

# A UNIFORM MODEL FOR KIRILLOV-RESHETIKHIN CRYSTALS II. ALCOVE MODEL, PATH MODEL, AND $P = X$

CRISTIAN LENART, SATOSHI NAITO, DAISUKE SAGAKI, ANNE SCHILLING,  
AND MARK SHIMOZONO

ABSTRACT. We establish the equality of the specialization  $P_\lambda(x; q, 0)$  of the Macdonald polynomial at  $t = 0$ , with the graded character  $X_\lambda(x; q)$  of a tensor product of “single-column” Kirillov-Reshetikhin (KR) modules for untwisted affine Lie algebras. This is achieved by providing an explicit crystal isomorphism between the quantum alcove model, which is naturally associated to Macdonald polynomials, and the projected level-zero affine Lakshmibai-Seshadri path model, which is intimately related to KR crystals.

## 1. INTRODUCTION

We prove the equality of the specialization  $P_\lambda(x; q, 0)$  of the Macdonald polynomial at  $t = 0$ , with the graded character  $X_\lambda(x; q)$  of a tensor product of “single-column” Kirillov-Reshetikhin (KR) modules [KR] for untwisted affine Lie algebras. The proof is to connect two known combinatorial models: the quantum alcove model coming from the Macdonald specialization [LL1], and a series of works by Naito and Sagaki [NS1, NS2, NS3, NS5, NS6] on the projections of level-zero affine Lakshmibai-Seshadri (LS) paths and  $X_\lambda(x; q)$ . The latter is combined with the prequel paper [LNSSS1], which gives a precise characterization of Littelmann’s level-zero weight poset [Li] in terms of the parabolic quantum Bruhat graph [BFP, P, LS] which originated from (small) quantum cohomology of partial flag manifolds.

The context of this project has its origins in Ion’s observation [Ion] that when the affine simple root  $\alpha_0$  is short (which includes the duals of untwisted affine root systems)  $P_\lambda(x; q, 0)$  is an affine Demazure character (see [Sa] for type  $A$ ). On the other hand, Fourier and Littelmann [FL] showed that for simply-laced affine Lie algebras, these Demazure characters are graded characters of tensor products of KR modules, and hence of local Weyl modules for current algebras, making use of results in [NS2]. Combining [Ion] and [FL] one deduces the equality  $P_\lambda = X_\lambda$  in the simply-laced cases.

Braverman and Finkelberg [BF2] have shown that for simply-laced affine root systems, the characters  $\Psi_\lambda(x; q)$  of the duals of the current algebra modules, called global Weyl modules, coincide with the characters of the spaces of global sections of line bundles on quasi-maps spaces, which arise in the study of quantum cohomology and quantum  $K$ -theory of the flag manifold. In simply-laced types the characters  $\Psi_\lambda(x; q)$  are equal to  $X_\lambda(x; q)$  (which is the graded character of a local Weyl module) times an explicit product of geometric series whose ratios are powers of  $q$  [CFK]. The characters  $\Psi_\lambda(x; q)$  are called  $q$ -Whittaker functions due to their appearance in the quantum group version of the Kostant-Whittaker reduction of Etingof [E] and Sevostyanov [Se]

---

2000 *Mathematics Subject Classification*. Primary 05E05. Secondary 33D52, 20G42.

*Key words and phrases*. Parabolic quantum Bruhat graph, Lakshmibai-Seshadri paths, Littelmann path model, crystal bases, Macdonald polynomials.

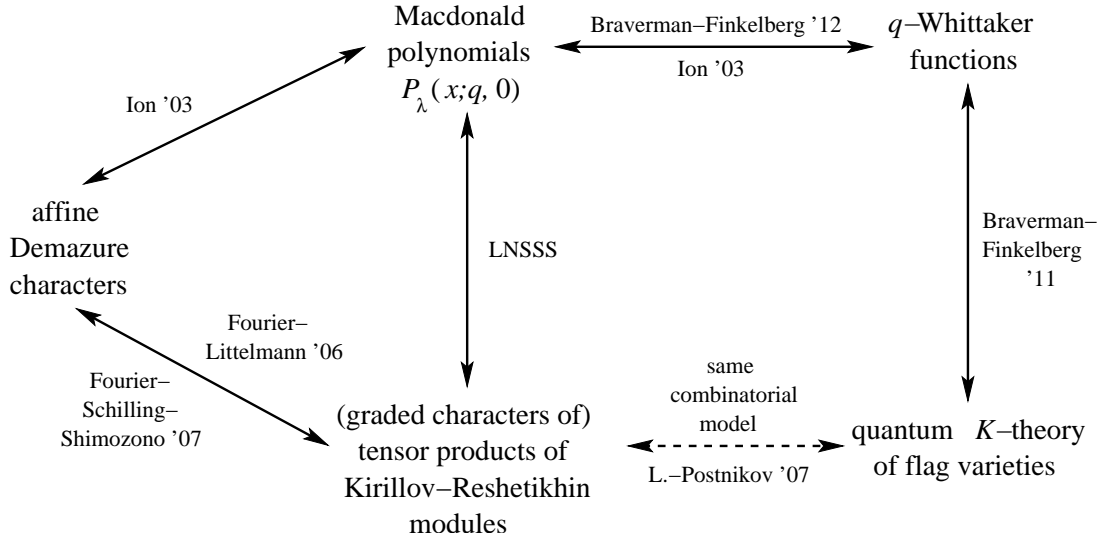


FIGURE 1. Outline

for the  $q$ -Toda integrable system. The characters  $\Psi_\lambda(x; q)$  are eigenfunctions of the  $q$ -Toda difference operators and their generating function yields the  $K$ -theoretic  $J$ -function of Givental and Lee [BF1].

Finally, the quantum alcove model arises in Lenart and Postnikov's conjectural description of the quantum product by a divisor in quantum  $K$ -theory [LP]. We summarize the connections discussed above in Figure 1.

Combinatorial models for all nonexceptional KR crystals (not just of column shape) were given in [FOS]. The quantum LS path model and the quantum alcove model uniformly describe tensor products of column shape KR crystals, for all untwisted affine types. More precisely, these models realize the root operators on the aforementioned tensor product, and also give efficient formulas for the corresponding energy function [HKOTY]. (The energy can be viewed as an affine grading on a tensor product of KR crystals [NS6, ST].) Another application of the quantum alcove model, currently under investigation in [LL2], is a uniform realization of the combinatorial  $R$ -matrix (i.e., the unique affine crystal isomorphism commuting factors in a tensor product of KR crystals).

There have been several developments related to the work in this paper. Based on our results, an interpretation of the  $q$ -Whittaker functions  $\Psi_\lambda(x; q)$  above is given in [INS, NS7], in terms of the crystal bases of level-zero extremal weight modules over quantum affine algebras. On another hand, our work was used in [CSSW] to provide the character of a stable level-one Demazure module associated to type  $B_n^{(1)}$  as an explicit combination of suitably specialized Macdonald polynomials. In addition, our results were used in a crucial way by Chari and Ion in [CI, Theorem 4.2] to show that Macdonald polynomials at  $t = 0$  are characters of local Weyl modules for current algebras. Based on this, they prove a Bernstein-Gelfand-Gelfand (BGG) reciprocity theorem for the category of representations of a current algebra. In a related work, Khoroshkin [Kho] exhibits a categorification of Macdonald polynomials, by realizing them as

the Euler characteristic of bigraded characters for certain complexes of modules over a current algebra. This realization simplifies considerably if BGG reciprocity holds (the mentioned complexes become actual modules concentrated in homological degree zero).

The paper is organized as follows. In Sections 2 and 3, we review the Lakshmibai-Seshadri (LS) and the quantum Lakshmibai-Seshadri path model, respectively. Theorem 3.3 shows that the set of projected level-zero affine LS paths  $\mathbb{B}(\lambda)_{\text{cl}}$  is the same as the set of quantum LS paths  $\text{QLS}(\lambda)$ , where  $\lambda$  is a (level-zero) dominant integral weight. This fact is also proven in [LNSS2] in a somewhat roundabout way, by providing an explicit description of the image of a quantum LS path under root operators and showing that the set of quantum LS paths is stable under the action of the root operators. (Quantum) LS paths carry a grading by a degree function (which is closely related to the energy function on KR crystals). We provide an explicit formula for the degree function of quantum LS paths in Theorem 4.6 in terms of the parabolic quantum Bruhat graph. For KR crystals, there exist the head and the tail energy functions. In Section 5, we relate the tail energy with the tail degree function using the Lusztig involution. It was conjectured in [HKOTT] and proven in [FOS1] for all nonexceptional types, which KR crystals are perfect. Since here we provide explicit models in terms of quantum LS paths of the single column KR crystals for exceptional types, we verify the conjectures of [HKOTT] for exceptional simply-laced types in Section 6 (except for two Dynkin nodes for type  $E_8^{(1)}$ ). In Section 7 the quantum alcove model and its crystal structure are defined. In Section 8, we show that there is a bijection between quantum alcove paths and quantum LS paths by exhibiting a forgetful map and its inverse. We show that up to Kashiwara operators  $f_0$  at the end of their strings, there is an affine crystal isomorphism between the quantum alcove paths and tensor products KR crystals. Section 9 contains the main application of this work: by showing that the energy/degree function under the affine crystal isomorphism maps to a height function in the alcove path model, we show that the character of tensor products of single column KR crystals is equal to the Macdonald polynomial evaluated at  $t = 0$  (see Corollary 9.10). We conclude in Section 10 with the proof of Lemmas from various sections.

We follow the same conventions and notation as in [LNSS1].

**Acknowledgments.** We would like to thank the Mathematisches Forschungsinstitut Oberwolfach for their support during the Research in Pairs program and ICERM for hosting the semester program “Automorphic Forms, Combinatorial Representation Theory and Multiple Dirichlet Series” in the spring 2013, where some of the ideas were conceived. We thank Dan Orr for valuable discussions. We used SAGE [Sage] and SAGE-COMBINAT [Sage-comb] in our work.

C.L. was partially supported by the NSF grant DMS–1101264. The hospitality and support of the Max-Planck-Institut für Mathematik in Bonn, where part of this work was carried out, is also gratefully acknowledged. S.N. was supported by Grant-in-Aid for Scientific Research (C), No. 24540010, Japan. D.S. was supported by Grant-in-Aid for Young Scientists (B) No.23740003, Japan. AS was partially supported by NSF grants DMS–1001256, OCI–1147247, and a grant from the Simons Foundation (#226108 to Anne Schilling). M.S. was partially supported by the NSF grant DMS-1200804.

## 2. LAKSHMIBAI-SESHADRI PATHS

In this section we review Lakshmibai-Seshadri paths and the corresponding affine crystal model. As summarized in Theorem 2.7 and Remark 2.8, the crystal of level-zero projected LS paths is isomorphic to KR crystals.

**2.1. Basic notation.** Let  $\mathfrak{g}_{\text{af}}$  be an untwisted affine Lie algebra over  $\mathbb{C}$  with Cartan matrix  $A = (a_{ij})_{i,j \in I}$ . The index set  $I_{\text{af}}$  of the Dynkin diagram of  $\mathfrak{g}_{\text{af}}$  is numbered as in [Kac, Section 4.8, Table Aff 1]. Take the distinguished vertex  $0 \in I_{\text{af}}$  as in [Kac], and set  $I := I_{\text{af}} \setminus \{0\}$ . Let  $\mathfrak{h}_{\text{af}} = \left( \bigoplus_{j \in I_{\text{af}}} \mathbb{C}\alpha_j^\vee \right) \oplus \mathbb{C}d$  denote the Cartan subalgebra of  $\mathfrak{g}_{\text{af}}$ , where  $\{\alpha_j^\vee\}_{j \in I_{\text{af}}} \subset \mathfrak{h}_{\text{af}}$  is the set of simple coroots, and  $d \in \mathfrak{h}_{\text{af}}$  is the scaling element (or degree operator). Also, we denote by  $\{\alpha_j\}_{j \in I_{\text{af}}} \subset \mathfrak{h}_{\text{af}}^* := \text{Hom}_{\mathbb{C}}(\mathfrak{h}_{\text{af}}, \mathbb{C})$  the set of simple roots, and by  $\Lambda_j \in \mathfrak{h}_{\text{af}}^*$ ,  $j \in I_{\text{af}}$ , the fundamental weights; note that  $\alpha_j(d) = \delta_{j,0}$  and  $\Lambda_j(d) = 0$  for  $j \in I_{\text{af}}$ . Let  $\delta = \sum_{j \in I_{\text{af}}} a_j \alpha_j \in \mathfrak{h}_{\text{af}}^*$  and  $c = \sum_{j \in I_{\text{af}}} a_j^\vee \alpha_j^\vee \in \mathfrak{h}_{\text{af}}$  denote the null root and the canonical central element of  $\mathfrak{g}_{\text{af}}$ , respectively. The dual weight lattice  $X_{\text{af}}^\vee$  and the weight lattice  $X_{\text{af}}$  are defined as follows:

$$(2.1) \quad X_{\text{af}}^\vee = \left( \bigoplus_{j \in I_{\text{af}}} \mathbb{Z}\alpha_j^\vee \right) \oplus \mathbb{Z}d \subset \mathfrak{h}_{\text{af}} \quad \text{and} \quad X_{\text{af}} = \left( \bigoplus_{j \in I_{\text{af}}} \mathbb{Z}\Lambda_j \right) \oplus \mathbb{Z}\delta \subset \mathfrak{h}_{\text{af}}^*.$$

It is clear that  $X_{\text{af}}$  contains  $Q_{\text{af}} := \bigoplus_{j \in I_{\text{af}}} \mathbb{Z}\alpha_j$ , and that  $X_{\text{af}} \cong \text{Hom}_{\mathbb{Z}}(X_{\text{af}}^\vee, \mathbb{Z})$ . Let  $\mathfrak{g}$  be the classical subalgebra of  $\mathfrak{g}_{\text{af}}$  and denote the finite weight lattice by  $X = \bigoplus_{i \in I} \mathbb{Z}\omega_i$ , where  $\omega_i$  are the fundamental weights associated to  $\mathfrak{g}$ . The natural projection  $\text{cl} : X_{\text{af}} \rightarrow X$  has kernel  $\mathbb{Z}\delta$  and sends  $\Lambda_i - a_i^\vee \Lambda_0 \mapsto \omega_i$  for  $i \in I$ .

Let  $W_{\text{af}}$  (resp.  $W$ ) be the affine (resp. finite) Weyl group with simple reflections  $r_i$  for  $i \in I_{\text{af}}$  (resp.  $i \in I$ ).  $W_{\text{af}}$  acts on  $X_{\text{af}}$  and  $X_{\text{af}}^\vee$  by

$$\begin{aligned} r_i \lambda &= \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i \\ r_i \mu &= \mu - \langle \mu, \alpha_i \rangle \alpha_i^\vee \end{aligned}$$

for  $i \in I_{\text{af}}$ ,  $\lambda \in X_{\text{af}}$ , and  $\mu \in X_{\text{af}}^\vee$ . We denote by  $\ell$  the length function in  $W_{\text{af}}$  (resp.  $W$ ).

The set of affine real roots (resp. roots) of  $\mathfrak{g}_{\text{af}}$  (resp.  $\mathfrak{g}$ ) are defined by  $\Phi^{\text{af}} = W_{\text{af}} \{\alpha_i \mid i \in I_{\text{af}}\}$  (resp.  $\Phi = W \{\alpha_i \mid i \in I\}$ ). The set of positive affine real (resp. positive) roots are the set  $\Phi^{\text{af}+} = \Phi^{\text{af}} \cap \bigoplus_{i \in I_{\text{af}}} \mathbb{Z}_{\geq 0} \alpha_i$  (resp.  $\Phi^+ = \Phi \cap \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ ). We have  $\Phi^{\text{af}} = \Phi^{\text{af}+} \sqcup \Phi^{\text{af}-}$  where  $\Phi^{\text{af}-} = -\Phi^{\text{af}+}$  and  $\Phi = \Phi^+ \sqcup \Phi^-$  where  $\Phi^- = -\Phi^+$ .

We have  $\delta = \alpha_0 + \theta$ , where  $\theta$  is the highest root for  $\mathfrak{g}$ , and

$$\Phi^{\text{af}+} = \Phi^+ \sqcup (\Phi + \mathbb{Z}_{>0} \delta).$$

The level of a weight  $\lambda \in X_{\text{af}}$  is defined by  $\text{lev}(\lambda) = \langle c, \lambda \rangle$ . Since the action of  $W_{\text{af}}$  on  $X_{\text{af}}$  is level-preserving, the sublattice  $X_{\text{af}}^0 \subset X_{\text{af}}$  of level-zero elements is  $W_{\text{af}}$ -stable. There is a section  $X \rightarrow X_{\text{af}}^0$  given by  $\omega_i \mapsto \Lambda_i - \text{lev}(\Lambda_i) \Lambda_0$  for  $i \in I$ .

Finally, we briefly review the level-zero poset (see [LNSS1, Definition 6.1]). Fix a dominant weight  $\lambda$  in the finite weight lattice  $X$  and let  $W_J$  is the stabilizer of  $\lambda$ . More precisely,  $W_J$  is the parabolic subgroup generated by  $r_i$  for  $i \in J$  where

$$J = \{i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0\}.$$

Let  $Q_J^\vee = \bigoplus_{i \in J} \mathbb{Z}\alpha_i^\vee$  be the associated coroot lattice,  $W^J$  the set of minimum-length coset representatives in  $W/W_J$ ,  $\Phi_J = \Phi_J^+ \sqcup \Phi_J^-$  the set of roots and positive/negative roots respectively, and  $\rho_J = \frac{1}{2} \sum_{\alpha \in \Phi_J^+} \alpha$ . We also use  $Q^\vee = Q_I^\vee$  and  $\rho = \rho_I$ .

We view  $X$  as a sublattice of  $X_{\text{af}}^0$ . Let  $X_{\text{af}}^0(\lambda)$  be the orbit of  $\lambda$  under the action of the affine Weyl group  $W_{\text{af}}$ .

**Definition 2.1.** (Level-zero weight poset [Li]) *A poset structure is defined on  $X_{\text{af}}^0(\lambda)$  as the transitive closure of the relation*

$$(2.2) \quad \mu < r_\beta \mu \quad \Leftrightarrow \quad \langle \beta^\vee, \mu \rangle > 0,$$

where  $\beta \in \Phi^{\text{af}+}$ . This poset is called the level-zero weight poset for  $\lambda$ .

**2.2. Definition of Lakshmibai-Seshadri paths.** In this subsection, we fix a dominant integral weight  $\lambda \in X$ . We recall the definition of Lakshmibai-Seshadri (LS) paths of shape  $\lambda$  from [Li, Section 4]. Let  $X_{\text{af}}^0(\lambda)$  be the level-zero weight poset for  $\lambda$  (see [LNSSS1, Definition 6.1]).

**Definition 2.2.** For  $\mu, \nu \in X_{\text{af}}^0(\lambda)$  with  $\nu > \mu$  and a rational number  $0 < b < 1$ , a  $b$ -chain for  $(\nu, \mu)$  is, by definition, a sequence  $\nu = \xi_0 \succ \xi_1 \succ \cdots \succ \xi_n = \mu$  of covers in  $X_{\text{af}}^0(\lambda)$  such that  $b\langle \gamma_k^\vee, \xi_k \rangle \in \mathbb{Z}$  for all  $k = 1, 2, \dots, n$ , where  $\gamma_k \in \Phi^{\text{af}+}$  is the corresponding positive real root for  $\xi_{k-1} \succ \xi_k$ .

**Definition 2.3.** An LS path of shape  $\lambda$  is, by definition, a pair  $\pi = (\underline{\nu}; \underline{b})$  of a sequence  $\underline{\nu} : \nu_1 > \nu_2 > \cdots > \nu_s$  of elements in  $X_{\text{af}}^0(\lambda)$  and a sequence  $\underline{b} : 0 = b_0 < b_1 < \cdots < b_s = 1$  of rational numbers satisfying the condition that there exists a  $b_u$ -chain for  $(\nu_u, \nu_{u+1})$  for each  $u = 1, 2, \dots, s-1$ .

Denote by  $\mathbb{B}(\lambda)$  the set of all LS paths of shape  $\lambda$ . We identify an element

$$\pi = (\nu_1, \nu_2, \dots, \nu_s; b_0, b_1, \dots, b_s) \in \mathbb{B}(\lambda)$$

with the following piecewise-linear, continuous map  $\pi : [0, 1] \rightarrow \mathbb{R} \otimes_{\mathbb{Z}} X_{\text{af}}^0$ :

$$(2.3) \quad \pi(t) = \sum_{q=1}^{p-1} (b_q - b_{q-1})\nu_q + (t - b_{p-1})\nu_p \quad \text{for } b_{p-1} \leq t \leq b_p, \quad 1 \leq p \leq s.$$

*Remark 2.4.* It follows from the definition of an LS path of shape  $\lambda$  that  $\pi_\nu := (\nu; 0, 1) \in \mathbb{B}(\lambda)$  for every  $\nu \in X_{\text{af}}^0(\lambda)$ , which corresponds to the straight line  $\pi_\nu(t) = t\nu$ ,  $t \in [0, 1]$ .

Recall that  $X_{\text{af}}^0/\mathbb{Z}\delta \cong X$ . Denote by

$$\text{cl} : \mathbb{R} \otimes_{\mathbb{Z}} X_{\text{af}}^0 \rightarrow \mathbb{R} \otimes_{\mathbb{Z}} X_{\text{af}}^0/\mathbb{R}\delta \cong \mathbb{R} \otimes_{\mathbb{Z}} X$$

the canonical projection; remark that  $\text{cl}(X_{\text{af}}^0(\lambda)) = W\lambda \cong W^J$  (see [LNSSS1, Lemma 3.1]). For  $\pi \in \mathbb{B}(\lambda)$ , we define  $\text{cl}(\pi)$  by:  $(\text{cl}(\pi))(t) := \text{cl}(\pi(t))$  for  $t \in [0, 1]$ ; note that  $\text{cl}(\pi)$  is a piecewise linear, continuous map from  $[0, 1]$  to  $\mathbb{R} \otimes_{\mathbb{Z}} X$ . Then we set

$$\mathbb{B}(\lambda)_{\text{cl}} := \{ \text{cl}(\pi) \mid \pi \in \mathbb{B}(\lambda) \};$$

an element of this set is called a projected level-zero LS path.

**2.3. Crystal structures on  $\mathbb{B}(\lambda)$  and  $\mathbb{B}(\lambda)_{\text{cl}}$ .** As in the previous subsection, let  $\lambda \in X$  be a dominant integral weight. We use the following notation:

$$(2.4) \quad \tilde{\alpha}_i := \begin{cases} \alpha_i & \text{if } i \neq 0, \\ -\theta & \text{if } i = 0, \end{cases} \quad s_i := \begin{cases} r_i & \text{if } i \neq 0, \\ r_\theta & \text{if } i = 0. \end{cases}$$

Following [Li], we give  $\mathbb{B}(\lambda)$  and  $\mathbb{B}(\lambda)_{\text{cl}}$  crystal structures with the weight lattices  $X_{\text{af}}^0$  and  $\text{cl}(X_{\text{af}}^0) \cong X$ , respectively. Here we focus on the crystal structure on  $\mathbb{B}(\lambda)_{\text{cl}}$ ; for the crystal structure on  $\mathbb{B}(\lambda)$ , in the argument below, replace  $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$  with  $\pi \in \mathbb{B}(\lambda)$ , and then replace  $\tilde{\alpha}_j \in \Phi$  and  $s_j \in W$  with  $\alpha_j \in \Phi^{\text{af}}$  and  $r_j \in W_{\text{af}}$ .

