \rangle) \\
&= \ell(v^\circ) + 1 - \chi \langle \beta^\vee \, , \, 2\rho - 2\rho_J \rangle \\
&= \ell(v^\circ) + 1 - \chi \langle w_0^J \beta^\vee \, , \, 2\rho - 2\rho_J \rangle
\end{aligned}$$

where the last equality holds by Lemma 4.4. This proves the existence of the required arrow in $\mathrm{QB}(W^J)$. □

Lemma 4.4. *For any $z \in W_J$,*

$$z(2\rho - 2\rho_J) = 2\rho - 2\rho_J. \tag{4.2}$$

Proof. $z \in W_J$ permutes the set $\Phi^+ \setminus \Phi_J^+$, whose sum is $2\rho - 2\rho_J$. □

5. Quantum Bruhat graph and the affine Bruhat order

In this section we consider the lift of the parabolic quantum Bruhat graph to the Bruhat order of the affine Weyl group (see Theorem 5.2). This is used in Section 5.5 to establish the Diamond Lemmas for the parabolic quantum Bruhat graph.

5.1. **Regular case.** The following result is [LS, Proposition 4.4].

Proposition 5.1. *Let $\mu \in Q^\vee$ be superantidominant and let $x = wt_{v\mu}$ with $w, v \in W$. Then $y = xr_{v\alpha+n\delta} \lessdot x$ if and only if one of the following hold.*

1. *$\ell(wv) = \ell(wvr_\alpha) - 1$ and $n = \langle \mu\,,\, \alpha\rangle$, giving $y = wr_{v\alpha}t_{v\mu}$.*
2. *$\ell(wv) = \ell(wvr_\alpha) - 1 + \langle \alpha^\vee\,,\, 2\rho\rangle$ and $n = 1 + \langle \mu\,,\, \alpha\rangle$, giving $y = wr_{v\alpha}t_{v(\mu+\alpha^\vee)}$.*
3. *$\ell(v) = \ell(vr_\alpha) + 1$ and $n = 0$, giving $y = wr_{v\alpha}t_{vr_\alpha\mu}$.*
4. *$\ell(v) = \ell(vr_\alpha) + 1 - \langle \alpha^\vee\,,\, 2\rho\rangle$ and $n = -1$ giving $y = wr_{v\alpha}t_{vr_\alpha(\mu+\alpha^\vee)}$.*

Note that if we impose the condition that both x and y are in W_{af}^- then $v = \mathrm{id}$ and only Cases (1) and (2) apply.

5.2. **Embeddings** $\mathrm{QB}(W^J) \hookrightarrow W_{\mathrm{af}}$**.** We shall give a parabolic analogue (Theorem 5.2 below) of Proposition 5.1 for W_{af}^-. Theorem 5.2 is proved in the same manner as Proposition 5.1 but the latter cannot be directly invoked to prove the former, since J-superantidominance does not imply superantidominance.

Let $\Omega_J \subset W_{\mathrm{af}}$ be the subset of elements of the form $w\pi_J(t_\mu)$ with $w \in W^J$ and $\mu \in Q^\vee$ strictly J-antidominant (see (3.15)) and J-adjusted. Define Ω_J^∞ similarly but with strict J-antidominance replaced by J-superantidominance. We have $\Omega_J^\infty \subset (W^J)_{\mathrm{af}} \cap W_{\mathrm{af}}^-$. Impose the Bruhat covers in Ω_J^∞ whenever the connecting root has classical part in $\Phi \setminus \Phi_J$. Then Ω_J^∞ is a subposet of the Bruhat poset W_{af}.

Theorem 5.2. *Every edge in $\mathrm{QB}(W^J)$ lifts to a downward Bruhat cover in Ω_J^∞, and every cover in Ω_J^∞ projects to an edge in $\mathrm{QB}(W^J)$. More precisely:*

1. *For any edge $\lfloor wr_\alpha \rfloor \xleftarrow{\alpha} w$ in $\mathrm{QB}(W^J)$, $z \in \Sigma_J$ (see (3.9)), and $\mu \in Q^\vee$ that is J-superantidominant and J-adjusted with $z = z_\mu$ (which exists by Lemma 3.15), there is a covering relation $y \lessdot x$ in Ω_J^∞ where*
$$x = wzt_\mu\,, \quad y = xr_{\widetilde{\alpha}} = wr_\alpha t_{\chi\alpha^\vee} zt_\mu\,, \quad \widetilde{\alpha} = z^{-1}\alpha + (\chi + \langle \mu\,,\, z^{-1}\alpha\rangle)\delta \in \Phi^{\mathrm{af}-}\,,$$
and χ is 0 or 1 according as the arrow in $\mathrm{QB}(W^J)$ is of Bruhat or quantum type respectively.
2. *Suppose $y \lessdot x$ is an arbitrary covering relation in Ω_J^∞. Then we can write $x = wzt_\mu$ with $w \in W^J$, $z = z_\mu \in W_J$, and $\mu \in Q^\vee$ J-superantidominant and J-adjusted, as well as $y = xr_\gamma$ with $\gamma = z^{-1}\alpha + n\delta \in \Phi^{\mathrm{af}}$, $\alpha \in \Phi^+ \setminus \Phi_J^+$, and $n \in \mathbb{Z}$. With the notation $\chi := n - \langle \mu\,,\, z^{-1}\alpha\rangle$, we have*
$$\chi \in \{0, 1\}\,, \quad \gamma = z^{-1}\alpha + (\chi + \langle \mu\,,\, z^{-1}\alpha\rangle)\delta \in \Phi^{\mathrm{af}-}\,;$$
furthermore, there is an edge $wr_\alpha z \xleftarrow{z^{-1}\alpha} wz$ in $\mathrm{QB}(W)$ and an edge $\lfloor wr_\alpha \rfloor \xleftarrow{\alpha} w$ in $\mathrm{QB}(W^J)$, where both edges are of Bruhat type if $\chi = 0$ and of quantum type if $\chi = 1$.

Remark 5.3. The affine Bruhat covering relation considered in part (2) is completely general, subject to both elements being in Ω_J^∞ and the transition root having classical part in $\Phi \setminus \Phi_J$.

Proof. (1) Since $\alpha \in \Phi^+ \setminus \Phi_J^+$ we have $z^{-1}\alpha \in \Phi^+ \setminus \Phi_J^+$. By Lemmas 3.7 and 3.16, $x \in (W^J)_{\mathrm{af}} \cap W_{\mathrm{af}}^-$. We have

$$\begin{aligned}
y &= x r_{\widetilde{\alpha}} = wzt_\mu r_{z^{-1}\alpha} t_{(\chi+\langle \mu\,,\, z^{-1}\alpha\rangle)\alpha^\vee} \\
&= w r_\alpha z t_{\chi z^{-1}\alpha^\vee} t_\mu \\
&= w r_\alpha t_{\chi\alpha^\vee} z t_\mu \\
\pi_J(y) &= \pi_J(w r_\alpha t_{\chi\alpha^\vee} z)\pi_J(t_\mu) \\
&= \pi_J(w r_\alpha t_{\chi\alpha^\vee}) z t_\mu \\
&= w r_\alpha t_{\chi\alpha^\vee} z t_\mu \\
&= y
\end{aligned}$$

using Proposition 3.5, the assumption on μ, and (2′) of the definition of $\mathrm{QB}(W^J)$ in the case $\chi = 1$. We conclude that $y \in (W^J)_{\mathrm{af}}$. Let $i \in I$. We have $y\alpha_i = wzr_{z_{\alpha}^{-1}}(\alpha_i - \langle \mu + \chi z^{-1}\alpha^\vee\,,\, \alpha_i\rangle\delta)$. If $i \notin J$ then the J-superantidominance of μ implies that $y\alpha_i \in \Phi^{\mathrm{af}+}$. Suppose $i \in J$. Then $\alpha_i \in \Phi_J^+$ and $y\alpha_i \in \Phi^{\mathrm{af}+}$ by the definition of $y \in (W^J)_{\mathrm{af}}$. We have shown that $y \in W_{\mathrm{af}}^-$. To prove $x \gtrdot y$ we need only show that $\ell(x) - \ell(y) = 1$. Suppose $\chi = 0$. Since y and x are in W_{af}^-, by [LS, Lemma 3.3] we have

$$\begin{aligned}
\ell(x) - \ell(y) &= \ell(t_\mu) - \ell(wz) - \ell(t_\mu) + \ell(wr_\alpha z) \\
&= -\ell(w) - \ell(z) + \ell(wr_\alpha) + \ell(z) \\
&= 1.
\end{aligned}$$

Suppose $\chi = 1$. We have $x = w\pi_J(t_\mu)$ and $y = \lfloor wr_\alpha \rfloor \pi_J(t_{\mu + z^{-1}\alpha^\vee})$. By Lemma 3.13 we have

$$\begin{aligned}
\ell(x) - \ell(y) &= -\ell(w) - \langle \mu\,,\, 2\rho - 2\rho_J\rangle + \ell(\lfloor wr_\alpha \rfloor) + \langle \mu + z^{-1}\alpha\,,\, 2\rho - 2\rho_J\rangle \\
&= 1 - \langle \alpha^\vee\,,\, 2\rho - 2\rho_J\rangle + \langle \alpha^\vee\,,\, z(2\rho - 2\rho_J)\rangle
\end{aligned}$$

by condition (2) of the case $\chi = 1$ of the arrow in $\mathrm{QB}(W^J)$. By Lemma 4.4 it follows that $\ell(x) - \ell(y) = 1$ as required.

(2) Let $n = \chi + \langle \mu\,,\, z^{-1}\alpha\rangle$ where $\chi \in \mathbb{Z}$.

We have $y = wr_\alpha z t_{\mu + \chi z^{-1}\alpha^\vee}$. Since $y \in W_{\mathrm{af}}^-$, $\mu + \chi z^{-1}\alpha^\vee$ is antidominant by [LS, Lemma 3.3]. By [LS, Lemma 3.2] we have

$$\begin{aligned}
1 &= \ell(x) - \ell(y) \\
&= (-\langle \mu\,,\, 2\rho\rangle - \ell(wz)) - \big(-\langle \mu + \chi z^{-1}\alpha^\vee\,,\, 2\rho\rangle - \ell(wzr_{z^{-1}\alpha})\big) \\
&= \ell(wzr_{z^{-1}\alpha}) - \ell(wz) + \chi\langle z^{-1}\alpha^\vee\,,\, 2\rho\rangle
\end{aligned}$$

By Lemma 4.1 we deduce that $\chi \in \{0, 1\}$.

Suppose $\chi = 0$. Then $y = wr_\alpha z t_\mu$ and $\ell(wzr_{z^{-1}\alpha}) - \ell(wz) = 1$, that is, $wz \lessdot wzr_{z^{-1}\alpha} = wr_\alpha z$. This gives the required Bruhat cover in $\mathrm{QB}(W)$. Since $y \in (W^J)_{\mathrm{af}}$ we have $\pi_J(y) = y$ and $wr_\alpha z t_\mu = \lfloor wr_\alpha z \rfloor \pi_J(t_\mu) = \lfloor wr_\alpha \rfloor z t_\mu$ using Proposition 3.5. We deduce that $wr_\alpha \in W^J$. By length-additivity it follows that $wr_\alpha \xleftarrow{\alpha} w$ is a Bruhat arrow in $\mathrm{QB}(W^J)$.

Otherwise we have $\chi = 1$. Then $y = wr_\alpha t_{\alpha^\vee} z t_\mu = wr_\alpha z t_{\mu + z^{-1}\alpha^\vee}$ and $\ell(wzr_{z^{-1}\alpha}) = \ell(wz) + 1 - \langle z^{-1}\alpha^\vee\,,\, 2\rho\rangle$, which yields the required quantum arrow in $\mathrm{QB}(W)$.

Since $y \in (W^J)_{\mathrm{af}}$ we have

$$\begin{aligned}
wr_\alpha t_{\alpha^\vee} z t_\mu = y = \pi(y) &= \pi_J(wr_\alpha t_{\alpha^\vee} z)\pi_J(t_\mu) \\
&= \pi_J(wr_\alpha t_{\alpha^\vee}) z t_\mu
\end{aligned}$$

from which we deduce that $wr_\alpha t_{\alpha^\vee} \in (W^J)_{\mathrm{af}}$ and that $\alpha^\vee$ is J-adjusted.

