A NEW COMBINATORIAL MODEL IN REPRESENTATION THEORY

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Abstract. The present paper is a survey of a simple combinatorial model for the irreducible characters of complex semisimple Lie algebras, and, more generally, of complex symmetrizable Kac-Moody algebras. This model, which will be referred to as the alcove path model, can be viewed as a discrete counterpart to the Littelmann path model. It allows us to give character formulas and a Littlewood-Richardson rule for decomposing tensor products of irreducible representations; it also leads to a nice description of crystal graphs, including a combinatorial realization of them as self-dual posets via a generalization of Schützenberger's involution. Overall, we can say that the alcove path model leads to an extensive generalization of the combinatorics of irreducible characters from Lie type A (where the combinatorics is based on Young tableaux, for instance) to arbitrary type.

1. Introduction

We have recently given a simple combinatorial model for the irreducible characters of a complex semisimple Lie group G and, more generally, for the *Demazure characters* [14]. For reasons explained below, we call our model the *alcove path model*. This was extended to complex symmetrizable Kac-Moody algebras in [15] (that is, to infinite root systems), and its combinatorics was investigated in more detail in [13]. The exposition in [14] was in the context of the equivariant K-theory of the generalized flag variety G/B; more precisely, we first derived a *Chevalley-type multiplication formula* in $K_T(G/B)$, and then we deduced from it our Demazure character formula. By contrast, the exposition in [15] was purely representation theoretic, being based on Stembridge's combinatorial model for Weyl characters [21].

Our model is based on enumerating certain saturated chains in the *Bruhat order* on the corresponding *Weyl group* W. This enumeration is determined by an *alcove path*, which is a sequence of adjacent alcoves for the *affine Weyl group* W_{aff} of the Langland's dual group $G^\vee$. Alcove paths correspond to reduced decompositions of elements in the affine Weyl group, as well as to certain sequences of positive roots defined by *interlacing conditions* [15], which extend the notion of a *reflection ordering* [4].

The alcove path model leads to an extensive generalization of the combinatorics of irreducible characters from Lie type A (where the combinatorics is based on Young tableaux, for instance) to arbitrary type. More precisely, we gave:

(1) cancellation free character formulas, including Demazure character formulas (Theorems 3.6);
(2) a *Littlewood-Richardson rule* for decomposing tensor products of irreducible representations (Theorem 5.3) and a branching rule;
(3) a combinatorial description of the crystal graphs corresponding to the irreducible representations (Corollary 5.4); this result includes a transparent proof, based on the Yang-Baxter equation, of the fact that the mentioned description does not depend on the choice involved in our model (Corollary 6.6);
(4) a combinatorial realization of a certain fundamental involution on the canonical basis (Theorem 7.8, see also Example 7.10); this involution exhibits the crystals as self-dual posets, corresponds to the action of the longest Weyl group element on an irreducible representation, and is a direct generalization of Schützenberger's involution on tableaux;

(5) an analog for arbitrary root systems, based on the Yang-Baxter equation, of Schützenberger's *sliding* algorithm, which is also known as *jeu de taquin* (Section 6); this algorithm has many applications to the representation theory of the Lie algebra of type A.

A future publication will be concerned with a direct generalization of the notion of the product of Young tableaux in the context of the product of crystals.

There are other models for the characters of the irreducible representations of G with highest weight λ, such as the *Littelmann path model*. Littelmann [16, 17, 19] showed that the characters can be described by counting certain continuous paths in $\mathfrak{h}_{\mathbb{R}}^*$. These paths are constructed recursively, by starting with an initial one, and by applying certain *root operators*. By making specific choices for the initial path, one can obtain special cases which have more explicit descriptions. For instance, a straight line initial path leads to the *Lakshmibai-Seshadri paths* (LS paths). These were introduced before Littelmann's work, in the context of *standard monomial theory* [12]. They have a nonrecursive description as weighted chains in the Bruhat order on the quotient W/W_λ of the corresponding Weyl group W modulo the stabilizer W_λ of the weight λ; therefore, we will use the term *LS chains* when referring to this description. Recently, Gaussent and Littelmann [6], motivated by the study of Mirković-Vilonen cycles, defined another combinatorial model for the irreducible characters of a complex semisimple Lie group. This model is based on *LS-galleries*, which are certain sequences of faces of alcoves for the corresponding affine Weyl group; these sequences are specified by several conditions (including some positivity and dimension conditions). According to [6], for each LS-gallery there is an associated Littelmann path and a saturated chain in the Bruhat order on W/W_λ. We explained in [15] the fact that we do not obtain Littelmann paths by applying the same procedure as in [6] (or similar ones) to our model.

It was shown in [15] that LS chains are a certain limiting case of a special case of our model. Note that LS chains cannot be entirely viewed as discrete objects since certain constructions related to them (such as the definition of root operators) involve their description as piecewise-linear paths. In relation to Littelmann paths in general, both LS-galleries and our model - which were developed independently - can be viewed as discrete counterparts. However, unlike LS-galleries, our model extends to the Kac-Moody case.

We believe that our model is more efficient in explicit computations than other known models due to its simplicity and its combinatorial nature. For instance, our model is equally simple for both regular and nonregular highest weights λ; indeed, instead of working with chains of cosets in W/W_λ (as in the case of LS chains) or with possibly lower dimensional faces of alcoves (as in the case of LS-galleries), we always work with chains in W and, in the case of a finite root system, with alcoves too. Compared to LS chains and LS-galleries, we also eliminate the need for making specific choices when selecting the corresponding chains in Bruhat order. We refer to Subsection 3.3 for a discussion about computational complexities.

Finally, we believe that the aspects of our model that were investigated in [13, 14, 15] represent just a small fraction of a rich combinatorial structure yet to be explored, which would generalize most of the combinatorics of Young tableaux.

2. Preliminaries

We recall some background information on finite root systems, affine Weyl groups, Demazure characters, and crystal graphs.

2.1. **Root systems.**

Let G be a connected, simply connected, simple complex Lie group. Fix a Borel subgroup B and a maximal torus T such that $G \supset B \supset T$. As usual, we denote by B^- be the opposite Borel subgroup, while N and N^- are the unipotent radicals of B and B^-, respectively. Let $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{n}$, and $\mathfrak{n}^-$ be the complex Lie algebras of G, T, N, and N^-, respectively. Let r be the rank of the Cartan subalgebra $\mathfrak{h}$. Let $\Phi \subset \mathfrak{h}^*$ be the corresponding irreducible *root system*, and let $\mathfrak{h}_{\mathbb{R}}^* \subset \mathfrak{h}^*$ be the real span of the roots. Let $\Phi^+ \subset \Phi$ be the set of positive roots corresponding to our choice of B. Then Φ is the disjoint union of Φ^+ and $\Phi^- := -\Phi^+$. We write $\alpha > 0$ (respectively, $\alpha < 0$) for $\alpha \in \Phi^+$ (respectively, $\alpha \in \Phi^-$), and we define $\mathrm{sgn}(\alpha)$ to be 1 (respectively -1). We also use the notation $|\alpha| := \mathrm{sgn}(\alpha)\alpha$. Let $\alpha_1, \dots, \alpha_r \in \Phi^+$ be the corresponding *simple roots*, which form a basis of $\mathfrak{h}^*_{\mathbb{R}}$. Let $\langle \cdot , \cdot \rangle$ denote the nondegenerate scalar product on $\mathfrak{h}^*_{\mathbb{R}}$ induced by the Killing form. Given a root α, the corresponding *coroot* is $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$. The collection of coroots $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$ forms the *dual root system.*

The *Weyl group* $W \subset \mathrm{Aut}(\mathfrak{h}^*_{\mathbb{R}})$ of the Lie group G is generated by the reflections $s_\alpha : \mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}$, for $\alpha \in \Phi$, given by

$$s_\alpha : \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \, \alpha.$$

In fact, the Weyl group W is generated by the *simple reflections* $s_1, \dots, s_r$ corresponding to the simple roots $s_i := s_{\alpha_i}$, subject to the *Coxeter relations:*

$$(s_i)^2 = 1 \quad \text{and} \quad (s_i s_j)^{m_{ij}} = 1 \quad \text{for any } i, j \in \{1, \dots, r\},$$

where m_{ij} is half of the order of the dihedral subgroup generated by s_i and s_j. An expression of a Weyl group element w as a product of generators $w = s_{i_1} \cdots s_{i_l}$ which has minimal length is called a *reduced decomposition* for w; its length $\ell(w) = l$ is called the *length* of w. The Weyl group contains a unique *longest element* $w_\circ$ with maximal length $\ell(w_\circ) = \#\Phi^+$. For $u, w \in W$, we say that u *covers* w, and write $u \gtrdot w$, if $w = u s_\beta$, for some $\beta \in \Phi^+$, and $\ell(u) = \ell(w) + 1$. The transitive closure "$>$" of the relation "$\gtrdot$" is called the *Bruhat order* on W.

The *weight lattice* Λ is given by

$$\Lambda := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for any } \alpha \in \Phi\}. \tag{2.1}$$

The weight lattice Λ is generated by the *fundamental weights* $\omega_1, \dots, \omega_r$, which are defined as the elements of the dual basis to the basis of simple coroots, i.e., $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. The set Λ^+ of *dominant weights* is given by

$$\Lambda^+ := \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for any } \alpha \in \Phi^+\}.$$

Let $\rho := \omega_1 + \cdots + \omega_r = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta$. The *height* of a coroot $\alpha^\vee \in \Phi^\vee$ is $\langle \rho, \alpha^\vee \rangle = c_1 + \cdots + c_r$ if $\alpha^\vee = c_1 \alpha_1^\vee + \cdots + c_r \alpha_r^\vee$. Since we assumed that Φ is irreducible, there is a unique *highest coroot* $\theta^\vee \in \Phi^\vee$ that has maximal height. (In other words, $\theta^\vee$ is the highest root of the dual root system $\Phi^\vee$. It should not be confused with the coroot of the highest root of Φ.) We will also use the *Coxeter number*, that can be defined as $h := \langle \rho, \theta^\vee \rangle + 1$.

2.2. **Affine Weyl groups.** In this subsection, we remind a few basic facts about affine Weyl groups and alcoves, cf. Humphreys [7, Chaper 4] for more details.

Let W_{aff} be the *affine Weyl group* for the Langland's dual group $G^\vee$. The affine Weyl group W_{aff} is generated by the affine reflections $s_{\alpha,k} : \mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}$, for $\alpha \in \Phi$ and $k \in \mathbb{Z}$, that reflect the space $\mathfrak{h}^*_{\mathbb{R}}$ with respect to the affine hyperplanes

$$H_{\alpha,k} := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \lambda, \alpha^\vee \rangle = k\}. \tag{2.2}$$

Explicitly, the affine reflection $s_{\alpha,k}$ is given by

$$s_{\alpha,k} : \lambda \mapsto s_\alpha(\lambda) + k\,\alpha = \lambda - (\langle \lambda, \alpha^\vee \rangle - k)\,\alpha.$$

The hyperplanes $H_{\alpha,k}$ divide the real vector space $\mathfrak{h}^*_{\mathbb{R}}$ into open regions, called *alcoves.* Each alcove A is given by inequalities of the form

$$A := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid m_\alpha < \langle \lambda, \alpha^\vee \rangle < m_\alpha + 1 \text{ for all } \alpha \in \Phi^+\},$$

where $m_\alpha = m_\alpha(A)$, $\alpha \in \Phi^+$, are some integers.

A proof of the following important property of the affine Weyl group can be found, e.g., in [7, Chapter 4].

Lemma 2.1. *The affine Weyl group W_{aff} acts simply transitively on the collection of all alcoves.*

The *fundamental alcove* $A_\circ$ is given by

$$A_\circ := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid 0 < \langle \lambda, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Phi^+\}.$$

Lemma 2.1 implies that, for any alcove A, there exists a unique element v_A of the affine Weyl group W_{aff} such that $v_A(A_\circ) = A$. Hence the map $A \mapsto v_A$ is a one-to-one correspondence between alcoves and elements of the affine Weyl group.