Let  $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$ . We see from [Li, Lemma 4.5 a)] that  $\eta(1) \in \text{cl}(X_{\text{af}}^0) \cong X$ . So we set

$$\text{wt}(\eta) := \eta(1) \in X.$$

Next we define root operators  $e_j$  and  $f_j$  for  $j \in I_{\text{af}} = I \sqcup \{0\}$  as follows (see [Li, Section 1]): Set

$$(2.5) \quad \begin{aligned} H(t) &= H_j^\eta(t) := \langle \tilde{\alpha}_j^\vee, \eta(t) \rangle \quad \text{for } t \in [0, 1], \\ m &= m_j^\eta := \min\{H_j^\eta(t) \mid t \in [0, 1]\}. \end{aligned}$$

It follows from [Li, Lemma 4.5 d)] that all local minima of  $H(t)$  are integers; in particular,  $m \in \mathbb{Z}_{\leq 0}$ . If  $m = 0$ , then  $e_j \eta := \mathbf{0}$ , where  $\mathbf{0}$  is an extra element not contained in  $\mathbb{B}(\lambda)_{\text{cl}}$ . If  $m \leq -1$ , then set

$$\begin{aligned} t_1 &:= \min\{t \in [0, 1] \mid H(t) = m\}, \\ t_0 &:= \max\{t \in [0, t_1] \mid H(t) = m + 1\}. \end{aligned}$$

*Remark 2.5.*

- (1) Recall that all local minima of  $H(t)$  are integers by [Li, Lemma 4.5 d)]. Hence we deduce that  $H(t)$  is strictly decreasing on  $[t_0, t_1]$ .
- (2) Because  $H(t)$  attains the minimum  $m$  at  $t = t_1$ , it follows immediately that  $H(t_1 + \varepsilon) \geq H(t_1)$  for sufficiently small  $\varepsilon > 0$ .
- (3) We deduce that  $H(t_0 - \varepsilon) \geq H(t_0)$  for sufficiently small  $\varepsilon > 0$ . Indeed, suppose that  $H(t_0 - \varepsilon) < H(t_0)$ . Then the minimum  $m'$  of  $H(t)$  on  $[0, t_0]$  is less than  $H(t_0) = m + 1$ . Since all local minima of  $H(t)$  are integers, we obtain  $m' = m$ . However, this contradicts the definition of  $t_1$ ; recall that  $t_0 < t_1$ .

Define  $e_j \eta$  for  $j \in I_{\text{af}}$  by:

$$(2.6) \quad (e_j \eta)(t) = \begin{cases} \eta(t) & \text{if } 0 \leq t \leq t_0, \\ \eta(t_0) + s_j(\eta(t) - \eta(t_0)) & \text{if } t_0 \leq t \leq t_1, \\ \eta(t) + \tilde{\alpha}_j & \text{if } t_1 \leq t \leq 1, \end{cases}$$

where  $s_j \in W$  is the reflection with respect to  $\tilde{\alpha}_j \in \Phi$ . We see from [Li, Corollary 2 a)] that  $e_j \eta \in \mathbb{B}(\lambda)_{\text{cl}}$ . The definition of  $f_j \eta \in \mathbb{B}(\lambda)_{\text{cl}} \cup \{\mathbf{0}\}$  is similar (see also [NS6, Section 2.2]). In addition, for  $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$  and  $j \in I_{\text{af}}$ , we set

$$(2.7) \quad \varepsilon_j(\eta) := \max\{n \geq 0 \mid e_j^n \eta \neq \mathbf{0}\}, \quad \varphi_j(\eta) := \max\{n \geq 0 \mid f_j^n \eta \neq \mathbf{0}\}.$$

We see from [Li, Section 2] that the set  $\mathbb{B}(\lambda)_{\text{cl}}$  together with the map  $\text{wt} : \mathbb{B}(\lambda)_{\text{cl}} \rightarrow X$ , the root operators  $e_j, f_j, j \in I_{\text{af}}$ , and the maps  $\varepsilon_j, \varphi_j, j \in I_{\text{af}}$ , becomes a crystal with  $\text{cl}(X_{\text{af}}^0(\lambda)) \cong X$  the weight lattice.

*Remark 2.6.* It is easily verified that

$$\begin{aligned} \text{wt}(\text{cl}(\pi)) &= \text{cl}(\text{wt}(\pi)) \quad \text{for } \pi \in \mathbb{B}(\lambda), \\ \text{cl}(e_j\pi) &= e_j \text{cl}(\pi) \quad \text{and} \quad \text{cl}(f_j\pi) = f_j \text{cl}(\pi) \quad \text{for } \pi \in \mathbb{B}(\lambda) \text{ and } j \in I_{\text{af}}, \\ \varepsilon_j(\text{cl}(\pi)) &= \varepsilon_j(\pi) \quad \text{and} \quad \varphi_j(\text{cl}(\pi)) = \varphi_j(\pi) \quad \text{for } \pi \in \mathbb{B}(\lambda) \text{ and } j \in I_{\text{af}}. \end{aligned}$$

We know the following theorem from [NS1, NS2, NS3].

**Theorem 2.7.**

- (1) For each  $i \in I$ , the crystal  $\mathbb{B}(\omega_i)_{\text{cl}}$  is isomorphic to the crystal basis of  $W(\omega_i)$ , the level-zero fundamental representation, introduced by Kashiwara [Kas].
- (2) The crystal graph of  $\mathbb{B}(\lambda)_{\text{cl}}$  is connected.
- (3) Let  $\mathbf{i} = (i_1, i_2, \dots, i_p)$  be an arbitrary sequence of elements of  $I$  (with repetitions allowed), and set  $\lambda_{\mathbf{i}} := \omega_{i_1} + \omega_{i_2} + \dots + \omega_{i_p}$ . Then, there exists an isomorphism  $\Psi_{\mathbf{i}} : \mathbb{B}(\lambda_{\mathbf{i}})_{\text{cl}} \xrightarrow{\sim} \mathbb{B}(\omega_{i_1})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_2})_{\text{cl}} \otimes \dots \otimes \mathbb{B}(\omega_{i_p})_{\text{cl}}$  of crystals.

*Remark 2.8.* It is known that the fundamental representation  $W(\omega_i)$  of level-zero is isomorphic to the Kirillov-Reshetikhin (KR) module  $W_1^{(i)}$  in the sense of [HKOTT, Section 2.3] (for the Drinfeld polynomials of  $W(\omega_i)$ , see [N, Remark 3.3]). Also we can prove that the crystal basis of  $W(\omega_i) \cong W_1^{(i)}$  is unique, up to a nonzero constant multiple (see also [NS4, Lemma 1.5.3]); we call this crystal basis a (one-column) KR crystal, and denote by  $B^{i,1}$ . By the theorem above, the crystal  $\mathbb{B}(\lambda)_{\text{cl}}$  of projected level-zero LS paths of shape  $\lambda$  is a model for the corresponding tensor product of KR crystals.

In this paper we use the Kashiwara convention for the tensor product. More precisely, for two crystals  $B_1$  and  $B_2$ , the tensor product  $B_1 \otimes B_2$  as a set is the Cartesian product of the two sets. For  $b = b_1 \otimes b_2 \in B_1 \otimes B_2$ , the weight function is simply  $\text{wt}(b) = \text{wt}(b_1) + \text{wt}(b_2)$ . In the Kashiwara convention the crystal operators are given by

$$f_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes f_i(b_2) & \text{if } \varepsilon_i(b_2) \geq \varphi_i(b_1), \\ f_i(b_1) \otimes b_2 & \text{otherwise,} \end{cases}$$

and similarly for  $e_i(b)$ , where  $\varepsilon_j$  and  $\varphi_j$  are defined as in (2.7).

### 3. QUANTUM LAKSHMIBAI-SESHADRI PATHS

In this section we review quantum Lakshmibai-Seshadri paths, which were defined in [LNSSS1] in terms of the parabolic quantum Bruhat graph. The main result of this section is Theorem 3.3, which shows that projected level-zero LS paths are quantum LS paths. In [LNSSS2] this is proved in a different fashion using root operators.

**3.1. The parabolic quantum Bruhat graph.** The quantum Bruhat graph was first introduced in a paper by Brenti, Fomin and Postnikov [BFP] motivated by work of Fomin, Gelfand and Postnikov [FGP] in type  $A$ . It later appeared in connection with the quantum cohomology of flag varieties in a paper by Fulton and Woodward [FW].

We denote by  $\text{QB}(W^J)$  the *parabolic quantum Bruhat graph*. Its vertex set is  $W^J$ . There are two kinds of directed edges. Both are labeled by some  $\alpha \in \Phi^+ \setminus \Phi_J^+$ . We use the notation  $[w]$  to indicate the minimum-length coset representative in the coset  $wW_J$ . For  $w \in W^J$  there is a directed edge  $w \xrightarrow{\alpha} [wr_\alpha]$  if  $\alpha \in \Phi^+ \setminus \Phi_J^+$  and one of the following holds:

- (1) (Bruhat edge)  $w \prec wr_\alpha$  is a covering relation in Bruhat order, that is,  $\ell(wr_\alpha) = \ell(w) + 1$ .  
(One may deduce that  $wr_\alpha \in W^J$ .)

- (2) (Quantum edge)

$$(3.1) \quad \ell(\lfloor wr_\alpha \rfloor) = \ell(w) + 1 - \langle \alpha^\vee, 2\rho - 2\rho_J \rangle.$$

We define the *weight* of an edge  $w \xrightarrow{\alpha} \lfloor wr_\alpha \rfloor$  in the parabolic quantum Bruhat graph to be either  $\alpha^\vee$  or 0, depending on whether it is a quantum edge or not, respectively. Then the weight of a directed path  $\mathbf{p}$ , denoted by  $\text{wt}(\mathbf{p})$ , is defined as the sum of the weights of its edges.

**3.2. Definition of quantum Lakshmibai-Seshadri paths.** In this subsection, we fix a dominant integral weight  $\lambda \in X$ . Set  $J := \{j \in I \mid \langle \alpha_j^\vee, \lambda \rangle = 0\}$ , so that  $W_J$  is the stabilizer of  $\lambda$ . Given a rational number  $b$ , we define  $\text{QB}_{b\lambda}(W^J)$  to be the subgraph of the parabolic quantum Bruhat graph  $\text{QB}(W^J)$  with the same vertex set but having only the edges:

$$(3.2) \quad x \xrightarrow{\gamma} y \quad \text{with} \quad \langle \gamma^\vee, b\lambda \rangle = b\langle \gamma^\vee, \lambda \rangle \in \mathbb{Z}.$$

**Definition 3.1.** A *quantum Lakshmibai-Seshadri (QLS) path* of shape  $\lambda$  is a pair  $\eta = (\underline{x}; \underline{b})$  of a sequence  $\underline{x} : x_1, x_2, \dots, x_s$  of elements in  $W^J$  with  $x_u \neq x_{u+1}$  for  $1 \leq u \leq s-1$  and a sequence  $\underline{b} : 0 = b_0 < b_1 < \dots < b_s = 1$  of rational numbers satisfying the condition that there exists a directed path from  $x_{u+1}$  to  $x_u$  in  $\text{QB}_{b_u\lambda}(W^J)$  for each  $1 \leq u \leq s-1$ .

Denote by  $\text{QLS}(\lambda)$  the set of QLS paths of shape  $\lambda$ . We use the notation  $x \xrightarrow{b} y$  to indicate that there exists a directed path from  $x$  to  $y$  in  $\text{QB}_{b\lambda}(W^J)$ ; so we can write the element

$$\eta = (x_1, x_2, \dots, x_s; b_0, b_1, \dots, b_s)$$

in  $\text{QLS}(\lambda)$  as follows:

$$(3.3) \quad x_1 \xleftarrow{b_1} x_2 \xleftarrow{b_2} \dots x_{s-2} \xleftarrow{b_{s-2}} x_{s-1} \xleftarrow{b_{s-1}} x_s.$$

Since  $W^J$  can be identified with  $W\lambda$  under the canonical bijection  $w \mapsto w\lambda$ , we will sometimes think of the elements  $x_i$  as weights. Moreover, we identify  $\eta$  with the following piecewise-linear, continuous map  $\eta : [0, 1] \rightarrow \mathbb{R} \otimes_{\mathbb{Z}} X$ :

$$(3.4) \quad \eta(t) = \sum_{q=1}^{p-1} (b_q - b_{q-1})x_q\lambda + (t - b_{p-1})x_p\lambda \quad \text{for } b_{p-1} \leq t \leq b_p, \quad 1 \leq p \leq s.$$

*Remark 3.2.* It follows from the definition of a QLS path of shape  $\lambda$  that  $\eta_x := (x; 0, 1) \in \text{QLS}(\lambda)$  for every  $x \in W^J$ , which corresponds to the straight line  $\eta_{x\lambda}(t) = tx\lambda$ ,  $t \in [0, 1]$ . We can easily see that  $\text{cl}(\pi_\nu) = \eta_{\text{cl}(\nu)}$  for  $\nu \in X_{\text{af}}^0(\lambda)$ ; recall that  $\text{cl}(X_{\text{af}}^0(\lambda)) = W\lambda$ .

**3.3. Relation between LS paths and QLS paths.** We now establish the correspondence between projected level-zero LS paths and quantum LS paths. As before,  $\lambda \in X$  is a fixed dominant integral weight.

**Theorem 3.3.**  $\mathbb{B}(\lambda)_{\text{cl}} = \text{QLS}(\lambda)$  under the identification  $\text{cl}(X_{\text{af}}^0(\lambda)) = W\lambda$ .

In order to prove this theorem, we need the following lemma.

**Lemma 3.4.** Let  $0 < b < 1$  be a rational number.

- (1) Let  $\mu, \nu \in X_{\text{af}}^0(\lambda)$ . If there exists a  $b$ -chain for  $(\nu, \mu)$ , then there exists a directed path from  $\text{cl}(\mu)$  to  $\text{cl}(\nu)$  in  $\text{QB}_{b\lambda}(W^J)$ .



(2) Let  $w, w' \in W^J$ . If there exists a directed path from  $w$  to  $w'$  in  $\text{QB}_{b\lambda}(W^J)$ , then for each  $\mu \in X_{\text{af}}^0(\lambda)$  with  $\text{cl}(\mu) = w$ , there exists a  $b$ -chain for  $(\nu, \mu)$  for some  $\nu \in X_{\text{af}}^0(\lambda)$  with  $\text{cl}(\nu) = w'$ .

*Proof.* (1) It suffices to show the assertion in the case that  $\nu$  is a cover of  $\mu$ , i.e.,  $\mu < \nu$  in  $X_{\text{af}}^0(\lambda)$ . Let  $\beta \in \Phi^{\text{af}+}$  be such that  $r_\beta \mu = \nu$  and  $b\langle \beta^\vee, \mu \rangle \in \mathbb{Z}$ . Then,  $\beta \in \Phi^+$  or  $\beta \in \delta - \Phi^+$  (see [LNSS1, Lemma 6.4(1)]). Set  $w := \text{cl}(\mu) \in W\lambda \cong W^J$ . If  $\beta \in \Phi^+$ , then it follows from [LNSS1, Theorem 6.5] that  $\gamma := w^{-1}\beta \in \Phi^+ \setminus \Phi_J^+$  and  $\text{cl}(\mu) = w \xrightarrow{\gamma} \text{cl}(\nu)$  in  $\text{QB}(W^J)$ . In addition, we see that  $b\langle \gamma^\vee, \lambda \rangle = b\langle \beta^\vee, \mu \rangle \in \mathbb{Z}$ , which implies that  $\text{cl}(\mu) = w \xrightarrow{\gamma} \text{cl}(\nu)$  in  $\text{QB}_{b\lambda}(W^J)$ . Similarly, if  $\beta \in \delta - \Phi^+$ , then it follows from [LNSS1, Theorem 6.5] that  $\gamma := w^{-1}(\beta - \delta) \in \Phi^+ \setminus \Phi_J^+$  and  $\text{cl}(\mu) = w \xrightarrow{\gamma} \text{cl}(\nu)$  in  $\text{QB}(W^J)$ . We see that  $b\langle \gamma^\vee, \lambda \rangle = b\langle \beta^\vee - c, \mu \rangle = b\langle \beta^\vee, \mu \rangle \in \mathbb{Z}$ , which implies that  $\text{cl}(\mu) = w \xrightarrow{\gamma} \text{cl}(\nu)$  in  $\text{QB}_{b\lambda}(W^J)$ . Thus we have proved part (1).

(2) Fix  $\mu \in X_{\text{af}}^0(\lambda)$  such that  $\text{cl}(\mu) = w$ . Assume that

$$w = x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} \dots \xrightarrow{\gamma_u} x_u = w'$$

is a directed path from  $w$  to  $w'$  in  $\text{QB}_{b\lambda}(W^J)$ . We show the assertion by induction on the length  $u$  of the directed path above. Assume first that  $u = 1$ ; for simplicity of notation, we set  $\gamma := \gamma_1$ . Set

$$\beta := \begin{cases} w\gamma & \text{if } w \xrightarrow{\gamma} w' \text{ is a Bruhat edge,} \\ \delta + w\gamma & \text{if } w \xrightarrow{\gamma} w' \text{ is a quantum edge.} \end{cases}$$

It follows from [LNSS1, Theorem 6.5] that  $\beta \in \Phi^{\text{af}+}$  and  $\mu < r_\beta \mu =: \nu$ . Also, we see that  $\text{cl}(\nu) = w'$ . In addition,  $b\langle \beta^\vee, \mu \rangle = b\langle \gamma^\vee, \lambda \rangle \in \mathbb{Z}$ . Thus,  $\mu < \nu$  is a  $b$ -chain for  $(\nu, \mu)$ . Assume that  $u \geq 2$ . By our induction hypothesis, there exists a  $b$ -chain for  $(\xi, \mu)$  for some  $\xi \in X_{\text{af}}^0(\lambda)$  with  $\text{cl}(\xi) = x_{u-1}$ . Also, by our induction hypothesis, there exists a  $b$ -chain for  $(\nu, \xi)$  for some  $\nu \in X_{\text{af}}^0(\lambda)$  with  $\text{cl}(\nu) = x_u = w'$ . Concatenating these  $b$ -chains, we obtain a  $b$ -chain for  $(\nu, \mu)$ . Thus we have proved the lemma.  $\square$

*Proof of Theorem 3.3.* First, let us show that  $\mathbb{B}(\lambda)_{\text{cl}} \subset \text{QLS}(\lambda)$ . Let

$$\pi = (\nu_1, \nu_2, \dots, \nu_{s-1}, \nu_s; b_0, b_1, b_2, \dots, b_{s-1}, b_s) \in \mathbb{B}(\lambda).$$

We show  $\text{cl}(\pi) \in \text{QLS}(\lambda)$  by induction on  $s$ . If  $s = 1$ , then the assertion is obvious by Remark 3.2. Assume that  $s > 1$ . Set

$$\pi' := (\nu_2, \dots, \nu_{s-1}, \nu_s; b_0, b_2, \dots, b_{s-1}, b_s).$$

Then we see that  $\pi' \in \mathbb{B}(\lambda)$ , and hence  $\text{cl}(\pi') \in \text{QLS}(\lambda)$  by our induction hypothesis. Write  $\text{cl}(\pi')$  as:

$$\text{cl}(\pi') := (y_1, y_2, \dots, y_u; c_0, c_1, \dots, c_{u-1}, c_u)$$

for some  $y_1, y_2, \dots, y_u \in W^J$  and  $0 = c_0 < c_1 < \dots < c_{u-1} < c_u = 1$ ; we should remark that  $0 < b_1 < b_2 \leq c_1$  and  $y_1 = \text{cl}(\nu_2)$ . The inequality  $b_2 \leq c_1$  comes from the fact that  $\text{cl}(\pi')(t) = t \text{cl}(\nu_2)$  for  $t \in [0, b_2]$  and  $\text{cl}(\pi')(t) = b_2 \text{cl}(\nu_2) + (t - b_2) \text{cl}(\nu_3)$  for  $t \in [b_2, b_3]$ . Therefore,

- (a) if  $\text{cl}(\nu_2) \neq \text{cl}(\nu_3)$ , then the first turning point  $c_1$  of  $\text{cl}(\pi')$  is equal to  $b_2$ .
- (b) if  $\text{cl}(\nu_2) = \text{cl}(\nu_3)$ , then the first turning point  $c_1$  of  $\text{cl}(\pi')$  is greater than  $b_2$ .

If  $\text{cl}(\nu_1) = \text{cl}(\nu_2)$ , then it follows immediately that  $\text{cl}(\pi) = \text{cl}(\pi')$ , and hence  $\text{cl}(\pi) \in \text{QLS}(\lambda)$ . Assume that  $\text{cl}(\nu_1) \neq \text{cl}(\nu_2) = y_1$ ; set  $x_1 := \text{cl}(\nu_1) \in W\lambda \cong W^J$ . Because there exists a  $b_1$ -chain for  $(\nu_1, \nu_2)$  by the definition of an LS path, we deduce from Lemma 3.4 (1) that there exists a directed path from  $y_1 = \text{cl}(\nu_2)$  to  $x_1 = \text{cl}(\nu_1)$  in  $\text{QB}_{b_1\lambda}(W^J)$ . Therefore, we see that

$$(x_1, y_1, y_2, \dots, y_u; c_0, b_1, c_1, \dots, c_{u-1}, c_u)$$

is a QLS path of shape  $\lambda$ , which is identical to  $\text{cl}(\pi)$ . Thus we obtain  $\text{cl}(\pi) \in \text{QLS}(\lambda)$ , as desired.