By Remark 3.6 and Lemma 3.11 we have $z_{\mu+z^{-1}\alpha^\vee} = z_\mu z_{z^{-1}\alpha^\vee} = z_\mu z_{\alpha^\vee} = z z_{\alpha^\vee}$. Since $\alpha^\vee$ is J-adjusted we have $wr_\alpha z = \pi_J(wr_\alpha) z_{\alpha^\vee} z = \pi_J(wr_\alpha) z_{\mu+z^{-1}\alpha^\vee}$ and the last product is length-additive. Therefore

$$\begin{aligned}
\ell(\pi_J(wr_\alpha)) &= \ell(wr_\alpha z) - \ell(z_{\mu+z^{-1}\alpha^\vee}) \\
&= \ell(wz) + 1 - \langle z^{-1}\alpha^\vee\,,\, 2\rho\rangle - \ell(z_{\mu+z^{-1}\alpha^\vee}) \\
&= \ell(w) + 1 + \ell(z_\mu) - \ell(z_{\mu+z^{-1}\alpha^\vee}) - \langle z^{-1}\alpha^\vee\,,\, 2\rho\rangle \\
&= \ell(w) + 1 + \langle z^{-1}\alpha^\vee\,,\, 2\rho_J\rangle - \langle z^{-1}\alpha^\vee\,,\, 2\rho\rangle \\
&= \ell(w) + 1 - \langle z^{-1}\alpha^\vee\,,\, 2\rho - 2\rho_J\rangle \\
&= \ell(w) + 1 - \langle \alpha^\vee\,,\, 2\rho - 2\rho_J\rangle
\end{aligned}$$

using Lemma 3.10, that μ and $\mu+z^{-1}\alpha^\vee$ are J-adjusted, and Lemma 4.4. This proves the existence of the required edge in $\mathrm{QB}(W^J)$. □

Example 5.4. Let $\mathfrak{g}$ be of type A_2 and $J = \{1\}$. Then $\mathrm{QB}(W^J)$ is given by

$$\begin{tikzcd}
& r_1r_2 \arrow[dd, dotted, "\alpha_2"] \\
r_2 \arrow[ur, "\alpha_1+\alpha_2"] & \\
& \mathrm{id} \arrow[ul, "\alpha_2"]
\end{tikzcd}$$

where the quantum arrow is dotted. In $\Omega_J \subset W_{\mathrm{af}}$, let $\mu = -6\omega_2^\vee$ and $\nu = -3\omega_2^\vee - \theta^\vee$. We have

$$\begin{tikzcd}
t_\mu = t_{-6\omega_2^\vee} \arrow[d, "6\delta-\alpha_2"'] & x = r_1t_\nu \arrow[d, "5\delta-r_1(\alpha_2)"'] \\
r_2t_\mu \arrow[d, "6\delta-\alpha_1-\alpha_2"'] & r_2(r_1t_\nu) \arrow[d, "4\delta-r_1(\alpha_1+\alpha_2)"'] \\
r_1r_2t_\mu \arrow[d, dotted, "5\delta-\alpha_2"'] & r_1r_2(r_1t_\nu) \arrow[d, dotted, "4\delta-r_1(\alpha_2)"'] \\
x = r_0r_1r_2t_\mu & r_0r_1r_2(r_1t_\nu) = t_{-3\omega_2^\vee}
\end{tikzcd}$$

We have a single chain running from $t_{-6\omega_2^\vee}$ down to $t_{-3\omega_2^\vee}$. The diagram is broken at $t_{-\nu}$, which appears at the bottom on the left and the top on the right. If the bottom element is removed from each side then one obtains an upside-down copy of $\mathrm{QB}(W^J)$. In this case the quantum arrows transition to a different copy of $\mathrm{QB}(W^J)$. The left hand copy has $z = \mathrm{id}$ and the right hand copy has $z = r_1$ where in this situation Σ_J is generated by r_1. The poset Ω_J^∞ is an infinite chain that wraps down onto the 3-cycle given by $\mathrm{QB}(W^J)$ with two flavors of lifts, one for $z = \mathrm{id}$ and the other for $z = r_1$.

Warning: generally not every affine cover is produced by left multiplication by a simple reflection, nor is a general quantum cover always induced by left multiplication by r_0 (although we shall see that left multiplication by r_0 always induces a quantum arrow).

We say that a walk in the directed graph $\mathrm{QB}(W^J)$ is locally-shortest if any segment of the walk not containing a repeated vertex is a shortest path.

Corollary 5.5. *Downward saturated chains in Ω_J^∞ project to locally-shortest walks in $\mathrm{QB}(W^J)$. Conversely, shortest paths in $\mathrm{QB}(W^J)$ are projections of downward saturated chains in Ω_J^∞.*

Proof. Say $x_0 \gtrdot x_1 \gtrdot \cdots \gtrdot x_N$ is a saturated Bruhat chain in Ω_J^∞. Let $\pi_J(x_i) = w_i z_i t_{\mu_i}$ where $w_i \in W^J$, $z_i \in W_J$, and $\mu_i \in Q^\vee$. Then Theorem 5.2 asserts that $w_0 \to w_1 \to \cdots \to w_N$ is a locally-shortest walk in $\mathrm{QB}(W^J)$.

Now let $u = u_0 \to u_1 \to \cdots \to u_N = u'$ be a shortest path in $\mathrm{QB}(W^J)$. We apply Theorem 5.2 to the edge $u_0 \to u_1$ with $\mu = \mu_0$ J-superantidominant and W_J-invariant. The element $x_0 = u_0 t_\mu$ lifts u_0 since $\pi_J(t_\mu) = t_\mu$. Then the Proposition produces a cocover $x_1 = u_1 z_1 t_{\mu_1}$ of x_0 with $z_1 \in \Sigma_J$. In general we have a descending Bruhat chain $x_0 \gtrdot x_1 \gtrdot \cdots \gtrdot x_{i-1} = u_{i-1} z_{i-1} t_{\mu_{i-1}}$ with $z_{i-1} \in \Sigma_J$ and we apply the Proposition to obtain a cocover $x_i = u_i z_i t_{\mu_i}$ of x_{i-1} with $z_i \in \Sigma_J$ and by induction the required affine chain is produced. □

Corollary 5.6. *For each $z \in \Sigma_J$ there is a copy of $\mathrm{QB}(W^J)$ inside $\mathrm{QB}(W)$, embedded by $w \mapsto wz$ such that the edge label α is sent to the root $z^{-1}\alpha$, and Bruhat and quantum edges are sent to the same kind of edge.*

Proof. For every $z \in \Sigma_J$, we take an edge $\lfloor w r_\alpha \rfloor \xleftarrow{\alpha} w$ in $\mathrm{QB}(W^J)$, lift it to $wzt_\mu \gtrdot w r_\alpha z t_\nu$ for some ν, and project to an edge $w r_\alpha z \xleftarrow{z^{-1}\alpha} wz$ in $\mathrm{QB}(W)$; the lift is based on Theorem 5.2 (1), and the projection on Theorem 5.2 (2). □

Remark 5.7. Lifting quantum edges causes a "phase shift" by an element $z \in \Sigma_J$. Theorem 5.2 is just general enough to lift in the presence of such a shift. If one tries to twist by a $z \in W_J$ that is not in Σ_J then the affine element of the form $x = wzt_\mu$ no longer lies in the set $(W^J)_{\mathrm{af}}$ and lifting the edge of $\mathrm{QB}(W^J)$ starting from x is not possible in general.

5.3. Trichotomy of cosets.

Lemma 5.8. [De] *Let W be a Weyl group, $W_J \subset W$ a parabolic subgroup, $v \in W^J$ and $r \in W$ a simple reflection. Then one of the following holds.*

1. *If $rv < v$ then $rv \in W^J$ and $rvW_J < vW_J$.*
2. *If $rv > v$ and $v^{-1}rv \in W_J$ then $rvW_J = vW_J$.*
3. *If $rv > v$ and $v^{-1}rv \notin W_J$ then $rv \in W^J$ and $rvW_J > vW_J$.*

Lemma 5.9. *Let $v \in W$ and $\alpha \in \Phi^+$. Let $\lambda \in X$ be a dominant weight (cf. Section 3.2 and the notation thereof, e.g., W_J is the stabilizer of λ).*

1. *Let $\langle \alpha^\vee , v\lambda \rangle < 0$. Then $v^{-1}\alpha \in \Phi^- \setminus \Phi_J^-$ and $r_\alpha v W_J < vW_J$.*
2. *Let $\langle \alpha^\vee , v\lambda \rangle = 0$. Then $v^{-1}\alpha \in \Phi_J$ and $r_\alpha v W_J = vW_J$.*
3. *Let $\langle \alpha^\vee , v\lambda \rangle > 0$. Then $v^{-1}\alpha \in \Phi^+ \setminus \Phi_J^+$ and $r_\alpha v W_J > vW_J$.*

The proof of the above lemma is easy using standard techniques for Weyl groups (see for example [BB, Proposition 2.5.1]).

5.4. Quantum edges induced by left multiplication by suitable reflections.

Proposition 5.10. [De] *Let $w \in W^J$ and $j \in I$. Then exactly one of the following holds.*

1. *$w^{-1}\alpha_j \in \Phi^- \setminus \Phi_J^-$. In this case $r_j w \in W^J$ and there is a Bruhat edge $w \xleftarrow{-w^{-1}\alpha} r_j w$ in $\mathrm{QB}(W^J)$.*

(2) $w^{-1}\alpha_j \in \Phi_J$. *In this case* $w^{-1}\alpha_j \in \Phi_J^+$ *and* $\lfloor r_j w \rfloor = w$.

(3) $w^{-1}\alpha_j \in \Phi^+ \setminus \Phi_J^+$. *In this case* $r_j w \in W^J$ *and there is a Bruhat edge* $r_j w \xleftarrow{w^{-1}\alpha_j} w$ *in* $\mathrm{QB}(W^J)$.

Proposition 5.11. *Let* $w \in W^J$. *Then exactly one of the following holds.*

(1) $w^{-1}\theta \in \Phi^- \setminus \Phi_J^-$. *In this case there is an edge* $\lfloor r_\theta w \rfloor \xleftarrow{-w^{-1}\theta} w$ *of quantum type in* $\mathrm{QB}(W^J)$.
(2) $w^{-1}\theta \in \Phi_J$. *In this case* $w^{-1}\theta \in \Phi_J^+$ *and* $\lfloor r_\theta w \rfloor = w$.
(3) $w^{-1}\theta \in \Phi^+ \setminus \Phi_J^+$. *In this case there is an edge* $w \xleftarrow{zw^{-1}\theta} \lfloor r_\theta w \rfloor$ *of quantum type in* $\mathrm{QB}(W^J)$, *where* $z = z_{w^{-1}\theta^\vee}$.

Proof. The three cases correspond to those in Lemma 5.9 with $v = w$ and $\alpha = \theta$. The conclusion in Case (2) is immediate from the definition of $w \in W^J$. By exchanging the roles of w and $\lfloor r_\theta w \rfloor$, it suffices to prove the existence of the edge in (1). Let $w^{-1}\theta \in \Phi^- \setminus \Phi_J^-$.

Choose any $\mu \in Q^\vee$ that is J-superantidominant and W_J-invariant. We have $\pi_J(t_\mu) = t_\mu$ and $x := wt_\mu \in W_{\mathrm{af}}^- \cap (W^J)_{\mathrm{af}}$. We have

$$x^{-1}\alpha_0 = -w^{-1}\theta + (1 + \langle \mu\,,\, -w^{-1}\theta \rangle)\delta \in \Phi^{\mathrm{af}-}\,, \tag{5.1}$$

since $w^{-1}\theta \in \Phi^-$. We conclude that $x > y := r_0 x = r_\theta w t_{\mu - w^{-1}\theta^\vee}$. Since $x \in W_{\mathrm{af}}^-$ it follows that $y \in W_{\mathrm{af}}^-$ as well. Let $\beta \in \Phi_J^{\mathrm{af}+}$. Suppose $y\beta \in \Phi^{\mathrm{af}-}$. Since $x \in (W^J)_{\mathrm{af}}$ we have $x\beta = r_0 y\beta \in \Phi^{\mathrm{af}+}$. Since r_0 has the sole inversion α_0, it follows that $x\beta = \alpha_0$ or $x^{-1}\alpha_0 = \beta \in \Phi_J^{\mathrm{af}+}$. But this contradicts (5.1). Therefore $y \in (W^J)_{\mathrm{af}}$.

By Theorem 5.2, the required quantum edge exists in $\mathrm{QB}(W^J)$. □

Corollary 5.12. *For every* $\gamma \in W\theta \cap (\Phi^+ \setminus \Phi_J^+)$, $z_{\gamma^\vee}\gamma^\vee$ *is* J*-adjusted and for every* $\gamma \in W\theta \cap \Phi^- \setminus \Phi_J^-$, $-\gamma^\vee$ *is* J*-adjusted.*

Proof. Follows from the existence of the edges in $\mathrm{QB}(W^J)$. □

5.5. **Diamond Lemmas for** $\mathrm{QB}(W^J)$**.** We recall the Diamond Lemma for Coxeter groups and the Bruhat order.

Lemma 5.13. [H] *Let* W *be any Coxeter group,* $v, w \in W$, *and* r *a simple reflection.*

(1) *Suppose* $v \lessdot w$, $rw < w$ *and* $v \neq rw$. *Then* $rv < v$ *and* $rv \lessdot rw$.
(2) *Suppose* $v \gtrdot w$, $rw > w$ *and* $v \neq rw$. *Then* $rv > v$ *and* $rv \gtrdot rw$.

In the following diagrams, a dotted (resp. plain) edge represents a quantum (resp. Bruhat) edge in $\mathrm{QB}(W^J)$. We always refer to the parabolic quantum Bruhat graph on W^J. Given $w \in W^J$ and $\gamma \in \Phi^+$, define $z, z' \in W_J$ by

$$r_\theta w = \lfloor r_\theta w \rfloor z\,, \qquad r_\theta \lfloor w r_\gamma \rfloor = \lfloor r_\theta \lfloor w r_\gamma \rfloor \rfloor z' = \lfloor r_\theta w r_\gamma \rfloor z'\,. \tag{5.2}$$

We are now ready to state the Diamond Lemmas for the parabolic quantum Bruhat graph.