Recall that $\theta^\vee \in \Phi^\vee$ is the highest coroot. Let $\theta \in \Phi^+$ be the corresponding root, and let $\alpha_0 := -\theta$. The fundamental alcove $A_\circ$ is, in fact, the simplex given by

$$A_\circ = \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid 0 < \langle \lambda, \alpha_i^\vee \rangle \text{ for } i = 1, \ldots, r, \text{ and } \langle \lambda, \theta^\vee \rangle < 1\}, \tag{2.3}$$

Lemma 2.1 also implies that the affine Weyl group is generated by the set of reflections $s_0, s_1, \ldots, s_r$ with respect to the walls of the fundamental alcove $A_\circ$, where $s_0 := s_{\alpha_0,-1}$ and $s_1, \ldots, s_r \in W$ are the simple reflections $s_i = s_{\alpha_i,0}$. Like the Weyl group, the affine Weyl group W_{aff} is a Coxeter group. As in the case of the Weyl group, a decomposition $v = s_{i_1} \cdots s_{i_l} \in W_{\text{aff}}$ is called *reduced* if it has minimal length; its length $\ell(v) = l$ is called the length of v.

We say that two alcoves A and B are *adjacent* if B is obtained by an affine reflection of A with respect to one of its walls. In other words, two alcoves are adjacent if they are distinct and have a common wall. For a pair of adjacent alcoves, let us write $A \xrightarrow{\beta} B$ if the common wall of A and B is of the form $H_{\beta,k}$ and the root $\beta \in \Phi$ points in the direction from A to B.

Let Z be the set of the elements of the lattice Λ/h that do not belong to any affine hyperplane $H_{\alpha,k}$ (recall that h is the Coxeter number). Each alcove A contains precisely one element ζ_A of the set Z (cf. [11, 14]); this will be called the *central point* of A. In particular, $\zeta_{A_\circ} = \rho/h$.

Proposition 2.2. [14] *For a pair of adjacent alcoves $A \xrightarrow{\alpha} B$, we have $\zeta_B - \zeta_A = \alpha/h$.*

2.3. **Demazure characters.** The *generalized flag variety* G/B is a smooth projective variety. It decomposes into a disjoint union of *Schubert cells* $X_w^\circ := BwB/B$ indexed by elements $w \in W$ of the Weyl group. The closures of Schubert cells $X_w := \overline{X_w^\circ}$ are called *Schubert varieties.* We have $u > w$ in the Bruhat order (defined above) if and only if $X_u \supset X_w$. Let $\mathcal{O}_{X_w}$ be the structure sheaf of the Schubert variety X_w. Let $\mathcal{L}_\lambda$ be the line bundle over G/B associated with the weight λ, that is, $\mathcal{L}_\lambda := G \times_B \mathbb{C}_{-\lambda}$, where B acts on G by right multiplication, and the B-action on $\mathbb{C}_{-\lambda} = \mathbb{C}$ corresponds to the character determined by $-\lambda$. (This character of T extends to B by defining it to be identically one on the commutator subgroup $[B, B]$.)

For a dominant weight $\lambda \in \Lambda^+$, let V_λ denote the finite dimensional irreducible representation of the Lie group G with highest weight λ. For $\lambda \in \Lambda^+$ and $w \in W$, the *Demazure module* $V_{\lambda,w}$ is the B-module that is dual to the space of global sections of the line bundle $\mathcal{L}_\lambda$ on the Schubert variety X_w:

$$V_{\lambda,w} := H^0(X_w, \mathcal{L}_\lambda)^*. \tag{2.4}$$

For the longest Weyl group element $w = w_\circ$, the space $V_{\lambda,w_\circ} = H^0(G/B, \mathcal{L}_\lambda)^*$ has the structure of a G-module. The classical *Borel-Weil theorem* says that $V_{\lambda,w_\circ}$ is isomorphic to the irreducible G-module V_λ.

Let $\mathbb{Z}[\Lambda]$ be the group algebra of the weight lattice Λ, which is isomorphic to the representation ring of T. The algebra $\mathbb{Z}[\Lambda]$ has a $\mathbb{Z}$-basis of formal exponents $\{e^\lambda \mid \lambda \in \Lambda\}$ with multiplication $e^\lambda \cdot e^\mu := e^{\lambda+\mu}$; in other words, $\mathbb{Z}[\Lambda] = \mathbb{Z}[e^{\pm\omega_1}, \cdots, e^{\pm\omega_r}]$ is the algebra of Laurent polynomials in r variables. The formal characters of the modules $V_{\lambda,w}$, called *Demazure characters*, are given by $ch(V_{\lambda,w}) = \sum_{\mu\in\Lambda} m_{\lambda,w}(\mu)\, e^\mu \in \mathbb{Z}[\Lambda]$, where $m_{\lambda,w}(\mu)$ is the multiplicity of the weight μ in $V_{\lambda,w}$. These characters generalize the characters of the irreducible representations $ch(V_\lambda) = ch(V_{\lambda,w_\circ})$. Demazure [3] gave a formula expressing the characters $ch(V_{\lambda,w})$ in terms of certain operators known as *Demazure operators.*

2.4. **Crystal graphs and Schützenberger's involution.** Let $U(\mathfrak{g})$ be the enveloping algebra of the Lie algebra $\mathfrak{g}$, which is generated by E_i, F_i, H_i, for $i = 1, \ldots, r$, subject to the Serre relations. Let $\mathcal{B}$ be the *canonical basis* of $U(\mathfrak{n}^-)$, and let $\mathcal{B}_\lambda := \mathcal{B} \cap V_\lambda$ be the canonical basis of the irreducible representation V_λ with highest weight λ. Let v_λ and v_λ^{low} be the highest and lowest weight vectors in $\mathcal{B}_\lambda$, respectively. Let $\widetilde{E}_i$, $\widetilde{F}_i$, for $i = 1, \ldots, r$, be Kashiwara's operators [8, 20]; these are also known as raising and lowering operators, respectively. The *crystal graph* of V_λ is the directed colored graph on $\mathcal{B}_\lambda$ defined by arrows $x \to y$ colored i for each $\widetilde{F}_i(x) = y$ or, equivalently, for each $\widetilde{E}_i(y) = x$. (In fact, Kashiwara introduced the notion of a crystal graph of an $U_q(\mathfrak{g})$-representation, where $U_q(\mathfrak{g})$ is the Drinfeld-Jimbo q-deformation of $U(\mathfrak{g})$, also known as a *quantum group*; using the quantum deformation, one can associate a crystal graph to a $\mathfrak{g}$-representation.) One can also define partial orders $\preceq_i$ on $\mathcal{B}_\lambda$ by

$$x \preceq_i y \quad \text{if} \quad x = \widetilde{F}_i^k(y) \text{ for some } k \geq 0\,.$$

We let $\preceq$ denote the partial order generated by all partial orders $\preceq_i$, for $i = 1, \ldots, r$. The poset $(\mathcal{B}_\lambda, \preceq)$ has maximum v_λ and minimum v_λ^{low}.

In order to proceed, we need the following general setup. Let V be a module over an associative algebra U and σ an automorphism of U. The twisted U-module V^σ is the same vector space V but with the new action $u * v := \sigma(u)v$ for $u \in U$ and $v \in V$. Clearly, $V^{\sigma\tau} = (V^\sigma)^\tau$ for every two automorphisms σ and τ of U. Furthermore, if V is a simple U-module, then so is V^σ. In particular, if $U = U(\mathfrak{g})$ and $V = V_\lambda$, then $(V_\lambda)^\sigma$ is isomorphic to $V_{\sigma(\lambda)}$ for some dominant weight $\sigma(\lambda)$. Thus there is an isomorphism of vector spaces $\sigma_\lambda : V_\lambda \to V_{\sigma(\lambda)}$ such that

$$\sigma_\lambda(uv) = \sigma(u)\sigma_\lambda(v)\,, \quad u \in U(\mathfrak{g})\,,\ v \in V_\lambda\,.$$

By Schur's lemma, σ_λ is unique up to a scalar multiple.

The longest Weyl group element $w_\circ$ defines an involution on the simple roots by $\alpha_i \mapsto \alpha_{i^*} := -w_\circ(\alpha_i)$. Consider the automorphisms of $U(\mathfrak{g})$ defined by

$$\phi(E_i) = F_i\,, \qquad \phi(F_i) = E_i\,, \qquad \phi(H_i) = -H_i\,, \tag{2.5}$$

$$\psi(E_i) = E_{i^*}\,, \quad \psi(F_i) = F_{i^*}\,, \quad \psi(H_i) = H_{i^*}\,, \tag{2.6}$$

and $\eta := \phi\psi$. Clearly, these three automorphisms together with the identity automorphism form a group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. It also easily follows from (2.5)-(2.6) that

$$\phi(\lambda) = \psi(\lambda) = -w_\circ(\lambda)\,, \quad \eta(\lambda) = \lambda\,.$$

We can normalize each of the maps ϕ_λ, ψ_λ, and η_λ by the requirement that

$$\phi_\lambda(v_\lambda) = v_{-w_\circ(\lambda)}^{low}\,, \quad \psi_\lambda(v_\lambda) = v_{-w_\circ(\lambda)}\,, \quad \eta_\lambda(v_\lambda) = v_\lambda^{low}\,. \tag{2.7}$$

(Of course, we also set Id_λ to be the identity map on V_λ.) By [20, Proposition 21.1.2], cf. also [1, Proposition 7.1], we have the following result.

Proposition 2.3. [1, 20] (1) *Each of the maps ϕ_λ and ψ_λ sends $\mathcal{B}_\lambda$ to $\mathcal{B}_{-w_\circ(\lambda)}$, while η_λ sends $\mathcal{B}_\lambda$ to itself.*

(2) *For every two (not necessarily distinct) elements σ, τ of the group $\{\mathrm{Id}, \phi, \psi, \eta\}$, we have $(\sigma\tau)_\lambda = \sigma_{\tau(\lambda)}\tau_\lambda$. In particular, the map η_λ is an involution.*

(3) *For every $i = 1, \ldots, r$, we have*

$$\phi_\lambda \widetilde{F}_i = \widetilde{E}_i \phi_\lambda\,, \quad \psi_\lambda \widetilde{F}_i = \widetilde{F}_{i^*} \psi_\lambda\,, \quad \eta_\lambda \widetilde{F}_i = \widetilde{E}_{i^*} \eta_\lambda\,. \tag{2.8}$$

In particular, the poset $(\mathcal{B}_\lambda, \preceq)$ is self-dual, and η_λ is the corresponding antiautomorphism.

Berenstein and Zelevinsky [1] showed that, in type A_{n-1} (that is, in the case of the Lie algebra $\mathfrak{sl}_n$), the operator η_λ is given by Schützenberger's *evacuation* procedure for semistandard Young tableaux (see e.g. [5]). More precisely, it is known that, for each partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq 0)$, the semistandard Young tableaux of shape λ and entries $1, \ldots, n$ parametrize the the canonical basis $\mathcal{B}_\lambda$ of V_λ. Hence, we can transfer the action of η_λ on $\mathcal{B}_\lambda$ to an action on the corresponding tableaux. As mentioned above, the latter action coincides with Schützenberger's evacuation map. One way to realize this map on a tableau T is the following three-step procedure.

(1) Rotate the tableau 180°, such that its row/column words get reversed.
(2) Complement the entries via the map $i \mapsto w_\circ(i) = n+1-i$, where $w_\circ$ is the longest element in the symmetric group S_n.
(3) Apply *jeu de taquin* to construct the *rectification* of the skew tableau obtained in the previous step, that is, successively apply Schützenberger's *sliding algorithm* for the inside corners of the mentioned tableau.

For convenience, we will call these steps: REVERSE, COMPLEMENT, SLIDE. They are illustrated in Figure 1 below.