Next, let us show the opposite inclusion, i.e.,  $\mathbb{B}(\lambda)_{\text{cl}} \supset \text{QLS}(\lambda)$ . Let

$$\eta = (x_1, x_2, \dots, x_{s-1}, x_s; b_0, b_1, b_2, \dots, b_{s-1}, b_s) \in \text{QLS}(\lambda).$$

We show by induction on  $s$  that there exists  $\pi \in \mathbb{B}(\lambda)$  such that  $\text{cl}(\pi) = \eta$ . If  $s = 1$ , then the assertion is obvious by Remark 3.2. Assume that  $s > 1$ . We see that

$$\eta' := (x_2, \dots, x_{s-1}, x_s; b_0, b_2, \dots, b_{s-1}, b_s)$$

is contained in  $\text{QLS}(\lambda)$ . Hence, by our induction hypothesis, there exists  $\pi' \in \mathbb{B}(\lambda)$  such that  $\text{cl}(\pi') = \eta'$ . Write  $\pi'$  as:

$$\pi' = (\mu_1, \mu_2, \dots, \mu_u; c_0, c_1, \dots, c_{u-1}, c_u)$$

for some  $\mu_1, \mu_2, \dots, \mu_u \in X_{\text{af}}^0(\lambda)$  and  $0 = c_0 < c_1 < \dots < c_{u-1} < c_u = 1$ ; we should remark that  $0 < b_1 < b_2 \leq c_1$  and  $\text{cl}(\mu_1) = x_2$ . Because there exists a directed path from  $x_2 = \text{cl}(\mu_1)$  to  $x_1$  in  $\text{QB}_{b_1\lambda}(W^J)$ , it follows from Lemma 3.4 (2) that there exists a  $b_1$ -chain for  $(\nu_1, \mu_1)$  for some  $\nu_1 \in X_{\text{af}}^0(\lambda)$  with  $\text{cl}(\nu_1) = x_1$ . Therefore,

$$\pi := (\nu_1, \mu_1, \mu_2, \dots, \mu_u; c_0, b_1, c_1, \dots, c_{u-1}, c_u) \in \mathbb{B}(\lambda).$$

It can be easily seen that  $\text{cl}(\pi) = \eta$ . Thus we have proved the opposite inclusion, thereby completing the proof of the theorem.  $\square$

#### 4. FORMULA FOR THE DEGREE FUNCTION

Throughout this section, we fix a dominant integral weight  $\lambda \in X$ , and set  $J := \{i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0\}$ . We define the degree function on projected level-zero LS paths in Section 4.2 and recall the relation with the energy function on KR crystals in Theorem 4.5. Theorem 4.6 is the main result of this section and provides an explicit expression for the degree function as sums of weights of shortest paths in the parabolic quantum Bruhat graph.

**4.1. Weights of directed paths.** We know the following proposition from [LNSS1, Proposition 8.1].

**Proposition 4.1.** *Let  $x, y \in W^J$ . Let  $\mathbf{p}$  and  $\mathbf{q}$  be a shortest and an arbitrary directed path from  $x$  to  $y$  in  $\text{QB}(W^J)$ , respectively. Then there exists  $h \in Q_+^\vee$  such that*

$$\text{wt}(\mathbf{q}) - \text{wt}(\mathbf{p}) \equiv h \pmod{Q_J^\vee}.$$

*In addition, if  $\mathbf{q}$  is also shortest, then  $\text{wt}(\mathbf{q}) \equiv \text{wt}(\mathbf{p}) \pmod{Q_J^\vee}$ .*

Let  $x, y \in W^J$ . By Proposition 4.1, the pairing of  $\lambda$  and the weight of a shortest directed path from  $x$  to  $y$  does not depend on the choice of a shortest directed path from  $x$  to  $y$ . Define  $\text{wt}_\lambda(x \Rightarrow y)$  to be the pairing of  $\lambda$  and the weight of a shortest directed path from  $x$  to  $y$ .

The following is a corollary to [LNSS1, Lemma 7.7].

**Corollary 4.2.** *Let  $w_1, w_2 \in W^J$ , and  $j \in I_{\text{af}}$ .*

(1) *If  $\langle \tilde{\alpha}_j^\vee, w_1 \lambda \rangle > 0$  and  $\langle \tilde{\alpha}_j^\vee, w_2 \lambda \rangle \leq 0$ , then*

$$\text{wt}_\lambda([s_j w_1] \Rightarrow w_2) = \text{wt}_\lambda(w_1 \Rightarrow w_2) - \delta_{j,0} \langle \tilde{\alpha}_j^\vee, w_1 \lambda \rangle.$$

(2) *If  $\langle \tilde{\alpha}_j^\vee, w_1 \lambda \rangle < 0$  and  $\langle \tilde{\alpha}_j^\vee, w_2 \lambda \rangle < 0$ , then*

$$\text{wt}_\lambda([s_j w_1] \Rightarrow [s_j w_2]) = \text{wt}_\lambda(w_1 \Rightarrow w_2) - \delta_{j,0} \langle \tilde{\alpha}_j^\vee, w_1 \lambda \rangle + \delta_{j,0} \langle \tilde{\alpha}_j^\vee, w_2 \lambda \rangle.$$

(3) *If  $\langle \tilde{\alpha}_j^\vee, w_1 \lambda \rangle \geq 0$  and  $\langle \tilde{\alpha}_j^\vee, w_2 \lambda \rangle < 0$ , then*

$$\text{wt}_\lambda(w_1 \Rightarrow [s_j w_2]) = \text{wt}_\lambda(w_1 \Rightarrow w_2) + \delta_{j,0} \langle \tilde{\alpha}_j^\vee, w_2 \lambda \rangle.$$

*Proof.* We give a proof only for part (1); the proofs for parts (2) and (3) are similar. Let  $\mathbf{p}$  be a shortest directed path from  $w_1$  to  $w_2$ . Then it follows from [LNSSS1, Lemma 7.7 (3) and (5)] that there exists a shortest directed path  $\mathbf{p}'$  from  $[s_j x]$  to  $w_2$  such that

$$\text{wt}(\mathbf{p}') = \text{wt}(\mathbf{p}) - \delta_{j,0} w_1^{-1} \tilde{\alpha}_j^\vee.$$

Hence,

$$\begin{aligned} \text{wt}_\lambda([s_j w_1] \Rightarrow w_2) &= \langle \text{wt}(\mathbf{p}'), \lambda \rangle = \langle \text{wt}(\mathbf{p}), \lambda \rangle - \delta_{j,0} \langle w_1^{-1} \tilde{\alpha}_j^\vee, \lambda \rangle \\ &= \text{wt}_\lambda(w_1 \Rightarrow w_2) - \delta_{j,0} \langle \tilde{\alpha}_j^\vee, w_1 \lambda \rangle. \end{aligned}$$

Thus we have proved the corollary.  $\square$

**4.2. Definition of the degree function.** Let us recall from [NS6, Section 3.1] the definition of the degree function

$$\text{Deg} = \text{Deg}_\lambda : \mathbb{B}(\lambda)_{\text{cl}} \rightarrow \mathbb{Z}_{\leq 0}.$$

Denote by  $\mathbb{B}_0(\lambda)$  the connected component of  $\mathbb{B}(\lambda)$  containing the straight line  $\pi_\lambda = (\lambda; 0, 1)$ . Also, for  $\pi = (\nu_1, \dots, \nu_s; b_0, \dots, b_s) \in \mathbb{B}(\lambda)$ , we set  $\iota(\pi) := \nu_1$ , and call it the initial direction of  $\pi$ ; note that  $\iota(\pi) = \pi(\varepsilon)/\varepsilon$  for sufficiently small  $\varepsilon > 0$ . We know from [NS6, Proposition 3.1.3] that for each  $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$ , there exists a unique  $\pi_\eta \in \mathbb{B}_0(\lambda)$  satisfying the conditions that  $\text{cl}(\pi_\eta) = \eta$  and  $\iota(\pi_\eta) \in \lambda - Q_+$ . Then it follows from [NS6, Lemma 3.1.1] that  $\pi_\eta(1) \in X_{\text{af}}^0$  is of the form:

$$\pi_\eta(1) = \lambda - \beta + K\delta$$

for some  $\beta \in Q_+$  and  $K \in \mathbb{Z}_{\geq 0}$ . We define the degree  $\text{Deg}(\eta) \in \mathbb{Z}_{\leq 0}$  of  $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$  by:

$$(4.1) \quad \text{Deg}(\eta) = -K \in \mathbb{Z}_{\leq 0}.$$

*Remark 4.3.* It is known (see, e.g., [NS6, Proposition 4.3.1]) that for each  $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$ , there exist  $j_1, j_2, \dots, j_k \in I_{\text{af}}$  such that  $e_{j_k} e_{j_2} \cdots e_{j_1} \eta = \eta_\lambda$ . Therefore, we deduce from [NS6, Lemma 3.2.1] that  $\text{Deg} = \text{Deg}_\lambda : \mathbb{B}(\lambda)_{\text{cl}} \rightarrow \mathbb{Z}_{\leq 0}$  is a unique function satisfying the following conditions:

- (i)  $\text{Deg}(\eta_\lambda) = 0$ ;
- (ii) for  $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$  and  $j \in I_{\text{af}}$  with  $e_j \eta \neq \mathbf{0}$ ,

$$(4.2) \quad \text{Deg}(e_j \eta) = \begin{cases} \text{Deg}(\eta) - 1 & \text{if } j = 0 \text{ and } \iota(e_0 \eta) = \iota(\eta), \\ \text{Deg}(\eta) - \langle \tilde{\alpha}_0^\vee, \iota(\eta) \rangle - 1 & \text{if } j = 0 \text{ and } \iota(e_0 \eta) = s_0(\iota(\eta)), \\ \text{Deg}(\eta) & \text{if } j \neq 0, \end{cases}$$

where  $\iota(\eta) := \eta(\varepsilon)/\varepsilon$  for sufficiently small  $\varepsilon > 0$ .

**4.3. Relation between the degree function and the energy function.** Write  $\lambda$  as  $\lambda = \omega_{i_1} + \omega_{i_2} + \cdots + \omega_{i_p}$  with  $i_1, i_2, \dots, i_p \in I$ . By Theorem 2.7 (3), there exists an isomorphism

$$(4.3) \quad \Psi : \mathbb{B}(\lambda)_{\text{cl}} \xrightarrow{\sim} \mathbb{B}(\omega_{i_1})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_2})_{\text{cl}} \otimes \cdots \otimes \mathbb{B}(\omega_{i_p})_{\text{cl}} =: \mathbb{B}$$

of crystals. Here we should recall from Remark 2.8 that  $\mathbb{B}(\omega_i)_{\text{cl}}$  is isomorphic to the one-column KR crystal  $B^{i,1}$ . Also, recall from Section 2.3 that we are using the Kashiwara convention for tensor products in this paper. So, following [HKOTY, Section 3] and [HKOTT, Section 3.3] (see also [SS] and [NS6, Section 4.1]), we define the energy function  $D = D_{\mathbb{B}} : \mathbb{B} \rightarrow \mathbb{Z}_{<0}$  on  $\mathbb{B}$  as follows. First, for each  $1 \leq k, l \leq p$ , there exists a unique isomorphism (called a combinatorial  $R$ -matrix)

$$R_{k,l} : \mathbb{B}(\omega_{i_k})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_l})_{\text{cl}} \xrightarrow{\sim} \mathbb{B}(\omega_{i_l})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_k})_{\text{cl}}$$

of crystals. Also, there exists a unique  $\mathbb{Z}$ -valued function (called a local energy function)  $H_{k,l} : \mathbb{B}(\omega_{i_k})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_l})_{\text{cl}} \rightarrow \mathbb{Z}$  satisfying the following conditions (H1) and (H2):

(H1) For  $\eta_k \otimes \eta_l \in \mathbb{B}(\omega_{i_k})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_l})_{\text{cl}}$  and  $j \in I_{\text{af}}$  such that  $e_j(\eta_k \otimes \eta_l) \neq \mathbf{0}$ ,

$$H_{k,l}(e_j(\eta_k \otimes \eta_l)) = \begin{cases} H_{k,l}(\eta_k \otimes \eta_l) + 1 & \text{if } j = 0, \text{ and if } e_0(\eta_k \otimes \eta_l) = e_0\eta_k \otimes \eta_l, e_0(\tilde{\eta}_l \otimes \tilde{\eta}_k) = e_0\tilde{\eta}_l \otimes \tilde{\eta}_k, \\ H_{k,l}(\eta_k \otimes \eta_l) - 1 & \text{if } j = 0, \text{ and if } e_0(\eta_k \otimes \eta_l) = \eta_k \otimes e_0\eta_l, e_0(\tilde{\eta}_l \otimes \tilde{\eta}_k) = \tilde{\eta}_l \otimes e_0\tilde{\eta}_k, \\ H_{k,l}(\eta_k \otimes \eta_l) & \text{otherwise,} \end{cases}$$

where we set  $\tilde{\eta}_l \otimes \tilde{\eta}_k := R_{k,l}(\eta_k \otimes \eta_l) \in \mathbb{B}(\omega_{i_l})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_k})_{\text{cl}}$ .

(H2)  $H_{k,l}(\eta_{\omega_{i_k}} \otimes \eta_{\omega_{i_l}}) = 0$ .

Now, for each  $1 \leq k < l \leq p$ , there exists a unique isomorphism

$$\begin{aligned} & \mathbb{B}(\omega_{i_k})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_{k+1}})_{\text{cl}} \otimes \cdots \otimes \mathbb{B}(\omega_{i_{l-1}})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_l})_{\text{cl}} \\ & \xrightarrow{\sim} \mathbb{B}(\omega_{i_l})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_k})_{\text{cl}} \otimes \cdots \otimes \mathbb{B}(\omega_{i_{l-2}})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_{l-1}})_{\text{cl}} \end{aligned}$$

of crystals, which is given by composition of combinatorial  $R$ -matrices. Given  $\eta_k \otimes \eta_{k+1} \otimes \cdots \otimes \eta_l \in \mathbb{B}(\omega_{i_k})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_{k+1}})_{\text{cl}} \otimes \cdots \otimes \mathbb{B}(\omega_{i_l})_{\text{cl}}$ , we define  $\eta_l^{(k)} \in \mathbb{B}(\omega_{i_l})_{\text{cl}}$  to be the first factor of the image of  $\eta_k \otimes \eta_{k+1} \otimes \cdots \otimes \eta_l$  under the above isomorphism of crystals. For convenience, we set  $\eta_l^{(l)} := \eta_l$  for  $\eta_l \in \mathbb{B}(\omega_{i_l})_{\text{cl}}$ ,  $1 \leq l \leq p$ . In addition, for each  $1 \leq k \leq p$ , take (and fix) an arbitrary element  $\eta_k^{\flat} \in \mathbb{B}(\omega_{i_k})_{\text{cl}}$  such that  $f_j \eta_k^{\flat} = \mathbf{0}$  for all  $j \in I$ . Then we define the energy function  $D = D_{\mathbb{B}} : \mathbb{B} = \mathbb{B}(\omega_{i_1})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_2})_{\text{cl}} \otimes \cdots \otimes \mathbb{B}(\omega_{i_p})_{\text{cl}} \rightarrow \mathbb{Z}$  by:

$$(4.4) \quad D(\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_p) = \sum_{1 \leq k < l \leq p} H_{k,l}(\eta_k \otimes \eta_l^{(k+1)}) + \sum_{k=1}^p H_{k,k}(\eta_k^{\flat} \otimes \eta_k^{(1)}).$$

*Remark 4.4.* The energy function  $D$  above corresponds to the ‘‘right’’ energy function  $D^R$  in [LeS, Section 2.4]; we should remark that the order of tensor products of crystals in [LeS] is ‘‘opposite’’ to that in this paper. In this paper, we call the energy defined in (4.4) the *head energy* since the tensor factors move towards the head (or first) tensor factor.

We know the following theorem from [NS6, Theorem 4.1.1].

**Theorem 4.5.** *Using the same notation as above, for every  $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$ , we have*

$$(4.5) \quad \text{Deg}(\eta) = D(\Psi(\eta)) - D_{\text{ext}},$$

where  $D_{\text{ext}} \in \mathbb{Z}$  is a constant defined by

$$D_{\text{ext}} := \sum_{k=1}^p H_{k,k}(\eta_k^b \otimes \eta_{\omega_{i_k}}).$$

**4.4. Formula for the degree function.** Let  $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$ . Because  $\mathbb{B}(\lambda)_{\text{cl}} = \text{QLS}(\lambda)$  by Theorem 3.3, we can write  $\eta$  as:

$$\eta = (x_1, x_2, \dots, x_s; b_0, b_1, \dots, b_s) \in \text{QLS}(\lambda)$$

for some  $x_1, x_2, \dots, x_s \in W^J$  and  $0 = b_0 < b_1 < \dots < b_s = 1$ ; note that  $\iota(\eta) = x_1\lambda$ .

**Theorem 4.6.** *With the same notation as above, we have the following equality:*

$$(4.6) \quad \text{Deg}(\eta) = - \sum_{k=1}^{s-1} (1 - b_k) \text{wt}_\lambda(x_{k+1} \Rightarrow x_k).$$

*Proof.* For  $\eta \in \text{QLS}(\lambda) = \mathbb{B}(\lambda)_{\text{cl}}$ , we define  $F(\eta)$  by the right-hand side of (4.6). It suffices to show that  $F$  satisfies conditions (i) and (ii) in Remark 4.3, i.e.,

- (i)  $F(\eta_\lambda) = 0$ ;
- (ii) for  $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$  and  $j \in I_{\text{af}}$  with  $e_j\eta \neq \mathbf{0}$ ,

$$(4.7) \quad F(e_j\eta) = \begin{cases} F(\eta) - 1 & \text{if } j = 0 \text{ and } \iota(e_0\eta) = \iota(\eta), \\ F(\eta) - \langle \tilde{\alpha}_0^\vee, \iota(\eta) \rangle - 1 & \text{if } j = 0 \text{ and } \iota(e_0\eta) = s_0(\iota(\eta)), \\ F(\eta) & \text{if } j \neq 0. \end{cases}$$

It is obvious that  $F$  satisfies condition (i). Let us show that  $F$  satisfies condition (ii). Let  $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$  and  $j \in I_{\text{af}}$  be such that  $e_j\eta \neq \mathbf{0}$ . We deduce that the point  $t_1 = \min\{t \in [0, 1] \mid H_j^\eta(t) = m_j^\eta\}$  is equal to  $b_u$  for some  $0 < u \leq s$ . Let  $0 < v \leq u$  be such that  $b_{v-1} \leq t_0 < b_v$ ; recall that  $t_0 = \max\{t \in [0, t_1] \mid H_j^\eta(t) = m_j^\eta + 1\}$ . It follows from the definition of the root operator  $e_j$  that  $e_j\eta \in \text{QLS}(\lambda)$  can be written as follows:

$$x_1 \xleftarrow{b_1} \dots x_{v-1} \xleftarrow{b_{v-1}} \underbrace{x_v \xleftarrow{t_0} [s_j x_v]}_{(a)} \xleftarrow{b_v} [s_j x_{v+1}] \xleftarrow{b_{v+1}} \dots \underbrace{[s_j x_u] \xleftarrow{b_u=t_1} x_{u+1}}_{(b)} \xleftarrow{b_{u+1}} \dots x_s.$$

Here, if  $b_{v-1} = t_0$ , then we drop (a) from the diagram above; note that in this case,  $x_{v-1} \neq [s_j x_v]$  since  $\langle \tilde{\alpha}_j^\vee, x_{v-1}\lambda \rangle \leq 0$  and  $\langle \tilde{\alpha}_j^\vee, s_j x_v \lambda \rangle = -\langle \tilde{\alpha}_j^\vee, x_v \lambda \rangle > 0$  by Remark 2.5 (1), (3). Also, if  $[s_j x_u] = x_{u+1}$ , then we replace (b) by  $x_{u+1}$  (or  $[s_j x_u]$ ) in the diagram above. We should remark that  $\iota(e_j\eta) = s_j\iota(\eta)$  if and only if  $m = m_j^\eta = 1$  (see (2.5)) and  $t_0 = b_0 = 0$ .

Now, by the definition of  $F$ , we have

$$(4.8) \quad F(e_j \eta) = - \left\{ \underbrace{\sum_{k=1}^{v-2} (1-b_k) \text{wt}_\lambda(x_{k+1} \Rightarrow x_k)}_{=:U_1} + R + \underbrace{\sum_{k=m}^{u-1} (1-b_k) \text{wt}_\lambda(\lfloor s_j x_{k+1} \rfloor \Rightarrow \lfloor s_j x_k \rfloor)}_{=:U_2} \right. \\ \left. + \underbrace{(1-b_u) \text{wt}_\lambda(x_{u+1} \Rightarrow \lfloor s_j x_u \rfloor)}_{=:U_3} + \sum_{k=u+1}^{s-1} (1-b_k) \text{wt}_\lambda(x_{k+1} \Rightarrow x_k) \right\},$$

where

$$(4.9) \quad R := \begin{cases} (1-b_{v-1}) \text{wt}_\lambda(x_v \Rightarrow x_{v-1}) \\ \quad + (1-t_0) \text{wt}_\lambda(\lfloor s_j x_v \rfloor \Rightarrow x_v) & \text{if } t_0 \neq b_{v-1}, \\ (1-b_{v-1}) \text{wt}_\lambda(\lfloor s_j x_v \rfloor \Rightarrow x_{v-1}) & \text{if } v > 1 \text{ and } t_0 = b_{v-1}, \\ 0 & \text{if } v = 1 \text{ and } t_0 = b_0 = 0, \end{cases}$$

and if  $u = s$  (resp.,  $v = 1$ ), then  $\text{wt}_\lambda(x_{u+1} \Rightarrow \lfloor s_j x_u \rfloor)$  in  $U_3$  (resp.,  $\text{wt}_\lambda(x_v \Rightarrow x_{v-1})$  in  $R$ ) is understood to be 0; notice that the equality (4.8) is valid even when  $\lfloor s_j x_u \rfloor = x_{u+1}$ . Also, in (4.9), notice that

$$(4.10) \quad \text{wt}_\lambda(\lfloor s_j x_v \rfloor \Rightarrow x_v) = -\delta_{j,0} \langle \tilde{\alpha}_j^\vee, x_v \lambda \rangle.$$

First, let us show that if  $v > 1$  and  $t_0 = b_{v-1}$  (cf. the second case of (4.9)), then

$$(4.11) \quad R = (1-b_{v-1}) \text{wt}_\lambda(x_v \Rightarrow x_{v-1}) - (1-t_0) \delta_{j,0} \langle \tilde{\alpha}_j^\vee, x_v \lambda \rangle.$$

Recall that  $\langle \tilde{\alpha}_j^\vee, x_{v-1} \lambda \rangle \leq 0$  and  $\langle \tilde{\alpha}_j^\vee, s_j x_v \lambda \rangle = -\langle \tilde{\alpha}_j^\vee, x_v \lambda \rangle > 0$  by Remark 2.5 (1), (3). Thus, applying Corollary 4.2 (1) to  $w_1 = \lfloor s_j x_v \rfloor$  and  $w_2 = x_{v-1}$ , we obtain

$$\begin{aligned} R &= (1-b_{v-1}) \text{wt}_\lambda(\lfloor s_j x_v \rfloor \Rightarrow x_{v-1}) \\ &= (1-b_{v-1}) \text{wt}_\lambda(x_v \Rightarrow x_{v-1}) - (1-b_{v-1}) \delta_{j,0} \langle \tilde{\alpha}_j^\vee, x_v \lambda \rangle \\ &= (1-b_{v-1}) \text{wt}_\lambda(x_v \Rightarrow x_{v-1}) - (1-t_0) \delta_{j,0} \langle \tilde{\alpha}_j^\vee, x_v \lambda \rangle, \end{aligned}$$

as desired. Combining (4.9), (4.10), and (4.11), we obtain

$$(4.12) \quad U_1 = \begin{cases} \sum_{k=1}^{v-1} (1-b_k) \text{wt}_\lambda(x_{k+1} \Rightarrow x_k) - (1-t_0) \delta_{j,0} \langle \tilde{\alpha}_j^\vee, x_v \lambda \rangle & \text{if } t_0 \neq 0, \\ 0 & \text{if } t_0 = 0. \end{cases}$$

Next, we remark that the function  $H_j^\eta(t)$  is strictly decreasing on  $[t_0, t_1]$  (see Remark 2.5 (1)), which implies that  $\langle \tilde{\alpha}_j^\vee, x_k \lambda \rangle < 0$  for all  $v \leq k \leq u$ . By Corollary 4.2 (2), we have

$$\text{wt}_\lambda(\lfloor s_j x_{k+1} \rfloor \Rightarrow \lfloor s_j x_k \rfloor) = \text{wt}_\lambda(x_{k+1} \Rightarrow x_k) - \delta_{j,0} \langle \tilde{\alpha}_j^\vee, x_{k+1} \lambda \rangle + \delta_{j,0} \langle \tilde{\alpha}_j^\vee, x_k \lambda \rangle$$

for each  $m \leq k \leq u-1$ . Therefore,

$$\begin{aligned}
U_2 &= \sum_{k=v}^{u-1} (1-b_k) \text{wt}_\lambda(x_{k+1} \Rightarrow x_k) - \delta_{j,0} \underbrace{\sum_{k=v}^{u-1} (1-b_k) \langle \tilde{\alpha}_j^\vee, x_{k+1} \lambda \rangle}_{=\sum_{k=v+1}^u (1-b_{k-1}) \langle \tilde{\alpha}_j^\vee, x_k \lambda \rangle} + \delta_{j,0} \sum_{k=v}^{u-1} (1-b_k) \langle \tilde{\alpha}_j^\vee, x_k \lambda \rangle \\
&= \sum_{k=v}^{u-1} (1-b_k) \text{wt}_\lambda(x_{k+1} \Rightarrow x_k) + \delta_{j,0} (1-b_v) \langle \tilde{\alpha}_j^\vee, x_v \lambda \rangle \\
(4.13) \quad & - \delta_{j,0} \sum_{k=v+1}^{u-1} (b_k - b_{k-1}) \langle \tilde{\alpha}_j^\vee, x_k \lambda \rangle - \delta_{j,0} (1-b_{u-1}) \langle \tilde{\alpha}_j^\vee, x_u \lambda \rangle.
\end{aligned}$$

Finally, let us show that

$$(4.14) \quad U_3 = (1-b_u) \text{wt}_\lambda(x_{u+1} \Rightarrow x_u) - \delta_{j,0} (1-b_u) \langle \tilde{\alpha}_j^\vee, x_u \lambda \rangle,$$

where if  $u = s$ , then  $\text{wt}_\lambda(x_{u+1} \Rightarrow x_u)$  is understood to be 0. If  $u = s$ , then the equality obviously holds. Assume that  $u < s$ . Then, since  $\langle \tilde{\alpha}_j^\vee, x_u \lambda \rangle < 0$  and  $\langle \tilde{\alpha}_j^\vee, x_{u+1} \lambda \rangle \geq 0$  by Remark 2.5 (2), the equality (4.14) follows immediately from Corollary 4.2 (3) (applied to  $w_1 = x_{u+1}$  and  $w_2 = x_u$ ).