Lemma 5.14. *Let* α *be a simple root in* Φ , $\gamma \in \Phi^+ \setminus \Phi_J^+$, *and* $w \in W^J$. *Then we have the following cases, in each of which the bottom two edges imply the top two edges in the left diagram, and the top two edges imply the bottom two edges in the right diagram.*

(1) *In both cases we assume $\gamma \neq w^{-1}\alpha$ and have $r_\alpha\lfloor wr_\gamma\rfloor = r_\alpha wr_\gamma = \lfloor r_\alpha wr_\gamma\rfloor$.*

$$\begin{tikzcd}[column sep=small]
& r_\alpha\lfloor wr_\gamma\rfloor & \\
r_\alpha w \arrow[ur, "\gamma"] & & \lfloor wr_\gamma\rfloor \arrow[ul, "\lfloor wr_\gamma\rfloor^{-1}\alpha"'] \\
& w \arrow[ul, "w^{-1}\alpha"] \arrow[ur, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& \lfloor wr_\gamma\rfloor & \\
w \arrow[ur, "\gamma"] & & r_\alpha\lfloor wr_\gamma\rfloor \arrow[ul, "-\lfloor wr_\gamma\rfloor^{-1}\alpha"'] \\
& r_\alpha w \arrow[ul, "-w^{-1}\alpha"] \arrow[ur, "\gamma"'] &
\end{tikzcd}
\tag{5.3}$$

(2) *Here we have $r_\alpha\lfloor wr_\gamma\rfloor = \lfloor r_\alpha wr_\gamma\rfloor$ in both cases.*

$$\begin{tikzcd}[column sep=small]
& r_\alpha\lfloor wr_\gamma\rfloor & \\
r_\alpha w \arrow[ur, dotted, "\gamma"] & & \lfloor wr_\gamma\rfloor \arrow[ul, "\lfloor wr_\gamma\rfloor^{-1}\alpha"'] \\
& w \arrow[ul, "w^{-1}\alpha"] \arrow[ur, dotted, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& \lfloor wr_\gamma\rfloor & \\
w \arrow[ur, dotted, "\gamma"] & & r_\alpha\lfloor wr_\gamma\rfloor \arrow[ul, "-\lfloor wr_\gamma\rfloor^{-1}\alpha"'] \\
& r_\alpha w \arrow[ul, "-w^{-1}\alpha"] \arrow[ur, dotted, "\gamma"'] &
\end{tikzcd}
\tag{5.4}$$

(3) *Here z, z' are defined as in* (5.2). *In subcase* (5.5) *(resp.* (5.6)*) we assume that $\langle w^{-1}\theta, \gamma^\vee\rangle$ is nonzero (resp. zero). In all cases, we have $wr_\gamma = \lfloor wr_\gamma\rfloor$.*

$$\begin{tikzcd}[column sep=small]
& \lfloor r_\theta wr_\gamma\rfloor & \\
\lfloor r_\theta w\rfloor \arrow[ur, dotted, "z\gamma"] & & \lfloor wr_\gamma\rfloor \arrow[ul, dotted, "-\lfloor wr_\gamma\rfloor^{-1}\theta"'] \\
& w \arrow[ul, dotted, "-w^{-1}\theta"] \arrow[ur, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& \lfloor wr_\gamma\rfloor & \\
w \arrow[ur, "\gamma"] & & \lfloor r_\theta wr_\gamma\rfloor \arrow[ul, dotted, "z'\lfloor wr_\gamma\rfloor^{-1}\theta"'] \\
& \lfloor r_\theta w\rfloor \arrow[ul, dotted, "zw^{-1}\theta"] \arrow[ur, dotted, "z\gamma"'] &
\end{tikzcd}
\tag{5.5}$$

$$\begin{tikzcd}[column sep=small]
& \lfloor r_\theta wr_\gamma\rfloor & \\
\lfloor r_\theta w\rfloor \arrow[ur, "z\gamma"] & & \lfloor wr_\gamma\rfloor \arrow[ul, dotted, "-\lfloor wr_\gamma\rfloor^{-1}\theta"'] \\
& w \arrow[ul, dotted, "-w^{-1}\theta"] \arrow[ur, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& \lfloor wr_\gamma\rfloor & \\
w \arrow[ur, "\gamma"] & & \lfloor r_\theta wr_\gamma\rfloor \arrow[ul, dotted, "z'\lfloor wr_\gamma\rfloor^{-1}\theta"'] \\
& \lfloor r_\theta w\rfloor \arrow[ul, dotted, "zw^{-1}\theta"] \arrow[ur, "z\gamma"'] &
\end{tikzcd}
\tag{5.6}$$

(4) *Here we assume $\gamma \neq -w^{-1}\theta$ in all cases, and z, z' are defined as in* (5.2). *In subcase* (5.7) *(resp.* (5.8)*) we assume that $\langle w^{-1}\theta, \gamma^\vee\rangle$ is nonzero (resp. zero).*

$$\begin{tikzcd}[column sep=small]
& \lfloor r_\theta wr_\gamma\rfloor & \\
\lfloor r_\theta w\rfloor \arrow[ur, "z\gamma"] & & \lfloor wr_\gamma\rfloor \arrow[ul, dotted, "-\lfloor wr_\gamma\rfloor^{-1}\theta"'] \\
& w \arrow[ul, dotted, "-w^{-1}\theta"] \arrow[ur, dotted, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& \lfloor wr_\gamma\rfloor & \\
w \arrow[ur, dotted, "\gamma"] & & \lfloor r_\theta wr_\gamma\rfloor \arrow[ul, dotted, "z'\lfloor wr_\gamma\rfloor^{-1}\theta"'] \\
& \lfloor r_\theta w\rfloor \arrow[ul, dotted, "zw^{-1}\theta"] \arrow[ur, "z\gamma"'] &
\end{tikzcd}
\tag{5.7}$$

$$
\begin{tikzcd}[column sep=small]
& \lfloor r_\theta w r_\gamma \rfloor & \\
\lfloor r_\theta w \rfloor \arrow[ur, dotted, "z\gamma"] & & \lfloor w r_\gamma \rfloor \arrow[ul, dotted, "-\lfloor w r_\gamma \rfloor^{-1}\theta"'] \\
& w \arrow[ul, dotted, "-w^{-1}\theta"] \arrow[ur, dotted, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& \lfloor w r_\gamma \rfloor & \\
w \arrow[ur, dotted, "\gamma"] & & \lfloor r_\theta w r_\gamma \rfloor \arrow[ul, dotted, "z'\lfloor w r_\gamma \rfloor^{-1}\theta"'] \\
& \lfloor r_\theta w \rfloor \arrow[ul, dotted, "z w^{-1}\theta"] \arrow[ur, dotted, "z\gamma"'] &
\end{tikzcd}
\tag{5.8}
$$

Remark 5.15. (1) The left diagram of (5.3) of Lemma 5.14 is the classical Diamond Lemma 5.13.

(2) The right diagrams in (5.3), (5.4), (5.5), (5.6), (5.7), and (5.8) are relabelings of the left diagrams in (5.3), (5.4), (5.7), (5.6), (5.5), and (5.8), respectively, and vice versa. For instance, we can obtain a right diagram by labeling w the leftmost vertex in the corresponding left diagram, and by recalculating all the other vertex and edge labels.

Proof of Lemma 5.14. By Proposition 4.3 only the left diagrams need to be established. In all cases, the bottom half of a diamond in $\mathrm{QB}(W^J)$ is lifted to the affine Bruhat order using Theorem 5.2 (1). There the diamond is completed using the usual Diamond Lemma 5.13 for the affine Weyl group. The affine diamond is pushed down to $\mathrm{QB}(W^J)$ using Theorem 5.2 (2).

Consider the left diagram in (5.4). By Theorem 5.2 (1), the quantum edge $\lfloor wr_\gamma \rfloor \xleftarrow{\gamma} w$ lifts to an affine Bruhat cover $y \lessdot x$ in Ω_J^∞ where $x = wt_\mu$, μ is J-superantidominant with $z_\mu = \mathrm{id}$, $y = wr_\gamma t_{\gamma^\vee+\mu} = xr_{\widetilde{\gamma}}$, and $\widetilde{\gamma} = \gamma + (1 + \langle \mu\,,\,\gamma\rangle)\delta \in \Phi^{\mathrm{af}-}$. Since $r_\alpha w \gtrdot w$ and $r_\alpha w \in W^J$, it follows that $r_\alpha x \lessdot x$. Moreover this covering relation is the affine lift into Ω_J^∞ of the Bruhat edge $r_\alpha w \xleftarrow{w^{-1}\alpha} w$. The elements $r_\alpha x$ and y are distinct since they have different translation components. By the Diamond Lemma 5.13 for the affine Weyl group, we have $r_\alpha x \gtrdot r_\alpha y$ and $y \gtrdot r_\alpha y$. The latter cover implies that $r_\alpha y \in \Omega_J^\infty$. Theorem 5.2 (2) yields the top half of the left diagram in (5.4).

Consider the bottom half of the left diagram in (5.5) (which is also the same half diagram in (5.6)). The quantum edge $\lfloor r_\theta w \rfloor \xleftarrow{-w^{-1}\theta} w$ lifts to the affine cover in Ω_J^∞ given by $r_0 x \lessdot x$ where $x = wt_\mu$ and $r_0 x = r_\theta w t_{\mu - w^{-1}\theta^\vee}$. The Bruhat edge $wr_\gamma \xleftarrow{\gamma} w$ lifts to the affine cover in Ω_J^∞ given by $wr_\gamma t_\mu = xr_{\widetilde{\gamma}} \lessdot wt_\mu$ where $\widetilde{\gamma} = \gamma + \langle \mu\,,\,\gamma\rangle\delta$. One may verify that $r_0 x \neq xr_{\widetilde{\gamma}}$. By the Diamond Lemma 5.13 for the affine Weyl group, we have $r_0 x \gtrdot r_0 x r_{\widetilde{\gamma}}$ and $xr_{\widetilde{\gamma}} \gtrdot r_0 x r_{\widetilde{\gamma}}$. Arguing as in the proof of Proposition 5.11 and using that $xr_{\widetilde{\gamma}} \in \Omega_J^\infty$, one may show that $r_0 x r_{\widetilde{\gamma}} \in \Omega_J^\infty$. By Theorem 5.2 (2) we obtain edges in $\mathrm{QB}(W^J)$ which complete the diamond, with the only remaining issue being the type of the edge $\lfloor r_\theta w \rfloor \to \lfloor r_\theta w r_\gamma \rfloor$. It is quantum or Bruhat depending on whether the translation elements in the affine lift $r_0 x \gtrdot r_0 x r_{\widetilde{\gamma}}$ are different or the same. Since $r_0 x r_{\widetilde{\gamma}} = r_\theta w r_\gamma t_{\mu - r_\gamma w^{-1}(\theta^\vee)}$ we see that the translation element changes in passing from $r_0 x$ to $r_0 x r_{\widetilde{\gamma}}$ if and only if $\langle w^{-1}(\theta^\vee)\,,\,\gamma\rangle \neq 0$, as required.

The cases for the diagrams (5.7) and (5.8) are similar to those for (5.5) and (5.6). □

6. The level-zero weight poset

In [Li], Littelmann introduced a poset related to Lakshmibai–Seshadri (LS) paths for arbitrary (not necessarily dominant) integral weights. We consider this poset for level-zero weights. Littelmann did not give a precise local description of it. Our main result in this section is a characterization of its cover relations in terms of the parabolic quantum Bruhat graph.

Fix a dominant weight λ in the finite weight lattice X (cf. Section 3.2 and the notation thereof, e.g., W_J is the stabilizer of λ). We view X as a sublattice of X_{af}^0. Let $X_{\mathrm{af}}^0(\lambda)$ be the orbit of λ under the action of the affine Weyl group W_{af}.

Definition 6.1. (Level-zero weight poset [Li]) *A poset structure is defined on $X_{\mathrm{af}}^0(\lambda)$ as the transitive closure of the relation*

$$\mu < r_\beta\mu \quad \Leftrightarrow \quad \langle \mu, \beta^\vee \rangle > 0\,, \tag{6.1}$$

where $\beta \in \Phi^{\mathrm{af}+}$. This poset is called the level-zero weight poset for λ.

Remarks 6.2.

(1) Assume that W_J is trivial, and we set $\mu = w\lambda$ for $w \in W_{\mathrm{af}}$. Then, for $\beta \in \Phi^{\mathrm{af}+}$, we have $\mu < r_\beta\mu$ in the level-zero weight poset if and only if $w^{-1}r_\beta \prec w^{-1}$ in the generic Bruhat order $\prec$ on W_{af} introduced by Lusztig [Lu]. Indeed, this equivalence follows from the definitions of these partial orders by using [Soe, Claim 4.14, page 96]. The generic Bruhat order also recently appeared in [La].

(2) We can define the poset $X_{\mathrm{af}}^0(-\lambda)$ on the orbit of the antidominant weight $-\lambda$ in the same way, using (6.1). The posets $X_{\mathrm{af}}^0(\lambda)$ and $X_{\mathrm{af}}^0(-\lambda)$ are dual isomorphic, in the sense that, for $\mu, \nu \in X_{\mathrm{af}}^0(\lambda)$, we have

$$\mu < \nu \quad \Leftrightarrow \quad -\mu > -\nu\,.$$

Therefore, all the statements in this section can be easily rephrased for $X_{\mathrm{af}}^0(-\lambda)$.

An example of $X_{\mathrm{af}}^0(\lambda)$ is given in Figure 6. As we can see from this example, $X_{\mathrm{af}}^0(\lambda)$ is not a graded poset in general.

Littelmann [Li] introduced a distance function on the level-zero weight poset. Namely, if $\mu \le \nu$ in $X_{\mathrm{af}}^0(\lambda)$, then $\mathrm{dist}(\mu,\nu)$[1] is the maximum length of a chain from μ to ν. Clearly, covers correspond to elements at distance 1.