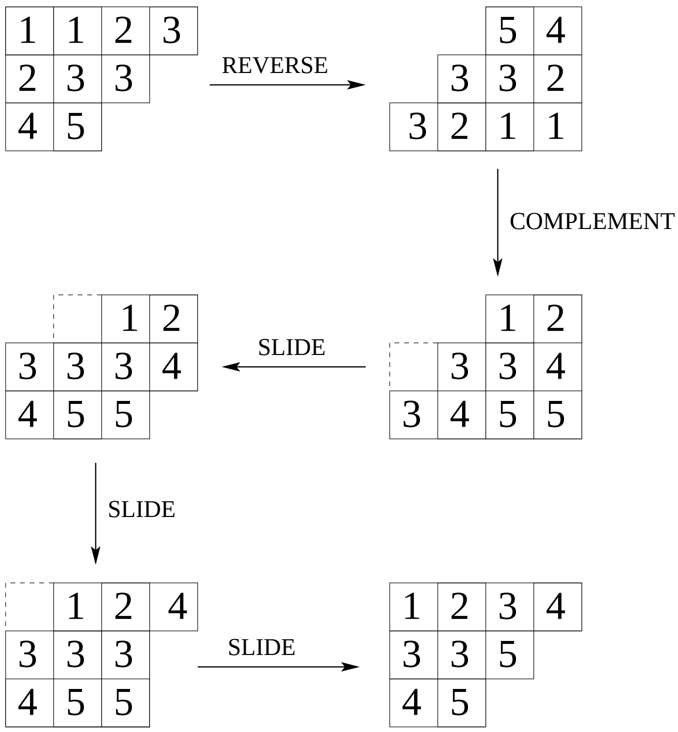

FIGURE 1. The evacuation map.

3. The Alcove Path Model

This section describes the basics of the model for the irreducible characters of semisimple Lie algebras that was introduced and investigated introduced in [13, 14, 15]. We refer to these papers for more details, including the proofs of the results mentioned in this survey. Although some of these results hold for infinite root systems (cf. [15]), the setup in this survey is that of a finite irreducible root system, as discussed in Section 2.

Our model is conveniently phrased in terms of several sequences, so let us mention some related notation. Given a totally ordered index set $I = \{i_1 < i_2 < \ldots < i_n\}$, a sequence $(a_{i_1}, a_{i_2}, \ldots, a_{i_n})$ is sometimes abbreviated to $\{a_j\}_{j \in I}$. We also let $[n] := \{1, 2, \ldots, n\}$.

3.1. **λ-chains.** The affine translations by weights preserve the set of affine hyperplanes $H_{\alpha,k}$, cf. (2.1) and (2.2). It follows that these affine translations map alcoves to alcoves. Let $A_\lambda = A_\circ + \lambda$ be the alcove obtained by the affine translation of the fundamental alcove $A_\circ$ by a weight $\lambda \in \Lambda$. Let v_λ be the corresponding element of W_{aff}, i.e,. v_λ is defined by $v_\lambda(A_\circ) = A_\lambda$. Note that the element v_λ may not be an affine translation itself.

Let us now fix a dominant weight λ. Let $v \mapsto \bar{v}$ be the homomorphism $W_{\text{aff}} \to W$ defined by ignoring the affine translation. In other words, $\bar{s}_{\alpha,k} = s_\alpha \in W$.

Definition 3.1. A λ-*chain of roots* is a sequence of positive roots $(\beta_1, \dots, \beta_n)$ which is determined as indicated below by a reduced decomposition $v_{-\lambda} = s_{i_1} \cdots s_{i_n}$ of $v_{-\lambda}$ as a product of generators of W_{aff}:

$$\beta_1 = \alpha_{i_1},\ \beta_2 = \bar{s}_{i_1}(\alpha_{i_2}),\ \beta_3 = \bar{s}_{i_1}\bar{s}_{i_2}(\alpha_{i_3}), \dots,\ \beta_n = \bar{s}_{i_1} \cdots \bar{s}_{i_{n-1}}(\alpha_{i_n})\,.$$

When the context allows, we will abbreviate "λ-chain of roots" to "λ-chain". The λ-*chain of reflections* associated with the above λ-chain of roots is the sequence $(\widehat{r}_1, \dots, \widehat{r}_n)$ of affine reflections in W_{aff} given by

$$\widehat{r}_1 = s_{i_1},\ \widehat{r}_2 = s_{i_1}s_{i_2}s_{i_1},\ \widehat{r}_3 = s_{i_1}s_{i_2}s_{i_3}s_{i_2}s_{i_1},\ \dots,\ \widehat{r}_n = s_{i_1} \cdots s_{i_n} \cdots s_{i_1}.$$

We will present two equivalent definitions of a λ-chain of roots.

Definition 3.2. An *alcove path* is a sequence of alcoves $(A_0, A_1, \dots, A_n)$ such that A_{i-1} and A_i are adjacent, for $i = 1, \dots, n$. We say that an alcove path is *reduced* if it has minimal length among all alcove paths from A_0 to A_n.

Given a finite sequence of roots $\Gamma = (\beta_1, \dots, \beta_n)$, we define the sequence of integers $(l_1^\emptyset, \dots, l_n^\emptyset)$ by $l_i^\emptyset := \#\{j < i \mid \beta_j = \beta_i\}$, for $i = 1, \dots, n$. We also need the following two conditions on Γ.

- (R1) The number of occurrences of any positive root α in Γ is $\langle \lambda, \alpha^\vee \rangle$.
- (R2) For each triple of positive roots (α, β, γ) with $\gamma^\vee = \alpha^\vee + \beta^\vee$, the subsequence of Γ consisting of α, β, γ is a concatenation of pairs (α, γ) and (β, γ) (in any order).

Theorem 3.3. [14] *The following statements are equivalent.*

- (a) *The sequence of roots $\Gamma = (\beta_1, \dots, \beta_n)$ is a λ-chain, and $(\widehat{r}_1, \dots, \widehat{r}_n)$ is the associated λ-chain of reflections.*
- (b) *We have a reduced alcove path $A_0 \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_n} A_n$ from $A_0 = A_\circ$ to $A_n = A_{-\lambda}$, and $\widehat{r}_i$ is the affine reflection in the common wall of A_{i-1} and A_i, for $i = 1, \dots, n$.*
- (c) *The sequence Γ satisfies conditions* (R1) *and* (R2) *above, and $\widehat{r}_i = s_{\beta_i, -l_i^\emptyset}$, for $i = 1, \dots, n$.*

We now describe a particular choice of a λ-chain. First note that constructing a λ-chain amounts to defining a total order on the index set

$$I := \{(\alpha, k) \mid \alpha \in \Phi^+,\ 0 \le k < \langle \lambda, \alpha^\vee \rangle\}\,,$$

such that condition (R2) above holds, where the sequence $\Gamma = \{\beta_i\}_{i \in I}$ is defined by $\beta_i = \alpha$ for $i = (\alpha, k)$. Fix a total order on the set of simple roots $\alpha_1 < \alpha_2 < \ldots < \alpha_r$. For each $i = (\alpha, k)$ in I, let $\alpha^\vee = c_1\alpha_1^\vee + \ldots + c_r\alpha_r^\vee$, and define the vector

$$v_i := \frac{1}{\langle \lambda, \alpha^\vee \rangle}(k, c_1, \dots, c_r)$$

in $\mathbb{Q}^{r+1}$. It turns out that the map $i \mapsto v_i$ is injective. Hence, we can define a total order on I by $i < j$ iff $v_i < v_j$ in the lexicographic order on $\mathbb{Q}^{r+1}$.

Proposition 3.4. [15] *Given the total order on I defined above, the sequence $\{\beta_i\}_{i \in I}$ defined by $\beta_i = \alpha$ for $i = (\alpha, k)$ is a λ-chain.*

3.2. **Admissible subsets.** For the remainder of this section, we fix a λ-chain $\Gamma = (\beta_1, \ldots, \beta_n)$. Let $r_i := s_{\beta_i}$. We now define the centerpiece of our combinatorial model for characters, which is our generalization of semistandard Young tableaux in type A.

Definition 3.5. An *admissible subset* is a subset of $[n]$ (possibly empty), that is, $J = \{j_1 < j_2 < \ldots < j_s\}$, such that we have the following saturated chain in the Bruhat order on W:

$$1 \lessdot r_{j_1} \lessdot r_{j_1} r_{j_2} \lessdot \ldots \lessdot r_{j_1} r_{j_2} \ldots r_{j_s} \,.$$

We denote by $\mathcal{A}(\Gamma)$ the collection of all admissible subsets corresponding to our fixed λ-chain Γ. Given an admissible subset J, we use the notation

$$\mu(J) := -\widehat{r}_{j_1} \ldots \widehat{r}_{j_s}(-\lambda)\,, \qquad w(J) := r_{j_1} \ldots r_{j_s} \,.$$

We call $\mu(J)$ the *weight* of the admissible subset J.

Theorem 3.6. [14, 15] (1) *We have the following character formula:*

$$ch(V_\lambda) = \sum_{J \in \mathcal{A}(\Gamma)} e^{\mu(J)} \,.$$

(2) *More generally, the following Demazure character formula holds for any $u \in W$:*

$$ch(V_{\lambda,u}) = \sum_J e^{-u\,\widehat{r}_{j_1} \cdots \widehat{r}_{j_s}(-\lambda)} \,,$$

where the summation is over all subsets $J = \{j_1 < \cdots < j_s\} \subseteq [n]$ such that

$$u \gtrdot u\, r_{j_1} \gtrdot u\, r_{j_1} r_{j_2} \gtrdot \cdots \gtrdot u\, r_{j_1} r_{j_2} \cdots r_{j_s}$$

is a saturated decreasing chain in the Bruhat order on the Weyl group W.

(3) *Assume that the λ-chain Γ has the property that the second occurence of a root can never be before the first occurence of another root. Then we also have the following Demazure character formula:*

$$ch(V_{\lambda,u}) = \sum_{\substack{J \in \mathcal{A}(\Gamma) \\ w(J) \le u}} e^{\mu(J)} \,.$$

Remark 3.7. Theorem 3.6 (3) is the analog of the Demazure character formula due to Littelmann [16], [18, Theorem 9.1]. Compared to the formula in Theorem 3.6 (2), the former has the advantage of realizing *all* Demazure characters $ch(V_{\lambda,u})$ (for a fixed λ) in terms of the *same* combinatorial objects, i.e., in terms of certain subsets of $\mathcal{A}(\Gamma)$.

Example 3.8. Consider $G = SL_n$ whose root system Φ is of type A_{n-1}, and whose Weyl group is the symmetric group S_n. We can identify the space $\mathfrak{h}_{\mathbb{R}}^*$ with the quotient space $V := \mathbb{R}^n / \mathbb{R}(1, \ldots, 1)$, where $\mathbb{R}(1, \ldots, 1)$ denotes the subspace in $\mathbb{R}^n$ spanned by the vector $(1, \ldots, 1)$. Let $\varepsilon_1, \ldots, \varepsilon_n \in V$ be the images of the coordinate vectors in $\mathbb{R}^n$. The positive roots are $\alpha_{ij} := \varepsilon_i - \varepsilon_j$ for $1 \le i < j \le n$, and the simple roots are $\alpha_i := \alpha_{i,i+1}$ for $i = 1, \ldots, n-1$. The longest root is $\theta = \alpha_{1n}$. The root system Φ is self-dual, that is, every coroot coincides with the corresponding root. The weight lattice is $\Lambda = \mathbb{Z}^n / \mathbb{Z}(1, \ldots, 1)$. We use the notation $[\lambda_1, \ldots, \lambda_n]$ for a weight, as the coset of $(\lambda_1, \ldots, \lambda_n)$ in $\mathbb{Z}^n$.