Substituting (4.12), (4.13), (4.14) into (4.8), we deduce that

$$\begin{aligned}
F(e_j \eta) &= - \underbrace{\sum_{k=1}^{u-1} (1-b_k) \text{wt}_\lambda(x_{k+1} \Rightarrow x_k)}_{=F(\eta)} - T \\
&+ \delta_{j,0} \underbrace{\left\{ (b_v - t_0) \langle \tilde{\alpha}_j^\vee, x_v \lambda \rangle + \sum_{k=v+1}^u (b_k - b_{k-1}) \langle \tilde{\alpha}_j^\vee, x_k \lambda \rangle \right\}}_{=:V},
\end{aligned}$$

where

$$T := \begin{cases} 0 & \text{if } t_0 \neq 0, \\ \delta_{j,0} \langle \tilde{\alpha}_j^\vee, x_1 \lambda \rangle = \delta_{j,0} \langle \tilde{\alpha}_j^\vee, \iota(\eta) \rangle & \text{if } t_0 = 0. \end{cases}$$

Here, observe that

$$V = H_j^\eta(b_u) - H_j^\eta(t_0) = H_j^\eta(t_1) - H_j^\eta(t_0) = m_j^\eta - (m_j^\eta + 1) = -1.$$

Thus we have shown that  $F$  satisfies (4.7), thereby completing the proof of Theorem 4.6.  $\square$

## 5. TAIL DEGREE FUNCTION AND TAIL ENERGY FUNCTION

Throughout this section, we fix a dominant integral weight  $\lambda \in X$ , and set  $J := \{i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0\}$ . Here we define the tail analogue of the head degree and energy function of the previous section using the Lusztig involution. In Theorem 5.5 we prove the tail analogue of the relation between the degree and energy function.

**5.1. Lusztig involution on  $\mathbb{B}(\lambda)_{\text{cl}}$ .** For each  $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$ , we define  $\eta^* : [0, 1] \rightarrow \mathbb{R} \otimes_{\mathbb{Z}} X$  by  $\eta^*(t) = \eta(1-t) - \eta(1)$  for  $t \in [0, 1]$ . We see that if  $\eta = (\mu_1, \mu_2, \dots, \mu_s; b_0, b_1, \dots, b_s)$ , then

$$(5.1) \quad \eta^* = (-\mu_s, -\mu_{s-1}, \dots, -\mu_1; 1-b_s, 1-b_{s-1}, \dots, 1-b_0);$$

note that  $-\mu_u \in W(-\lambda) = W(-w_\circ\lambda)$  for  $1 \leq u \leq s$ , where  $w_\circ$  is the longest Weyl group element. It is easily checked that  $\eta^* \in \mathbb{B}(-w_\circ\lambda)_{\text{cl}}$ . Also, we see that  $\text{wt}(\eta^*) = \eta^*(1) = -\eta(1) = -\text{wt}(\eta)$ , and from [Li, Lemma 2.1 e)] that

$$(5.2) \quad (e_j\eta)^* = f_j\eta^*, \quad (f_j\eta)^* = e_j\eta^* \quad \text{for } j \in I_{\text{af}}.$$

Let us denote by  $\sigma : I \rightarrow I$  the Dynkin diagram automorphism for  $\mathfrak{g}$  induced by the longest element  $w_\circ \in W$ , i.e.,  $w_\circ\alpha_j = -\alpha_{\sigma(j)}$  for  $j \in I$ . If we set  $\sigma(0) := 0$ , then  $\sigma : I_{\text{af}} \rightarrow I_{\text{af}}$  is the Dynkin diagram automorphism for the affine Lie algebra  $\mathfrak{g}_{\text{af}}$ . Now,  $\sigma$  acts as  $-w_\circ$  on the integral weight lattice  $X$  and also on the Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ ; note that  $\sigma(\theta) = \theta$ , and

$$\sigma s_j = s_{\sigma(j)}\sigma \quad \text{on } X \text{ and on } \mathfrak{h}$$

for  $j \in I_{\text{af}}$ . Hence there exists a group automorphism, denoted also by  $\sigma$ , of the Weyl group  $W$  such that  $\sigma(s_j) = s_{\sigma(j)}$  for all  $j \in I_{\text{af}}$ ; notice that  $\ell(\sigma(w)) = \ell(w)$  for  $w \in W$ , and  $\sigma(r_\alpha) = r_{\sigma(\alpha)}$  for  $\alpha \in \Phi_+$ . The following lemma is easily shown.

**Lemma 5.1.**

- (1) If  $w \in W^J$ , then  $\sigma(w) \in W^{\sigma(J)}$ .
- (2) Let  $0 \leq b < 1$  be a rational number. For  $w_1, w_2 \in W^J$  and  $\beta \in \Phi^+ \setminus \Phi_J^+$ ,

$$w_1 \xrightarrow{\beta} w_2 \quad \text{in } \text{QB}_{b\lambda}(W^J) \quad \iff \quad \sigma(w_1) \xrightarrow{\sigma(\beta)} \sigma(w_2) \quad \text{in } \text{QB}_{b\sigma(\lambda)}(W^{\sigma(J)}).$$

In addition, the types (i.e., Bruhat or quantum) of these two edges coincide.

For  $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$ , we define  $\sigma(\eta)$  by  $(\sigma(\eta))(t) = \sigma(\eta(t))$  for  $t \in [0, 1]$ . Then,  $\sigma(\eta) \in \mathbb{B}(\sigma(\lambda))_{\text{cl}} = \mathbb{B}(-w_\circ\lambda)_{\text{cl}}$ . Indeed, if  $\eta \in \mathbb{B}(\lambda)_{\text{cl}} = \text{QLS}(\lambda)$  is of the form

$$(5.3) \quad \eta = (x_1, x_2, \dots, x_s; b_0, b_1, \dots, b_s)$$

with  $x_1, x_2, \dots, x_s \in W^J$ , then

$$(5.4) \quad \sigma(\eta) = (\sigma(x_1), \sigma(x_2), \dots, \sigma(x_s); b_0, b_1, \dots, b_s);$$

we can easily check by using Lemma 3.4 that  $\sigma(\eta) \in \text{QLS}(\sigma(\lambda)) = \mathbb{B}(\sigma(\lambda))_{\text{cl}}$ . In addition, we have

$$\text{wt}(\sigma(\eta)) = \sigma(\text{wt}(\eta)), \quad \sigma(e_j\eta) = e_{\sigma(j)}\sigma(\eta), \quad \sigma(f_j\eta) = f_{\sigma(j)}\sigma(\eta)$$

for  $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$  and  $j \in I_{\text{af}}$ .

For each  $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$ , we set  $S(\eta) := \sigma(\eta^*)$ ; by the argument above, we see that  $S(\eta) \in \mathbb{B}(\lambda)_{\text{cl}}$ . Furthermore, it is easily checked that  $S$  is an involution on  $\mathbb{B}(\lambda)_{\text{cl}}$ , which we call the *Lusztig involution* (see also [LeS] for the affine version of the Lusztig involution in type  $C$ ) such that

$$(5.5) \quad \text{wt}(S(\eta)) = -\sigma(\text{wt}(\eta)), \quad S(e_j\eta) = f_{\sigma(j)}S(\eta), \quad S(f_j\eta) = e_{\sigma(j)}S(\eta)$$

for  $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$  and  $j \in I_{\text{af}}$ . We remark that if  $\eta$  is of the form (5.3), then

$$(5.6) \quad \begin{aligned} S(\eta) &= (\lfloor \sigma(x_s w_\circ) \rfloor, \lfloor \sigma(x_{s-1} w_\circ) \rfloor, \dots, \lfloor \sigma(x_1 w_\circ) \rfloor; 1-b_s, 1-b_{s-1}, \dots, 1-b_0) \\ &= (\lfloor w_\circ x_s \rfloor, \lfloor w_\circ x_{s-1} \rfloor, \dots, \lfloor w_\circ x_1 \rfloor; 1-b_s, 1-b_{s-1}, \dots, 1-b_0). \end{aligned}$$



**5.2. Tail degree function.** We define the tail degree function  $\text{Deg}_\lambda^{\text{tail}} : \mathbb{B}(\lambda) \rightarrow \mathbb{Z}_{\leq 0}$  by  $\text{Deg}_\lambda^{\text{tail}}(\eta) = \text{Deg}_\lambda(S(\eta))$  for  $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$ , where  $\text{Deg}_\lambda : \mathbb{B}(\lambda)_{\text{cl}} \rightarrow \mathbb{Z}_{\leq 0}$  is the degree function defined in Section 4.2.

*Remark 5.2.* Let  $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$ . As in [NS6, Proposition 3.1.3], we can show that there exists a unique  $\pi_\eta^L \in \mathbb{B}_0(\lambda)$  satisfying the conditions that  $\text{cl}(\pi_\eta^L) = \eta$  and  $\kappa(\pi_\eta^L) \in \lambda - Q_+$ , where for  $\pi = (\nu_1, \dots, \nu_s; b_0, \dots, b_s) \in \mathbb{B}(\lambda)$ , we set  $\kappa(\pi) = \nu_s$ , and call it the final direction of  $\pi$ . We see that  $\pi_\eta^L(1) \in X_{\text{af}}^0$  is of the form:  $\pi_\eta^L(1) = \lambda - \beta + K\delta$  for some  $\beta \in Q_+$  and  $K \in \mathbb{Z}_{\leq 0}$ . Then,  $K = \text{Deg}_\lambda^{\text{tail}}(\eta)$ .

**Proposition 5.3.** *Let  $\eta = (x_1, x_2, \dots, x_s; b_0, b_1, \dots, b_s) \in \mathbb{B}(\lambda)_{\text{cl}} = \text{QLS}(\lambda)$ . Then,*

$$\text{Deg}_\lambda^{\text{tail}}(\eta) = - \sum_{k=1}^{s-1} b_k \text{wt}_\lambda(x_{k+1} \Rightarrow x_k).$$

*Proof.* It follows from Theorem 4.6 and (5.6) that

$$\begin{aligned} \text{Deg}_\lambda^{\text{tail}}(\eta) &= - \sum_{k=1}^{s-1} \{1 - (1 - b_{s-k})\} \text{wt}_\lambda([w_\circ x_{s-k}] \Rightarrow [w_\circ x_{s-k+1}]) \\ &= - \sum_{k=1}^{s-1} b_k \text{wt}_\lambda([w_\circ x_k] \Rightarrow [w_\circ x_{k+1}]). \end{aligned}$$

Also, it follows from [LNSSS1, Proposition 4.3 (3)] that if the weight of a shortest directed path from  $[w_\circ x_k]$  to  $[w_\circ x_{k+1}]$  is equal to  $\xi^\vee \in Q^\vee$ , then the weight of a shortest directed path from  $x_{k+1}$  to  $x_k$  is equal to  $w_\circ^J \xi^\vee \in Q^\vee$ , where  $w_\circ^J$  is the longest element of  $W_J$ . Since  $w_\circ^J \lambda = \lambda$  by the definition of  $J$ , we have  $\text{wt}_\lambda([w_\circ x_k] \Rightarrow [w_\circ x_{k+1}]) = \text{wt}_\lambda(x_{k+1} \Rightarrow x_k)$ . Thus we have proved the proposition.  $\square$

**5.3. Tail energy function.** As in Section 4.3, write  $\lambda$  as  $\lambda = \omega_{i_1} + \omega_{i_2} + \dots + \omega_{i_p}$  with  $i_1, i_2, \dots, i_p \in I$ , and let

$$\Psi : \mathbb{B}(\lambda)_{\text{cl}} \xrightarrow{\sim} \mathbb{B}(\omega_{i_1})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_2})_{\text{cl}} \otimes \dots \otimes \mathbb{B}(\omega_{i_p})_{\text{cl}} = \mathbb{B}$$

be the isomorphism of crystals; recall again that  $\mathbb{B}(\omega_i)_{\text{cl}} \cong B^{i,1}$  as crystals. Following [LeS, Section 2.4], we define the tail energy function  $D^{\text{tail}} = D_{\mathbb{B}}^{\text{tail}} : \mathbb{B} \rightarrow \mathbb{Z}_{\leq 0}$  on  $\mathbb{B}$  as follows. For each  $1 \leq k < l \leq p$ , there exists a unique isomorphism

$$\begin{aligned} &\mathbb{B}(\omega_{i_k})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_{k+1}})_{\text{cl}} \otimes \dots \otimes \mathbb{B}(\omega_{i_{l-1}})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_l})_{\text{cl}} \\ &\xrightarrow{\sim} \mathbb{B}(\omega_{i_{k+1}})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_{k+2}})_{\text{cl}} \otimes \dots \otimes \mathbb{B}(\omega_{i_l})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_k})_{\text{cl}} \end{aligned}$$

of crystals, which is given by composition of combinatorial  $R$ -matrices. Given  $\eta_k \otimes \eta_{k+1} \otimes \dots \otimes \eta_l \in \mathbb{B}(\omega_{i_k})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_{k+1}})_{\text{cl}} \otimes \dots \otimes \mathbb{B}(\omega_{i_l})_{\text{cl}}$ , we define  $\eta_k^{(l)} \in \mathbb{B}(\omega_{i_k})_{\text{cl}}$  to be the last factor of the image of  $\eta_k \otimes \eta_{k+1} \otimes \dots \otimes \eta_l$  under the above isomorphism of crystals. In addition, for each  $1 \leq k \leq p$ , take (and fix) an arbitrary element  $\eta_k^\sharp \in \mathbb{B}(\omega_{i_k})_{\text{cl}}$  such that  $e_j \eta_k^\sharp = \mathbf{0}$  for all  $j \in I$ . Then we define the tail energy function  $D^{\text{tail}} = D_{\mathbb{B}}^{\text{tail}} : \mathbb{B} = \mathbb{B}(\omega_{i_1})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_2})_{\text{cl}} \otimes \dots \otimes \mathbb{B}(\omega_{i_p})_{\text{cl}} \rightarrow \mathbb{Z}$  by:

$$(5.7) \quad D^{\text{tail}}(\eta_1 \otimes \eta_2 \otimes \dots \otimes \eta_p) = \sum_{1 \leq k < l \leq p} H_{k,l}(\eta_k^{(l-1)} \otimes \eta_l) + \sum_{k=1}^p H_{k,k}(\eta_k^{(p)} \otimes \eta_k^\sharp).$$

Also, we define a constant  $D_{\text{ext}}^{\text{tail}} \in \mathbb{Z}$  by

$$(5.8) \quad D_{\text{ext}}^{\text{tail}} = \sum_{k=1}^p H_{k,k}(\eta_{\omega_{\circ}\omega_{i_k}} \otimes \eta_k^{\sharp}).$$

Now, set  $\tilde{\mathbb{B}} := \mathbb{B}(\omega_{i_p})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_{p-1}})_{\text{cl}} \otimes \cdots \otimes \mathbb{B}(\omega_{i_1})_{\text{cl}}$ , and define  $S : \mathbb{B} \rightarrow \tilde{\mathbb{B}}$  by:

$$(5.9) \quad S(\eta_1 \otimes \cdots \otimes \eta_p) = S(\eta_p) \otimes \cdots \otimes S(\eta_1)$$

for  $\eta_1 \otimes \cdots \otimes \eta_p \in \mathbb{B} = \mathbb{B}(\omega_{i_1})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_2})_{\text{cl}} \otimes \cdots \otimes \mathbb{B}(\omega_{i_p})_{\text{cl}}$ . Then we deduce that

$$(5.10) \quad \text{wt}(S(\boldsymbol{\eta})) = -\sigma(\text{wt}(\boldsymbol{\eta})), \quad S(e_j \boldsymbol{\eta}) = f_{\sigma(j)} S(\boldsymbol{\eta}), \quad S(f_j \boldsymbol{\eta}) = e_{\sigma(j)} S(\boldsymbol{\eta})$$

for  $\boldsymbol{\eta} \in \mathbb{B}$  and  $j \in I_{\text{af}}$ .

**Proposition 5.4** (cf. [LeS, Proposition 2.6]). *For every  $\boldsymbol{\eta} \in \mathbb{B} = \mathbb{B}(\omega_{i_1})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_2})_{\text{cl}} \otimes \cdots \otimes \mathbb{B}(\omega_{i_p})_{\text{cl}}$ , we have*

$$D_{\mathbb{B}}^{\text{tail}}(\boldsymbol{\eta}) - D_{\text{ext}}^{\text{tail}} = D_{\tilde{\mathbb{B}}}^{\text{tail}}(S(\boldsymbol{\eta})) - D_{\text{ext}}.$$

*Proof.* Observe that  $f_j S(\eta_k^{\sharp}) = \mathbf{0}$  for every  $j \in I$  by (5.5). Because the right-hand side of the equation above does not depend on the choice of  $\eta_k^{\sharp} \in \mathbb{B}(\omega_{i_k})_{\text{cl}}$  by Theorem 4.5, we may assume that  $\eta_k^{\flat} = S(\eta_k^{\sharp})$  for each  $1 \leq k \leq p$ . Also, notice that  $\eta_{\omega_{i_k}} = S(\eta_{\omega_{\circ}\omega_{i_k}})$  for  $1 \leq k \leq p$ . We can show the assertion in exactly the same way as [LeS, Proposition 2.6]. For simplicity of notation, we set  $\mathbb{B}_k := \mathbb{B}(\omega_{i_k})_{\text{cl}}$  and  $\tilde{\mathbb{B}}_k := \mathbb{B}(\omega_{i_{p-k+1}})_{\text{cl}}$  for  $1 \leq k \leq p$ ; notice that

$$\mathbb{B} = \mathbb{B}_1 \otimes \cdots \otimes \mathbb{B}_p, \quad \tilde{\mathbb{B}} = \tilde{\mathbb{B}}_1 \otimes \cdots \otimes \tilde{\mathbb{B}}_p.$$

For each  $1 \leq k < l \leq p$ , consider the following diagram:

$$(5.11) \quad \begin{array}{ccc} \mathbb{B}_k \otimes \mathbb{B}_{k+1} \otimes \cdots \otimes \mathbb{B}_l & \xrightarrow{\sim} & \mathbb{B}_{k+1} \otimes \cdots \otimes \mathbb{B}_l \otimes \mathbb{B}_k \\ \downarrow S & & \downarrow S \\ \tilde{\mathbb{B}}_{p-l+1} \otimes \cdots \otimes \tilde{\mathbb{B}}_{p-k} \otimes \tilde{\mathbb{B}}_{p-k+1} & \xrightarrow{\sim} & \tilde{\mathbb{B}}_{p-k+1} \otimes \tilde{\mathbb{B}}_{p-l+1} \otimes \cdots \otimes \tilde{\mathbb{B}}_{p-k}. \end{array}$$

Here the vertical  $S$ 's are defined in the same manner as (5.9). Because the same commutative relations as (5.10) hold for these  $S$ 's, we deduce from the connectedness of the crystals appearing in the diagram above (see Theorem 2.7 (2), (3)) that the diagram above is commutative. Also, we can show in exactly the same manner as for [LeS, (2.6)] that for each  $1 \leq k, l \leq p$ ,

$$(5.12) \quad H_{k,l}(\eta_k \otimes \eta_l) = H_{l,k}(S(\eta_l) \otimes S(\eta_k)) \quad \text{for every } \eta_k \otimes \eta_l \in \mathbb{B}_{k,l}.$$

Now, let  $\boldsymbol{\eta} = \eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_p \in \mathbb{B}$ , and set  $\tilde{\eta}_{p-k+1} := S(\eta_k) \in \tilde{\mathbb{B}}_{p-k+1}$  for  $1 \leq k \leq p$ ; note that  $S(\boldsymbol{\eta}) = \tilde{\eta}_1 \otimes \cdots \otimes \tilde{\eta}_p$ . By the commutative diagram (5.11), we see that

$$S(\eta_k^{(l)}) = \tilde{\eta}_{p-k+1}^{(p-l+1)} \quad \text{for } 1 \leq k < l \leq p.$$

Hence we obtain

$$H_{l,k}(\tilde{\eta}_{p-l+1} \otimes \tilde{\eta}_{p-k+1}^{(p-l+2)}) = H_{l,k}(S(\eta_l) \otimes S(\eta_k^{(l-1)})) = H_{k,l}(\eta_k^{(l-1)} \otimes \eta_l) \quad \text{by (5.12);}$$

remark that  $H_{l,k}$  is the local energy function on  $\tilde{\mathbb{B}}_{p-l+1} \otimes \tilde{\mathbb{B}}_{p-k+1}$ . Similarly,

$$H_{k,k}(\eta_k^{\flat} \otimes \tilde{\eta}_{p-k+1}^{(1)}) = H_{k,k}(S(\eta_k^{\sharp}) \otimes S(\eta_k^{(p)})) = H_{k,k}(\eta_k^{(p)} \otimes \eta_k^{\sharp}).$$