Lemma 6.3. [Li, Lemma 4.1] *Let $\mu, \nu \in X_{\mathrm{af}}^0(\lambda)$.*

(1) *If $\mu \le \nu$ and α is a simple root in Φ^{af} such that $\langle \mu, \alpha^\vee \rangle \ge 0$ but $\langle \nu, \alpha^\vee \rangle < 0$, then $\mu \le r_\alpha\nu$ and $\mathrm{dist}(\mu, r_\alpha\nu) < \mathrm{dist}(\mu,\nu)$.*

(2) *If $\mu \le \nu$ and α is a simple root in Φ^{af} such that $\langle \mu, \alpha^\vee \rangle > 0$ but $\langle \nu, \alpha^\vee \rangle \le 0$, then $r_\alpha\mu \le \nu$ and $\mathrm{dist}(r_\alpha\mu, \nu) < \mathrm{dist}(\mu,\nu)$.*

(3) *If $\mu \le \nu$ and α is a simple root in Φ^{af} such that $\langle \mu, \alpha^\vee \rangle, \langle \nu, \alpha^\vee \rangle > 0$ (respectively $\langle \mu, \alpha^\vee \rangle, \langle \nu, \alpha^\vee \rangle < 0$), then $\mathrm{dist}(\mu,\nu) = \mathrm{dist}(r_\alpha\mu, r_\alpha\nu)$.*

We label a cover $\mu \lessdot \nu = r_\beta\mu$ of $X_{\mathrm{af}}^0(\lambda)$ by the corresponding positive real root β. Preliminary results about the covers of $X_{\mathrm{af}}^0(\lambda)$ were obtained by Naito and Sagaki.

Lemma 6.4.

(1) [NS4, Remark 2.10 and Lemma 2.11]. *For untwisted types, a necessary condition for $\mu < \nu$ to be a cover in $X_{\mathrm{af}}^0(\lambda)$ is that $\nu = r_\beta\mu$ with $\beta \in \Phi^+$ or $\beta \in \delta - \Phi^+$.*

(2) [NS4, Remark 2.10 (2)] *Let $\mu, \nu \in X_{\mathrm{af}}^0(\lambda)$ be such that $\nu = r_\alpha\mu$ for a simple root α in Φ^{af} such that $\langle \mu, \alpha^\vee \rangle > 0$. Then $\mathrm{dist}(\mu,\nu) = 1$.*

We consider the standard projection map cl from $X_{\mathrm{af}}^0(\lambda)$ to the orbit of λ under the finite Weyl group (by factoring out the δ part). We identify $W\lambda \simeq W/W_J \simeq W^J$, and consider on W^J the parabolic quantum Bruhat graph structure. Note that, by contrast with $X_{\mathrm{af}}^0(\lambda)$, the edges of the latter are labeled by positive roots $\gamma \in \Phi^+$ (of the finite root system) corresponding to right multiplication by r_γ. We use solid arrows to denote covers in the Bruhat order, whereas dotted arrows denote quantum edges in the parabolic quantum Bruhat graph on W^J.

Our main result is that the level-zero weight poset is an affine lift of the corresponding parabolic quantum Bruhat graph. This is illustrated in Figure 6, where the edges of the (parabolic) Bruhat graph (i.e., the slice of the level-zero weight poset with no δ, onto which we project) are shown in red. Projecting all vertices onto the red part, one obtains the quantum Bruhat graph of Figure 1.

[1]The notation in [Li] is $\mathrm{dist}(\nu,\mu)$.

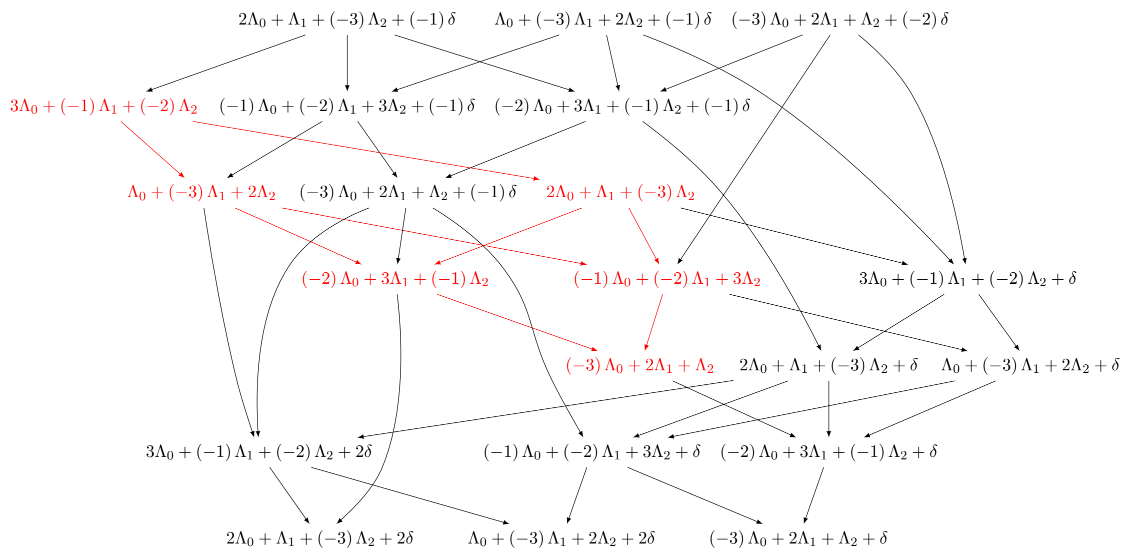

FIGURE 3. Slice of the level-zero weight poset for type $A_2^{(1)}$ and weight $2\Lambda_1 + \Lambda_2 - 3\Lambda_0$.

Theorem 6.5. *Let $\mu \in X^0_{\mathrm{af}}(\lambda)$ and $w := \mathrm{cl}(\mu) \in W^J$. If $\mu \lessdot \nu$ is a cover in $X^0_{\mathrm{af}}(\lambda)$ labeled by $\beta \in \Phi^{\mathrm{af}+}$, then $w \to \mathrm{cl}(\nu)$ is an up (respectively down) edge in the parabolic quantum Bruhat graph on W^J labeled by $w^{-1}\beta \in \Phi^+ \setminus \Phi^+_J$ (respectively $w^{-1}(\beta - \delta)$), depending on $\beta \in \Phi^+$ (respectively $\beta \in \delta - \Phi^+$). Conversely, if $w \xrightarrow{\gamma} wr_\gamma = w'$ (respectively $w \overset{\gamma}{\dashrightarrow} \lfloor wr_\gamma \rfloor = w'$) in the parabolic quantum Bruhat graph for $\gamma \in \Phi^+ \setminus \Phi^+_J$, then there exists a cover $\mu \lessdot \nu$ in $X^0_{\mathrm{af}}(\lambda)$ labeled by $w\gamma$ (respectively $\delta + w\gamma$) with $\mathrm{cl}(\nu) = w'$.*

The proof of Theorem 6.5 is given in the remainder of this section.

6.1. **Outline of the proof.** Let us begin by giving a brief outline of the proof. To relate the cover relations in the level-zero weight poset $X^0_{\mathrm{af}}(\lambda)$ and the edges in the parabolic quantum Bruhat graph, we use the so-called Diamond Lemma on $X^0_{\mathrm{af}}(\lambda)$ to successively move a cover $\mu \lessdot r_\beta\mu$ "closer" to a cover $\mu \lessdot r_\alpha\mu$ for a simple root α in Φ^{af}. For simple roots, the statement of Theorem 6.5 is proved in Section 6.2. The Diamond Lemma in the level-zero weight poset is the subject of Section 6.3. Recall that the Diamond Lemmas for the parabolic quantum Bruhat graph were treated in Section 5.5. In Section 6.4 we prove some further statements related to the Diamond Lemmas for the parabolic quantum Bruhat graph that we need for our arguments. We conclude in Section 6.5 with the main argument, based on matching the diamond reductions in the level-zero weight poset with those in the parabolic quantum Bruhat graph.

6.2. **Results for simple roots.** In this section, we characterize a cover relation $\mu \lessdot \nu$ in $X^0_{\mathrm{af}}(\lambda)$ when μ and ν are related by an affine simple reflection.

We start with a simple lemma. Since some versions of it will be needed beyond this section, we collect all of them here.

Lemma 6.6. *Let α be a simple root, β a positive root (both in Φ^{af}), and $\mu = wt_\tau\lambda$ with $w \in W^J$ and $\tau \in Q^\vee$. Let $\gamma \in \Phi^+$ be given by $\beta = \pm w\gamma + k\delta$.*

(1) *We have*

$$\mathrm{cl}(\mu) = w\,, \qquad \mathrm{cl}(r_\beta\mu) = \lfloor wr_\gamma \rfloor\,.$$

(2) *If $\alpha \neq \alpha_0$, assume that $r_\alpha w \in W^J$, i.e., $\mathrm{cl}(\mu) \neq \mathrm{cl}(r_\alpha \mu)$. Then we have*

$$\mathrm{cl}(r_\alpha \mu) = \begin{cases} r_\alpha w = \lfloor r_\alpha w \rfloor & \text{if } \alpha \neq \alpha_0 \\ \lfloor r_\theta w \rfloor & \text{if } \alpha = \alpha_0 \,. \end{cases}$$

(3) *If $\alpha \neq \alpha_0$, assume that $r_\alpha \lfloor w r_\gamma \rfloor \in W^J$, i.e., $\mathrm{cl}(r_\beta \mu) \neq \mathrm{cl}(r_\alpha r_\beta \mu)$. Then we have*

$$\mathrm{cl}(r_\alpha r_\beta \mu) = \begin{cases} r_\alpha \lfloor w r_\gamma \rfloor = \lfloor r_\alpha \lfloor w r_\gamma \rfloor \rfloor & \text{if } \alpha \neq \alpha_0 \\ \lfloor r_\theta w r_\gamma \rfloor & \text{if } \alpha = \alpha_0 \,. \end{cases}$$

Proof. We have

$$\mu = w t_\tau \lambda = w\lambda - \langle \lambda, \tau \rangle \delta \,. \tag{6.2}$$

So $\mathrm{cl}(\mu) = w$. Similarly, we have

$$\mathrm{cl}(r_\beta \mu) = \mathrm{cl}(r_{w\gamma} t_{\pm k w \gamma^\vee} w t_\tau(\lambda)) = \lfloor w r_\gamma \rfloor \,, \quad \text{and} \quad \mathrm{cl}(r_\alpha \mu) = \begin{cases} \lfloor r_\alpha w \rfloor & \text{if } \alpha \neq \alpha_0 \\ \lfloor r_\theta w \rfloor & \text{if } \alpha = \alpha_0 \,. \end{cases}$$

In addition, if $\alpha \neq \alpha_0$, then $\lfloor r_\alpha w \rfloor$ can only be w or $r_\alpha w$, by Lemma 5.8; but the first case cannot happen by the assumptions of the lemma. The calculation of $\mathrm{cl}(r_\alpha r_\beta \mu)$ is similar, by also noting that, if $\alpha \neq \alpha_0$, then $\lfloor r_\alpha w r_\gamma \rfloor = \lfloor r_\alpha \lfloor w r_\gamma \rfloor \rfloor$. □

Lemma 6.7. *Let $\mu, \nu \in X^0_{\mathrm{af}}(\lambda)$ be such that $\nu = r_\alpha \mu$ for a simple root α in Φ^{af}. Then $\mu \lessdot \nu$ is a cover in $X^0_{\mathrm{af}}(\lambda)$ if and only if $\mathrm{cl}(\mu) \to \mathrm{cl}(\nu)$ is an up (respectively down) edge in the parabolic quantum Bruhat graph on W^J labeled by $w^{-1}\alpha$ (respectively $-w^{-1}\theta$), where $w = \mathrm{cl}(\mu) \in W^J$, depending on $\alpha \neq \alpha_0$ (respectively $\alpha = \alpha_0$).*

Proof. Since α is a simple root, we have by Lemma 6.4 (2) that $\mathrm{dist}(\mu, \nu) = 1$ if $\mu < \nu$. So in this case $\mu \lessdot \nu$ is equivalent to $\mu < \nu$. Letting $\mu = w t_\tau(\lambda)$ with $w \in W^J$ and $\tau \in Q^\vee$, we have $\mathrm{cl}(\mu) = w$, by Lemma 6.6 (1). Let us first assume that $\alpha \neq \alpha_0$. Then

$$\langle \mu, \alpha^\vee \rangle = \langle w\lambda, \alpha^\vee \rangle = \langle \lambda, w^{-1}\alpha^\vee \rangle,$$

where for the first equality we used (6.2). Hence

$$\mu < r_\alpha \mu \quad \Leftrightarrow \quad \langle \mu, \alpha^\vee \rangle > 0 \quad \Leftrightarrow \quad w^{-1}\alpha \in \Phi^+ \setminus \Phi^+_J \quad \Leftrightarrow \quad w \lessdot r_\alpha w \;\text{ in } W^J \,,$$

where the last equivalence is based on Lemma 5.9. The last condition is equivalent to $\mathrm{cl}(\mu) \to \mathrm{cl}(\nu)$ being an up edge in the parabolic quantum Bruhat graph, by Lemma 6.6 (2). This proves the claim for $\alpha \neq \alpha_0$.

Now assume $\alpha = \alpha_0$. Similarly to before

$$\langle \mu, \alpha^\vee \rangle = \langle w\lambda, \alpha^\vee \rangle = \langle w\lambda, -\theta^\vee \rangle = -\langle \lambda, w^{-1}\theta^\vee \rangle,$$

where we used $\alpha_0 = -\theta + \delta$, or $\alpha_0^\vee + \theta^\vee = c$. Hence

$$\mu > r_\alpha \mu \quad \Leftrightarrow \quad \langle \mu, \alpha^\vee \rangle > 0 \quad \Leftrightarrow \quad w^{-1}\theta \in \Phi^- \setminus \Phi^-_J .$$

By Proposition 5.11, the last condition is equivalent to the fact that $w \xrightarrow{-w^{-1}\theta} \lfloor r_\theta w \rfloor$ is a down edge in the parabolic quantum Bruhat graph. Also note that $\mathrm{cl}(r_\alpha \mu) = \lfloor r_\theta w \rfloor$, by Lemma 6.6 (2). This proves the claim for $\alpha = \alpha_0$. □

6.3. **The Diamond Lemma in the level-zero weight poset.** In this section we investigate the Diamond Lemma in the level-zero weight poset $X^0_{\mathrm{af}}(\lambda)$.