Suppose that $n = 3$ and $\lambda = 2\,\omega_1 + \omega_2 = 3\varepsilon_1 + \varepsilon_2$. Ordering the simple roots $(\alpha_{23} < \alpha_{12})$, Proposition 3.4 gives the following λ-chain:

$$(\beta_1, \ldots, \beta_6) = (\alpha_{12},\ \alpha_{13},\ \alpha_{23},\ \alpha_{13},\ \alpha_{12},\ \alpha_{13}).$$

This is associated with the reduced decomposition $v_{-\lambda} = s_1 s_2 s_1 s_0 s_1 s_2$ in the affine Weyl group. The corresponding λ-chain of reflections is

$$(\widehat{r}_1, \ldots, \widehat{r}_6) = (s_{\alpha_{12},0},\ s_{\alpha_{13},0},\ s_{\alpha_{23},0},\ s_{\alpha_{13},-1},\ s_{\alpha_{12},-1},\ s_{\alpha_{13},-2}).$$

Assume that we want to compute the Demazure character $ch(V_{\lambda,u})$ for $u = s_2 s_1$. There are five saturated chains in Bruhat order descending from u: (empty chain), $(u \gtrdot u s_{\alpha_{12}} = s_2)$, $(u \gtrdot u s_{\alpha_{13}} = s_1)$,

$(u \gtrdot us_{\alpha_{12}} \gtrdot us_{\alpha_{12}}s_{\alpha_{23}} = 1)$, $(u \gtrdot us_{\alpha_{13}} \gtrdot us_{\alpha_{13}}s_{\alpha_{12}} = 1)$. Thus, the Demazure character formula in Theorem 3.6 (2) requires us to sum over the following subsequences in the λ-chain $(\beta_1, \ldots, \beta_6)$:

$$\text{(empty subsequence)},\ (\alpha_{12}),\ (\alpha_{13}), (\alpha_{12},\, \alpha_{23}),\ (\alpha_{13},\, \alpha_{12}).$$

The sequence $(\beta_1, \ldots, \beta_6)$ contains one empty subsequence, two subsequences of the form (α_{12}), three subsequences of the form (α_{13}), one subsequence of the form $(\alpha_{12}, \alpha_{23})$, and two subsequence of the form $(\alpha_{13}, \alpha_{12})$. Hence, we have

$$\begin{aligned} ch(V_{[3,1,0],s_2s_1}) &= e^{-u(-\lambda)} + e^{-u\widehat{r}_1(-\lambda)} + e^{-u\widehat{r}_5(-\lambda)} + e^{-u\widehat{r}_2(-\lambda)} + e^{-u\widehat{r}_4(-\lambda)} + e^{-u\widehat{r}_6(-\lambda)} + \\ &+ e^{-u\widehat{r}_1\widehat{r}_3(-\lambda)} + e^{-u\widehat{r}_2\widehat{r}_5(-\lambda)} + e^{-u\widehat{r}_4\widehat{r}_5(-\lambda)}. \end{aligned}$$

We can explicitly write this expression as

$$\begin{aligned} &ch(V_{[3,1,0],s_2s_1}) = \\ &\quad e^{[1,0,3]} + e^{[3,0,1]} + e^{[2,0,2]} + e^{[1,3,0]} + e^{[1,2,1]} + e^{[1,1,2]} + e^{[3,1,0]} + e^{[2,2,0]} + e^{[2,1,1]}. \end{aligned}$$

Example 3.9. Suppose that the root system Φ is of type G_2. The positive roots are $\gamma_1 = \alpha_1,\ \gamma_2 = 3\alpha_1 + \alpha_2,\ \gamma_3 = 2\alpha_1 + \alpha_2,\ \gamma_4 = 3\alpha_1 + 2\alpha_2,\ \gamma_5 = \alpha_1 + \alpha_2,\ \gamma_6 = \alpha_2$. The corresponding coroots are $\gamma_1^\vee = \alpha_1^\vee,\ \gamma_2^\vee = \alpha_1^\vee + \alpha_2^\vee,\ \gamma_3^\vee = 2\alpha_1^\vee + 3\alpha_2^\vee,\ \gamma_4^\vee = \alpha_1^\vee + 2\alpha_2^\vee,\ \gamma_5^\vee = \alpha_1^\vee + 3\alpha_2^\vee,\ \gamma_6^\vee = \alpha_2^\vee$.

Suppose that $\lambda = \omega_2$. Proposition 3.4 gives the following ω_2-chain:

$$(\beta_1, \ldots, \beta_{10}) = (\gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_2, \gamma_5, \gamma_3, \gamma_4, \gamma_5, \gamma_3)\,.$$

Thus, we have $\widehat{r}_1 = s_{\gamma_6,0}$, $\widehat{r}_2 = s_{\gamma_5,0}$, $\widehat{r}_3 = s_{\gamma_4,0}$, $\widehat{r}_4 = s_{\gamma_3,0}$, $\widehat{r}_5 = s_{\gamma_2,0}$, $\widehat{r}_6 = s_{\gamma_5,1}$, $\widehat{r}_7 = s_{\gamma_3,1}$, $\widehat{r}_8 = s_{\gamma_4,1}$, $\widehat{r}_9 = s_{\gamma_5,2}$, $\widehat{r}_{10} = s_{\gamma_3,2}$. There are six saturated chains in the Bruhat order (starting at the identity) on the corresponding Weyl group that can be retrieved as subchains of the ω_2-chain. We indicate each such chain and the corresponding admissible subsets in $\{1, \ldots, 10\}$.

(1) 1: $\{\}$;
(2) $1 < s_{\gamma_6}$: $\{1\}$;
(3) $1 < s_{\gamma_6} < s_{\gamma_6}s_{\gamma_5}$: $\{1,2\}$, $\{1,6\}$, $\{1,9\}$;
(4) $1 < s_{\gamma_6} < s_{\gamma_6}s_{\gamma_5} < s_{\gamma_6}s_{\gamma_5}s_{\gamma_4}$: $\{1,2,3\}$, $\{1,2,8\}$, $\{1,6,8\}$;
(5) $1 < s_{\gamma_6} < s_{\gamma_6}s_{\gamma_5} < s_{\gamma_6}s_{\gamma_5}s_{\gamma_4} < s_{\gamma_6}s_{\gamma_5}s_{\gamma_4}s_{\gamma_3}$: $\{1,2,3,4\}$, $\{1,2,3,7\}$, $\{1,2,3,10\}$, $\{1,2,8,10\}$, $\{1,6,8,10\}$;
(6) $1 < s_{\gamma_6} < s_{\gamma_6}s_{\gamma_5} < s_{\gamma_6}s_{\gamma_5}s_{\gamma_4} < s_{\gamma_6}s_{\gamma_5}s_{\gamma_4}s_{\gamma_3} < s_{\gamma_6}s_{\gamma_5}s_{\gamma_4}s_{\gamma_3}s_{\gamma_2}$: $\{1,2,3,4,5\}$.

The weight of each admissible subset is now easy to compute. We are lead to the expression for the character $ch(V_{\omega_2})$ as the following sum over admissible subsets:

$$ch(V_{\omega_2}) = e^{\omega_2} + e^{\widehat{r}_1(\omega_2)} + e^{\widehat{r}_1\,\widehat{r}_2(\omega_2)} + e^{\widehat{r}_1\,\widehat{r}_6(\omega_2)} + e^{\widehat{r}_1\,\widehat{r}_9(\omega_2)} + \cdots + e^{\widehat{r}_1\,\widehat{r}_6\,\widehat{r}_8\,\widehat{r}_{10}(\omega_2)} + e^{\widehat{r}_1\,\widehat{r}_2\,\widehat{r}_3\,\widehat{r}_4\,\widehat{r}_5(\omega_2)}.$$

3.3. **Computational complexities.** In this subsection, we compare the computational complexity of our model with that of LS-paths constructed via root operators.

Fix a root system of rank r with N positive roots, a dominant weight λ, and a Weyl group element u of length l. We want to determine the character of the Demazure module $V_{\lambda,u}$. Let d be its dimension, and let L be the length of the affine Weyl group element $v_{-\lambda}$ (that is, the number of affine hyperplanes separating the fundamental alcove $A_\circ$ and $A_\circ - \lambda$). Note that $L = 2(\lambda, \rho^\vee)$, where $\rho^\vee = \frac{1}{2}\sum_{\beta\in\Phi^+}\beta^\vee$. We claim that the complexity of the character formula in Theorem 3.6 (2) is $O(d\,lL)$. Indeed, we start by determining an alcove path via the method underlying Proposition 3.4, which involves sorting a sequence of L rational numbers. The complexity is $O(L\,\log L)$, and note that $\log L$ is, in general, much smaller than d (see below for some examples). Whenever we examine some subword of the word of length L we fixed at the beginning, we have to check at most $L-1$ ways to add an extra reflection at the end. On the other hand, in each case, we have to check whether, upon multiplying by the corresponding nonaffine reflection, the length decreases by precisely 1. The complexity of the latter operation is $O(l)$, based on the Strong Exchange Condition [7, Theorem 5.8]. Then, for each "good" subword, we have to do a calculation, namely applying at most $2l$ affine reflections to $-\lambda$. In fact, it is fairly easy to implement this algorithm.

Now let us examine at the complexity of the algorithm based on root operators for constructing the LS-paths associated with λ. In other words, we are looking at the complexity of constructing the corresponding crystal graph. We have to generate the whole crystal graph first, and then figure out which paths give weights for the Demazure module. For each path, we can apply r root operators. Each path has at most N linear steps, so applying a root operator has complexity $O(N)$. But now we have to check whether the result is a path already determined, so we have to compare the obtained path with the other paths (that were already determined) of the same rank in the crystal graph (viewed as a ranked poset). This has complexity $O(NM)$, where M is the maximum number of elements of the same rank. Since we have at most $N+1$ ranks, M is at least $d/(N+1)$. In conclusion, the complexity is $O(drNM)$, which is at least $O(d^2 r)$.

Let us get a better picture of how the two results compare. Assume we are in a classical type, and let us first take λ to be the i-th fundamental weight, with i fixed, plus $u = w_\circ$. Clearly l is $O(r^2)$, L is $O(r)$, and d is $O(r^i)$, so the complexity of our formula is $O(r^{i+3})$. For LS-paths, we get at least $O(r^{2i+1})$. So the ratio between the complexity in the model based on LS-paths and our model is at least $O(r^{i-2})$.

Let us also take $\lambda = \rho$. In this case $d = 2^N$, and a simple calculation shows that L is $O(r^3)$. Our formula has complexity $O(2^N r^5)$, while the model based on LS-paths has complexity at least $O(2^{2N} r)$. So the ratio between the complexities is at least $O(2^N/r^4)$, where N is $r(r+1)/2$, r^2, and r^2-r in types A, B/C, and D, respectively.

4. Related Structures and Properties

In the first two subsections, we present two alternative ways of viewing admissible subsets, which are closely related to the equivalent definitions of λ-chains in Theorem 3.3 (b) and (c). We conclude this section with some combinatorial properties of admissible subsets.

4.1. **Galleries.**

Definition 4.1. A *gallery* is a sequence $\gamma = (F_0 = \{0\}, A_0 = A_\circ, F_1, A_1, F_2, \ldots, F_n, A_n, F_\infty = \{\mu\})$ such that $A_0, \ldots, A_n$ are alcoves; F_i is a codimension one common face of the alcoves A_{i-1} and A_i, for $i = 1, \ldots, n$; and F_∞ is a vertex of the last alcove A_n. The weight μ is called the *weight* of the gallery and is denoted by $\mu(\gamma)$. The folding operator ϕ_i is the operator which acts on a gallery by leaving its initial segment from A_0 to A_{i-1} intact and by reflecting the remaining tail in the affine hyperplane containing the face F_i. In other words, we define

$$\phi_i(\gamma) := (F_0, A_0, F_1, A_1, \ldots, A_{i-1}, F_i' = F_i, A_i', F_{i+1}', A_{i+1}', \ldots, A_n', F_\infty')\,;$$

here $A_j' := \widehat{t}_i(A_j)$ for $j \in \{i, \ldots, n\}$, $F_j' := \widehat{t}_i(F_j)$ for $j \in \{i, \ldots, n\} \cup \{\infty\}$, and $\widehat{t}_i$ is the affine reflection in the hyperplane containing F_i, as in Theorem 3.3.