Furthermore, since  $S(\eta_{w_\circ\omega_{i_k}}) = \eta_{\omega_{i_k}}$ , we have

$$H_{k,k}(\eta_k^\flat \otimes \eta_{\omega_{i_k}}) = H_{k,k}(S(\eta_k^\sharp) \otimes S(\eta_{w_\circ\omega_{i_k}})) = H_{k,k}(\eta_{w_\circ\omega_{i_k}} \otimes \eta_k^\sharp).$$

Combining these, we deduce that

$$\begin{aligned} D_{\mathbb{B}}^{\text{tail}}(\boldsymbol{\eta}) &= D_{\mathbb{B}}^{\text{tail}}(\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_p) \\ &= \sum_{1 \leq k < l \leq p} H_{k,l}(\eta_k^{(l-1)} \otimes \eta_l) + \sum_{k=1}^p H_{k,k}(\eta_k^{(p)} \otimes \eta_k^\sharp) \\ &= \sum_{1 \leq k < l \leq p} H_{l,k}(\tilde{\eta}_{p-l+1} \otimes \tilde{\eta}_{p-k+1}^{(p-l+2)}) + \sum_{k=1}^p H_{k,k}(\eta_k^\flat \otimes \tilde{\eta}_{p-k+1}^{(1)}) \\ &= D_{\tilde{\mathbb{B}}}(\tilde{\eta}_1 \otimes \tilde{\eta}_2 \otimes \cdots \otimes \tilde{\eta}_p) = D_{\tilde{\mathbb{B}}}(S(\boldsymbol{\eta})) \end{aligned}$$

and

$$D_{\text{ext}}^{\text{tail}} = \sum_{k=1}^p H_{k,k}(\eta_{w_\circ\omega_{i_k}} \otimes \eta_k^\sharp) = \sum_{k=1}^p H_{k,k}(\eta_k^\flat \otimes \eta_{\omega_{i_k}}) = D_{\text{ext}}.$$

Thus we have proved the proposition.  $\square$

#### 5.4. Relation between the tail degree function and the tail energy function.

**Theorem 5.5.** *Keep the notation and setting at the beginning of Section 5.3. For each  $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$ , we have*

$$\text{Deg}_\lambda^{\text{tail}}(\eta) = D_{\mathbb{B}}^{\text{tail}}(\Psi(\eta)) - D_{\text{ext}}^{\text{tail}}.$$

*Proof.* As in the previous subsection, set  $\tilde{\mathbb{B}} := \mathbb{B}(\omega_{i_p})_{\text{cl}} \otimes \mathbb{B}(\omega_{i_{p-1}})_{\text{cl}} \otimes \cdots \otimes \mathbb{B}(\omega_{i_1})_{\text{cl}}$ , and let  $\tilde{\Psi} : \mathbb{B}(\lambda)_{\text{cl}} \xrightarrow{\sim} \tilde{\mathbb{B}}$  be the isomorphism of crystals given by Theorem 2.7 (3). By the connectedness of the crystals (see Theorem 2.7 (2), (3)) and (5.5), (5.10), we have the following commutative diagram:

$$(5.13) \quad \begin{array}{ccc} \mathbb{B}(\lambda)_{\text{cl}} & \xrightarrow{\Psi} & \mathbb{B} \\ \downarrow S & & \downarrow S \\ \mathbb{B}(\lambda)_{\text{cl}} & \xrightarrow{\tilde{\Psi}} & \tilde{\mathbb{B}} \end{array}$$

Now, we see that

$$\begin{aligned} \text{Deg}_\lambda^{\text{tail}}(\eta) &= \text{Deg}_\lambda(S(\eta)) = D_{\tilde{\mathbb{B}}}(\tilde{\Psi}(S(\eta))) - D_{\text{ext}} \quad \text{by Theorem 4.5} \\ &= D_{\tilde{\mathbb{B}}}(S(\Psi(\eta))) - D_{\text{ext}} \quad \text{by (5.13)} \\ &= D_{\mathbb{B}}^{\text{tail}}(\Psi(\eta)) - D_{\text{ext}}^{\text{tail}} \quad \text{by Proposition 5.4.} \end{aligned}$$

Thus we have proved the theorem.  $\square$

## 6. PERFECTNESS AND CLASSICAL DECOMPOSITION

The notion of perfectness plays an important role for level-zero crystals. It ensures for example that the Kyoto path model is applicable, which gives a model for highest weight affine crystals as a semi-infinite tensor product of Kirillov–Reshetikhin crystals. Let us define perfect crystals, see for example [HK]. Given a crystal  $\mathcal{B}$  and  $b \in \mathcal{B}$ , we need the definition

$$\varepsilon(b) = \sum_{i \in I_{\text{af}}} \varepsilon_i(b) \Lambda_i \quad \text{and} \quad \varphi(b) = \sum_{i \in I_{\text{af}}} \varphi_i(b) \Lambda_i$$

with  $\varepsilon_i(b)$  and  $\varphi_i(b)$  as defined in (2.7). Furthermore, denote by  $\overline{X}_{\text{af}}^{+\ell} = \{\lambda \in \overline{X}_{\text{af}}^+ \mid \text{lev}(\lambda) = \ell\}$  the set of dominant weights of level  $\ell$ , where  $\overline{X}_{\text{af}}^+ := \bigoplus_{i \in I_{\text{af}}} \mathbb{Z}_{\geq 0} \Lambda_i$ .

**Definition 6.1.** *For a positive integer  $\ell > 0$ , a crystal  $\mathcal{B}$  is called a perfect crystal of level  $\ell$ , if the following conditions are satisfied:*

- (1)  $\mathcal{B}$  is isomorphic to the crystal graph of a finite-dimensional  $U'_q(\mathfrak{g})$ -module.
- (2)  $\mathcal{B} \otimes \mathcal{B}$  is connected.
- (3) There exists a  $\lambda \in \bigoplus_{i \in I_{\text{af}}} \mathbb{Z} \Lambda_i$ , such that  $\text{wt}(\mathcal{B}) \subset \lambda + \sum_{i \in I} \mathbb{Z}_{\leq 0} \alpha_i$  and there is a unique element in  $\mathcal{B}$  of classical weight  $\lambda$ .
- (4)  $\forall b \in \mathcal{B}$ ,  $\text{lev}(\varepsilon(b)) \geq \ell$ .
- (5)  $\forall \Lambda \in \overline{X}_{\text{af}}^{+\ell}$ , there exist unique elements  $b_\Lambda, b^\Lambda \in \mathcal{B}$ , such that

$$\varepsilon(b_\Lambda) = \Lambda = \varphi(b^\Lambda).$$

We denote by  $\mathcal{B}_{\min}$  the set of minimal elements in  $\mathcal{B}$ , namely

$$\mathcal{B}_{\min} = \{b \in \mathcal{B} \mid \text{lev}(\varepsilon(b)) = \ell\}.$$

Note that condition (5) of Definition 6.1 ensures that  $\varepsilon, \varphi : \mathcal{B}_{\min} \rightarrow \overline{X}_{\text{af}}^{+\ell}$  are bijections.

Recall from Section 2.1 that  $\delta = \sum_{j \in I_{\text{af}}} a_j \alpha_j \in \mathfrak{h}_{\text{af}}^*$  and  $c = \sum_{j \in I_{\text{af}}} a_j^\vee \alpha_j^\vee \in \mathfrak{h}_{\text{af}}$ . Define  $c_r = \max\{\frac{a_r}{a_r^\vee}, a_0^\vee\}$ .

**Conjecture 6.2.** [HKOTT, Conjecture 2.1] *The Kirillov-Reshetikhin crystal  $B^{r,s}$  is perfect if and only if  $\frac{s}{c_r}$  is an integer. If  $B^{r,s}$  is perfect, its level is  $\frac{s}{c_r}$ .*

For all nonexceptional types this conjecture was proven in [FOS1]. Given the explicit models for  $B^{r,1}$  for all untwisted types in this paper and their implementation into SAGE [Sage, Sage-comb], we have verified Conjecture 6.2 also for untwisted exceptional types when  $s = 1$ . For type  $G_2^{(1)}$ , perfectness was also treated in [Y].

**Theorem 6.3.** *Conjecture 6.2 holds for  $B^{r,1}$  for types  $G_2^{(1)}, F_4^{(1)}, E_6^{(1)}, E_7^{(1)}$  for all Dynkin nodes, and type  $E_8^{(1)}$  for all nodes (except possibly 5, 8 in the labeling of [HKOTT]). In addition, the graded classical decompositions of [HKOTY, Appendix A] were verified (except for type  $E_8^{(1)}$ ).*

For the other nodes in type  $E_8^{(1)}$  the program is currently too slow to test it.

*Proof.* Point (1) of Definition 6.1 follows from Remark 2.8. Point (2) can be deduced from [Kas]. Points (3)-(5) were checked explicitly on the computer using the implementation of level-zero LS paths in SAGE [Sage, Sage-comb] (version sage-6.1 or higher), see for example

```
sage: C = CartanType(['E', 6, 1])
sage: R = RootSystem(C)
sage: La = R.weight_space().basis()
sage: LS = CrystalOfProjectedLevelZeroLSPaths(La[1])
sage: LS.is_perfect()
```

This showed that  $B^{r,1}$  is perfect

- for all nodes of type  $E_{6,7}^{(1)}$  and the nodes specified in the theorem for type  $E_8^{(1)}$ ;
- the first 2 nodes of  $F_4^{(1)}$  (long roots);
- the second node of  $G_2^{(1)}$  (long root).

This confirms the perfectness claim of the theorem. The graded classical decompositions of [HKOTY, Appendix A] were also confirmed by computer.  $\square$

## 7. THE QUANTUM ALCOVE MODEL

Now let us recall the *quantum alcove model* [LL1]. Throughout this section we refer to roots and weights in the corresponding finite lattices. Fix a dominant integral weight  $\lambda \in X$ .

**7.1. The objects of the model.** We say that two alcoves are adjacent if they are distinct and have a common wall. Given a pair of adjacent alcoves  $A$  and  $B$ , we write  $A \xrightarrow{\beta} B$  for  $\beta \in \Phi$  if the common wall is orthogonal to  $\beta$  and  $\beta$  points in the direction from  $A$  to  $B$ . Recall that alcoves are separated by hyperplanes of the form

$$H_{\beta,l} = \{\mu \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \beta^\vee, \mu \rangle = l\},$$

where  $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R} \otimes X$ . We denote by  $r_{\beta,l}$  the affine reflection in this hyperplane.

**Definition 7.1.** [LP] *An alcove path is a sequence of alcoves  $\Pi = (A_0, A_1, \dots, A_m)$  such that  $A_{j-1}$  and  $A_j$  are adjacent, for  $j = 1, \dots, m$ . We say that  $\Pi$  is reduced if it has minimal length among all alcove paths from  $A_0$  to  $A_m$ .*

Let  $A_\lambda = A_o + \lambda$  be the translation of the fundamental alcove  $A_o$  by the weight  $\lambda$ . The fundamental alcove is defined as

$$A_o = \{\mu \in \mathfrak{h}_{\mathbb{R}}^* \mid 0 < \langle \alpha^\vee, \mu \rangle < 1 \text{ for all } \alpha \in \Phi^+\}.$$

**Definition 7.2.** [LP] *The sequence of roots  $(\beta_1, \beta_2, \dots, \beta_m)$  is called a  $\lambda$ -chain if*

$$A_0 = A_o \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \dots \xrightarrow{-\beta_m} A_m = A_{-\lambda}$$

*is a reduced alcove path.*

A reduced alcove path  $\Pi = (A_0 = A_o, A_1, \dots, A_m = A_{-\lambda})$  can be identified with the corresponding total order on the hyperplanes  $H_{\beta,-l}$ , to be called  $\lambda$ -hyperplanes, which separate  $A_o$  from  $A_{-\lambda}$  (i.e., are subject to  $\beta \in \Phi^+$  and  $0 \leq l < \langle \beta^\vee, \lambda \rangle$ ); we refer here to the sequence  $H_{\beta_i, -l_i}$  for  $i = 1, \dots, m$ , where  $H_{\beta_i, -l_i}$  contains the common wall of  $A_{i-1}$  and  $A_i$ . Note also that a  $\lambda$ -chain  $(\beta_1, \dots, \beta_m)$  determines the corresponding reduced alcove path. Indeed, we can recover the corresponding sequence  $(l_1, \dots, l_m)$ , to be called the height sequence, by setting  $l_i := |\{j < i \mid \beta_j = \beta_i\}|$ . Therefore, we will sometimes refer to the sequence of  $\lambda$ -hyperplanes considered above as a  $\lambda$ -chain. Let  $r_i := r_{\beta_i}$  and  $\widehat{r}_i := r_{\beta_i, -l_i}$ .

*Remark 7.3.* An alcove path corresponds to the choice of a reduced word for the affine Weyl group element sending  $A_\circ$  to  $A_{-\lambda}$  [LP, Lemma 5.3]. Another equivalent definition of an alcove path/ $\lambda$ -chain, based on a root interlacing condition which generalizes a similar condition characterizing reflection orderings, can be found in [LP1, Definition 4.1 and Proposition 10.2].

We will work with a special choice of a  $\lambda$ -chain in [LP1, Section 4], which we now recall.

**Proposition 7.4.** [LP1] *Given a total order  $I = \{1 < 2 < \dots < r\}$  on the set of Dynkin nodes, one may express a coroot  $\beta^\vee = \sum_{i=1}^r c_i \alpha_i^\vee$  in the  $\mathbb{Z}$ -basis of simple coroots. Consider the total order on the set of  $\lambda$ -hyperplanes defined by the lexicographic order on their images in  $\mathbb{Q}^{r+1}$  under the map*

$$(7.1) \quad H_{\beta, -l} \mapsto \frac{1}{\langle \beta^\vee, \lambda \rangle} (l, c_1, \dots, c_r).$$

*This map is injective, thereby endowing the set of  $\lambda$ -hyperplanes with a total order, which is a  $\lambda$ -chain. We call it the lexicographic (lex)  $\lambda$ -chain.*

The objects of the quantum alcove model are defined next.

**Definition 7.5.** [LL1] *Given a  $\lambda$ -chain  $\Gamma = (\beta_1, \dots, \beta_m)$ , a finite subset  $A = \{j_1 < j_2 < \dots < j_s\}$  of  $[m] := \{1, \dots, m\}$  (possibly empty) is an admissible subset if we have the following path in the quantum Bruhat graph on  $W$ :*

$$(7.2) \quad 1 \xrightarrow{\beta_{j_1}} r_{j_1} \xrightarrow{\beta_{j_2}} r_{j_1} r_{j_2} \xrightarrow{\beta_{j_3}} \dots \xrightarrow{\beta_{j_s}} r_{j_1} r_{j_2} \dots r_{j_s}.$$

*The weight of  $A$  (not necessarily admissible) is defined by*

$$(7.3) \quad \text{wt}(A) := -\widehat{r}_{j_1} \dots \widehat{r}_{j_s}(-\lambda).$$

*We let  $\mathcal{A}(\Gamma)$  be the collection of all admissible subsets of  $[m]$ .*

*Remark 7.6.* If we restrict to admissible subsets for which the path (7.2) has no down steps, we recover the classical alcove model in [LP, LP1].

**7.2. Root operators in the quantum alcove model.** We continue to use the notation in Section 7. Fix a  $\lambda$ -chain  $\Gamma = (\beta_1, \dots, \beta_m)$  and the corresponding reduced alcove path  $\Pi$ . In this section, we recall from [LL1] the construction of (combinatorial) root operators in the quantum alcove model, namely on the collection  $\mathcal{A}(\Gamma)$  of admissible subsets of  $[m]$ .

Let  $A = \{j_1 < j_2 < \dots < j_s\}$  be an arbitrary subset of  $[m]$ . The elements of  $A$  are called *folding positions*. We “fold”  $\Pi$  in the hyperplanes corresponding to these positions and obtain a “folded alcove path”. Like  $\Pi$ , this can be recorded by a sequence of roots, namely  $\Gamma(A) = (\gamma_1, \gamma_2, \dots, \gamma_m)$ ; here

$$(7.4) \quad \gamma_i := r_{j_1} r_{j_2} \dots r_{j_k}(\beta_i),$$

with  $j_k$  the largest folding position less than  $i$ . We define  $\gamma_\infty := r_{j_1} r_{j_2} \dots r_{j_s}(\rho)$ . Upon folding, the hyperplane separating the alcoves  $A_{i-1}$  and  $A_i$  in  $\Pi$  is mapped to

$$(7.5) \quad H_{|\gamma_i|, -l_i^A} = \widehat{r}_{j_1} \widehat{r}_{j_2} \dots \widehat{r}_{j_k}(H_{\beta_i, -l_i}),$$

for some  $l_i^A$ , which is defined by this relation. Here we write  $|\alpha| := \text{sgn}(\alpha)\alpha$ , where  $\text{sgn}(\alpha)$  is the sign of the root  $\alpha$ .

Given  $A \subseteq [m]$  and  $\alpha \in \Phi$ , we will use the following notation:

$$I_\alpha = I_\alpha(A) := \{i \in [m] \mid \gamma_i = \pm\alpha\}, \quad \widehat{I}_\alpha = \widehat{I}_\alpha(A) := I_\alpha \cup \{\infty\},$$

and  $l_\alpha^\infty := \langle \text{sgn}(\alpha)\alpha^\vee, \text{wt}(A) \rangle$ . The following graphical representation of the heights  $l_i^A$  for  $i \in I_\alpha$  and  $l_\alpha^\infty$  is useful for defining the root operators. Let

$$\widehat{I}_\alpha = \{i_1 < i_2 < \cdots < i_n \leq m < i_{n+1} = \infty\} \quad \text{and} \quad \varepsilon_i := \begin{cases} 1 & \text{if } i \notin A, \\ -1 & \text{if } i \in A. \end{cases}$$

If  $\alpha \in \Phi^+$ , we define the continuous piecewise linear function  $g_\alpha : [0, n + \frac{1}{2}] \rightarrow \mathbb{R}$  by

$$(7.6) \quad g_\alpha(0) = -\frac{1}{2}, \quad g'_\alpha(t) = \begin{cases} \text{sgn}(\gamma_{i_k}) & \text{if } t \in (k-1, k - \frac{1}{2}), k = 1, \dots, n \\ \varepsilon_{i_k} \text{sgn}(\gamma_{i_k}) & \text{if } t \in (k - \frac{1}{2}, k), k = 1, \dots, n \\ \text{sgn}(\langle \alpha^\vee, \gamma_\infty \rangle) & \text{if } t \in (n, n + \frac{1}{2}). \end{cases}$$

If  $\alpha \in \Phi^-$ , we define  $g_\alpha$  to be the graph obtained by reflecting  $g_{-\alpha}$  in the  $x$ -axis. By [LP1][Propositions 5.3 and 5.5], for any  $\alpha$  we have

$$(7.7) \quad \text{sgn}(\alpha)l_{i_k}^A = g_\alpha\left(k - \frac{1}{2}\right), k = 1, \dots, n, \quad \text{and} \quad \text{sgn}(\alpha)l_\alpha^\infty := \langle \alpha^\vee, \text{wt}(A) \rangle = g_\alpha\left(n + \frac{1}{2}\right).$$

Let  $A$  now be an admissible subset, so  $A \in \mathcal{A}(\Gamma)$ . Let  $\delta_{i,j}$  be the Kronecker delta function. Fix  $p$  in  $I_{\text{af}}$ , so  $\tilde{\alpha}_p$  is a simple root if  $p > 0$ , or  $-\theta$  if  $p = 0$ , see (2.4). Let  $M$  be the maximum of  $g_{\tilde{\alpha}_p}$ . Let  $m$  be the minimum index  $i$  in  $\widehat{I}_{\tilde{\alpha}_p}$  for which we have  $\text{sgn}(\tilde{\alpha}_p)l_i^A = M$ . It was proved in [LL1] that, if  $M \geq \delta_{p,0}$ , then either  $m \in A$  or  $m = \infty$ ; furthermore, if  $M > \delta_{p,0}$ , then  $m$  has a predecessor  $k$  in  $\widehat{I}_{\tilde{\alpha}_p}$ , and we have  $k \notin A$ . We define

$$(7.8) \quad f_p(A) := \begin{cases} (A \setminus \{m\}) \cup \{k\} & \text{if } M > \delta_{p,0} \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Now we define  $e_p$ . Again let  $M := \max g_{\tilde{\alpha}_p}$ . Assuming that  $M > \langle \tilde{\alpha}_p^\vee, \text{wt}(A) \rangle$ , let  $k$  be the maximum index  $i$  in  $I_{\tilde{\alpha}_p}$  for which we have  $\text{sgn}(\tilde{\alpha}_p)l_i^A = M$ , and let  $m$  be the successor of  $k$  in  $\widehat{I}_{\tilde{\alpha}_p}$ . Assuming also that  $M \geq \delta_{p,0}$ , it was proved in [LL1] that  $k \in A$ , and either  $m \notin A$  or  $m = \infty$ . Define

$$(7.9) \quad e_p(A) := \begin{cases} (A \setminus \{k\}) \cup \{m\} & \text{if } M > \langle \tilde{\alpha}_p^\vee, \text{wt}(A) \rangle \text{ and } M \geq \delta_{p,0} \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

In the above definitions, we use the convention that  $A \setminus \{\infty\} = A \cup \{\infty\} = A$ . For an example, we refer to [LL1][Examples 3.6-3.7].

The following theorem about the (combinatorial) root operators was proved in [LL1].

**Theorem 7.7.** [LL1][Theorem 3.8]

- (1) If  $A$  is an admissible subset and if  $f_p(A) \neq \mathbf{0}$ , then  $f_p(A)$  is also an admissible subset. Similarly for  $e_p(A)$ . Moreover,  $f_p(A) = A'$  if and only if  $e_p(A') = A$ .
- (2) We have  $\text{wt}(f_p(A)) = \text{wt}(A) - \tilde{\alpha}_p$ . Moreover, if  $M \geq \delta_{p,0}$ , then

$$\varphi_p(A) = M - \delta_{p,0}, \quad \varepsilon_p(A) = M - \langle \tilde{\alpha}_p^\vee, \text{wt}(A) \rangle,$$

while otherwise  $\varphi_p(A) = \varepsilon_p(A) = 0$ .

*Remark 7.8.* Let  $A = \{j_1 < \cdots < j_s\}$  be an admissible subset, and  $w_i := r_{j_1}r_{j_2}\cdots r_{j_i}$ . Let  $M, m, k$  be as in the above definition of  $f_p(A)$ , assuming  $M > \delta_{p,0}$ . Now assume that  $m \neq \infty$ , and let  $a < b$  be such that

$$j_a < k < j_{a+1} < \cdots < j_b = m < j_{b+1};$$

if  $a = 0$  or  $b + 1 > s$ , then the corresponding indices  $j_a$ , respectively  $j_{b+1}$ , are missing. In the proof of Theorem 7.7 in [LL1], it was shown that  $f_p$  has the effect of changing the path in the quantum Bruhat graph

$$1 = w_0 \rightarrow \dots \rightarrow w_a \rightarrow w_{a+1} \rightarrow \dots \rightarrow w_{b-1} \rightarrow w_b \rightarrow \dots \rightarrow w_s$$

corresponding to  $A$  into the following path corresponding to  $f_p(A)$ :

$$1 = w_0 \rightarrow \dots \rightarrow w_a \rightarrow s_p w_a \rightarrow s_p w_{a+1} \rightarrow \dots \rightarrow s_p w_{b-1} = w_b \rightarrow \dots \rightarrow w_s,$$

see (2.4). The case  $m = \infty$  is similar.