Lemma 6.8. *Let $\mu \in X^0_{\mathrm{af}}(\lambda)$ and $\mu < r_\beta \mu$ in $X^0_{\mathrm{af}}(\lambda)$, where $\beta \in \Phi^{\mathrm{af}+}$. Then there exists a simple root α in Φ^{af} (in fact, $\alpha \neq \alpha_0$ if $\beta \in \Phi$) such that $\langle \beta, \alpha^\vee \rangle > 0$, and either*

$$(1) \quad \mu \lessdot r_\alpha \mu \qquad \textit{or} \qquad (2) \quad r_\alpha r_\beta \mu \lessdot r_\beta \mu$$

is a cover in $X^0_{\mathrm{af}}(\lambda)$.

Proof. We pick a simple root α in the decomposition of β such that $\langle \beta, \alpha^\vee \rangle > 0$. This clearly exists, and in fact $\alpha \neq \alpha_0$ if $\beta \in \Phi$.

By Definition 6.1, we have $\langle \mu, \beta^\vee \rangle > 0$. We claim that either

$$\langle \mu, \alpha^\vee \rangle > 0 \qquad \text{or} \qquad \langle \mu, -r_\beta \alpha^\vee \rangle = -\langle r_\beta \mu, \alpha^\vee \rangle > 0\,. \tag{6.3}$$

Indeed, the reflection formula

$$r_\beta \alpha^\vee = \alpha^\vee - \langle \beta, \alpha^\vee \rangle \beta^\vee$$

implies that $\alpha^\vee - r_\beta \alpha^\vee$ is a positive multiple of $\beta^\vee$. Now (6.3) follows since $\langle \mu, \beta^\vee \rangle > 0$. We conclude the proof by combining (6.3) with Lemma 6.4 (2). □

Next we state the Diamond Lemma for the level-zero weight poset.

Lemma 6.9. *Let α be a simple root, $\beta \neq \alpha$ a positive root (both in Φ^{af}), and $\mu \in X^0_{\mathrm{af}}(\lambda)$. In the left diagram, the bottom two covers imply the top two covers, while the top two covers imply the bottom two covers in the right diagram.*

$$\begin{tikzcd}
 & r_\alpha r_\beta \mu & & & & r_\beta \mu & \\
r_\alpha \mu \arrow[ur, "r_\alpha \beta"] & & r_\beta \mu \arrow[ul, "\alpha"'] & & \mu \arrow[ur, "\beta"] & & r_\alpha r_\beta \mu \arrow[ul, "\alpha"'] \\
 & \mu \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] & & & & r_\alpha \mu \arrow[ul, "\alpha"] \arrow[ur, "r_\alpha \beta"'] &
\end{tikzcd} \tag{6.4}$$

Proof. We start by assuming that the bottom two arrows are covers in the left diagram. Set $\nu := r_\beta \mu$. By definition, we have $\langle \mu, \alpha^\vee \rangle > 0$ and $\langle \mu, \beta^\vee \rangle > 0$. We first show that $\langle \nu, \alpha^\vee \rangle > 0$, which implies that we have the cover $\nu \lessdot r_\alpha \nu$, by Lemma 6.4 (2). Indeed, if $\langle \nu, \alpha^\vee \rangle \leq 0$, then Lemma 6.3 (2) would imply $\mathrm{dist}(r_\alpha \mu, \nu) < \mathrm{dist}(\mu, \nu)$; since $\mathrm{dist}(\mu, \nu) = 1$, it would follow that $\nu = r_\alpha \mu$, which is impossible, since $\alpha \neq \beta$.

Turning to the remaining edge of the diamond, we clearly have $r_\alpha \mu < r_\alpha \nu$, as

$$r_\alpha \nu = r_{r_\alpha \beta}(r_\alpha \mu) \quad \text{and} \quad \langle r_\alpha \mu, r_\alpha \beta^\vee \rangle = \langle \mu, \beta^\vee \rangle > 0\,;$$

note that $r_\alpha \beta$ is a positive root, as $\alpha \neq \beta$. The hypotheses of Lemma 6.3 (3) apply, so we have $1 = \mathrm{dist}(\mu, \nu) = \mathrm{dist}(r_\alpha \mu, r_\alpha \nu)$. We conclude that we have the cover $r_\alpha \mu \lessdot r_\alpha \nu$.

The proof for the right diagram is similar, where we now assume that the top two arrows are covers. More precisely, in order to prove that the bottom arrows are covers, we use Lemma 6.3 (1) for the left one, and then Lemma 6.3 (3) for the right one. □

6.4. **More on the Diamond Lemmas for the PQBG.** Recall the Diamond Lemmas for the parabolic quantum Bruhat graph on W^J from Section 5.5. Recall that given $w \in W^J$ and $\gamma \in \Phi^+$, define $z, z' \in W_J$ by

$$r_\theta w = \lfloor r_\theta w \rfloor z\,, \qquad r_\theta \lfloor w r_\gamma \rfloor = \lfloor r_\theta \lfloor w r_\gamma \rfloor \rfloor z' = \lfloor r_\theta w r_\gamma \rfloor z'\,.$$

We need an analogue of Lemma 6.8.

Lemma 6.10. *Let $w \in W^J$, and let $w \xrightarrow{\gamma} wr_\gamma$ or $w \overset{\gamma}{\dashrightarrow} \lfloor wr_\gamma \rfloor$ in the parabolic quantum Bruhat graph (where $\gamma \in \Phi^+ \setminus \Phi_J^+$). Define $\beta \in \Phi^{\mathrm{af}+}$ by*

$$\beta := \begin{cases} w\gamma & \text{if } w \xrightarrow{\gamma} wr_\gamma \\ \delta + w\gamma & \text{if } w \overset{\gamma}{\dashrightarrow} \lfloor wr_\gamma \rfloor \end{cases}. \tag{6.5}$$

There exists an affine simple root α (in fact, $\alpha \neq \alpha_0$ if $w \xrightarrow{\gamma} wr_\gamma$), such that $\langle \beta, \alpha^\vee \rangle > 0$, and we have the edge in the parabolic quantum Bruhat graph indicated either in case (1) *or* (2) *below, where z' is defined as in* (5.2)*:*

$$(1)\ \begin{cases} w \xrightarrow{w^{-1}\alpha} r_\alpha w & \text{if } \alpha \neq \alpha_0 \\ w \overset{-w^{-1}\theta}{\dashrightarrow} \lfloor r_\theta w \rfloor & \text{if } \alpha = \alpha_0\,, \end{cases} \qquad (2)\ \begin{cases} r_\alpha \lfloor wr_\gamma \rfloor \xrightarrow{-\lfloor wr_\gamma \rfloor^{-1}\alpha} \lfloor wr_\gamma \rfloor & \text{if } \alpha \neq \alpha_0 \\ \lfloor r_\theta wr_\gamma \rfloor \overset{z'\lfloor wr_\gamma \rfloor^{-1}\theta}{\dashrightarrow} \lfloor wr_\gamma \rfloor & \text{if } \alpha = \alpha_0\,. \end{cases}$$

Remark 6.11. In (6.5), if $w \overset{\gamma}{\dashrightarrow} \lfloor wr_\gamma \rfloor$, then $w\gamma \in \Phi^-$ for the following reason. Observe that $\ell(\lfloor wr_\gamma \rfloor) \leq \ell(w) - 1$ since $\gamma \in \Phi^+ \setminus \Phi_J^+$. Now suppose, by contradiction, that $w\gamma \in \Phi^+$. Then, we have $wr_\gamma > w$ in the usual Bruhat order on W. Therefore, by [BB, Proposition 2.5.1], we obtain $\lfloor wr_\gamma \rfloor \geq \lfloor w \rfloor = w$, which implies that $\ell(\lfloor wr_\gamma \rfloor) \geq \ell(w)$. This is a contradiction. This proves that $w\gamma \in \Phi^-$.

Proof. Let $\mu := w\lambda$, where $\lambda \in X_{\mathrm{af}}^0$ is the fixed dominant element in the finite weight lattice whose stabilizer is W_J. We claim that $\mu < r_\beta \mu$ in $X_{\mathrm{af}}^0(\lambda)$, which means that $\langle \mu, \beta^\vee \rangle > 0$. Indeed, since $\gamma \in \Phi^+ \setminus \Phi_J^+$, it follows from (6.5) that in both cases we have

$$\langle \mu, \beta^\vee \rangle = \langle w\lambda, w\gamma^\vee \rangle = \langle \lambda, \gamma^\vee \rangle > 0\,.$$

We now apply Lemma 6.8 to deduce the existence of a simple root α in Φ^{af} (in fact, $\alpha \neq \alpha_0$ if $w \xrightarrow{\gamma} wr_\gamma$) such that $\langle \beta, \alpha^\vee \rangle > 0$, and either

$$(1) \quad \mu \lessdot r_\alpha \mu \qquad \text{or} \qquad (2) \quad r_\alpha r_\beta \mu \lessdot r_\beta \mu$$

in $X_{\mathrm{af}}^0(\lambda)$. By Lemma 6.7 and Lemma 6.6, cases (1) and (2) can be rephrased as cases (1) and (2) in the lemma to be proved, respectively. ☐

Note that we do not need all the cases of the diamond Lemma 5.14 for the PQBG, for instance the one where all four edges are down edges. By stating that we have a certain edge in the parabolic quantum Bruhat graph, we implicitly assume that both its vertices are in W^J.

6.5. **Main argument.** We address separately the direct ($\Rightarrow$) and the converse ($\Leftarrow$) statements. Recall that the height of a root is the sum of the coefficients in its expansion in the basis of simple roots.

Proof of ($\Rightarrow$) *in Theorem* 6.5. Consider the cover $\mu \lessdot \nu = r_\beta \mu$ in $X_{\mathrm{af}}^0(\lambda)$ labeled by β, and let $w := \mathrm{cl}(\mu)$. We proceed by induction on the height of β. If β is a simple root, the conclusion follows directly from Lemma 6.7. If β is not a simple root, we apply Lemma 6.8; this gives an affine simple root $\alpha \neq \beta$ with $\langle \beta, \alpha^\vee \rangle > 0$, which also satisfies condition (1) or (2) in the mentioned lemma. Depending on these two cases, by Lemma 6.9, we have one of the two diamonds in (6.4) (in $X_{\mathrm{af}}^0(\lambda)$). Let $\beta' := r_\alpha \beta$. We will need the fact that β and β' are in Φ^+ or $\delta - \Phi^+$ (not necessarily both in the same set), by Lemma 6.4 (1).

Assume that we have the left diamond in (6.4), as the reasoning is completely similar for the right diamond (we simply interchange the statements of the form "bottom implies top" and "top implies bottom" provided by Lemmas 6.9 and 5.14). Lemma 6.7 tells us that, by projecting its edges pointing northwest (labeled by the simple root α) via the map cl, we obtain two up edges or two down edges in the parabolic quantum Bruhat graph (depending on $\alpha \neq \alpha_0$ or $\alpha = \alpha_0$, respectively). Moreover, by Lemma 6.6, the four vertices of the projected diamond and its top left edge are labeled as in left diamond in (5.3) (or (5.4), which has the same labels), and (5.7), respectively, where γ is defined as in Lemma 6.6; indeed, if γ' is defined with respect to β' and $r_\alpha w$ as γ is defined with respect to β and w in Lemma 6.6, then $\gamma' = \gamma$ in the first case, and $\gamma' = z(\gamma)$ in the second case. Since $\langle \beta, \alpha^\vee \rangle > 0$, the height of β' is strictly smaller than the height of β; so by induction we know that the top left edge of the projected diamond is an up or down edge in the parabolic quantum Bruhat graph, depending on $\beta' \in \Phi^+$ or $\beta' \in \delta - \Phi^+$, respectively.

By Lemma 6.8, we have one of the following three cases:

$$(\beta \in \Phi^+, \alpha \neq \alpha_0)\,, \qquad (\beta \in \delta - \Phi^+, \alpha \neq \alpha_0)\,, \qquad (\beta \in \delta - \Phi^+, \alpha = \alpha_0)\,. \tag{6.6}$$

By calculating $\beta' = r_\alpha \beta$, we deduce that, in the mentioned three cases, we have

$$\beta' \in \Phi^+\,, \qquad \beta' \in \delta - \Phi^+\,, \qquad \beta' \in \Phi^+\,, \tag{6.7}$$

respectively. For the last computation, let $\beta = \delta - \overline{\beta}$ and write

$$\beta' = r_{\alpha_0}(\delta - \overline{\beta}) = r_\theta t_{-\theta^\vee}(\delta - \overline{\beta}) = -r_\theta \overline{\beta} + (1 - \langle \overline{\beta}, \theta^\vee \rangle)\delta\,; \tag{6.8}$$

here the coefficient of δ needs to be 0 or 1, as noted above, but the second case cannot happen since

$$\langle \overline{\beta}, \theta^\vee \rangle = \langle \beta, \alpha^\vee \rangle \neq 0\,. \tag{6.9}$$

Hence, in the three cases in (6.6) and (6.7), the top two edges of the projected diamond (and their vertices) are as in the left diamonds in (5.3), (5.4), and (5.7), respectively. By Remark 5.15, these three diamonds coincide, up to relabeling, with the right diamonds in (5.3), (5.4), and (5.5), respectively. Therefore, we can apply the statements in Lemma 5.14 associated with the latter diamonds (stating that their top two edges imply their bottom two edges) to deduce that the projection of the edge $\mu \lessdot \nu$ is as claimed, namely an up edge in the first case, and a down edge in the last two cases (in the parabolic quantum Bruhat graph). Note that the condition $\gamma \neq |w^{-1}\alpha|$ needed in the first case is satisfied since $\beta = |w\gamma|$ in this case and $\beta \neq \alpha$; here $|\alpha| = \pm\alpha$ depending on whether α is positive or negative. In addition, the condition $\langle w^{-1}\theta, \gamma^\vee \rangle = \langle \theta, w\gamma^\vee \rangle \neq 0$ needed in the third case is precisely (6.9). This concludes the induction step. $\square$

Now let us turn to the converse statement.