The galleries defined above are special cases of the generalized galleries in [6].

Recall that our fixed λ-chain $\Gamma = (\beta_1, \ldots, \beta_n)$ determines a reduced alcove path $A_0 = A_\circ \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_n} A_n = A_{-\lambda}$. This alcove path determines, in turn, an obvious gallery

$$\gamma(\emptyset) = (F_0, A_0, F_1, \ldots, F_n, A_n, F_\infty)$$

of weight $-\lambda$.

Definition 4.2. Given a subset $J = \{j_1 < \cdots < j_s\} \subseteq [n]$, we associate with it the gallery $\gamma(J) := \phi_{j_1} \cdots \phi_{j_s}(\gamma(\emptyset))$. If J is an admissible subset, we call $\gamma(J)$ an *admissible gallery*.

Remarks 4.3. (1) The weight of the gallery $\gamma(J)$, i.e. $\mu(\gamma(J))$, is $-\mu(J)$.

(2) In order to define the gallery $\gamma(J)$, we augmented the index set $[n]$ corresponding to the fixed λ-chain by adding a new minimum 0 and a new maximum ∞. The same procedure is applied when the initial index set is an arbitrary (finite) totally ordered set.

4.2. **Chains of roots.**

Definition 4.4. A *chain of roots* is an object of the form

$$\Gamma = ((\gamma_1, \gamma_1'), \dots, (\gamma_n, \gamma_n'), \gamma_\infty)\,, \tag{4.1}$$

where (γ_i, γ_i') are pairs of roots with $\gamma_i' = \pm\gamma_i$, for $i = 1, \dots, n$, and γ_∞ is a weight. Given a chain of roots Γ and i in $[n]$, we let $t_i := s_{\gamma_i}$ and we define

$$\phi_i(\Gamma) := ((\delta_1, \delta_1'), \dots, (\delta_n, \delta_n'), \delta_\infty))\,,$$

where $\delta_\infty := t_i(\gamma_\infty)$ and

$$(\delta_j, \delta_j') := \begin{cases} (\gamma_j, \gamma_j') & \text{if } j < i \\ (\gamma_j, t_i(\gamma_j')) & \text{if } j = i \\ (t_i(\gamma_j), t_i(\gamma_j')) & \text{if } j > i\,. \end{cases}$$

Our fixed λ-chain $\Gamma = (\beta_1, \dots, \beta_n)$ determines the chain of roots

$$\Gamma(\emptyset) := ((\beta_1, \beta_1), \dots, (\beta_n, \beta_n), \rho)\,;$$

recall that ρ was defined in Subsection 2.1.

Definition 4.5. Given a subset $J = \{j_1 < \cdots < j_s\} \subseteq [n]$, we associate with it the chain of roots $\Gamma(J) := \phi_{j_1} \cdots \phi_{j_s}(\Gamma(\emptyset))$. If J is an admissible subset, we call $\Gamma(J)$ an *admissible folding* (of $\Gamma(\emptyset)$).

Remark 4.6. We can also define folding operators on subsets J of $[n]$ by $\phi_i \,:\, J \mapsto J \triangle \{i\}$, where $\triangle$ denotes the symmetric difference of sets. The folding operators ϕ_i on J, $\gamma(J)$, and $\Gamma(J)$ are commuting involutions (for $J \subseteq [n]$), and their actions are compatible. Throughout this paper, we use J, $\gamma(J)$, and $\Gamma(J)$ interchangeably. We will call the elements of J the *folding positions* in $\gamma(J)$ and $\Gamma(J)$.

Given a fixed subset J of $[n]$, we will now discuss the relationship between the gallery $\gamma(J)$ and the chain of roots $\Gamma(J)$.

Let $\gamma = (F_0, A_0, F_1, \dots, F_n, A_n, F_\infty)$ be an arbitrary gallery. Let $\widehat{t}_i$ be the affine reflection in the common wall of A_{i-1} and A_i, as usual. We associate with γ a chain of roots $\Gamma(\gamma) = ((\gamma_1, \gamma_1'), \dots, (\gamma_n, \gamma_n'), \gamma_\infty)$ as follows:

$$\gamma_i := h(\zeta_{A_{i-1}} - \zeta_{\widehat{t}_i(A_{i-1})})\,, \quad \gamma_i' := h(\zeta_{\widehat{t}_i(A_i)} - \zeta_{A_i})\,, \quad \gamma_\infty := h(\zeta_{A_n} - \mu(\gamma))\,; \tag{4.2}$$

here h is the Coxeter number, $i = 1, \dots, n$, and ζ_A is the central point of the alcove A, as defined in Subsection 2.2. By Proposition 2.2, we have

$$\widehat{t}_i(A_{i-1}) \xrightarrow{\gamma_i} A_{i-1}\,, \qquad A_i \xrightarrow{\gamma_i'} \widehat{t}_i(A_i)\,. \tag{4.3}$$

On the one hand, $\Gamma(\gamma)$ uniquely determines the gallery γ. On the other hand, we have $\Gamma(J) = \Gamma(\gamma(J))$.

Remark 4.7. In [15], we also associated with an admissible subset J a certain piecewise-linear path $\pi(J)$. This is closely related to $\gamma(J)$ and $\Gamma(J)$; essentially, it is obtained from the path joining the central points of the alcoves in the gallery $\gamma(\emptyset)$ via the folding operators used to construct $\gamma(J)$ from $\gamma(\emptyset)$. However, the path $\pi(J)$ is *not* a Littelmann path in general.

4.3. Combinatorial properties. Let J be a fixed admissible subset, and let

$$\gamma(J) = (F_0, A_0, F_1, \ldots, F_n, A_n, F_\infty)\,, \quad \Gamma(J) = ((\gamma_1, \gamma_1'), \ldots, (\gamma_n, \gamma_n'), \gamma_\infty)\,.$$

Let us also fix a simple root α_p. We associate with J the sequence of integers $L(J) = (l_1, \ldots, l_n)$ defined by $F_i \subset H_{-|\gamma_i|, l_i}$ for $i = 1, \ldots, n$. Note that $L(\emptyset) = (l_1^\emptyset, \ldots, l_n^\emptyset)$, as defined in Subsection 3.1. We also define $l_\infty^p := \langle \mu(J), \alpha_p^\vee \rangle$, which means that $F_\infty \subset H_{-\alpha_p, l_\infty^p}$. Finally, we let

$$I(J,p) := \{i \in [n] \mid \gamma_i = \pm\alpha_p\}\,, \quad L(J,p) := (\{l_i\}_{i \in I(J,p)}, l_\infty^p)\,, \quad M(J,p) := \max L(J,p)\,. \tag{4.4}$$

It turns out that $M(J,p) \geq 0$.

Let $I(J,p) = \{i_1 < i_2 < \ldots < i_m\}$. We associate with J and p the sequence $\Sigma(J,p) = (\sigma_1, \ldots, \sigma_{m+1})$, where $\sigma_j := (\operatorname{sgn}(\gamma_{i_j}), \operatorname{sgn}(\gamma_{i_j}'))$ for $j = 1, \ldots, m$, and $\sigma_{m+1} := \operatorname{sgn}(\langle \gamma_\infty, \alpha_p^\vee \rangle)$. We now present some properties of the sequence $\Sigma(J,p)$, which will be used later, and which reflect the combinatorics of admissible subsets, as discussed in [15].

Proposition 4.8. [15] *The sequence $\Sigma(J,p)$ has the following properties:*

- (S1) $\sigma_j \in \{(1,1), (-1,-1), (1,-1)\}$ *for* $j = 1, \ldots, m$;
- (S2) $j = 0$ *or* $\sigma_j = (1,1)$ *implies* $\sigma_{j+1} \in \{(1,1), (1,-1), 1\}$.

The sequence $\Sigma(J,p)$ determines a continuous piecewise-linear function $g_{J,p} : [0, m+\frac{1}{2}] \to \mathbb{R}$ as shown below. By a step (h,k) of a function f at $x = a$, we understand that $f(a+h) = f(a) + k$, and that f is linear between a and $a+h$. We set $g_{J,p}(0) = -\frac{1}{2}$ and, by scanning $\Sigma(J,p)$ from left to right while ignoring brackets, we impose the following condition: the ith entry ± 1 corresponds to a step $(\frac{1}{2}, \pm\frac{1}{2})$ of $g_{J,p}$ at $x = \frac{i-1}{2}$, respectively.

Proposition 4.9. [15] *The function $g_{J,p}$ encodes the sequence $L(J,p)$ as follows:*

$$l_{i_j} = g_{J,p}\left(j - \frac{1}{2}\right),\ j = 1, \ldots, m\,, \quad \text{and} \quad l_\infty^p = g_{J,p}\left(m + \frac{1}{2}\right)\,.$$

Example 4.10. Assume that the entries of $\Gamma(J)$ indexed by the elements of $I(J,p)$ are $(\alpha_p, -\alpha_p)$, $(-\alpha_p, -\alpha_p)$, (α_p, α_p), (α_p, α_p), $(\alpha_p, -\alpha_p)$, $(-\alpha_p, -\alpha_p)$, $(\alpha_p, -\alpha_p)$, (α_p, α_p), in this order; also assume that $\operatorname{sgn}(\langle \gamma_\infty, \alpha_p^\vee \rangle) = 1$. The graph of $g_{J,p}$ is shown in Figure 2; this graph is separated into segments corresponding to the entries of the sequence $\Sigma(J,p)$.

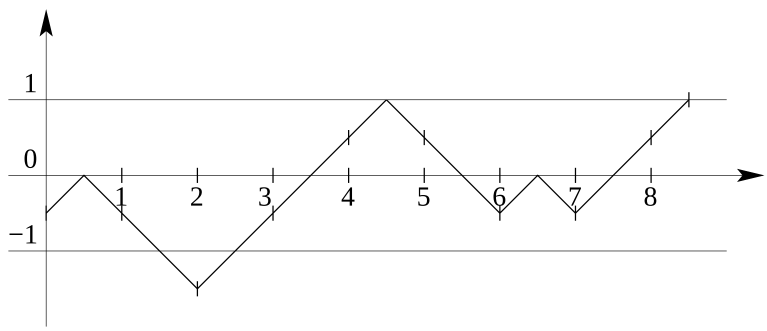

FIGURE 2. The graph of the function $g_{J,p}$ in Example 4.10.

5. Root operators

We now define partial operators known as *root operators* on the collection $\mathcal{A}(\Gamma)$ of admissible subsets corresponding to our fixed λ-chain. They are associated with a fixed simple root α_p, and are traditionally denoted by F_p (also called a lowering operator) and E_p (also called a raising operator). The notation is the one introduced in the previous section.