## 8. THE BIJECTION BETWEEN PROJECTED LS PATHS AND THE QUANTUM ALCOVE MODEL

The main result of this section is the crystal isomorphism between the quantum LS paths of Section 3 and the quantum alcove model of Section 7 as stated in Theorem 8.10.

**8.1. The forgetful map.** Fix a dominant integral weight  $\lambda$ , and recall from Sections 3.2 and 7.1 the notation related to quantum LS paths and the quantum alcove model, respectively. We will also use the notation  $\text{QLS}(-\lambda)$  for  $\text{QLS}(-w_\circ\lambda)$ .

We will now define a forgetful map from the quantum alcove model based on a lex  $\lambda$ -chain  $\Gamma_{\text{lex}} = (\beta_1, \dots, \beta_m)$  (see Proposition 7.4), namely from  $\mathcal{A}(\lambda) := \mathcal{A}(\Gamma_{\text{lex}})$ , to the set of quantum LS paths  $\text{QLS}(-\lambda)$ . Given an index  $i \in [m]$ , we let  $t_i := l_i / \langle \beta_i^\vee, \lambda \rangle$ , where  $l_i$  is the height defined in Section 7. Note that  $0 \leq t_1 \leq t_2 \leq \dots \leq t_m$ , by the definition of  $\Gamma$ . Consider an admissible subset  $A = \{j_1 < j_2 < \dots < j_s\}$ , and let

$$\{0 = a_0 < a_1 < \dots < a_p\} := \{t_{j_1} \leq t_{j_2} \leq \dots \leq t_{j_s}\} \cup \{0\}.$$

Let  $0 = n_0 \leq n_1 < \dots < n_{p+1} = s$  be such that  $t_{j_h} = a_k$  if and only if  $n_k < h \leq n_{k+1}$ , for  $k = 0, \dots, p$ . Define Weyl group elements  $u_h$  for  $h = 0, \dots, s$  and  $w_k$  for  $k = 0, \dots, p$  by  $u_0 := 1$ ,  $u_h := r_{j_1} \dots r_{j_h}$ , and  $w_k := u_{n_{k+1}}$ . Let also  $\mu_k := w_k(\lambda)$ . For any  $k = 1, \dots, p$ , we have the following path in the quantum Bruhat graph  $\text{QB}(W)$ :

$$(8.1) \quad w_{k-1} = u_{n_k} \xrightarrow{\beta_{n_k+1}} u_{n_{k+1}} \xrightarrow{\beta_{n_k+2}} \dots \xrightarrow{\beta_{n_k+1}} u_{n_{k+1}} = w_k.$$

We claim that this is a path in  $\text{QB}_{a_k\lambda}(W)$ . Indeed, for  $n_k < h \leq n_{k+1}$ , we have

$$a_k \langle \beta_{j_h}^\vee, \lambda \rangle = l_{j_h} \in \mathbb{Z}_{\geq 0},$$

by the definition of  $a_k = t_{j_h}$ . By Lemma 8.1 below, for each edge  $u_i \rightarrow u_{i+1}$  in the path (8.1) there is a path from  $[u_i]$  to  $[u_{i+1}]$  in  $\text{QB}_{a_k\lambda}(W^J)$ . Therefore, we have  $[w_{k-1}] \xrightarrow{a_k} [w_k]$ , or equivalently  $-\mu_{k-1} \xleftarrow{a_k} -\mu_k$ . We conclude that

$$(8.2) \quad -\mu_0 \xleftarrow{a_1} -\mu_1 \xleftarrow{a_2} \dots \xleftarrow{a_p} -\mu_p$$

is an LS path in  $\text{QLS}(-\lambda)$ . We denote it by  $\Pi(A)$ , and the dual LS path defined in (5.1) by  $\Pi^*(A)$ .

We now state Lemma 8.1, which is the main ingredient in the above construction. We need the  $b$ -Bruhat order on  $W_{\text{af}}$ , denoted  $<_b$ , which is defined by a condition completely similar to (3.2) applied to the covers in  $W_{\text{af}}$ . This lemma will be proved in Section 10.1 below.

**Lemma 8.1.** *Let  $w \xrightarrow{\gamma} wr_\gamma$  be an edge in  $\text{QB}_{b\lambda}(W)$  for some  $b$ , which is viewed as a path  $\mathbf{q}$ . Then there exists a path  $\mathbf{p}$  from  $[w]$  to  $[wr_\gamma]$  in  $\text{QB}_{b\lambda}(W^J)$  (possibly of length 0), such that  $\text{wt}(\mathbf{p}) \equiv \text{wt}(\mathbf{q}) \pmod{Q_J^\vee}$ .*



*Remarks 8.2.*

- (1) The special case of the lemma corresponding to the  $b$ -Bruhat order on  $W$  and  $W^J$  (i.e., the subgraphs of  $\text{QB}_{b\lambda}(W)$  and  $\text{QB}_{b\lambda}(W^J)$  with no down edges) was proved in [LeSh, Lemma 4.16]. For  $b = 0$ , i.e., the usual Bruhat order, the latter result is well-known; see, e.g., [BB, Proposition 2.5.1].
- (2) Based on the strong connectivity of the quantum Bruhat graph (cf. Theorem 8.3 below), the lemma implies the same property for the parabolic quantum Bruhat graph. Note that this was proved by different methods (and, in fact, in a slightly stronger form) in [LNSSS1, Lemma 6.12].

**8.2. The inverse map.** Next we prove that the forgetful map in Section 8.1, from the quantum alcove model to quantum LS paths, is a bijection, by exhibiting the inverse map. We will use the *shellability* of the quantum Bruhat graph  $\text{QB}(W)$  with respect to a reflection ordering on the positive roots [Dy], which we now recall. We denote by  $\ell(v \rightarrow w)$  the length of a shortest directed path from  $v$  to  $w$  in  $\text{QB}(W)$ .

**Theorem 8.3.** [BFP] *Fix a reflection ordering on  $\Phi^+$ .*

- (1) *For any pair of elements  $v, w \in W$ , there is a unique path from  $v$  to  $w$  in the quantum Bruhat graph  $\text{QB}(W)$  such that its sequence of edge labels is strictly increasing (resp., decreasing) with respect to the reflection ordering.*
- (2) *The path in (1) has the smallest possible length  $\ell(v \rightarrow w)$  and is lexicographically minimal (resp., maximal) among all shortest paths from  $v$  to  $w$ .*

In [LeSh, Section 4.3], we constructed a reflection ordering  $<_\lambda$  on  $\Phi^+$  which depends on  $\lambda$ . The bottom of the order  $<_\lambda$  consists of the roots in  $\Phi^+ \setminus \Phi_J^+$ . For two such roots  $\alpha$  and  $\beta$ , define  $\alpha < \beta$  whenever the hyperplane  $H_{(\alpha,0)}$  precedes  $H_{(\beta,0)}$  in the lex  $\lambda$ -chain (see Proposition 7.4). This forms an *initial section* [Dy] of  $<_\lambda$ . The top of the order  $<_\lambda$  consists of the positive roots for the Weyl group  $W_J$ , and we fix any reflection ordering for them. We refer to the reflection ordering  $<_\lambda$  throughout this section.

*Remark 8.4.* Given a  $\lambda$ -hyperplane  $H_{\beta,-l}$ , we call the first component of the vector associated with it in (7.1), namely  $l/\langle \beta^\vee, \lambda \rangle$ , the *relative height* of  $H_{\beta,-l}$ . It is not hard to see that, in the lex  $\lambda$ -chain, the order on the  $\lambda$ -hyperplanes  $H_{\beta,-l}$  with the same relative height is given by the order  $<_\lambda$  on the corresponding roots  $\beta$ . We will use this fact implicitly below.

Recall from [LNSSS1, Proposition 7.2] that the distance  $\ell(v \rightarrow x)$  has a unique minimum as a function of  $x \in wW_J$ , for fixed  $v, w \in W$ . We refer also to [LNSSS1, Theorem 7.1], stating that the mentioned minimum is, in fact, the minimum of the coset  $wW_J$  with respect to the  $v$ -tilted Bruhat order  $\preceq_v$  on  $W$  [BFP]; therefore, it makes sense to denote it by  $\min(wW_J, \preceq_v)$ , although we will not use this stronger result.

**Lemma 8.5.** *Consider  $\sigma, \tau \in W^J$  and  $w_J \in W_J$ . Let  $\tau w'_J = \min(\tau W_J, \preceq_{\sigma w_J})$ .*

- (1) *There is a unique path in  $\text{QB}(W)$  from  $\sigma w_J$  to some  $x \in \tau W_J$  whose edge labels are increasing and lie in  $\Phi^+ \setminus \Phi_J^+$ . This path ends at  $\tau w'_J$ .*
- (2) *Assume that there is a path from  $\sigma$  to  $\tau$  in  $\text{QB}_{b\lambda}(W^J)$  for some  $b \in \mathbb{Q}$ . Then the path in (1) from  $\sigma w_J$  to  $\tau w'_J$  is in  $\text{QB}_{b\lambda}(W)$ .*

*Proof.* The first part is just the content of [LNSSS1, Lemmas 7.4 and 7.5], based on the results recalled above. For the second part, we start by considering a second path in  $\text{QB}(W)$  from  $\sigma w_J$  to  $\tau w'_J$ , beside the one given by (1). This path is formed by concatenating the following:

- a path from  $\sigma w_J$  to  $\sigma$  with only down edges and all edge labels in  $\Phi_J^+$  (for instance simple roots in  $\Phi_J$ );
- a path from  $\sigma$  to  $\tau$  constructed from the path in  $\text{QB}(W^J)$  between the same two elements by replacing each edge  $u \xrightarrow{\alpha} [ur_\alpha]$  with  $u \xrightarrow{\alpha} ur_\alpha$  (cf. [LNSSS1, Condition (2') in Section 4.2]) followed by a path from  $ur_\alpha$  to  $[ur_\alpha]$  with only down edges and all edge labels in  $\Phi_J^+$  (for instance simple roots in  $\Phi_J$ );
- a path from  $\tau$  to  $\tau w'_J$  with only up edges and all edge labels in  $\Phi_J^+$ .

By Theorem 8.3 (2), we know that the first of the two paths above is a shortest one (from  $\sigma w_J$  to  $\tau w'_J$ ). Furthermore, by the hypothesis, the second path is in  $\text{QB}_{b\lambda}(W)$  (any edge in  $\text{QB}(W)$  labeled by a root in  $\Phi_J^+$  is by default in  $\text{QB}_{b\lambda}(W)$ ). So we can apply Lemma 8.6 below and deduce that the first path is also in  $\text{QB}_{b\lambda}(W)$ .  $\square$

Let us now state Lemma 8.6, which will be proved in Section 10.2 below.

**Lemma 8.6.** *Consider two paths in  $\text{QB}(W)$  between some  $v$  and  $w$ . Assume that the first one is a shortest path, while the second one is in  $\text{QB}_{b\lambda}(W)$ , for some  $b$ . Then the first path is in  $\text{QB}_{b\lambda}(W)$  as well.*

We now construct the inverse of the forgetful map in Section 8.1. We begin with a quantum LS path in  $\text{QLS}(-\lambda)$ , which is written in the form

$$(8.3) \quad \sigma_0 \xrightarrow{a_1} \sigma_1 \xrightarrow{a_2} \dots \xrightarrow{a_p} \sigma_p,$$

where  $\sigma_i \in W^J$  and  $0 = a_0 < a_1 < \dots < a_p < 1$ , cf. (8.2). We will now associate with it an admissible subset (see Definition 7.5), i.e., a lex increasing sequence of  $\lambda$ -hyperplanes, and the corresponding path in  $\text{QB}(W)$  defined in (7.2).

We start by defining the sequence  $w_{-1}, w_0, \dots, w_p$  in  $W$  recursively by  $w_{-1} = 1$ , and by  $w_i = \min(\sigma_i W_J, \preceq_{w_{i-1}})$  for  $i = 0, \dots, p$ . Note that  $w_0 = \sigma_0$ . For each  $i = 0, \dots, p$ , consider the unique path in  $\text{QB}(W)$  with increasing edge labels (with respect to  $<_\lambda$ , cf. Theorem 8.3) from  $w_{i-1}$  to  $w_i$ . Note that the path corresponding to  $i = 0$  is, in fact, a saturated chain in the Bruhat order on  $W^J$ . By (8.3) and Lemma 8.5, for any  $i$ , the edges of the corresponding path are in  $\text{QB}_{a_i\lambda}(W)$  and no edge label is in  $\Phi_J^+$ . We define the path in the quantum alcove model corresponding to the quantum LS path (8.3) by concatenating the paths constructed above, for  $i = 0, \dots, p$ . The corresponding sequence of  $\lambda$ -hyperplanes is defined by associating with a label  $\beta$  in the path from  $w_{i-1}$  to  $w_i$  the  $\lambda$ -hyperplane  $(\beta, -a_i\langle\beta^\vee, \lambda\rangle)$ ; this is indeed a  $\lambda$ -hyperplane:  $a_i\langle\beta^\vee, \lambda\rangle$  is an integer, and we have  $0 \leq a_i\langle\beta^\vee, \lambda\rangle < \langle\beta^\vee, \lambda\rangle$ , as  $0 \leq a_i < 1$  and  $\beta \in \Phi^+ \setminus \Phi_J^+$ , so  $\langle\beta^\vee, \lambda\rangle > 0$ . The constructed sequence of  $\lambda$ -hyperplanes is lex increasing because the sequence  $(a_i)$  is increasing and the edge labels in the path from  $w_{i-1}$  to  $w_i$  increase (with respect to  $<_\lambda$ ). So we constructed an admissible sequence, which is now associated with the quantum LS path in (8.3).

**Proposition 8.7.** *The forgetful map  $A \mapsto \Pi^*(A)$  is a weight-preserving bijection from  $\mathcal{A}(\lambda)$  to  $\text{QLS}(\lambda)$ .*

*Proof.* We need to show that the maps in Sections 8.1 and 8.2 are mutually inverse. The crucial fact to check is that the map  $\Pi$  followed by the backward one is the identity. This follows from

the uniqueness part in Lemma 8.5 (1). To prove that the map  $\Pi^*$  preserves weights, we note that the proof of [LeSh][Proposition 4.18] carries through in our setup.  $\square$

**8.3. The crystal isomorphism of  $\mathcal{A}(\lambda)$  and  $\mathbb{B}$ .** We will now prove that, up to the  $f_0$  arrows at the end of a string, we can view  $\mathcal{A}(\lambda)$  as a model for the tensor product of KR crystals  $\mathbb{B}$  via the bijection  $\tilde{\Psi} := \Psi \circ \Pi^*$ , see (4.3).

**Definition 8.8.** *Let  $b \rightarrow f_i(b)$  be an arrow in  $\mathbb{B}$ . It is called a Demazure arrow if  $i \neq 0$ , or  $i = 0$  and  $\varepsilon_0(b) \geq 1$ . It is called a dual Demazure arrow if  $i \neq 0$ , or  $i = 0$  and  $\varphi_0(b) \geq 2$ .*

*Remark 8.9.* In the case when all of the tensor factors of  $\mathbb{B}$  are perfect crystals (see Definition 6.1), the subgraph of  $\mathbb{B}$  consisting of the dual Demazure arrows is connected. Recall the discussion in Section 6 about which column shape KR crystals are perfect.

We now state the main result of this section, relating the crystal structures in the quantum LS path model and the quantum alcove model.

**Theorem 8.10.** *Consider the root operator  $e_p$  (and the corresponding map  $\varepsilon_p$ ) for quantum LS paths, as defined in Section 2.3, and the root operator  $f_p$  in the quantum alcove model defined in Section 7.2. Given  $A$  in  $\mathcal{A}(\lambda)$ , we have  $f_p(A) \neq \mathbf{0}$  if and only if  $\varepsilon_p(\Pi(A)) > \delta_{p,0}$ ; in this case, we have*

$$e_p(\Pi(A)) = \Pi(f_p(A)).$$

*Proof.* The proof of the similar result for the classical alcove model, namely [LP1][Theorem 9.4], carries through (cf. Remark 7.6). The main fact underlying this proof is the similarity between the definition (2.6) of  $e_p$  for quantum LS paths, and the change under  $f_p$  of the relevant path in the quantum Bruhat graph, which is explained in Remark 7.8; note that in both cases the reflection  $s_p$  is applied to a segment of the corresponding path.

To be more precise, the proof is based on deforming the path  $\Pi(A)$  to a path  $\widehat{\Pi}_\varepsilon(A)$  between the same endpoints (where  $\varepsilon$  is a sufficiently small positive real number), such that the latter does not pass through the intersection of two or more  $\lambda$ -hyperplanes (here we exclude the endpoints). The path  $\widehat{\Pi}_\varepsilon(A)$  encodes the same information as the “folded alcove path” corresponding to  $A$  or, equivalently, the sequence of roots  $\Gamma(A)$ , see Section 7.2. Therefore, the action of  $e_p$  on  $\widehat{\Pi}_\varepsilon(A)$  and of  $f_p$  on  $A$  (where the latter is based on  $\Gamma(A)$ ) are equivalent. The proof concludes by taking the limit  $\varepsilon \rightarrow 0$ , under which  $\widehat{\Pi}_\varepsilon(A)$  goes to  $\Pi(A)$ .

We will now point out the additional elements in the proof. First, some results invoked in the proof of [LP1][Theorem 9.4] need to be replaced, as follows: [LP1][Corollary 6.11] with [LL1][Propositions 3.15 and 3.18], [LP1][Corollary 6.12] with [LL1][Propositions 3.16 and 3.19], and [LP1][Proposition 7.3] with Remark 7.8. Other than this, there is just one notable addition to the proof, which has to do with the case  $p = 0$ . Consider the number  $M$  in the definition of  $f_0(A)$ , and assume for the moment that  $M \geq 1$ . By the same reasoning as in [LP1], we can see that the minimum of the function  $t \mapsto \langle \tilde{\alpha}_0^\vee, \widehat{\Pi}_\varepsilon(A)(t) \rangle$  is  $-M$ . Therefore, as discussed in [LNSS2][Section 2.2], cf. also [Li][Lemma 2.1 (c)], the maximum number of times  $e_0$  can be applied to  $\widehat{\Pi}_\varepsilon(A)$  is  $M$ . Meanwhile, the maximum number of times  $f_0$  can be applied to  $A$  is  $M - 1$ , by Theorem 7.7 (2). In the remaining case, namely  $M < 1$ , we have  $f_0(A) = \mathbf{0}$ , and the minimum of the function mentioned above is 0, so  $e_0$  is not defined on  $\widehat{\Pi}_\varepsilon(A)$ . We conclude that  $f_0(A) \neq \mathbf{0}$  if and only if  $\varepsilon_0(\widehat{\Pi}_\varepsilon(A)) \geq 2$ . The rest of the argument is identical to the one in [LP1].  $\square$

*Remarks 8.11.*

(1) The forgetful map  $\Pi$  from the quantum alcove model to the quantum LS path model is a very natural map. Therefore, we think of the former model as a mirror image of the latter, via this bijection. If we use the mentioned identification to construct the non-dual Demazure arrows in the quantum alcove model, we quickly realize that, in general, the constructions are considerably more involved than (7.8)-(7.9), see [LL1][Example 4.9].

(2) Although the quantum alcove model so far misses the non-dual Demazure arrows, it has the advantage of being a discrete model. Therefore, combinatorial methods are applicable, for instance in proving the independence of the model from the choice of an initial alcove path (or  $\lambda$ -chain of roots), see below, including the application in Remark 8.14 (2). This should be compared with the subtle continuous arguments used for the similar purpose in the Littelmann path model [Li].

Based on (5.2), we immediately obtain the following corollary of Theorems 2.7, 3.3, 8.10, and Proposition 8.7.

**Corollary 8.12.** *The bijection  $\tilde{\Psi} := \Psi \circ \Pi^*$  is a weight-preserving affine crystal isomorphism from  $\mathcal{A}(\lambda)$  to the subgraph of  $\mathbb{B}$  consisting of the dual Demazure arrows.*

Recall that the set  $\mathcal{A}(\lambda) = \mathcal{A}(\Gamma_{\text{lex}})$  in Corollary 8.12 is based on a lex  $\lambda$ -chain  $\Gamma_{\text{lex}}$ . Thus, we can conjecture the following stronger version of Corollary 8.12; below we discuss more evidence, as well as future related work.

**Conjecture 8.13.** *Given any  $\lambda$ -chain  $\Gamma$ , there is a weight-preserving affine crystal isomorphism between  $\mathcal{A}(\Gamma)$  and the subgraph of  $\mathbb{B}$  consisting of the dual Demazure arrows.*

We plan to prove this conjecture in [LL2] by using Corollary 8.12 as the starting point. Then, given two  $\lambda$ -chains  $\Gamma$  and  $\Gamma'$ , we would construct a bijection between  $\mathcal{A}(\Gamma)$  and  $\mathcal{A}(\Gamma')$  preserving the dual Demazure arrows and the weights of the vertices; this would mean that the quantum alcove model does not depend on the choice of a  $\lambda$ -chain. This construction will be based on generalizing to the quantum alcove model the so-called Yang-Baxter moves in [Le1]. As a result, we would obtain a collection of a priori different bijections between  $\mathbb{B}$  and  $\mathcal{A}(\Gamma)$ .

*Remarks 8.14.*

(1) We believe that the bijections mentioned above would be identical. In fact, this would clearly be the case if all the tensor factors of  $\mathbb{B}$  are perfect crystals. Indeed, since the subgraph of  $\mathbb{B}$  consisting of the dual Demazure arrows is connected, there is no more than one isomorphism between it and  $\mathcal{A}(\Gamma)$ .

(2) In the case when all the tensor factors of  $\mathbb{B}$  are perfect crystals, a corollary of the work in [LL2] would be the following application of the quantum alcove model, cf. Remark 8.14 (1). By making specific choices for the  $\lambda$ -chains  $\Gamma$  and  $\Gamma'$ , the bijection between  $\mathcal{A}(\Gamma)$  and  $\mathcal{A}(\Gamma')$  mentioned above would give a uniform realization of the combinatorial  $R$ -matrix (i.e., the unique affine crystal isomorphism commuting factors in a tensor product of KR crystals). In fact, we believe that this statement would hold in full generality, rather than just the perfect case.

(3) In [LL1] we proved that Conjecture 8.13 is true in types  $A$  and  $C$ , for certain  $\lambda$ -chains different from the lex ones, via certain bijections constructed in [Le2]. Here we used the realization of the corresponding crystal  $\mathbb{B}$  in terms of Kashiwara-Nakashima (KN) columns [KN].