Proof of ($\Leftarrow$) *in Theorem* 6.5. Assume that $\mathrm{cl}(\mu) = w$ and we have the edge in the parabolic quantum Bruhat graph $w \xrightarrow{\gamma} wr_\gamma = w'$ or $w \overset{\gamma}{\dashrightarrow} \lfloor wr_\gamma \rfloor = w'$. Defining β as in (6.5), we claim that $\nu := r_\beta \mu$ satisfies the conditions in the theorem. Indeed, note first that $\mathrm{cl}(\nu) = w'$, by Lemma 6.6 (1). We now proceed by induction on the height of β. If β is an affine simple root, the conclusion follows directly from Lemma 6.7. If β is not a simple root, we apply Lemma 6.10; this gives an affine simple root $\alpha \neq \beta$ satisfying $\langle \beta, \alpha^\vee \rangle > 0$ and either condition (1) or (2) in the mentioned lemma. Assume that condition (1) holds, as the reasoning is completely similar if condition (2) holds (we simply interchange the statements of the form "bottom implies top" and "top implies bottom" provided by Lemmas 5.14 and 6.9).

By Lemma 6.10, we have one of the following three cases:

$$(\beta \in \Phi^+, \alpha \neq \alpha_0)\,, \qquad (\beta \in \delta - \Phi^+, \alpha \neq \alpha_0)\,, \qquad (\beta \in \delta - \Phi^+, \alpha = \alpha_0)\,. \tag{6.10}$$

By Lemma 5.14, we have the left diamonds in (5.3), (5.4), and (5.7), respectively. Note that the conditions $\gamma \neq w^{-1}\alpha$ and $\gamma \neq -w^{-1}\theta$ needed in the first and third cases, respectively, are satisfied since $\beta \neq \alpha$, where we recall the definition of β in (6.5); in addition, the condition $\langle w^{-1}\theta, \gamma^\vee \rangle \neq 0$ needed in the third case follows from $\langle \beta, \alpha^\vee \rangle > 0$, cf. (6.9) above. Let β' be defined as in (6.5) for the top left edge of these diamonds. It is not hard to check that in all cases $\beta' = r_\alpha \beta$. For instance, letting $\beta = \delta - \overline{\beta}$ in the third case (where $\overline{\beta} = -w\gamma \in \Phi^+$), we have

$$\beta' = \lfloor r_\theta w \rfloor z(\gamma) = r_\theta w \gamma = -r_\theta \overline{\beta} = r_{\alpha_0}(\delta - \overline{\beta})\,;$$

here the last equality follows from (6.8) and (6.9) above, as well as the well-known fact that $\langle \overline{\beta}, \theta^\vee \rangle$ can only be 0 or 1 if $\overline{\beta} \neq \theta$ (which is clearly true).

Since $\langle \beta, \alpha^\vee \rangle > 0$, the height of $\beta' = r_\alpha \beta$ is strictly smaller than the height of β. Therefore, we can use induction (together with the calculation of $\mathrm{cl}(r_\alpha \mu)$ from Lemma 6.6 (2)) to deduce that we have a cover

$$r_\alpha \mu \xrightarrow{\beta'} r_{\beta'} r_\alpha \mu = r_\alpha r_\beta \mu$$

in $X^0_{\mathrm{af}}(\lambda)$. On the other hand, by Lemma 6.6, we can see that $r_\beta \mu$ and $r_\alpha r_\beta \mu$ project to the vertices of the top right edge of the left diamonds in (5.3), (5.4), and (5.7), depending on the case. Therefore, by Lemma 6.7, we also have the cover

$$r_\beta \mu \xrightarrow{\alpha} r_\alpha r_\beta \mu$$

in $X^0_{\mathrm{af}}(\lambda)$. We now proved that we have the top two edges in the left diamond in (6.4). As $\beta \neq \alpha$, we can now apply the statement of Lemma 6.9 corresponding to the right diamond in (6.4) (which is just a relabeling of the left one) to deduce that we have the cover $\mu \lessdot r_\beta \mu$ labeled by β in $X^0_{\mathrm{af}}(\lambda)$. This concludes the induction step. □

6.6. **Connectivity of the parabolic quantum Bruhat graph and quantum length.** In this subsection we show that the parabolic quantum Bruhat graph is strongly connected when using only simple reflections. For the quantum Bruhat graph, this result is [HST, Theorem 4.2].

We use the following notation:

$$\widetilde{\alpha}_i := \begin{cases} \alpha_i & \text{if } i \neq 0, \\ -\theta & \text{if } i = 0, \end{cases} \qquad s_i := \begin{cases} r_i & \text{if } i \neq 0, \\ r_\theta & \text{if } i = 0. \end{cases}$$

Also, in this subsection we do not draw quantum edges in the parabolic quantum Bruhat graph by dotted lines.

Lemma 6.12. *For each u, $v \in W^J$, there exist a sequence $u = x_0, x_1, \ldots, x_n = v$ of elements of W^J and a sequence $i_1, i_2, \ldots, i_n \in I \cup \{0\}$ such that $x_{k+1} = \lfloor s_{i_{k+1}} x_k \rfloor$ with $x_k^{-1} \widetilde{\alpha}_{i_{k+1}} \in \Phi^+ \setminus \Phi^+_J$ for each $0 \leq k \leq n-1$.*

Remark 6.13. Keep the notation in the lemma above. We see from Lemma 6.7 and Lemma 6.4 (2) that

$$u = x_0 \xrightarrow{x_0^{-1}\widetilde{\alpha}_{i_1}} x_1 \xrightarrow{x_1^{-1}\widetilde{\alpha}_{i_2}} \cdots\cdots \xrightarrow{x_{n-2}^{-1}\widetilde{\alpha}_{i_{n-1}}} x_{n-1} \xrightarrow{x_{n-1}^{-1}\widetilde{\alpha}_{i_n}} x_n = v$$

in the parabolic quantum Bruhat graph. In particular, the parabolic quantum Bruhat graph is strongly connected using only simple reflections (i.e., for each u, $v \in W^J$, there exists a directed path from u to v in the parabolic quantum Bruhat graph, where the edges correspond to multiplying on the left by simple reflections).

We are now ready to define the notion of *quantum length* of an element in W. This will be used in the proof of the tilted Bruhat Theorem 7.1.

Definition 6.14. *Let $u \in W$. We see from Lemma* 6.12 *(with $J = \emptyset$ and $v = e$, where e is the identity in W) and Remark* 6.13 *that there exist a sequence $u = x_0, x_1, \ldots, x_n = v$ of elements of W and a sequence $i_1, i_2, \ldots, i_n \in I \cup \{0\}$ such that*

$$u = x_0 \xrightarrow{x_0^{-1}\widetilde{\alpha}_{i_1}} x_1 \xrightarrow{x_1^{-1}\widetilde{\alpha}_{i_2}} \cdots\cdots \xrightarrow{x_{n-2}^{-1}\widetilde{\alpha}_{i_{n-1}}} x_{n-1} \xrightarrow{x_{n-1}^{-1}\widetilde{\alpha}_{i_n}} x_n = e$$

in the quantum Bruhat graph. We define the quantum length $q\ell(u)$ of u to be the minimal of the length n of such sequences.

Proof of Lemma 6.12. Let λ be a dominant weight such that $\{j \in I \mid \langle \alpha_j^\vee , \lambda \rangle = 0\} = J$; note that the stabilizer of λ in W is identical to W_J, and hence $W\lambda \cong W/W_J = W^J$. Set $\mu := u\lambda$ and $\nu := v\lambda$. We see from [AK, Lemma 1.4] that there exists $i_1, i_2, \dots, i_n \in I \cup \{0\}$ such that

$$\begin{cases} s_{i_n} \cdots s_{i_2} s_{i_1} \mu = \nu, \\ \langle \alpha_{i_{k+1}}^\vee , s_{i_k} \cdots s_{i_2} s_{i_1} \mu \rangle > 0 \quad \text{for all } 0 \le k \le n-1. \end{cases}$$

For each $0 \le k \le n$, we define $x_k \in W^J$ to be the minimal coset representative for the coset containing $s_{i_k} \cdots s_{i_2} s_{i_1}$; note that $x_0 = u$ and $x_n = v$. It is obvious that $x_{k+1} = \lfloor s_{i_{k+1}} x_k \rfloor$ for every $0 \le k \le n-1$. Also, because

$$\langle \alpha_{i_{k+1}}^\vee , x_k \lambda \rangle = \langle \alpha_{i_{k+1}}^\vee , s_{i_k} \cdots s_{i_2} s_{i_1} \mu \rangle > 0,$$

it follows immediately that $x_k^{-1} \widetilde{\alpha}_{i_{k+1}} \in \Phi^+ \setminus \Phi_J^+$. Thus we have proved the lemma. □

7. Tilted Bruhat theorem

Given $u \in W$ the *u-tilted Bruhat order* on W [BFP] is defined by $w_1 \preccurlyeq_u w_2$ if there is a shortest path in the quantum Bruhat graph $\mathrm{QB}(W)$ from u to w_2 that passes through w_1. More precisely, if we denote by $\ell(w_1 \to w_2)$ the length of a shortest directed path from w_1 to w_2 in the quantum Bruhat graph $\mathrm{QB}(W)$, then for $u, w_1, w_2 \in W$,

$$w_1 \preccurlyeq_u w_2 \iff \ell(u \to w_2) = \ell(u \to w_1) + \ell(w_1 \to w_2).$$

It was shown in [BFP] that this is a partial order. In [LS, Theorem 4.8] it was reproved by showing that $(W, \preccurlyeq)$ is (dual to) an induced subposet of the affine Bruhat order.

Here we prove a property of the u-tilted Bruhat order with respect to any parabolic subgroup $W_J \subset W$ of the finite Weyl group.

Theorem 7.1 (Tilted Bruhat Theorem)**.** *For every $u, z \in W$ and any parabolic subgroup $W_J \subset W$, the coset zW_J contains a unique $\preccurlyeq_u$-minimal element.*

The tilted Bruhat theorem is a quantum Bruhat graph analogue of the Deodhar lift [De] (see also [LeSh, Proposition 3.1]) which states that if $\tau \in W/W_J$ and $v \in W$ such that $vW_J \le \tau$ in W/W_J, then the set

$$\{w \in W \mid v \le w \text{ and } wW_J = \tau\}$$

has a Bruhat-minimum.

We start by stating a weaker version of Theorem 7.1, which is easily proved.

Proposition 7.2. *Fix $u, z \in W$. There exists a unique element $x \in zW_J$ such that the distance $\ell(u \to x)$ attains its minimum value.*

Proposition 7.2 suffices for our main application in [LNSSS], namely for bijecting the models for KR crystals based on projected LS-path and quantum Bruhat chains. However, an explicit construction of this bijection depends on an algorithm for determining $x = x_0 \in zW_J$ minimizing $\ell(u \to x)$; such an algorithm is given in the proof of Theorem 7.1. The proof of Proposition 7.2 relies on the *shellability* of the quantum Bruhat graph with respect to a reflection ordering on the positive roots [Dy], which we now recall.

Theorem 7.3. [BFP] *Fix a reflection ordering on Φ^+.*

(1) *For any pair of elements $v, w \in W$, there is a unique path from v to w in the quantum Bruhat graph $\mathrm{QB}(W)$ such that its sequence of edge labels is strictly increasing (resp., decreasing) with respect to the reflection ordering.*

(2) *The path in* (1) *has the smallest possible length $\ell(v \to w)$ and is lexicographically minimal (resp., maximal) among all shortest paths from v to w.*

The proof of Proposition 7.2 is immediate once we have the following two easy lemmas. These are in terms of a reflection ordering whose top (also called an initial section) consists of the roots in $\Phi^+ \setminus \Phi_J^+$, while its bottom is a reflection ordering on Φ_J^+. Such an order was constructed in [LeSh, Section 4.3] in terms of a dominant weight λ whose stabilizer is W_J. The roots in $\Phi^+ \setminus \Phi_J^+$ are ordered according to the lexicographic order on their images in $\mathbb{Q}^r$ via the injective map

$$\alpha \mapsto \frac{1}{\langle \lambda, \alpha^\vee \rangle}(c_1, \ldots, c_r)\,,$$

where $\alpha^\vee = c_1\alpha_1^\vee + \cdots + c_r\alpha_r^\vee$ expresses $\alpha^\vee$ in the basis of simple coroots (on which we fix an order). For the roots in Φ_J^+, we choose any reflection ordering.

Lemma 7.4. *Assume that $\ell(u \to x)$, as a function of $x \in zW_J$, has a minimum at $x = x_0$. Then the path from u to x_0 with increasing edge labels has all its labels in $\Phi^+ \setminus \Phi_J^+$.*

Proof. The mentioned path has length $\ell(u \to x_0)$, by Theorem 7.3 (2). Assume that it has at least one label in Φ_J^+. By the structure of our particular reflection ordering, all of these labels must be at the end of the path. This means that the tail of the path starting with some $x_1 \neq x_0$ consists entirely of elements in zW_J. Since $\ell(u \to x_1) < \ell(u \to x_0)$, we reached a contradiction. □

Lemma 7.5. *Assume that the paths with increasing edge labels from u to two elements x_0, x_1 in zW_J have all labels in $\Phi^+ \setminus \Phi_J^+$. Then $x_0 = x_1$.*

Proof. Assume $x_0 \neq x_1$. The induced subgraph of $\mathrm{QB}(W)$ on zW_J, to be denoted $\mathrm{QB}(zW_J)$, is isomorphic to $\mathrm{QB}(W_J)$ under the map $w \mapsto \lfloor z \rfloor w$ for $w \in W_J$ (this is immediate from definitions and the length-additive factorization of the elements in zW_J). Thus, by Theorem 7.3 (1), we can consider the path from x_0 to x_1 in $\mathrm{QB}(zW_J)$ with increasing edge labels (in Φ_J^+). By concatenating this path with the one from u to x_0 in the hypothesis (whose labels are in $\Phi^+ \setminus \Phi_J^+$), we obtain a path with increasing edge labels from u to x_1. But this path is clearly different from the one in the hypothesis between the same vertices. This contradicts the uniqueness statement in Theorem 7.3 (1). □

Proof of Proposition 7.2. This is immediate by combining Lemmas 7.4 and 7.5. □

Next we prepare for the proof of the tilted Bruhat Theorem 7.1.