We first consider F_p on the admissible subset J. This is defined whenever $M(J,p) > 0$. Let $m = m_F(J,p)$ be defined by

$$m_F(J,p) := \begin{cases} \min\{i \in I(J,p) \mid l_i = M(J,p)\} & \text{if this set is nonempty} \\ \infty & \text{otherwise.} \end{cases}$$

Let $k = k_F(J,p)$ be the predecessor of m in $I(J,p) \cup \{\infty\}$, which always exists. It turns out that $m \in J$ if $m \neq \infty$, but $k \notin J$ (cf. Proposition 5.1 below). Finally, we set

$$F_p(J) := (J \setminus \{m\}) \cup \{k\}\,. \tag{5.1}$$

Proposition 5.1. [15] *Given the above setup, the following hold.*

1. *If $m \neq \infty$, then $\gamma'_m = -\gamma_m = -\alpha_p$. We also have $\gamma_k = \gamma'_k = \alpha_p$ and $l_k = M(J,p) - 1$.*
2. *We have $\mu(F_p(J)) = \mu(J) - \alpha_p$.*
3. *We have $w(F_p(J)) = w(J)$ if $m \neq \infty$, and $w(F_p(J)) = s_p w(J)$ otherwise.*

Let us now define a partial inverse E_p to F_p. The operator E_p is defined on the admissible subset J whenever $M(J,p) > \langle \mu(J), \alpha_p^\vee \rangle$. Let $k = k_E(J,p)$ be defined by

$$k_E(J,p) := \max\{i \in I(J,p) \mid l_i = M(J,p)\}\,;$$

the above set turns out to be always nonempty. Let $m = m_E(J,p)$ be the successor of k in $I(J,p) \cup \{\infty\}$. It turns out that $k \in J$ but $m \notin J$ (cf. Proposition 5.2 below). Finally, we set

$$E_p(J) := (J \setminus \{k\}) \cup (\{m\} \setminus \{\infty\})\,. \tag{5.2}$$

Proposition 5.2. [15] *Given the above setup, the following hold.*

1. *We have $\gamma'_k = -\gamma_k = -\alpha_p$. If $m \neq \infty$, then $\gamma_m = \gamma'_m = -\alpha_p$, and $l_m = M(J,p) - 1$.*
2. *We have $\mu(E_p(J)) = \mu(J) + \alpha_p$.*
3. *We have $w(E_p(J)) = w(J)$ if $m \neq \infty$, and $w(E_p(J)) = s_p w(J)$ otherwise.*

In [15], we showed that the above root operators satisfy the axioms of the combinatorial model for Weyl characters in [21]. As a consequence, we deduced the Littlewood-Richardson rule below for decomposing tensor products of irreducible representations. The approach via Stembridge's general model was already applied to LS chains in [21, Section 8]. Compared to the proofs in [6, 17, 19], this approach has the advantage of making a part of the proof independent of a particular model for Weyl characters, by using a system of axioms for such models. For simplicity, we denote the character $ch(V_\lambda)$ by $\chi(\lambda)$.

Theorem 5.3. [15] *We have*

$$\chi(\lambda) \cdot \chi(\nu) = \sum \chi(\nu + \mu(J))\,,$$

where the summation is over all J in $\mathcal{A}(\Gamma)$ satisfying $\langle \nu + \mu(J), \alpha_p^\vee \rangle \geq M(J,p)$ for all $p = 1, \ldots, r$.

Similarly to Kashiwara's operators (see Subsection 2.4), the root operators above define a directed colored graph structure and a poset structure on the set $\mathcal{A}(\Gamma)$ of admissible subsets corresponding to a fixed λ-chain Γ. The following result was proved in [13, 15].

Theorem 5.4. [13, 15] *For any λ-chain Γ, the directed colored graph on the set $\mathcal{A}(\Gamma)$ defined by the root operators is isomorphic to the crystal graph of the irreducible representation V_λ with highest weight λ. Under this isomorphism, the weight of an admissible subset gives the weight space in which the corresponding element of the canonical basis lies.*

To be more precise, in [15] we proved the above theorem for the special λ-chain provided by Proposition 3.4. The general result then follows from Corollary 6.6 below (proved in [13]); this states that the directed colored graph structure on $\mathcal{A}(\Gamma)$ is independent of the λ-chain Γ. Based on Theorem 5.4, we will now identify the elements of the canonical basis with the corresponding admissible subsets.

Define an action of a simple reflection s_p on an admissible subset J by

$$s_p(J) := F_p^{\langle \mu(J), \alpha_p^\vee \rangle}(J)\,. \tag{5.3}$$

Up to the isomorphism in Theorem 5.4, this action coincides with the one on crystals defined by Kashiwara in [9] and [10, Theorem 11.1]; hence it leads to an action of the full Weyl group W.

Corollary 5.5. [13] *Equation* (5.3) *defines a W-action on admissible subsets. We have $\mu(w(J)) = w(\mu(J))$ for all w in W and all admissible subsets J.*

Example 5.6. Figure 3 displays the galleries $\gamma(J)$ corresponding to the admissible subsets J in Example 3.9, the associated paths $\pi(J)$ mentioned in Remark 4.7, as well as the action of the root operators F_p on J. For each path, we shade the fundamental alcove, mark the origin by a white dot "$\circ$", and mark the endpoint of a black dot "$\bullet$". Since some linear steps in $\pi(J)$ might coincide, we display slight deformations of these paths, so that no information is lost in their graphical representations. As discussed above, the weights of the irreducible representation V_{ω_2} are obtained by changing the signs of the endpoints of the paths $\pi(J)$ (marked by black dots). The roots in the corresponding admissible foldings $\Gamma(J)$ can also be read off, as discussed above. At each step, a path $\pi(J)$ either crosses a wall of the affine Coxeter arrangement or bounces off a wall. The associated admissible subset J is the set of indices of bouncing steps in the path.

6. Yang-Baxter Moves

In this section, we define the analog of Schützenberger's sliding algorithm in our model, which we call a *Yang-Baxter move*, for reasons explained below.

We start with some results on dihedral subgroups of Weyl groups. Let $\overline{W}$ be a dihedral Weyl group of order $2q$, that is, a Weyl group of type $A_1 \times A_1$, A_2, B_2, or G_2 (with $q = 2, 3, 4, 6$, respectively). Let $\overline{\Phi}$ be the corresponding root system with simple roots α, β. The sequence

$$\beta_1 := \alpha,\ \ \beta_2 := s_\alpha(\beta),\ \ \beta_3 := s_\alpha s_\beta(\alpha),\ \ \ldots,\ \ \beta_{q-1} := s_\beta(\alpha),\ \ \beta_q := \beta \tag{6.1}$$

is a *reflection ordering* on the positive roots of $\overline{\Phi}$ (cf. [4]). As an illustration, we present the Bruhat order on the Weyl group of type G_2 in Figure 4. Here, as well as throughout this paper, we label a cover $v \lessdot vs_\gamma$ in Bruhat order by the corresponding root γ.

Let us now consider an arbitrary Weyl group W with a dihedral reflection subgroup $\overline{W}$ and corresponding root systems $\Phi \supseteq \overline{\Phi}$. The roots of $\overline{\Phi}$ are denoted as in (6.1), as we let $r_i := s_{\beta_i}$, as above.

Proposition 6.1. *For each pair of elements $u < w$ in the same (left) coset of W modulo $\overline{W}$, there is a unique saturated increasing chain in Bruhat order from u to w whose labels form a subsequence of* (6.1).

It is known that any element w of W can be written uniquely as $w = \lfloor w \rfloor \overline{w}$, where $\lfloor w \rfloor$ is the minimal representative of the left coset $w\overline{W}$, and $\overline{w} \in \overline{W}$. Let

$$u \lessdot ur_{j_1} \lessdot ur_{j_1}r_{j_2} \lessdot \ldots \lessdot ur_{j_1}\ldots r_{j_k} = w\,,$$

be the chain in Bruhat order from u to w provided by Proposition 6.1, where $k = \ell(w) - \ell(u)$. Clearly, the set of indices $\{j_1 < j_2 < \ldots < j_k\}$ only depends on $\overline{u}$ and $\overline{w}$, so we will denote it by $J(\overline{u}, \overline{w})$.

We obtain another reflection ordering by reversing the sequence (6.1). Let us denote the corresponding subset of $[q]$ by $J'(\overline{u}, \overline{w})$. We are interested in passing from the chain between u and w compatible with

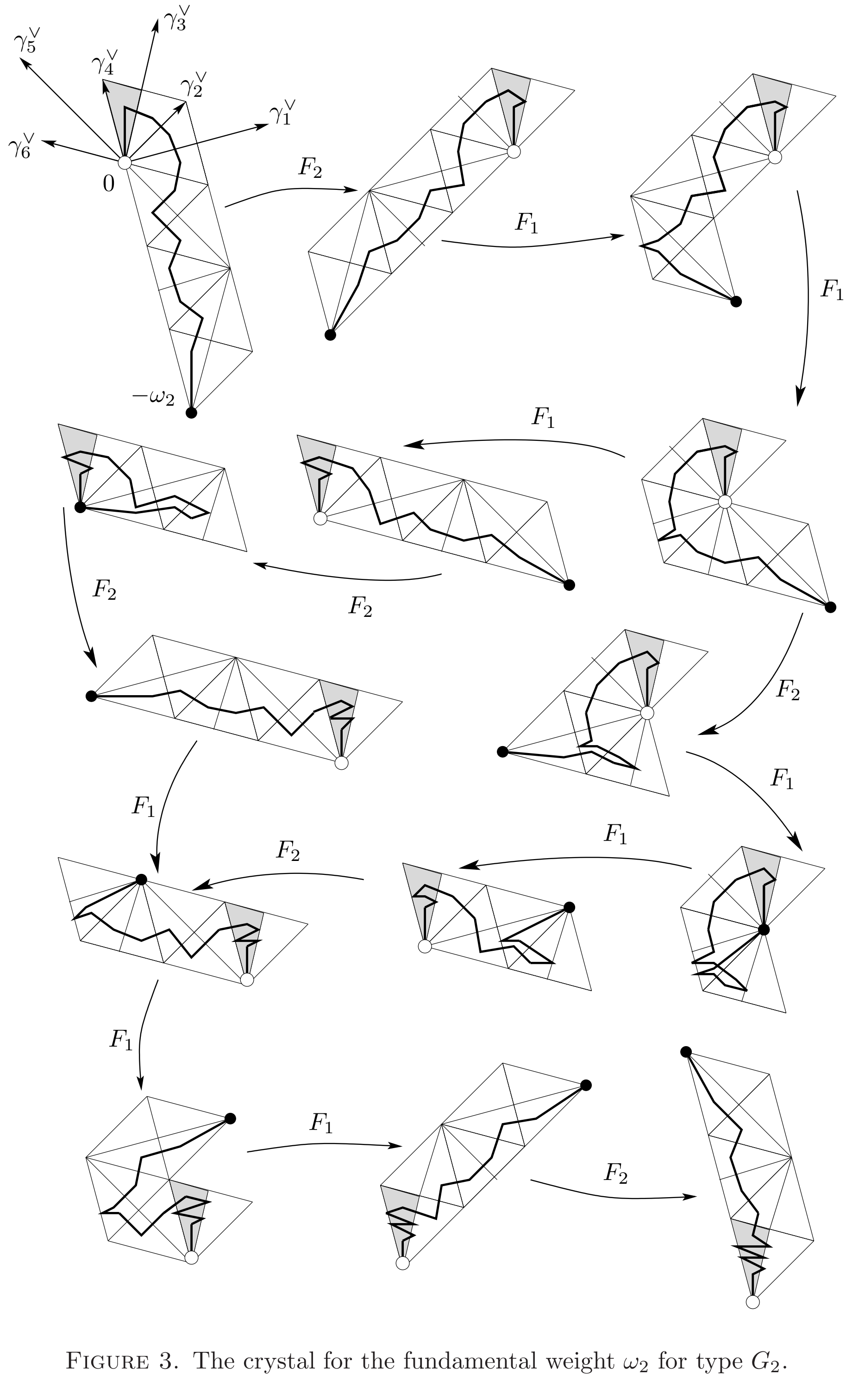

FIGURE 3. The crystal for the fundamental weight ω_2 for type G_2.

the ordering (6.1) to the chain compatible with the reverse ordering. If we fix $a := \ell(\overline{u})$ and $b := \ell(\overline{w})$,

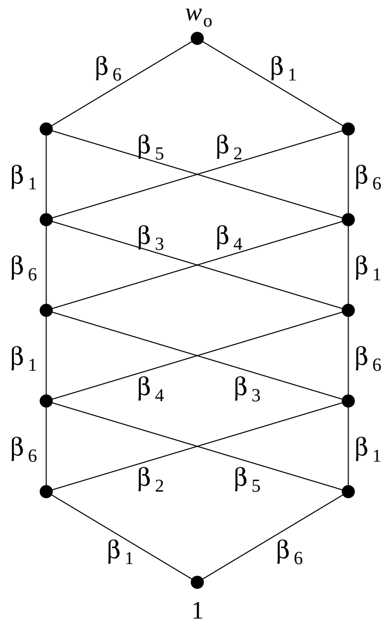

FIGURE 4. The Bruhat order on the Weyl group of type G_2.

we can realize the passage from $J(\overline{u},\overline{w})$ to $J'(\overline{u},\overline{w})$ via the involution $Y_{q,a,b}$ described below.