9. THE ENERGY FUNCTION IN THE QUANTUM ALCOVE MODEL AND  $P = X$ 

We use the notation in Section 7. Given the lex  $\lambda$ -chain  $\Gamma = (\beta_1, \dots, \beta_m)$  with height sequence  $(l_1, \dots, l_m)$ , we define the complementary height sequence  $(\tilde{l}_1, \dots, \tilde{l}_m)$  by  $\tilde{l}_i := \langle \beta_i^\vee, \lambda \rangle - l_i$ . In other words,  $\tilde{l}_i = |\{j \geq i \mid \beta_j = \beta_i\}|$ .

**Definition 9.1.** *Given  $A = \{j_1 < \dots < j_s\} \in \mathcal{A}(\Gamma)$ , we let  $A^- := \{j_i \in A \mid r_{j_1} \cdots r_{j_{i-1}} > r_{j_1} \cdots r_{j_i}\}$  (in other words, we record the quantum steps in the path (7.2)). We also define*

$$(9.1) \quad \text{height}(A) := \sum_{j \in A^-} \tilde{l}_j.$$

For examples, we refer to [Le2].

Our goal is to show that the bijection in Section 8.1 translates the height statistic in the quantum alcove model to the (tail) energy statistic in the quantum LS path model given in Proposition 5.3. The main result we need is the following lemma, whose proof is given in Section 10.3.

**Lemma 9.2.** *Let  $\sigma$  and  $\tau$  be in  $W^J$  and  $v \in \sigma W_J$ ,  $w \in \tau W_J$ . Consider a shortest path  $\mathbf{p}$  from  $\sigma$  to  $\tau$  in  $\text{QB}(W^J)$ , as well as a shortest path  $\mathbf{q}$  from  $v$  to  $w$  in  $\text{QB}(W)$ . Then  $\langle \text{wt}(\mathbf{p}), \lambda \rangle = \langle \text{wt}(\mathbf{q}), \lambda \rangle$ .*

We can now state our main result.

**Theorem 9.3.** *Consider an admissible subset  $A$  in  $\mathcal{A}(\lambda)$ , and the corresponding quantum LS path  $\Pi(A)$  in  $\text{QLS}(-\lambda)$ , written in the form (8.3), where  $\sigma_i \in W^J$  and  $0 = a_0 < a_1 < \dots < a_p$ . Then we have*

$$\text{height}(A) = \sum_{i=1}^p (1 - a_i) \text{wt}_\lambda(\sigma_{i-1} \Rightarrow \sigma_i).$$

*Proof.* Recall from Section 8.2 that  $A$  (in fact, the corresponding path (7.2) in  $\text{QB}(W)$ ) can be reconstructed from the quantum LS path by first defining recursively a sequence  $w_i \in \sigma_i W_J$ ,  $i = 0, \dots, p$  (and  $w_{-1} = 1$ ), and then by concatenating the unique paths  $\mathbf{q}_i$  with increasing edge labels (with respect to  $<_\lambda$ , cf. Section 8.2) between  $w_{i-1}$  and  $w_i$ , for  $i = 0, \dots, p$ . By Theorem 8.3 (2), the paths  $\mathbf{q}_i$  are shortest ones. Therefore, by Lemma 9.2, we have

$$(9.2) \quad \text{wt}_\lambda(\sigma_{i-1} \Rightarrow \sigma_i) = \langle \text{wt}(\mathbf{q}_i), \lambda \rangle \quad \text{for } i = 1, \dots, p;$$

we also have  $\text{wt}(\mathbf{q}_0) = 0$ .

Consider a quantum edge in some path  $\mathbf{q}_i$ , and let  $\beta_j$  be the root in the lex  $\lambda$ -chain labeling it (so  $j \in A^-$ ). As discussed in Section 8.2, we have

$$(9.3) \quad a_i \langle \beta_j^\vee, \lambda \rangle = l_j, \quad \text{so } (1 - a_i) \langle \beta_j^\vee, \lambda \rangle = \tilde{l}_j.$$

By noting that  $\text{wt}(\mathbf{q}_i)$  is the sum of  $\beta_j^\vee$  for all such  $\beta_j$ , and by combining (9.1), (9.2), and (9.3), the statement of the theorem follows.  $\square$

**Corollary 9.4.** *Keep the notation of Sections 5 and 8.3. For each  $A \in \mathcal{A}(\lambda)$ , we have*

$$D_{\mathbb{B}}^{\text{tail}}(\tilde{\Psi}(A)) - D_{\text{ext}}^{\text{tail}} = -\text{height}(A).$$

*Proof.* The statement follows directly from Proposition 5.3, Theorem 5.5, and Theorem 9.3, where we also use the duality between  $\text{QLS}(\lambda)$  and  $\text{QLS}(-\lambda)$  via the sign change and reversal map  $t \mapsto 1 - t$  of Section 5.1.  $\square$

We conjecture the following strengthening of Corollary 9.4, cf. Conjecture 8.13. This will be addressed in [LL2], in the setup discussed previously in connection to Conjecture 8.13.

**Conjecture 9.5.** *Corollary 9.4 holds for  $\mathcal{A}(\Gamma)$  where  $\Gamma$  is an arbitrary  $\lambda$ -chain, with  $\tilde{\Psi}$  replaced with one of the isomorphisms in Conjecture 8.13.*

*Remark 9.6.* In [LeS] the energy function in types  $A$  and  $C$  was realized in terms of a statistic in the model based on KN columns, which is known as charge. Furthermore, in [Le2] it was shown that this statistic is the translation of the height statistic via the bijections constructed there (also mentioned in Remark 8.14 (3)), between the corresponding quantum alcove model and models based on KN columns. This should be compared with Corollary 9.4 and Conjecture 9.5, where the constant  $D_{\text{ext}}^{\text{tail}}$  is 0 in these cases.

The following is due to Ion [Ion, Thm. 4.2] for the dual of an untwisted affine root system.

**Lemma 9.7.** *For  $\lambda$  dominant,*

$$(9.4) \quad P_\lambda(x; q, 0) = E_{w_\circ\lambda}(x; q, 0)$$

where  $E_\mu$  is the nonsymmetric Macdonald polynomial [Ma2].

*Proof.* Applying [Ma2, (5.7.8)] and its notation, at  $t = 0$  we have  $\xi_\mu \rightarrow 0$  if  $\mu$  is not the unique antidominant element  $w_\circ\lambda$  in the finite Weyl group orbit of  $\lambda$ . Indeed, letting  $v(\mu) = r_{i_1}r_{i_2}\cdots r_{i_p}$  be a reduced expression of the shortest element  $v(\mu)$  in the finite Weyl group such that  $v(\mu)\mu$  is antidominant, we obtain

$$\xi_\mu = \prod_{k=1}^p \frac{tq^{-\langle \beta_k^\vee, \mu \rangle} - t^{-\langle v(\mu)\beta_k^\vee, \rho \rangle}}{q^{-\langle \beta_k^\vee, \mu \rangle} - t^{-\langle v(\mu)\beta_k^\vee, \rho \rangle}},$$

where  $\beta_k := r_{i_p}\cdots r_{i_{k+1}}\alpha_{i_k}$  for  $1 \leq k \leq p$ ; here we note that  $\langle \beta_k^\vee, \mu \rangle > 0$  and  $\langle v(\mu)\beta_k^\vee, \rho \rangle < 0$  for all  $1 \leq k \leq p$  since the elements  $\beta_k$ ,  $1 \leq k \leq p$ , comprise the inversion set  $S(v(\mu))$  for  $v(\mu)$ .  $\square$

**Proposition 9.8.** *For  $\lambda$  dominant,*

$$(9.5) \quad P_\lambda(x; q, 0) = \sum_{A \in \mathcal{A}(\lambda)} q^{\text{height}(A)} x^{\text{wt}(A)}.$$

*Proof.* By Lemma 9.7 and restating [OS, Corollary 4.4] in our setting for the quantum alcove model of Section 7.1, we obtain

$$P_\lambda(x; q, 0) = \sum_{A \in \mathcal{A}(-w_\circ\lambda)} q^{\text{height}(A)} x^{-\text{wt}(A)}.$$

Since  $P_\lambda(x; q, t)$  is symmetric it follows that

$$P_\lambda(x; q, 0) = \sum_{A \in \mathcal{A}(-w_\circ\lambda)} q^{\text{height}(A)} x^{-w_\circ \text{wt}(A)}.$$

It suffices to show that

$$(9.6) \quad \sum_{A \in \mathcal{A}(-w_\circ\lambda)} q^{\text{height}(A)} x^{-w_\circ \text{wt}(A)} = \sum_{A \in \mathcal{A}(\lambda)} q^{\text{height}(A)} x^{\text{wt}(A)}.$$

Recall from Proposition 8.7 that the forgetful map  $\Pi^* : \mathcal{A}(\lambda) \rightarrow \text{QLS}(\lambda)$  is a weight-preserving bijection. Using the duality between  $\text{QLS}(\lambda)$  and  $\text{QLS}(-\lambda)$  via the sign change and reversal map  $t \mapsto 1 - t$  of §5.1, we see from Theorem 9.3 and Proposition 5.3 that

$$-\text{height}(A) = \text{Deg}_\lambda^{\text{tail}}(\Pi^*(A)) \quad \text{for all } A \in \mathcal{A}(\lambda).$$

From this we deduce that the right hand side of (9.6) is equal to

$$(9.7) \quad \sum_{A \in \mathcal{A}(\lambda)} q^{\text{height}(A)} x^{\text{wt}(A)} = \sum_{A \in \mathcal{A}(\lambda)} q^{-\text{Deg}_\lambda^{\text{tail}}(\Pi^*(A))} x^{\text{wt}(\Pi^*(A))} = \sum_{\eta \in \text{QLS}(\lambda)} q^{-\text{Deg}_\lambda^{\text{tail}}(\eta)} x^{\text{wt}(\eta)}.$$

Similarly, the left hand side of (9.6) is equal to

$$(9.8) \quad \sum_{A \in \mathcal{A}(-w_\circ \lambda)} q^{\text{height}(A)} x^{-w_\circ \text{wt}(A)} = \sum_{\eta \in \text{QLS}(-w_\circ \lambda)} q^{-\text{Deg}_{-w_\circ \lambda}^{\text{tail}}(\eta)} x^{-w_\circ \text{wt}(\eta)}.$$

Let  $\sigma$  be the Dynkin diagram automorphism defined by  $-w_\circ \alpha_i = \alpha_{\sigma(i)}$  for all  $i \in I$ . We obtain a bijection  $\sigma : \text{QLS}(\lambda) \rightarrow \text{QLS}(\sigma(\lambda))$ ,  $\eta \mapsto \sigma(\eta)$  with  $\sigma(\eta)$  defined by (5.4); note that  $\text{wt}(\sigma(\eta)) = \sigma(\text{wt}(\eta))$  for all  $\eta \in \text{QLS}(\lambda)$ . Writing  $\eta \in \text{QLS}(\lambda)$  as  $\eta = (x_1, \dots, x_s; b_0, b_1, \dots, b_s)$ , then from Proposition 5.3 and Lemma 5.1 we have

$$\begin{aligned} \text{Deg}_\lambda^{\text{tail}}(\eta) &= - \sum_{k=1}^{s-1} b_k \text{wt}_\lambda(x_{k+1} \Rightarrow x_k) \\ &= - \sum_{k=1}^{s-1} b_k \text{wt}_{\sigma(\lambda)}(\sigma(x_{k+1}) \Rightarrow \sigma(x_k)) = \text{Deg}_{\sigma(\lambda)}^{\text{tail}}(\sigma(\eta)). \end{aligned}$$

Using this relation and the equality  $\sigma(\lambda) = -w_\circ \lambda$ , we compute the right-hand side of (9.7) as follows:

$$\begin{aligned} \sum_{\eta \in \text{QLS}(\lambda)} q^{-\text{Deg}_\lambda^{\text{tail}}(\eta)} x^{\text{wt}(\eta)} &= \sum_{\eta \in \text{QLS}(\lambda)} q^{-\text{Deg}_{\sigma(\lambda)}^{\text{tail}}(\sigma(\eta))} x^{\sigma(\text{wt}(\sigma(\eta)))} \\ &= \sum_{\eta \in \text{QLS}(\sigma(\lambda))} q^{-\text{Deg}_{\sigma(\lambda)}^{\text{tail}}(\eta)} x^{\sigma \text{wt}(\eta)} = \sum_{\eta \in \text{QLS}(-w_\circ \lambda)} q^{-\text{Deg}_{-w_\circ \lambda}^{\text{tail}}(\eta)} x^{-w_\circ \text{wt}(\eta)}, \end{aligned}$$

which is just the right-hand side of (9.8).  $\square$

*Remark 9.9.*

- (1) The formula (9.5) holds for any  $\lambda$ -chain.
- (2) Proposition 9.8 is stated in [Le2, Thm. 2.6, Prop. 2.7] but the above argument is needed to complete its proof.

Now define the graded character corresponding to the KR crystal  $\mathbb{B}$  (see for example [HKOTT, HKOTY]) by

$$(9.9) \quad X_\lambda(x; q) := \sum_{b \in \mathbb{B}} q^{D_{\mathbb{B}}^{\text{tail}}(b) - D_{\text{ext}}^{\text{tail}} \text{wt}(b)},$$

where  $\text{wt}(b)$  is the weight of the crystal element  $b$ . From Proposition 8.7 and Corollary 9.4, we immediately derive our main result.

**Corollary 9.10.** *We have*

$$P_\lambda(x; q^{-1}, 0) = X_\lambda(x; q).$$

*Remark 9.11.* In type  $A$ , the Macdonald polynomial at  $t = 0$  can be expanded in terms of Schur functions with Kostka-Foulkes polynomials as the transition matrix [Ma, Chapter III.6]. These in turn can be expressed as one-dimensional configuration sums  $X$  [NY], which implies the  $P = X$  result in type  $A$ . In all simply-laced types it was known by combining the results in [Ion] and [FL], which equate a certain affine Demazure character with  $P$  and  $X$ , respectively. It was also known in type  $C$  by [Le2, LeS].

## 10. PROOFS OF THE LEMMAS IN SECTIONS 8.1, 8.2, AND 9

**10.1. Proof of Lemma 8.1.** In the proof of this lemma, a dotted (resp. plain) edge represents a quantum (resp. Bruhat) edge in  $\text{QB}(W)$  or  $\text{QB}(W^J)$ , while a dashed edge can be of both types. Define  $\beta \in \Phi^{\text{af}+}$  by

$$(10.1) \quad \beta := \begin{cases} w\gamma & \text{if } w\gamma \in \Phi^+ \\ \delta + w\gamma & \text{if } w\gamma \in \Phi^- . \end{cases}$$

As in the proof of one of the main results in [LNSSS1], namely Theorem 6.5 (more precisely, the converse statement), we proceed by induction on the height of  $\beta$  (i.e., the sum of the coefficients in its expansion in the basis of affine simple roots). The base case, when  $\beta$  is an affine simple root, is treated in the following lemma.

**Lemma 10.1.** *In  $\text{QB}_{b\lambda}(W)$  we have an edge  $w \xrightarrow{w^{-1}\alpha} r_\alpha w$  for a finite simple root  $\alpha$  with  $w^{-1}\alpha \notin \Phi_J$  (resp.  $w \xrightarrow{w^{-1}\theta} r_\theta w$ , where  $w^{-1}\theta \notin \Phi_J$ ) if and only if in  $\text{QB}_{b\lambda}(W^J)$  we have  $[w] \xrightarrow{[w]^{-1}\alpha} r_\alpha [w]$  (resp.  $[w] \xrightarrow{[w]^{-1}\theta} [r_\theta w]$ ).*

*Proof.* Let us first ignore the parameter  $b$  (or just assume  $b = 0$ ). By the trichotomy of cosets in [LNSSS1, Propositions 5.10 and 5.11], there is a simple way to test whether we have the mentioned edges in  $\text{QB}(W)$ , namely  $w^{-1}(\eta) \in \Phi^\pm \setminus \Phi_J^\pm$ , where  $\eta$  is the simple root  $\alpha$  or  $\theta$ , respectively; similarly for the mentioned edges in  $\text{QB}(W^J)$ , with  $w$  replaced by  $[w]$ . The proof is completed by noting that

$$w^{-1}\eta \in \Phi^\pm \setminus \Phi_J^\pm \Leftrightarrow [w]^{-1}\eta \in \Phi^\pm \setminus \Phi_J^\pm ,$$

where  $\eta$  can be any root, in fact; indeed, writing  $w = [w]w_J$ , we have  $[w]^{-1} = w_J w^{-1}$ , and we know that the elements of  $W_J$  permute  $\Phi^\pm \setminus \Phi_J^\pm$ . For an arbitrary  $b$  (and  $\eta$ ), we observe that

$$b\langle w^{-1}\eta^\vee, \lambda \rangle = b\langle \eta^\vee, w\lambda \rangle = b\langle \eta^\vee, [w]\lambda \rangle = b\langle [w]^{-1}\eta^\vee, \lambda \rangle .$$

□

We need the following result from [LNSSS1], which we recall.

**Lemma 10.2.** [LNSSS1, Lemma 6.10] *Let  $w \in W$ , and let  $\gamma \in \Phi^+ \setminus \Phi_J^+$ . Define  $\beta \in \Phi^{\text{af}+}$  as in (10.1). There exists an affine simple root  $\alpha$  (in fact,  $\alpha \neq \alpha_0$  if  $w\gamma \in \Phi^+$ ) such that  $\langle \alpha^\vee, \beta \rangle > 0$ , and we have the edge in  $\text{QB}(W^J)$  indicated either in case (1) or (2) below, where  $z$  is defined by  $r_\theta [wr_\gamma] = [r_\theta [wr_\gamma]]z = [r_\theta wr_\gamma]z$ :*

$$(1) \quad \begin{cases} [w] \xrightarrow{[w]^{-1}\alpha} r_\alpha [w] & \text{if } \alpha \neq \alpha_0 \\ [w] \xrightarrow{[w]^{-1}\theta} [r_\theta w] & \text{if } \alpha = \alpha_0 , \end{cases} \quad (2) \quad \begin{cases} r_\alpha [wr_\gamma] \xrightarrow{[wr_\gamma]^{-1}\alpha} [wr_\gamma] & \text{if } \alpha \neq \alpha_0 \\ [r_\theta wr_\gamma] \xrightarrow{z[wr_\gamma]^{-1}\theta} [wr_\gamma] & \text{if } \alpha = \alpha_0 . \end{cases}$$



We also need the following lemma.

**Lemma 10.3.** *Consider any one of the diamonds in the parabolic quantum Bruhat graph  $\text{QB}(W^J)$  listed in [LNSS1, Lemma 5.14]. If one of the two paths (of length 2) is in  $\text{QB}_{b\lambda}(W^J)$ , for some fixed  $b$ , then the other one is too.*

*Proof.* By [LNSS1, Lemma 5.14], we know that, up to sign and left multiplication by elements of  $W_J$ , the pairs of labels on the two paths are  $\{w^{-1}\eta, \gamma\}$  and  $\{\gamma, r_\gamma w^{-1}\eta\}$ , for some  $\gamma \in \Phi^+ \setminus \Phi_J^+$  and  $w \in W^J$ , while  $\eta$  is a finite simple root or  $\theta$ . The equivalence of the integrality conditions with respect to  $b$  for the two pairs follows from the simple calculation

$$b\langle r_\gamma w^{-1}\eta^\vee, \lambda \rangle = b\langle w^{-1}\eta^\vee + c\gamma^\vee, \lambda \rangle = b\langle w^{-1}\eta^\vee, \lambda \rangle + c(b\langle \gamma^\vee, \lambda \rangle),$$

where  $c$  is an integer. On another hand, it is clear that mapping roots via elements of  $W_J$  preserves the integrality condition.  $\square$

*Proof of Lemma 8.1.* We can assume that  $\gamma \notin \Phi_J$ , as otherwise the statement is obvious. As stated above, we proceed by induction on the height of the affine root  $\beta$ . If  $\beta$  is an affine simple root, the conclusion follows directly from Lemma 10.1. Otherwise, we apply Lemma 10.2 for  $\text{QB}(W^J)$ ; this gives an affine simple root  $\alpha$  satisfying

$$(10.2) \quad \alpha \neq \beta, \quad \langle \alpha^\vee, \beta \rangle > 0,$$

and either condition (1) or (2) in the mentioned lemma. Assume that condition (1) holds, as the reasoning is completely similar if condition (2) holds. By Lemma 10.1, we have

$$(10.3) \quad \begin{array}{ll} w \xrightarrow{w^{-1}\alpha} r_\alpha w & \text{if } \alpha \neq \alpha_0, \text{ where } w^{-1}\alpha \notin \Phi_J, \text{ and} \\ w \xrightarrow{-w^{-1}\theta} r_\theta w & \text{if } \alpha = \alpha_0, \text{ where } w^{-1}\theta \notin \Phi_J. \end{array}$$

By Lemma 10.2, we have one of the following three cases:

$$(10.4) \quad (\beta \in \Phi^+, \alpha \neq \alpha_0), \quad (\beta \in \delta - \Phi^+, \alpha \neq \alpha_0), \quad (\beta \in \delta - \Phi^+, \alpha = \alpha_0).$$

By [LNSS1, Lemma 5.14], known as the ‘‘diamond lemma’’, we have the left diamonds in [LNSS1, Eqs. (5.3), (5.4), and (5.7)], respectively. Note that all the necessary conditions for applying the diamond lemma are checked as in the proof of the converse statement of [LNSS1, Theorem 6.5]. We can represent the diamonds in the three cases (10.4) using the single diagram below, where  $\eta := \alpha$  if  $\alpha \neq \alpha_0$ , and  $\eta := \theta$ , otherwise.

$$(10.5) \quad \begin{array}{ccc} r_\eta w - \overset{\gamma}{\succ} & r_\eta w r_\gamma \\ \uparrow & \uparrow \\ |w^{-1}\eta| & |r_\gamma w^{-1}\eta| \\ \uparrow & \uparrow \\ w - \overset{\gamma}{\succ} & w r_\gamma \end{array}$$

Recall that the bottom edge is viewed as a path  $\mathbf{q}$ ; similarly, we view the top edge as a path  $\mathbf{q}'$ , and we clearly have  $\text{wt}(\mathbf{q}) = \text{wt}(\mathbf{q}')$ .

Define  $\beta'$  for the top edge of the diamond (10.5) in the same way as  $\beta$  was defined for the bottom one in (10.1). As in the proof of the converse statement of [LNSS1, Theorem 6.5], we can check in all three cases (10.4) that  $\beta' = r_\alpha \beta$ . Since  $\langle \alpha^\vee, \beta \rangle > 0$ , this implies that the height



In this case, the edge  $x_i \dashrightarrow y_i$  is in  $\text{QB}_{b\lambda}(W^J)$ , because it coincides with the edge  $y_{i-1} \dashrightarrow y_i$ , which has this property by the induction hypothesis, cf. (10.6). By Lemma 10.3, all vertical edges are also in  $\text{QB}_{b\lambda}(W^J)$ , in particular the rightmost one. By Lemma 10.1, the edge  $wr_\gamma \dashrightarrow r_\eta wr_\gamma$  in (10.5) is in  $\text{QB}_{b\lambda}(W)$ . By applying Lemma 10.3 to (10.5) this time, we conclude that the edge  $w \dashrightarrow r_\eta w$  is in  $\text{QB}_{b\lambda}(W)$ . But we showed in (10.3) that  $w^{-1}\eta \notin \Phi_J$ , so by Lemma 10.1 the edge  $[w] \dashrightarrow [r_\eta w]$  is in  $\text{QB}_{b\lambda}(W^J)$ . We now define  $\mathbf{p}$  as the following path in  $\text{QB}_{b\lambda}(W^J)$ :

$$\mathbf{p} : x_0 = [w] \dashrightarrow [r_\eta w] = y_0 \dashrightarrow \cdots \dashrightarrow y_{i-1} = x_i \dashrightarrow \cdots \dashrightarrow x_n = [wr_\gamma] .$$

We then prove that  $\text{wt}(\mathbf{p}) \equiv \text{wt}(\mathbf{q}) \pmod{Q_J^\vee}$  in a completely similar way to Case 1, which concludes the induction step.