7.1. **Preliminaries.**

We use the following notation:

$$\widetilde{\alpha}_i := \begin{cases} \alpha_i & \text{if } i \neq 0, \\ -\theta & \text{if } i = 0, \end{cases} \qquad s_i := \begin{cases} r_i & \text{if } i \neq 0, \\ r_\theta & \text{if } i = 0. \end{cases}$$

Also, we denote the identity of W by e.

Remark 7.6. Let $w \in W$, and $i \in I \cup \{0\}$. If $w^{-1}\widetilde{\alpha}_i$ is positive, then we have

$$w \xrightarrow{w^{-1}\widetilde{\alpha}_i} s_i w$$

in the quantum Bruhat graph by Theorem 6.5. Here, this arrow is an up arrow (resp., down arrow) if $i \neq 0$ (resp., $i = 0$).

Lemma 7.7. *Let $w_1, w_2 \in W$, and let $i \in I \cup \{0\}$. Assume that $w_1^{-1}\widetilde{\alpha}_i$ is positive, and $w_2^{-1}\widetilde{\alpha}_i$ is negative.*

(1) *If there exists a directed path from w_1 to w_2 of length a, then there exists a directed path from $s_i w_1$ to w_2 of length $a - 1$.*

(2) *If there exists a directed path from w_1 to w_2 of length a, then there exists a directed path from w_1 to $s_i w_2$ of length $a-1$.*

Proof. We give a proof only for part (1); part (2) can be shown similarly. We prove the assertion by induction on a. Assume that $a = 1$; we show that $s_i w_1 = w_2$. Let γ be the positive root such that $w_1 \xrightarrow{\gamma} w_2 = w_1 r_\gamma$ (this arrow is either an up arrow or a down arrow). Then, by Remark 7.6, we have in the quantum Bruhat graph,

$$\begin{tikzcd} s_i w_1 & \\ w_1 \arrow[u, "w_1^{-1}\widetilde{\alpha}_i"] \arrow[r, "\gamma"'] & w_2 \end{tikzcd}$$

where the arrow from w_1 to $s_i w_1$ is an up arrow (resp., a down arrow) if $i \neq 0$ (resp., $i = 0$). Suppose that $\gamma \neq w_1^{-1}\widetilde{\alpha}_i$. Then, by Lemma 5.14 (use the left diagram in part (1) or (2) if $i \neq 0$, and use the left diagram in part (3) or (4) if $i = 0$; note that $z = e$), we have

$$\begin{tikzcd} s_i w_1 \arrow[r, "\gamma"] & s_i w_2 \\ w_1 \arrow[u, "w_1^{-1}\widetilde{\alpha}_i"] \arrow[r, "\gamma"'] & w_2 \arrow[u, "w_2^{-1}\widetilde{\alpha}_i"'] \end{tikzcd}$$

We should note that the arrow from w_2 to $s_i w_2$ in the diagram above is an up arrow (resp., a down arrow) if $i \neq 0$ (resp., if $i = 0$). However, we deduce from Remark 7.6, along with the assumption that $w_2^{-1}\widetilde{\alpha}_i$ is negative, that if $i \neq 0$ (resp., $i = 0$), then there exists an up (resp., down) arrow from $s_i w_2$ to w_2, which is a contradiction. Thus we have $\gamma = w_1^{-1}\widetilde{\alpha}_i$, and hence $s_i w_1 = w_1 r_\gamma = w_2$.

Now, assume that $a \geq 2$, and let

$$w_1 = x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_a} x_a = w_2$$

be a directed path from w_1 to w_2 of length a. If $\gamma_1 = w_1^{-1}\widetilde{\alpha}_i$, then $x_1 = s_i w_1$. Thus we have a directed path

$$s_i w_1 = x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_a} x_a = w_2$$

of length $a-1$. Assume that $\gamma_1 \neq w_1^{-1}\widetilde{\alpha}_i$. By the same argument as above, we have

$$\begin{tikzcd} s_i w_1 \arrow[r, "\gamma_1"] & s_i x_1 & & \\ w_1 \arrow[u, "w_1^{-1}\widetilde{\alpha}_i"] \arrow[r, "\gamma_1"'] & x_1 \arrow[u, "x_1^{-1}\widetilde{\alpha}_i"'] \arrow[r, "\gamma_2"'] & \cdots \arrow[r, "\gamma_a"'] & x_a = w_2 \end{tikzcd}$$

Because the arrow from x_1 to $s_i x_1$ in the diagram above is an up arrow (resp., a down arrow) if $i \neq 0$ (resp., if $i = 0$), it follows that $x_1^{-1}\widetilde{\alpha}_i$ is positive. Applying the induction hypothesis to x_1 and w_2, we obtain a directed path from $s_i x_1$ to w_2 of length $a-2$:

$$\begin{tikzcd} s_i w_1 \arrow[r, "\gamma_1"] & s_i x_1 \arrow[rrd, dotted, "{}^{\exists}\text{directed path of length } a-2"] & & \\ w_1 \arrow[u, "w_1^{-1}\widetilde{\alpha}_i"] \arrow[r, "\gamma_1"'] & x_1 \arrow[u, "x_1^{-1}\widetilde{\alpha}_i"'] \arrow[r, "\gamma_2"'] & \cdots \arrow[r, "\gamma_a"'] & x_a = w_2 \end{tikzcd}$$

Concatenating this directed path with $s_i w_1 \xrightarrow{\gamma_1} s_i x_1$, we obtain a directed path from $s_i w_1$ to w_2 of length $a-1$. Thus we have proved part (1) of the lemma. □

7.2. **Proof of the tilted Bruhat Theorem 7.1.** We prove Theorem 7.1 by induction on $q\ell(u)$. If $q\ell(u) = 0$, then $u = e$. We know from [BFP, p.435] that the e-tilted Bruhat order $\preccurlyeq_e$ on W is just the Bruhat order on W. Hence, for each $z \in W$, the minimal coset representative in zW_J is the unique $\preccurlyeq_e$-minimal element. Therefore the assertion holds.

Assume that $q\ell(u) > 0$. Let $u = x_0, x_1, \ldots, x_n = e$ be a sequence of elements in W satisfying the condition in Lemma 6.12, with $n = q\ell(u)$. Put $v := x_1$; note that $q\ell(v) = q\ell(u) - 1$. Thus the inductive assumption is:

Theorem 7.1 is true for this v (and arbitrary $z \in W$).

Assume that $v = s_i u$ for some $i \in I \cup \{0\}$. Since $u^{-1}\widetilde{\alpha}_i$ is positive, it follows from Remark 7.6 that

$$u \xrightarrow{u^{-1}\widetilde{\alpha}_i} v = s_i u, \tag{7.1}$$

where this arrow is an up arrow (resp., a down arrow) if $i \neq 0$ (resp., $i = 0$).

Case 1. Assume that $z^{-1}\widetilde{\alpha}_i \in \Delta^- \setminus \Delta_J^-$; note that $(zy)^{-1}\widetilde{\alpha}_i$ is negative for all $y \in W_J$.

By the inductive assumption, there exists a unique minimal element in the coset zW_J with respect to $\preccurlyeq_v$, which we denote by $\min(zW_J, \preccurlyeq_v)$. Let $x \in W_J$ be such that

$$\min(zW_J, \preccurlyeq_v) = zx.$$

Let us show that $zx \in zW_J$ is a unique minimal element in the coset zW_J with respect to $\preccurlyeq_u$, that is,

$$\min(zW_J, \preccurlyeq_u) = zx.$$

Let $y \in W_J$ be an arbitrary element in W_J. There exists a shortest directed path from v to zy that passes through zx:

$$v \to \cdots \to zx \to \cdots \to zy.$$

Concatenating $u \to v$ of (7.1) and this directed path, we obtain a directed path

$$u \to v \to \cdots \to zx \to \cdots \to zy \tag{7.2}$$

of length $\ell(v \to zy) + 1$. Let us show that this directed path is shortest. Suppose that $\ell(u \to zy) < \ell(v \to zy) + 1$. Recall that $u^{-1}\widetilde{\alpha}_i$ is positive, and $(zy)^{-1}\widetilde{\alpha}_i$ is negative. By Lemma 7.7 (1), we obtain a directed path from $s_i u = v$ to zy whose length is equal to $\ell(u \to zy) - 1$. Hence,

$$\ell(v \to zy) \leq \ell(u \to zy) - 1 < \ell(v \to zy) + 1 - 1 = \ell(v \to zy),$$

which is a contradiction. Therefore, the directed path (7.2) is shortest.

Case 2. Assume that $z^{-1}\widetilde{\alpha}_i \in \Delta^+ \setminus \Delta_J^+$; note that $(zy)^{-1}\widetilde{\alpha}_i$ is positive for all $y \in W_J$, which implies that $zy \to s_i zy$ by Remark 7.6.

By the inductive assumption, there exists a unique minimal element in the coset $s_i zW_J$ with respect to $\preccurlyeq_v$, which we denote by $\min(s_i zW_J, \preccurlyeq_v)$. Let $x \in W_J$ be such that

$$\min(s_i zW_J, \preccurlyeq_v) = s_i zx.$$

Let us show that $zx \in zW_J$ is a unique minimal element in the coset zW_J with respect to $\preccurlyeq_u$;

$$\min(zW_J, \preccurlyeq_u) = zx.$$

Let $y \in W_J$ be an arbitrary element in W_J. We construct a directed path from u to zy that passes through zx as follows: First, we construct a directed path from u to zx. Concatenating $u \to v$ of (7.1) and a shortest directed path from v to $s_i zx$, we obtain a directed path from u to $s_i zx$ of length $\ell(v \to s_i zx) + 1$:

$$u \to v \to \cdots \to s_i zx$$

Because $u^{-1}\widetilde{\alpha}_i$ is positive and $(s_izx)^{-1}\widetilde{\alpha}_i$ is negative, it follows from Lemma 7.7 (2) that there exists a directed path from u to zx of length $\ell(v \to s_izx) + 1 - 1 = \ell(v \to s_izx)$:

$$\begin{tikzcd}
 & & zx \arrow[d] \\
u \arrow[r] \arrow[urr, dotted, "{}^{\exists}\text{directed path of length } \ell(v \to s_izx)"] & \cdots \arrow[r] & s_izx
\end{tikzcd}$$

Next, we construct a directed path from zx to zy. Concatenating $zx \to s_izx$ and a shortest directed path from s_izx to s_izy, we obtain a directed path from zx to s_izy of length $\ell(s_izx \to s_izy) + 1$:

$$\begin{tikzcd}
 & & zx \arrow[d] \arrow[drr, dotted, "\text{Concatenation of directed paths}"] & & \\
u \arrow[r] \arrow[urr, dotted, "\text{directed path of length } \ell(v \to s_izx)"] & \cdots \arrow[r] & s_izx \arrow[r] & \cdots \arrow[r] & s_izy
\end{tikzcd}$$

Because $(zx)^{-1}\widetilde{\alpha}_i$ is positive and $(s_izy)^{-1}\widetilde{\alpha}_i$ is negative, it follows from Lemma 7.7 (2) that there exists a directed path from zx to zy of length $\ell(s_izx \to s_izy) + 1 - 1 = \ell(s_izx \to s_izy)$.

$$\begin{tikzcd}
 & & zx \arrow[d] \arrow[drr, dotted] \arrow[rr, dotted, "{}^{\exists}\text{directed path of length } \ell(s_izx \to s_izy)"] & & zy \arrow[d] \\
u \arrow[r] \arrow[urr, dotted, "\text{directed path of length } \ell(v \to s_izx)"] & \cdots \arrow[r] & s_izx \arrow[r] & \cdots \arrow[r] & s_izy
\end{tikzcd}$$

Concatenating the directed paths above, we obtain a directed path from u to zy of length $\ell(v \to s_izx) + \ell(s_izx \to s_izy) = \ell(v \to s_izy)$ (recall that $s_izx \preccurlyeq_v s_izy$ by the definition of $x \in W_J$) that passes through zx.

Let us show that this directed path is shortest. Suppose that $\ell(u \to zy) < \ell(v \to s_izy)$. Concatenating a shortest directed path from u to zy and the directed path $zy \to s_izy$, we obtain a directed path from u to s_izy of the form:

$$\underbrace{u \to \cdots \to zy}_{\text{shortest}} \to s_izy;$$

note that its length is $\ell(u \to zy) + 1$. Because $u^{-1}\widetilde{\alpha}_i$ is positive, and $(s_izy)^{-1}\widetilde{\alpha}_i$ is negative, it follows from Lemma 7.7 (1) that there exists a directed path from $s_iu = v$ to s_izy of length $\ell(u \to zy) + 1 - 1 = \ell(u \to zy)$. Since $\ell(u \to zy) < \ell(v \to s_izy)$, this is a contradiction.

Case 3. Assume that $z^{-1}\widetilde{\alpha}_i \in \Delta_J$; note that $s_izW_J = zW_J$.