Case 0: $\emptyset \leftrightarrow \emptyset$ if $a = b$.

Case 1.1: $\{1\} \leftrightarrow \{q\}$ if $0 \le a = b-1 \le q-1$.

Case 1.2: $\{q-a\} \leftrightarrow \{a+1\}$ if $0 < a = b-1 < q-1$.

Case 2.1: $\{1, a+2, a+3, \ldots, b\} \leftrightarrow \{a+1, a+2, \ldots, b-1, q\}$ if $0 \le a < a+2 \le b < q$.

Case 2.2: $\{1, a+2, a+3, \ldots, b-1, q\} \leftrightarrow \{a+1, a+2, \ldots, b\}$ if $0 < a < a+2 \le b \le q$.

Case 3: $[q] \leftrightarrow [q]$ if $a = 0$ and $b = q$.

Let us now consider an index set

$$I := \{\overline{1} < \ldots < \overline{t} < 1 < \ldots < q < \overline{t+1} < \ldots < \overline{n}\},$$

and let $\overline{I} := \{\overline{1}, \ldots, \overline{n}\}$. Let $\Gamma = \{\beta_i\}_{i \in I}$ be a λ-chain, denote $r_i := s_{\beta_i}$ as before, and let $\Gamma' = \{\beta'_i\}_{i \in I}$ be the sequence of roots defined by

$$\beta'_i = \begin{cases} \beta_{q+1-i} & \text{if } i \in I \setminus \overline{I} \\ \beta_i & \text{otherwise}. \end{cases}$$

In other words, the sequence Γ' is obtained from the λ-chain Γ by reversing a certain segment. Now assume that $\{\beta_1, \ldots, \beta_q\}$ are the positive roots of a rank two root system $\overline{\Phi}$ (without repetition). Let $\overline{W}$ be the corresponding dihedral reflection subgroup of the Weyl group W.

Proposition 6.2. [14] (1) *The sequence* Γ' *is also a* λ*-chain, and the sequence* $\beta_1, \ldots, \beta_q$ *is a reflection ordering.*

(2) *We can obtain any* λ*-chain for a fixed dominant weight* λ *from any other* λ*-chain by moves of the form* $\Gamma \to \Gamma'$.

Let us now map the admissible subsets in $\mathcal{A}(\Gamma)$ to those in $\mathcal{A}(\Gamma')$. Given $J \in \mathcal{A}(\Gamma)$, let

$$\overline{J} := J \cap \overline{I}, \quad u := w(J \cap \{\overline{1}, \ldots, \overline{t}\}), \quad \text{and} \quad w := w(J \cap (\{\overline{1}, \ldots, \overline{t}\} \cup [q])). \tag{6.2}$$

Also let

$$u = \lfloor u \rfloor \overline{u}\,, \quad w = \lfloor w \rfloor \overline{w}\,, \quad a := \ell(\overline{u})\,, \quad \text{and} \quad b := \ell(\overline{w})\,, \tag{6.3}$$

as above. It is clear that we have a bijection $Y : \mathcal{A}(\Gamma) \to \mathcal{A}(\Gamma')$ given by

$$Y(J) := \overline{J} \cup Y_{q,a,b}(J \setminus \overline{J})\,. \tag{6.4}$$

We call the moves $J \mapsto Y(J)$ Yang-Baxter moves (see Remark 6.4 (1)). Clearly, a Yang-Baxter move preserves the Weyl group element $w(\,\cdot\,)$ associated to an admissible subset, that is, $w(Y(J)) = w(J)$. In addition, Theorem 6.3 below holds.

Theorem 6.3. [13] *The map Y preserves the weight of an admissible subset. In other words, $\mu(Y(J)) = \mu(J)$ for all admissible subsets J.*

Remark 6.4. Consider the ring $K := \mathbb{Z}[\Lambda/h] \otimes \mathbb{Z}[W]$, where $\mathbb{Z}[W]$ is the group algebra of the Weyl group W, and $\mathbb{Z}[\Lambda/h]$ is the group algebra of $\Lambda/h := \{\lambda/h \mid \lambda \in \Lambda\}$ (i.e., of the weight lattice shrunk h times, h being the Coxeter number defined in Subsection 2.1). In [14], we defined certain $\mathbb{Z}[\Lambda/h]$-linear operators R_α on K, for $\alpha \in \Phi$, and proved that they satisfy the *Yang-Baxter equation* in the sense of Cherednik [2]. The main application was to show that, given a λ-chain $\Gamma = (\beta_1, \ldots, \beta_n)$, we have

$$R_{\beta_n} \ldots R_{\beta_1}(1) = \sum_{J \in \mathcal{A}(\Gamma)} e^{\mu(J)} w(J)\,. \tag{6.5}$$

Due to the Yang-Baxter property, the right-hand side of the above formula does not change when we replace the λ-chain Γ by Γ', as defined above. The Yang-Baxter moves described above implement the passage from Γ to Γ' at the level of the individual terms in (6.5); this justifies the terminology.

We now present the main result related to Yang-Baxter moves.

Theorem 6.5. [13] *The root operators commute with the Yang-Baxter moves, that is, a root operator F_p is defined on an admissible subset J if and only if it is defined on $Y(J)$ and we have*

$$Y(F_p(J)) = F_p(Y(J))\,.$$

Theorem 6.5 asserts that the map Y above is an isomorphism between $\mathcal{A}(\Gamma)$ and $\mathcal{A}(\Gamma')$ as directed colored graphs. Given two arbitrary λ-chains Γ and Γ', we know from Proposition 6.2 (2) that they can be related by a sequence of λ-chains $\Gamma = \Gamma_0, \Gamma_1, \ldots, \Gamma_m = \Gamma'$ to which correspond Yang-Baxter moves $Y_1, \ldots, Y_m$. Hence the composition $Y_m \ldots Y_1$ is an isomorphism between $\mathcal{A}(\Gamma)$ and $\mathcal{A}(\Gamma')$ as directed colored graphs. Since every directed graph $\mathcal{A}(\Gamma)$ has a unique source (cf. [15, Proposition 6.9]), its automorphism group as a directed colored graph consists only of the identity. Thus, we have the following corollary of Theorem 6.5.

Corollary 6.6. [13] *Given two arbitrary λ-chains Γ and Γ', the directed colored graph structures on $\mathcal{A}(\Gamma)$ and $\mathcal{A}(\Gamma')$ are isomorphic. This isomorphism is unique and, therefore, is given by the composition of Yang-Baxter moves corresponding to* any *sequence of λ-chains relating Γ and Γ'.*

We have given a transparent combinatorial explanation for the independence of the directed colored graph defined by our root operators from the chosen λ-chain. Similarly, it was proved in [17] that the directed colored graph structure on Littelmann paths generated by the corresponding root operators is independent of the initial path. However, this proof, which is based on continuous arguments, is less transparent.

7. Schützenberger's Involution

In this section, we present an explicit description of the involution η_λ in Subsection 2.4 in the spirit of Schützenberger's evacuation. We will show that the role of jeu de taquin in the definition of the evacuation map is played by the Yang-Baxter moves.

Throughout the remainder of this paper, we fix an index set $I := \{\overline{1} < \ldots < \overline{q} < 1 < \ldots < n\}$ and a λ-chain $\Gamma = \{\beta_i\}_{i \in I}$ such that $l_i^{\emptyset} = 0$ if and only if $i \in \overline{I} := \{\overline{1} < \ldots < \overline{q}\}$. In other words, the second occurence of a root can never be before the first occurence of another root. We will also write $\Gamma := (\beta_{\overline{1}}, \ldots, \beta_{\overline{q}}, \beta_1, \ldots, \beta_n)$. Let us recall the notation $r_i := s_{\beta_i}$ for $i \in I$.

Given a Weyl group element w, we denote by $\lfloor w \rfloor$ and $\lceil w \rceil$ the minimal and the maximal representatives of the coset wW_λ, respectively (where W_λ is the stabilizer of the weight λ). Let $w_\circ^\lambda$ be the longest element of W_λ. Based on the discussion in Subsection 3.1, it is easy to see that we have the saturated increasing chain in Bruhat order

$$1 \lessdot r_{\overline{1}} \lessdot r_{\overline{1}} r_{\overline{2}} \lessdot \ldots \lessdot r_{\overline{1}} \ldots r_{\overline{q}}$$

from 1 to $\lfloor w_\circ \rfloor = w_\circ w_\circ^\lambda$. Hence the set $J_{\min} := \overline{I}$ is an admissible subset.

Proposition 7.1. [13] *The admissible subset $J_{\min}$ is the minimum of the poset $\mathcal{A}(\Gamma)$.*

Definition 7.2. Let J be an admissible subset. Let $J \cap \overline{I} = \{\overline{j}_1 < \ldots < \overline{j}_a\}$ and $J \setminus \overline{I} = \{j_1 < \ldots < j_s\}$. The *initial key* $\kappa_0(J)$ and the *final key* $\kappa_1(J)$ of J are the Weyl group elements defined by

$$\kappa_0(J) := r_{\overline{j}_1} \ldots r_{\overline{j}_a}\,, \qquad \kappa_1(J) := w(J) = \kappa_0(J) r_{j_1} \ldots r_{j_s}\,.$$

Remark 7.3. The keys $\kappa_0(J)$ and $\kappa_1(J)$ are the analogs of the left and right keys of a semistandard Young tableau, respectively. They are also analogs of the final and the initial direction of an LS chain (cf., e.g., [16]).

We will now present an appropriate way to "reverse" a λ-chain and an associated admissible subset. We associate with our fixed λ-chain Γ another sequence $\Gamma^{\mathrm{rev}} := \{\beta_i'\}_{i \in I}$ by

$$\beta_i' := \begin{cases} \beta_i & \text{if } i \in \overline{I} \\ w_\circ^\lambda(\beta_{n+1-i}) & \text{otherwise}\,. \end{cases}$$

In other words, we have

$$\Gamma^{\mathrm{rev}} = (\beta_{\overline{1}}, \ldots, \beta_{\overline{q}}, w_\circ^\lambda(\beta_n), w_\circ^\lambda(\beta_{n-1}), \ldots, w_\circ^\lambda(\beta_1))\,. \tag{7.1}$$

Proposition 7.4. [13] *Γ^{rev} is a λ-chain.*

Let $r_i' := s_{\beta_i'}$ for $i \in I$. Fix an admissible subset

$$J = \{\overline{j}_1 < \ldots < \overline{j}_a < j_1 < \ldots < j_s\} \tag{7.2}$$

in $\mathcal{A}(\Gamma)$, where $\{\overline{j}_1 < \ldots < \overline{j}_a\} \subseteq \overline{I}$ and $\{j_1 < \ldots < j_s\} \subseteq I \setminus \overline{I}$. Let $u := \kappa_0(J)$ and $w := \kappa_1(J)$. We have the increasing saturated chain

$$1 \lessdot r_{\overline{j}_1} \lessdot r_{\overline{j}_1} r_{\overline{j}_2} \lessdot \ldots \lessdot r_{\overline{j}_1} \ldots r_{\overline{j}_a} = u \lessdot u r_{j_1} \lessdot u r_{j_1} r_{j_2} \lessdot \ldots \lessdot u r_{j_1} \ldots r_{j_s} = w\,. \tag{7.3}$$

According to [4], there is a unique saturated increasing chain in Bruhat order of the form

$$1 \lessdot r'_{\overline{k}_1} \lessdot r'_{\overline{k}_1} r'_{\overline{k}_2} \lessdot \ldots \lessdot r'_{\overline{k}_1} \ldots r'_{\overline{k}_b} = \lfloor w_\circ w \rfloor = w_\circ w w_\circ^\lambda\,,$$

where $\{\overline{k}_1 < \overline{k}_2 < \ldots < \overline{k}_b\} \subseteq \overline{I}$. Define

$$J^{\mathrm{rev}} := \{\overline{k}_1 < \ldots < \overline{k}_b < k_1 < \ldots < k_s\}\,, \tag{7.4}$$

where $k_i := n + 1 - j_{s+1-i}$ for $i = 1, \ldots, s$. Note that $\beta'_{k_i} = w_\circ^\lambda(\beta_{j_{s+1-i}})$ for $i = 1, \ldots, s$.