**Case 3.** The last case to consider is the one when  $r_\gamma w^{-1}\eta \in \Phi_J$ . We still have the edge

$$wr_\gamma \dashrightarrow \overset{|r_\gamma w^{-1}\eta|}{\dashrightarrow} r_\eta wr_\gamma$$

in  $\text{QB}_{b\lambda}(W)$ , because  $\langle r_\gamma w^{-1}\eta, \lambda \rangle = 0$ . So we can reason as in the previous paragraph in order to prove that the edge  $[w] \dashrightarrow [r_\eta w]$  is in  $\text{QB}_{b\lambda}(W^J)$ . We now define  $\mathbf{p}$  as the following path in  $\text{QB}_{b\lambda}(W^J)$ :

$$\mathbf{p} : [w] \dashrightarrow [r_\eta w] = y_0 \dashrightarrow y_1 \dashrightarrow \cdots \dashrightarrow y_n = x_n = [wr_\gamma] .$$

Note that this is the only case when the induction step produces a path of a different length (more precisely, longer by 1) based on the path in the induction hypothesis.

Now let us prove that  $\text{wt}(\mathbf{p}) \equiv \text{wt}(\mathbf{q}) \pmod{Q_J^\vee}$ . If  $\eta \neq \theta$ , then the first edge of  $\mathbf{p}$  is a Bruhat edge, so  $\text{wt}(\mathbf{p}) = \text{wt}(\mathbf{p}')$ . Applying the induction hypothesis and the fact that  $\text{wt}(\mathbf{q}) = \text{wt}(\mathbf{q}')$  concludes the induction step in this case. If  $\eta = \theta$ , then by the same reasoning as in Case 1, we deduce

$$\text{wt}(\mathbf{p}') = \text{wt}(\mathbf{q}') = 0, \quad \text{wt}(\mathbf{q}) = \gamma^\vee, \quad w\gamma \in \Phi^- .$$

We conclude that  $\text{wt}(\mathbf{p}) \equiv -w^{-1}\theta^\vee \pmod{Q_J^\vee}$  (cf. Case 1), so we need to prove that  $-w^{-1}\theta^\vee \equiv \gamma^\vee \pmod{Q_J^\vee}$ . This follows from

$$\Phi_J^\vee \ni r_\gamma w^{-1}\theta^\vee = w^{-1}\theta^\vee - \underbrace{\langle w^{-1}\theta^\vee, \gamma \rangle}_{=-1} \gamma^\vee = w^{-1}\theta^\vee + \gamma^\vee .$$

Here we derived the fact that  $\langle w^{-1}\theta^\vee, \gamma \rangle = -1$  as in Case 1.  $\square$

**10.2. Proof of Lemma 8.6.** We require some notation and results from [LS]. Let  $Q$  be the finite root lattice,  $Q^\vee$  the coroot lattice,  $Q^{\vee+} = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i^\vee$  the positive cone of coroots, and  $W_{\text{af}}^-$  the set of minimum coset representatives in  $W_{\text{af}}/W$ . We have  $W_{\text{af}} \cong W \ltimes Q^\vee$ ; denote by  $t_\mu$  the image of  $\mu \in Q^\vee$  in  $W_{\text{af}}$ . Write  $y \lessdot x$  for the covering relation in the (strong) Bruhat order on  $W_{\text{af}}$ . For an affine real root  $\beta$  let  $r_\beta \in W_{\text{af}}$  be the associated reflection and write  $r_i$  for  $\beta = \alpha_i$  a simple root. For  $M \in \mathbb{Z}_{>0}$  say that  $\lambda \in Q^\vee$  is  $M$ -superantidominant if  $\langle \lambda, \alpha \rangle \leq -M$  for every positive root  $\alpha$ . We fix once and for all a sufficiently large  $M \in \mathbb{Z}_{>0}$  ( $M = 2|W| + 2$  is sufficient).

**Lemma 10.4.** [LS, Lem. 3.3] *Let  $w \in W$  and  $\lambda \in Q^\vee$ . Then  $wt_\lambda \in W_{\text{af}}^-$  if and only if  $\lambda$  is antidominant (that is,  $\langle \lambda, \alpha_i \rangle \leq 0$  for all  $i \in I$ ) and  $w \in W^\lambda$  where  $W^\lambda$  is the set of minimum length coset representatives for  $W/W_\lambda$  and  $W_\lambda$  is the stabilizer in  $W$  of  $\lambda$ .*

**Proposition 10.5.** [LS, Prop. 4.4] *Let  $\lambda \in Q^\vee$  be superantidominant and let  $x = wt_{v\lambda}$  with  $v, w \in W$ . Then  $y = xr_{v\alpha+n\delta} < x$  if and only if one of the following conditions holds:*

- (i)  $\ell(wv) = \ell(wvr_\alpha) - 1$  and  $n = \langle \lambda, \alpha \rangle$ , giving  $y = wr_{v\alpha}t_{v\lambda}$ ;
- (ii)  $\ell(wv) = \ell(wvr_\alpha) + \langle \alpha^\vee, 2\rho \rangle - 1$  and  $n = \langle \lambda, \alpha \rangle + 1$ , giving  $y = wr_{v\alpha}t_{v(\lambda+\alpha^\vee)}$ ;
- (iii)  $\ell(v) = \ell(vr_\alpha) + 1$  and  $n = 0$ , giving  $y = wr_{v\alpha}t_{vr_\alpha\lambda}$ ;
- (iv)  $\ell(v) = \ell(vr_\alpha) - \langle \alpha^\vee, 2\rho \rangle + 1$  and  $n = -1$ , giving  $y = wr_{v\alpha}t_{vr_\alpha(\lambda+\alpha^\vee)}$ .

We start with the following lemma.

**Lemma 10.6.** *Assume that in  $W_{\text{af}}^-$  we have*

$$vt_\mu > wt_\nu, \quad vt_\mu >_b wt_{\nu'}, \quad \text{where } \nu' - \nu \in Q^{\vee+},$$

*and  $\mu, \nu, \nu' \in Q^\vee$  are superantidominant. Then  $vt_\mu >_b wt_\nu$ , and in fact any saturated chain between these elements is a chain in  $b$ -Bruhat order.*

*Proof.* We claim that  $wt_\nu > wt_{\nu'}$  using a downward chain in  $W_{\text{af}}^-$ . It suffices to prove this when  $\nu' - \nu = \alpha_i^\vee$  for some  $i \in I$ . Suppose this is the case. Suppose first that  $wr_i < w$ . By Proposition 10.5 we have  $wt_\nu > wr_it_{\nu+\alpha_i^\vee} > wt_{\nu+\alpha_i^\vee}$  as required. Otherwise we have  $wr_i > w$ . Then by Proposition 10.5 we have  $wt_\nu > wr_it_\nu > wt_{\nu+\alpha_i^\vee}$  as required.

Knowing this, using Proposition 10.5 we pick a downward saturated chain from  $vt_\mu$  to  $wt_\nu$ , followed by one from  $wt_\nu$  to  $wt_{\nu'}$ , all in  $W_{\text{af}}^-$ . By the hypothesis, there is a downward saturated chain in  $b$ -Bruhat order from  $vt_\mu$  to  $wt_{\nu'}$ . By [LeSh, Lemma 4.15], we know that the first chain is in  $b$ -Bruhat order too, which concludes the proof.  $\square$

*Proof of Lemma 8.6.* By Proposition 10.5 we can lift both paths to downward saturated chains in  $W_{\text{af}}^-$  starting at  $vt_\mu$ , where  $\mu$  is a fixed superantidominant weight. Denote the endpoints of the two chains by  $wt_{\mu+\kappa}$  and  $wt_{\mu+\kappa'}$ , respectively. Recall that  $\kappa$  and  $\kappa'$  are the sums of the coroots corresponding to (the labels of) the down steps in the paths in  $\text{QB}(W)$  which are lifted. Since the first path in  $\text{QB}(W)$  is a shortest one, by [Po, Lemma 1], we have  $\kappa' - \kappa \in Q^{\vee+}$ . Furthermore, by the hypothesis, the second chain in  $W_{\text{af}}^-$  is in  $b$ -Bruhat order. Thus the hypotheses of Lemma 10.6 are all satisfied, so we conclude that the first chain in  $W_{\text{af}}^-$  is also in  $b$ -Bruhat order, and therefore the first path in  $\text{QB}(W)$  is in  $\text{QB}_{b\lambda}(W)$ .  $\square$

**10.3. Proof of Lemma 9.2.** Let us first recall Proposition 4.1, which is the parabolic generalization of a lemma due to Postnikov [Po].

*Proof of Lemma 9.2.* By Lemma 8.1, we can construct a path from  $\mathbf{p}'$  from  $\sigma$  to  $\tau$  in  $\text{QB}(W^J)$  with

$$(10.9) \quad \text{wt}(\mathbf{p}') \equiv \text{wt}(\mathbf{q}) \pmod{Q_J^\vee};$$

namely, we simply concatenate the paths in  $\text{QB}(W^J)$  that correspond, by the mentioned lemma, to each edge of  $\mathbf{q}$ , cf. the construction of the forgetful map in Section 8.1. By Lemma 4.1, we have

$$(10.10) \quad \langle \text{wt}(\mathbf{p}'), \lambda \rangle \geq \langle \text{wt}(\mathbf{p}), \lambda \rangle.$$

We then exhibit a path  $\mathbf{q}'$  from  $v$  to  $w$  like in the proof of Lemma 8.5 (on which the construction of the inverse map in Section 8.2 is based); we refer to this proof for the details. Namely, we concatenate the following:

- a path from  $v$  to  $\sigma$  with only down edges and all edge labels in  $\Phi_J^+$ ;
- a path from  $\sigma$  to  $\tau$  constructed based on  $\mathbf{p}$ ;
- a path from  $\tau$  to  $w$  with only up edges and all edge labels in  $\Phi_J^+$ .

Note that

$$(10.11) \quad \langle \text{wt}(\mathbf{q}'), \lambda \rangle = \langle \text{wt}(\mathbf{p}), \lambda \rangle,$$

since all the edges in the first segment of  $\mathbf{q}'$ , as well as the extra edges introduced in the second segment, have labels orthogonal to  $\lambda$ . On another hand, by Lemma 4.1 (in fact, we only need here the original version [Po, Lemma 1 (3)]), we have

$$(10.12) \quad \langle \text{wt}(\mathbf{q}'), \lambda \rangle \geq \langle \text{wt}(\mathbf{q}), \lambda \rangle.$$

The proof is concluded by combining (10.9), (10.10), (10.11), and (10.12).  $\square$

#### REFERENCES

- [BB] A. Björner and F. Brenti. *Combinatorics of Coxeter groups*. Graduate Texts in Mathematics Vol. 231, Springer, New York, 2005.
- [BF1] A. Braverman and M. Finkelberg. Semi-infinite Schubert varieties and quantum  $K$ -theory of flag manifolds. preprint [arXiv:1111.2266](#).
- [BF2] A. Braverman and M. Finkelberg. Weyl modules and  $q$ -Whittaker functions. preprint [arXiv:1203.1583](#).
- [BFP] F. Brenti, S. Fomin, and A. Postnikov. Mixed Bruhat operators and Yang-Baxter equations for Weyl groups. *Int. Math. Res. Notices*, no. 8, 419–441, 1999.
- [CFK] V. Chari, G. Fourier, and T. Khandai. A categorical approach to Weyl modules. *Transform. Groups*, 15:517–549, 2010.
- [CI] V. Chari and B. Ion. BGG reciprocity for current algebras. preprint [arXiv:1307.1440](#).
- [CSSW] V. Chari, L. Schneider, P. Shereen, and J. Wand. Modules with Demazure flags and character formulae. preprint [arXiv:1310.5191](#).
- [Dy] M. J. Dyer. Hecke algebras and shellings of Bruhat intervals. *Compositio Math.*, 89:91–115, 1993.
- [E] P. Etingof. Whittaker functions on quantum groups and  $q$ -deformed Toda operators. Differential topology, infinite-dimensional Lie algebras, and applications, pp. 9–25, Amer. Math. Soc. Transl. Ser. 2, 194, Amer. Math. Soc., Providence, RI, 1999.
- [FGP] S. Fomin, S. Gelfand, and A. Postnikov. Quantum Schubert polynomials. *J. Amer. Math. Soc.*, 10:565–596, 1997.
- [FL] G. Fourier and P. Littelmann. Tensor product structure of affine Demazure modules and limit constructions. *Nagoya Math. J.*, 182:171–198, 2006.
- [FOS] G. Fourier, M. Okado, and A. Schilling. Kirillov-Reshetikhin crystals for nonexceptional types. *Adv. Math.*, 222:1080–1116, 2009.
- [FOS1] G. Fourier, M. Okado, and A. Schilling. Perfectness of Kirillov-Reshetikhin crystals for nonexceptional types. *Contemp. Math.*, 506:127–143, 2010.
- [FW] W. Fulton and C. Woodward. On the quantum product of Schubert classes. *J. Algebraic Geom.*, 13:641–661, 2004.
- [HKOTT] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Z. Tsuboi. Paths, crystals and fermionic formulae. in “MathPhys Odyssey 2001, Integrable Models and Beyond” (M. Kashiwara and T. Miwa, Eds.), Prog. Math. Phys. Vol. 23, pp. 205–272, Birkhäuser, Boston, 2002.
- [HKOTY] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Y. Yamada. Remarks on fermionic formula. in “Recent Developments in Quantum Affine Algebras and Related Topics” (N. Jing and K.C. Misra, Eds.), Contemp. Math. Vol. 248, pp. 243–291, Amer. Math. Soc., Providence, RI, 1999.
- [HK] J. Hong and S.J. Kang. *Introduction to Quantum Groups and Crystal Bases*, volume 42 of *Graduate Studies in Mathematics*. Amer. Math. Soc., 2000.

- [Ion] B. Ion. Nonsymmetric Macdonald polynomials and Demazure characters. *Duke Math. J.*, 116:299–318, 2003.
- [INS] M. Ishii, S. Naito, and D. Sagaki. Semi-infinite Lakshmibai-Seshadri path model for level-zero extremal weight modules over quantum affine algebras. preprint [arXiv:1402.3884](#).
- [Kac] V. G. Kac. *Infinite Dimensional Lie Algebras*, 3rd Edition, Cambridge University Press, Cambridge, UK, 1990.
- [Kas] M. Kashiwara. On level-zero representations of quantized affine algebras. *Duke Math. J.*, 112:117–175, 2002.
- [KN] M. Kashiwara and T. Nakashima. Crystal graphs for representations of the  $q$ -analogue of classical Lie algebras. *J. Algebra*, 165:295–345, 1994.
- [Kho] A. Khoroshkin. Highest weight categories and Macdonald polynomials. preprint [arXiv:1312.7053](#).
- [KR] A. Kirillov and N. Reshetikhin. Representations of Yangians and multiplicities of the inclusion of the irreducible components of the tensor product of representations of simple Lie algebras. *J. Sov. Math.*, 52:3156–3164, 1990.
- [KMOU] A. Kuniba, K. C. Misra, M. Okado, and J. Uchiyama. Demazure modules and perfect crystals. *Comm. Math. Phys.*, 192:555–567, 1998.
- [LS] T. Lam and M. Shimozono. Quantum cohomology of  $G/P$  and homology of affine Grassmannian. *Acta Math.*, 204:49–90, 2010.
- [Le1] C. Lenart. On the combinatorics of crystal graphs, I. Lusztig’s involution. *Adv. Math.*, 211:324–340, 2007.
- [Le2] C. Lenart. From Macdonald polynomials to a charge statistic beyond type  $A$ . *J. Combin. Theory Ser. A*, 119:683–712, 2012.
- [LL1] C. Lenart and A. Lubovsky. A generalization of the alcove model and its applications. [arXiv:1112.2216](#). 24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012), pp. 875–886, *Discrete Math. Theor. Comput. Sci. Proc.*, AR, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2012.
- [LL2] C. Lenart and A. Lubovsky. A uniform realization of the combinatorial  $R$ -matrix. In preparation.
- [LNSSS1] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono. A uniform model for Kirillov-Reshetikhin crystals I. Lifting the parabolic quantum Bruhat graph. *Int. Math. Res. Not.*, DOI [10.1093/imrn/rnt263](#).
- [LNSSS2] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono. Quantum Lakshmibai-Seshadri paths and root operators. preprint [arXiv:1308.3529](#). To appear in *Proceedings of the 5th Mathematical Society of Japan Seasonal Institute: Schubert Calculus*, Osaka, Japan, 2014.
- [LP] C. Lenart and A. Postnikov. Affine Weyl groups in  $K$ -theory and representation theory. *Int. Math. Res. Not.*, 1–65, 2007. Art. ID rnm038.
- [LP1] C. Lenart and A. Postnikov. A combinatorial model for crystals of Kac-Moody algebras. *Trans. Amer. Math. Soc.*, 360:4349–4381, 2008.
- [LeS] C. Lenart and A. Schilling. Crystal energy via the charge in types  $A$  and  $C$ . *Math. Zeitschrift*, 273:401–426, 2013.
- [LeSh] C. Lenart and M. Shimozono. Equivariant  $K$ -Chevalley rules for Kac-Moody flag manifolds. *Amer. J. Math.*, 136:1–39, 2014.
- [Li] P. Littelmann. Paths and root operators in representation theory. *Ann. of Math. (2)*, 142:499–525, 1995.
- [Ma] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford Mathematical Monographs. Oxford University Press, Oxford, second edition, 1995.
- [Ma2] I. G. Macdonald. *Affine Hecke algebras and orthogonal polynomials*. Cambridge Tracts in Mathematics, 157. Cambridge University Press, Cambridge, 2003.
- [NS1] S. Naito and D. Sagaki. Path model for a level-zero extremal weight module over a quantum affine algebra. *Int. Math. Res. Not.*, no. 32, 1731–1754, 2003.
- [NS2] S. Naito and D. Sagaki. Crystal of Lakshmibai-Seshadri paths associated to an integral weight of level zero for an affine Lie algebra. *Int. Math. Res. Not.*, no. 14, 815–840, 2005.
- [NS3] S. Naito and D. Sagaki. Path model for a level-zero extremal weight module over a quantum affine algebra. II. *Adv. Math.*, 200:102–124, 2006.
- [NS4] S. Naito and D. Sagaki. Construction of perfect crystals conjecturally corresponding to Kirillov-Reshetikhin modules over twisted quantum affine algebras. *Commun. Math. Phys.*, 263:749–787, 2006.

- [NS5] S. Naito and D. Sagaki. Crystal structure on the set of Lakshmibai-Seshadri paths of an arbitrary level-zero shape. *Proc. London Math. Soc.*, 96:582–622, 2008.
- [NS6] S. Naito and D. Sagaki. Lakshmibai-Seshadri paths of level-zero weight shape and one-dimensional sums associated to level-zero fundamental representations. *Compos. Math.*, 144:1525–1556, 2008.
- [NS7] S. Naito and D. Sagaki. Demazure subcrystals of crystal bases of level-zero extremal weight modules over quantum affine algebras. preprint [arXiv:1404.2436](https://arxiv.org/abs/1404.2436).
- [N] H. Nakajima. Extremal weight modules of quantum affine algebras. in “Representation Theory of Algebraic Groups and Quantum Groups” (T. Shoji et al., Eds.), Adv. Stud. Pure Math. Vol. 40, pp. 343–369, Math. Soc. Japan, 2004.
- [NY] A. Nakayashiki and Y. Yamada. Kostka polynomials and energy functions in solvable lattice models. *Selecta Math. (N.S.)*, 3:547–599, 1997.
- [OS] D. Orr and M. Shimozono. Specializations of nonsymmetric Macdonald-Koornwinder polynomials. preprint [arXiv:1310.0279](https://arxiv.org/abs/1310.0279)
- [P] D. Peterson. Quantum cohomology of  $G/P$ . Lecture notes, Massachusetts Institute of Technology, Cambridge, MA, Spring 1997.
- [Po] A. Postnikov. Quantum Bruhat graph and Schubert polynomials. *Proc. Amer. Math. Soc.*, 133:699–709, 2004.
- [Sage] Stein, W. A., and others. *Sage Mathematics Software (Version 5.4)*. The Sage Development Team, 2012. <http://www.sagemath.org>.
- [Sage-comb] The Sage-Combinat community. *Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, 2008-2012*. <http://combinat.sagemath.org>.
- [Sa] Y. Sanderson. On the connection between Macdonald polynomials and Demazure characters. *J. Algebraic Combin.*, 11:269–275, 2000.
- [Se] A. Sevostyanov. Regular nilpotent elements and quantum groups. *Comm. Math. Phys.*, 204:1–16, 1999.
- [SS] A. Schilling and M. Shimozono.  $X = M$  for symmetric powers. *J. Algebra*, 295:562–610, 2006.
- [ST] A. Schilling and P. Tingley. Demazure crystals, Kirillov-Reshetikhin crystals, and the energy function. *Electronic J. Combin.*, 19(2) P2, 2012.
- [Y] S. Yamane. Perfect crystals of  $U_q(G_2^{(1)})$ . *J. Algebra* 210(2):440–486, 1998.

(Cristian Lenart) DEPARTMENT OF MATHEMATICS AND STATISTICS, STATE UNIVERSITY OF NEW YORK AT ALBANY, ALBANY, NY 12222, U.S.A.

*E-mail address:* [clenart@albany.edu](mailto:clenart@albany.edu)

*URL:* <http://www.albany.edu/~lenart/>

(Satoshi Naito) DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, 2-12-1 OH-OKAYAMA, MEGURO-KU, TOKYO 152-8551, JAPAN

*E-mail address:* [naito@math.titech.ac.jp](mailto:naito@math.titech.ac.jp)

(Daisuke Sagaki) INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, TSUKUBA, IBARAKI 305-8571, JAPAN

*E-mail address:* [sagaki@math.tsukuba.ac.jp](mailto:sagaki@math.tsukuba.ac.jp)

(Anne Schilling) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, ONE SHIELDS AVENUE, DAVIS, CA 95616-8633, U.S.A.

*E-mail address:* [anne@math.ucdavis.edu](mailto:anne@math.ucdavis.edu)

*URL:* <http://www.math.ucdavis.edu/~anne>

(Mark Shimozono) DEPARTMENT OF MATHEMATICS, MC 0151, 460 MCBRYDE HALL, VIRGINIA TECH, 225 STANGER ST., BLACKSBURG, VA 24061 USA

*E-mail address:* [mshimo@vt.edu](mailto:mshimo@vt.edu)