By the inductive assumption, there exists a unique minimal element in the coset zW_J with respect to $\preccurlyeq_v$, which we denote by $\min(zW_J, \preccurlyeq_v)$. Let $x \in W_J$ be such that

$$\min(zW_J, \preccurlyeq_v) = zx.$$

Subcase 3.1. Assume that $(zx)^{-1}\widetilde{\alpha}_i \in \Delta_J^+$. Let us show that

$$\min(zW_J, \preccurlyeq_u) = zx.$$

Take an arbitrary $y \in W_J$.

3.1.1. Assume first that $(zy)^{-1}\widetilde{\alpha}_i \in \Delta_J^-$. Then we can check in exactly the same way as in Case 1 that concatenating $u \to v$ of (7.1) and a shortest directed path from v to zy that passes through zx gives a shortest directed path from u to zy:

$$\underbrace{u \to}_{(7.1)} v \underbrace{\to \cdots \to zx \to \cdots \to zy}_{\text{shortest}}.$$

3.1.2. Assume next that $(zy)^{-1}\widetilde{\alpha}_i \in \Delta_J^+$. Concatenating $u \to v$ of (7.1) and a shortest directed path from v to zx, we obtain a directed path from u to zx of length $\ell(v \to zx) + 1$:

$$\underbrace{u \to}_{(7.1)} v \underbrace{\to \cdots \to zx}_{\text{shortest}}.$$

Because $(zx)^{-1}\widetilde{\alpha}_i$ is positive, and $(s_izy)^{-1}\widetilde{\alpha}_i$ is negative, we see by applying Lemma 7.7 (2) to a shortest directed path from zx to s_izy that there exists a directed path from zx to zy of length $\ell(zx \to s_izy) - 1$:

$$\begin{tikzcd} & & & & & zy \arrow[d] \\ u \arrow[r] & v \arrow[r] & \cdots \arrow[r] & zx \arrow[r] \arrow[urr, dotted, "{}^{\exists}\text{directed path of length } \ell(zx \to s_izy) - 1"] & \cdots \arrow[r] & s_izy \end{tikzcd}$$

Concatenating these directed paths, we obtain a directed path from u to zy that passes through zx; its length is equal to

$$\begin{aligned} (\ell(v \to zx) + 1) + (\ell(zx \to s_izy) - 1) &= \ell(v \to zx) + \ell(zx \to s_izy) \\ &= \ell(v \to s_izy); \end{aligned}$$

recall that $zx \preccurlyeq_v s_izy$. We can show in exactly the same way as in Case 2 that this directed path is shortest.

Subcase 3.2. Assume that $(zx)^{-1}\widetilde{\alpha}_i \in \Delta_J^-$. Let us show that

$$\min(zW_J, \preccurlyeq_u) = s_izx.$$

Take an arbitrary $y \in W_J$.

3.2.1. Assume that $(zy)^{-1}\widetilde{\alpha}_i \in \Delta_J^-$. Concatenating $u \to v$ of (7.1) and a shortest directed path from v to zx, we obtain a directed path from u to zx of length $\ell(v \to zx) + 1$:

$$\underbrace{u \to}_{(7.1)} v \underbrace{\to \cdots \to zx}_{\text{shortest}}.$$

Because $u^{-1}\widetilde{\alpha}_i$ is positive, and $(zx)^{-1}\widetilde{\alpha}_i$ is negative, it follows from Lemma 7.7 (2) that there exists a directed path from u to s_izx of length $\ell(v \to zx) + 1 - 1 = \ell(v \to zx)$:

$$\begin{tikzcd} & & s_izx \arrow[d] \\ u \arrow[r] \arrow[urr, dotted, "{}^{\exists}\text{directed path of length } \ell(v \to zx)"] & \cdots \arrow[r] & zx \end{tikzcd}$$

Concatenating this directed path, $s_izx \to zx$, and a shortest directed path from zx to zy, we obtain a directed path from u to zy that passes through s_izx:

$$\begin{tikzcd}
& & s_izx \arrow[d] \arrow[rrd, dotted, "\text{Concatenation of directed paths}"] & & \\
u \arrow[r] \arrow[rru, dotted, "\substack{\text{directed path}\\ \text{of length } \ell(v\to zx)}"] & \cdots \arrow[r] & zx \arrow[r] & \cdots \arrow[r] & zy
\end{tikzcd}$$

The length of this directed path is equal to

$$\ell(v \to zx) + 1 + \ell(zx \to zy) = \ell(v \to zy) + 1;$$

recall that $zx \preccurlyeq_v zy$. We can show in exactly the same way as the argument in Case 1 that this directed path is shortest.

3.2.2. Assume that $(zy)^{-1}\widetilde{\alpha}_i \in \Delta_J^+$. By the same argument as in 3.2.1, we have

$$\begin{tikzcd}
& & s_izx \arrow[d] \\
u \arrow[r] \arrow[rru, dotted, "\substack{{}^{\exists}\text{directed path}\\ \text{of length } \ell(v\to zx)}"] & \cdots \arrow[r] & zx
\end{tikzcd}$$

Concatenating $s_izx \to zx$ and a shortest directed path from zx to s_izy, we obtain a directed path from s_izx to s_izy of length $\ell(zx \to s_izy) + 1$:

$$\begin{tikzcd}
& & s_izx \arrow[d] \arrow[rrd, dotted, "\text{Concatenation of directed paths}"] & & \\
u \arrow[r] \arrow[rru, dotted, "\substack{\text{directed path}\\ \text{of length } \ell(v\to zx)}"] & \cdots \arrow[r] & zx \arrow[r] & \cdots \arrow[r] & s_izy
\end{tikzcd}$$

Since$(s_izx)^{-1}\widetilde{\alpha}_i$ is positive, and $(s_izy)^{-1}\widetilde{\alpha}_i$ is negative, it follows from Lemma 7.7 (2) that there exists a directed path from s_izx to zy of length $\ell(zx \to s_izy) + 1 - 1 = \ell(zx \to s_izy)$:

$$\begin{tikzcd}
& & s_izx \arrow[d] \arrow[rr, dotted, "\substack{{}^{\exists}\text{directed path}\\ \text{of length } \ell(zx\to s_izy)}"] \arrow[rrd, dotted] & & zy \arrow[d] \\
u \arrow[r] \arrow[rru, dotted, "\substack{\text{directed path}\\ \text{of length } \ell(v\to zx)}"] & \cdots \arrow[r] & zx \arrow[r] & \cdots \arrow[r] & s_izy
\end{tikzcd}$$

Concatenating these directed paths, we obtain a directed path from u to zy that passes through s_izx; its length is equal to

$$\ell(v \to zx) + \ell(zx \to s_izy) = \ell(v \to s_izy) \qquad (\because\ zx \preccurlyeq_v s_izy).$$

We can show in exactly the same way as the argument in Case 2 that this directed path is shortest. Thus we have proved Theorem 7.1. $\square$

References

[AK] T. Akasaka, M. Kashiwara, Finite-dimensional representations of quantum affine algebras. *Publ. Res. Inst. Math. Sci.*, 33:839–867, 1997.

[BB] A. Björner and F. Brenti. Combinatorics of Coxeter groups. Graduate Texts in Mathematics Vol. 231, Springer, New York, 2005.

[BFP] F. Brenti, S. Fomin, A. Postnikov, Mixed Bruhat operators and Yang-Baxter equations for Weyl groups. *Int. Math. Res. Not.* no. 8, 419–441, 1999.

[BMO] A. Braverman, D. Maulik, A. Okounkov, Quantum cohomology of the Springer resolution. *Adv. Math.*, 227:421–458, 2011.

[De] V. Deodhar, A splitting criterion for the Bruhat orderings on Coxeter groups. *Comm. Algebra*, 15:1889–1894, 1987.

[Dy] M. J. Dyer, Hecke algebras and shellings of Bruhat intervals. *Compositio Math.*, 89:91–115, 1993.

[FL1] G. Fourier, P. Littelmann, Tensor product structure of affine Demazure modules and limit constructions. *Nagoya Math. J.* 182:171–198, 2006.

[FL2] G. Fourier, P. Littelmann, Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions. *Adv. Math.*, 211:566–593, 2007.

[FOS] G. Fourier, M. Okado, A. Schilling, Kirillov–Reshetikhin crystals for nonexceptional types. *Adv. Math.*, 222:1080–1116, 2009.

[FSS] G. Fourier, A. Schilling, M. Shimozono, Demazure structure inside Kirillov–Reshetikhin crystals. *J. Algebra*, 309:386–404, 2007.

[FW] W. Fulton, C. Woodward, On the quantum product of Schubert classes. *J. Algebraic Geom.*, 13:641–661, 2004.

[H] J. E. Humphreys, Reflection groups and Coxeter groups. Cambridge Studies in Advanced Mathematics Vol. 29, Cambridge University Press, Cambridge, 1990.

[HST] F. Hivert, A. Schilling, N. M. Thiéry, Hecke group algebras as quotients of affine Hecke algebras at level 0. *J. Combin. Theory Ser. A*, 116:844–863, 2009.

[Ion] B. Ion, Nonsymmetric Macdonald polynomials and Demazure characters. *Duke Math. J.*, 116:299–318, 2003.

[Kac] V. Kac, Infinite dimensional Lie algebras. Third edition. Cambridge University Press, Cambridge, 1990.

[KMOU] A. Kuniba, K. .C. Misra, M. Okado, J. Uchiyama, Demazure modules and perfect crystals. *Comm. Math. Phys.* 192:555–567, 1998.

[KMOTU] A. Kuniba, K. .C. Misra, M. Okado, T. Takagi, J. Uchiyama, Crystals for Demazure modules of classical affine Lie algebras. *J. Algebra* 208:185–215, 1998.

[La] M. Lanini, On the stable moment graph of an affine Kac–Moody algebra. preprint arXiv:1210.3218.

[Li] P. Littelmann, Paths and root operators in representation theory. *Ann. of Math.*, (2) 142:499-525, 1995.

[Le] C. Lenart, From Macdonald polynomials to a charge statistic beyond type *A*. *J. Combin. Theory Ser. A*, 119:683–712, 2012.

[LeL] C. Lenart, A. Lubovsky, A generalization of the alcove model and its applications. 24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012), 875–886, *Discrete Math. Theor. Comput. Sci. Proc.*, AR, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2012.

[LP] C. Lenart, A. Postnikov, Affine Weyl groups in K-theory and representation theory. *Int. Math. Res. Not.* no. 12, 1–65, 2007. Art. ID rnm038.

[LP1] C. Lenart, A. Postnikov, A combinatorial model for crystals of Kac-Moody algebras. *Trans. Amer. Math. Soc.*, 360:4349–4381, 2008.

[LeS] C. Lenart, A. Schilling, Crystal energy via the charge in types A and C. *Math. Zeitschrift*, to appear, doi:10.1007/s00209-012-1011-2.

[LkSe] V. Lakshmibai, C. S. Seshadri, Standard monomial theory. Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), 279–322, Manoj Prakashan, Madras, 1991.

[LS] T. Lam, M. Shimozono, Quantum cohomology of G/P and homology of affine Grassmannian. *Acta Math.*, 204:49–90, 2010.

[LeSh] C. Lenart, M. Shimozono, Equivariant K-Chevalley rules for Kac-Moody flag manifolds. preprint arXiv:1203.3237.

[LNSSS] C. Lenart, S. Naito, D. Sagaki, A. Schilling, M. Shimozono, A uniform model for Kirillov-Reshetikhin crystals II. in preparation.

[Lu] G. Lusztig, Hecke algebras and Jantzen's generic decomposition patterns. *Adv. Math.*, 37:121–164, 1980.

[M] A.-L. Mare, Polynomial representatives of Schubert classes in $QH^*(G/B)$. *Math. Res. Lett.*, 9:757–769, 2002.

[NS1] S. Naito, D. Sagaki, Path model for a level-zero extremal weight module over a quantum affine algebra. *Int. Math. Res. Not.* no. 32, 1731–1754, 2003.

[NS2] S. Naito, D. Sagaki, Path model for a level-zero extremal weight module over a quantum affine algebra. II. *Adv. Math.*, 200:102–124, 2006.

[NS3] S. Naito, D. Sagaki, Lakshmibai-Seshadri paths of level-zero shape and one-dimensional sums associated to level-zero fundamental representations. *Compos. Math.*, 144:1525–1556, 2008.

[NS4] S. Naito, D. Sagaki, Crystal structure on the set of Lakshmibai-Seshadri paths of an arbitrary level-zero shape. *Proc. Lond. Math. Soc.*, 96:582–622, 2008.

[Na1] K. Naoi, Weyl modules, Demazure modules and finite crystals for non-simply laced type. *Adv. Math.*, 229:875–934, 2012.

[Na2] K. Naoi, Demazure crystals and tensor products of perfect Kirillov-Reshetikhin crystals with various levels. preprint arXiv:1108.3139.

[P] D. Peterson, Quantum cohomology of G/P. Lecture notes, Massachusetts Institute of Technology, Cambridge, MA, Spring 1997.

[RY] A. Ram, M. Yip, A combinatorial formula for Macdonald polynomials. *Adv. Math.*, 226:309–331, 2011.

[Sa] W. A. Stein and others. Sage Mathematics Software (Version 5.4). *The Sage Development Team*, 2012. `http://www.sagemath.org`.

[Sa-comb] The Sage-Combinat community. Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, 2008-2012. `http://combinat.sagemath.org`.

[ST] A. Schilling, P. Tingley, Demazure crystals, Kirillov–Reshetikhin crystals, and the energy function. *The Electronic Journal of Combinatorics*, 19:P2, 2012.

[Soe] W. Soergel, Kazhdan-Lusztig polynomials and a combinatoric for tilting modules. *Represent. Theory*, 1:83–114, 1997.

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