Proposition 7.5. [13] *J^{rev} is an admissible subset in $\mathcal{A}(\Gamma^{\mathrm{rev}})$. We have*

$$\kappa_0(J^{\mathrm{rev}}) = \lfloor w_\circ \kappa_1(J) \rfloor\,, \quad \kappa_1(J^{\mathrm{rev}}) = \lfloor w_\circ \kappa_0(J) \rfloor\,, \tag{7.5}$$

as well as $(J^{\mathrm{rev}})^{\mathrm{rev}} = J$.

We now present a direct way to obtain the gallery $\gamma(J^{\rm rev})$ from $\gamma(J)$. Let us write

$$\gamma(J) = (F_{\overline{0}}, A_{\overline{0}}, F_{\overline{1}}, \ldots, F_{\overline{q}}, A_0, F_1, A_1, \ldots, A_n, F_\infty)\,;$$

the corresponding augmented index set is $\{\overline{0} < \overline{1} < \ldots < \overline{q} = 0 < 1 < \ldots < n < \infty\}$. Let $\mu := -\mu(J)$, that is, $F_\infty = \{\mu\}$. Now define another gallery in the following way:

$$\gamma^\omega := (F'_{\overline{0}}, A'_{\overline{0}}, F'_{\overline{1}}, \ldots, F'_{\overline{q}}, A'_0, F'_1, A'_1, \ldots, A'_n, F'_\infty)\,.$$

The notation is as follows:

- ω is the map on $\mathfrak{h}^*_{\mathbb{R}}$ defined by $x \mapsto -w_\circ(x-\mu)$;
- $A'_i := \omega(A_{n-i})$ for $i = 0, \ldots, n$, $F'_i := \omega(F_{n+1-i})$ for $i = 1, \ldots, n$, and $F'_\infty = \{w_\circ(\mu)\}$;
- $(F'_{\overline{0}}, A'_{\overline{0}}, F'_{\overline{1}}, \ldots, F'_{\overline{q}})$ is the initial segment of the gallery $\gamma(J^{\rm rev})$.

In fact, it is not obvious that γ^ω is a gallery, but the justification is not hard either.

Proposition 7.6. [13] *The gallery γ^ω coincides with $\gamma(J^{\rm rev})$. In particular, we have $\mu(J^{\rm rev}) = w_\circ(\mu(J))$.*

We will now present the main result related to the map $J \mapsto J^{\rm rev}$, which involves its commutation with the root operators.

Theorem 7.7. [13] *A root operator F_p is defined on the admissible subset J if and only if E_{p^*} is defined on $J^{\rm rev}$, and we have*

$$(F_p(J))^{\rm rev} = E_{p^*}(J^{\rm rev})\,.$$

We can summarize the construction in this section (based on Propositions 7.4 and 7.5) as follows: given the λ-chain Γ (for a fixed dominant weight λ), we defined the λ-chain $\Gamma^{\rm rev}$, and given $J \in \mathcal{A}(\Gamma)$, we defined $J^{\rm rev} \in \mathcal{A}(\Gamma^{\rm rev})$. Hence we can map $J^{\rm rev}$ to an admissible subset $J^* \in \mathcal{A}(\Gamma)$ using Yang-Baxter moves, as it is described in Section 6 and it is recalled below. To be more precise, let $R : \mathcal{A}(\Gamma) \to \mathcal{A}(\Gamma^{\rm rev})$ denote the bijection $J \mapsto J^{\rm rev}$. On the other hand, we know from Proposition 6.2 (2) that the λ-chains $\Gamma^{\rm rev}$ and Γ can be related by a sequence of λ-chains $\Gamma^{\rm rev} = \Gamma_0, \Gamma_1, \ldots, \Gamma_m = \Gamma$ to which correspond Yang-Baxter moves $Y_1, \ldots, Y_m$. By Corollary 6.6, the composition $Y := Y_m \ldots Y_1$ does not depend on the sequence of intermediate λ-chains, and it defines a bijection from $\mathcal{A}(\Gamma^{\rm rev})$ to $\mathcal{A}(\Gamma)$. We let $J^* := YR(J)$ and conclude that it is a bijection on $\mathcal{A}(\Gamma)$. The main result of this section now follows directly from Theorems 6.5 and 7.7.

Theorem 7.8. [13] *The bijection $J \mapsto J^*$ constructed above coincides with the fundamental involution η_λ on the canonical basis. In other words, a root operator F_p is defined on the admissible subset J if and only if E_{p^*} is defined on J^*, and we have*

$$(J_{\min})^* = J_{\max}\,, \quad (J_{\max})^* = J_{\min}\,, \quad \text{and} \quad (F_p(J))^* = E_{p^*}(J^*)\,, \;\; \text{for } p = 1, \ldots, r\,. \tag{7.6}$$

In particular, the map $J \mapsto J^*$ expresses combinatorially the self-duality of the poset $\mathcal{A}(\Gamma)$.

Remark 7.9. The above construction is analogous to the definition of Schützenberger's evacuation map (see, for instance, [5]). Below, we recall from Subsection 2.4 the three-step procedure defining this map and we discuss the analogy with our construction in the case of each step.

(1) REVERSE: We rotate a given semistandard Young tableau by 180°. This corresponds to reversing its word, in the same way as we reversed the direction of our gallery, cf. Proposition 7.6.
(2) COMPLEMENT: We complement each entry via the map $i \mapsto w_\circ(i)$, where $w_\circ$ is the longest element in the corresponding symmetric group. This corresponds to using $w_\circ$ for the arbitrary Weyl group in the definition (7.4) of $J^{\rm rev}$.
(3) SLIDE: We apply jeu de taquin on the obtained skew tableau. This corresponds to the Yang-Baxter moves $Y_1, \ldots, Y_m$ discussed above.

Example 7.10. Consider the Lie algebra $\mathfrak{sl}_3$ of type A_2. We use the setup in Example 3.8.

Consider the dominant weight $\lambda = 4\varepsilon_1 + 2\varepsilon_2$ and the following λ-chain:

$$\begin{array}{ccccccccc} & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 \\ \Gamma = (& \alpha_{12}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & \underline{\alpha_{12}}, & \alpha_{13}, & \underline{\alpha_{23}}, & \alpha_{13}) \,. \end{array}$$

Here we indicated the index corresponding to each root, using the notation at the beginning of this section; more precisely, we have $I = \{\overline{1} < \overline{2} < \overline{3} < 1 < 2 < 3 < 4 < 5\}$ and $\overline{I} = \{\overline{1} < \overline{2} < \overline{3}\}$. By the defining relation (7.1), we have

$$\begin{array}{ccccccccc} & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 \\ \Gamma^{\rm rev} = (& \underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & \underline{\alpha_{23}}, & \alpha_{13}, & \underline{\alpha_{12}}, & \alpha_{13}) \,. \end{array}$$

Consider the admissible subset $J = \{2, 4\}$. This is indicated above by the underlined roots in Γ. In order to define $J^{\rm rev}$, cf. (7.4), we need to compute

$$\kappa_0(J^{\rm rev}) = w_\circ w(J) = (s_{\alpha_{12}} s_{\alpha_{23}} s_{\alpha_{12}})(s_{\alpha_{12}} s_{\alpha_{23}}) = s_{\alpha_{12}} \,.$$

Hence we have $J^{\rm rev} = \{\overline{1}, 2, 4\}$. This is indicated above by the underlined positions in $\Gamma^{\rm rev}$.

In order to transform the λ-chain $\Gamma^{\rm rev}$ into Γ, we need to perform a single Yang-Baxter move; this consists of reversing the order of the bracketed roots below:

$$\begin{array}{cccccccccc} & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 & \\ \Gamma^{\rm rev} = (& \underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & (\underline{\alpha_{23}}, & \alpha_{13}, & \underline{\alpha_{12}}), & \alpha_{13}) & \longrightarrow \\ & \overline{1} & \overline{2} & \overline{3} & 1 & 2 & 3 & 4 & 5 & \\ \Gamma \quad = (& \underline{\alpha_{12}}, & \alpha_{13}, & \alpha_{23}, & \alpha_{13}, & (\alpha_{12}, & \underline{\alpha_{13}}, & \underline{\alpha_{23}}), & \alpha_{13}) \,. & \end{array}$$

The underlined roots indicate the way in which the Yang-Baxter move $J^{\rm rev} \mapsto Y(J^{\rm rev}) = J^*$ works. All we need to know is that there are two saturated chains in Bruhat order between the permutations u and w, cf. the notation in (6.2):

$$u = s_{\alpha_{12}} \lessdot s_{\alpha_{12}} s_{\alpha_{23}} \lessdot s_{\alpha_{12}} s_{\alpha_{23}} s_{\alpha_{12}} = w \,, \quad u = s_{\alpha_{12}} \lessdot s_{\alpha_{12}} s_{\alpha_{13}} \lessdot s_{\alpha_{12}} s_{\alpha_{13}} s_{\alpha_{23}} = w \,.$$

The first chain is retrieved as a subchain of $\Gamma^{\rm rev}$ and corresponds to $J^{\rm rev}$, while the second one is retrieved as a subchain of Γ and corresponds to J^*. Hence we have $J^* = \{\overline{1}, 3, 4\}$.

We can give an intrinsic explanation for the fact that the map $J \mapsto J^*$ is an involution on $\mathcal{A}(\Gamma)$; this explanation is only based on the results in Sections 6 and 7, so it does not rely on Proposition 2.3 (2). Let us first recall the bijections $R : \mathcal{A}(\Gamma) \to \mathcal{A}(\Gamma^{\rm rev})$ and $Y : \mathcal{A}(\Gamma^{\rm rev}) \to \mathcal{A}(\Gamma)$ defined above. We claim that $YR = R^{-1}Y^{-1}$, which would prove that the composition YR is an involution. In the same way as we proved Theorem 7.8 (that is, as a direct consequence of Theorems 6.5 and 7.7), we can verify that the composition $R^{-1}Y^{-1}$ satisfies the conditions in (7.6). Since these conditions uniquely determine the corresponding map from $\mathcal{A}(\Gamma)$ to itself, our claim follows.

Remark 7.11. According to the above discussion, we have a second way of realizing the fundamental involution η_λ on the canonical basis, namely as $R^{-1}Y^{-1}$. In some sense, this is the analog of the construction of the evacuation map based on the *promotion* operation, in which the sliding operations precede the complementation (see, for instance, [5, p. 184]).

We have the following corollary of Propositions 7.5 and 7.6.

Corollary 7.12. [13] *For any $J \in \mathcal{A}(\Gamma)$, we have*

$$\mu(J^*) = w_\circ(\mu(J)) \,, \quad \kappa_0(J^*) = \lfloor w_\circ \kappa_1(J) \rfloor \,, \quad \kappa_1(J^*) = \lfloor w_\circ \kappa_0(J) \rfloor \,. \tag{7.7}$$

Remark 7.13. It would be interesting to check whether the triple $(\mu(J),\, \kappa_0(J),\, \kappa_1(J))$ uniquely determines J. This would be an analog of Dyer's result [4] mentioned several times above. If this is true, then the involution $J \mapsto J^*$ is also determined by the conditions (7.7), as opposed to (7.6). We would thus have a short proof of Theorem 7.8 that does not depend on Theorems 6.5 and 7.7, but only on Corollary 7.12.

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