THE ALCOVE PATH MODEL AND TABLEAUX

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Abstract. The second author and Postnikov have recently constructed a simple combinatorial model for the characters of the irreducible representations of a complex semisimple Lie group, that is referred to as the alcove path model. In this paper we relate the alcove path model to the the semistandard Young tableaux in type A and the Kashiwara-Nakashima tableaux in type C. More explicitly, we construct bijections between the objects in the alcove path model (certain saturated chains in the Bruhat order on the corresponding Weyl group) and the corresponding tableaux. We show that this bijection preserves the corresponding crystal structures, and we give applications to Demazure characters and basis constructions.

1. Introduction

The second author and Postnikov have recently constructed a simple combinatorial model for the characters of the irreducible representations of a complex semisimple Lie group G and, more generally, for the Demazure characters [9]. For reasons explained below, this model is called the alcove path model. This was extended to complex symmetrizable Kac-Moody algebras in [10] (that is, to infinite root systems), and its combinatorics was investigated in more detail in [7].

There are other models in this area, such as: semistandard Young tableaux [2, 14, 16] and Kashiwara-Nakashima tableaux [15], Littelmann paths [11, 12, 13, 4], LS-galleries [3], the model in [1] based on Lusztig's parametrization of canonical bases, some models based on geometric constructions etc. The alcove path model has advantages related to its generality, simplicity, combinatorial nature, and other applications, such as Demazure modules (which form a filtration of the irreducible modules) and Schubert calculus. In particular, it leads to a far-reaching generalization of the type A combinatorics of Young tableaux. The second author has developed a Maple software package for manipulations based on the alcove path model [8].

The alcove path model is based on enumerating certain saturated chains in the Bruhat order on the corresponding Weyl group W. This enumeration is determined by an alcove path, which is a sequence of adjacent alcoves for the affine Weyl group W_{aff} of the Langlands dual group $G^\vee$. Alcove paths correspond to decompositions of elements in the affine Weyl group. Based on this model, one can extend to arbitrary root systems a considerable part of the very rich combinatorics of semistandard Young tableaux, which are classical combinatorial objects in the representation theory of SL_n. More precisely, we have:

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(1) cancellation free character formulas, including Demazure character formulas;
(2) a Littlewood-Richardson rule for decomposing tensor products of irreducible representations and a branching rule;
(3) a combinatorial description of the crystal graphs corresponding to the irreducible representations;
(4) a combinatorial realization of certain fundamental involution on the canonical basis, which exhibits the crystals as self-dual posets, corresponds to the action of the longest Weyl group element on an irreducible representation, and generalizes Schützenberger's involution on tableaux;
(5) a generalization to arbitrary root systems of Schützenberger's sliding algorithm (also known as jeu de taquin), which has many applications to the representation theory of the Lie algebra of type A.

The goal of this paper is to relate the alcove path model to the semistandard Young tableaux in type A and the Kashiwara-Nakashima tableaux in type C. In type A, we construct an explicit bijection between the objects in the two models that is compatible with the corresponding crystal structures. Applications will include an efficient construction of a basis for an irreducible representation. We also construct the bijection in type C to Kashiwara-Nakashima tableaux, and describe an algorithm for constructing its inverse. We are currently working on extending the crystal property and the basis construction from type A to type C. Future applications will also include Demazure characters.

2. Background and Notation

2.1. **Tableaux.** In this section we recall the background on semistandard Young tableaux, and we refer the reader to [2, 14, 16] for more details. We also recall the tableaux of Kashiwara and Nakashima in type C [15].

A Young diagram is a sequence of left justified boxes in rows with the lengths of rows weakly decreasing. A *Young tableau* is then a filling of a Young diagram where numbers are placed in each box such that the entries are weakly increasing across rows and strictly increasing down columns.

The shape λ of a tableau T is given by $\lambda = (\lambda_1, ..., \lambda_k)$ where λ_i is the length of the i^{th} row and k is the number of rows. The shape of the conjugate tableau is given by $\lambda' = (\lambda'_1, ..., \lambda'_m)$ where λ'_i is the length of the i^{th} column and m is the number of columns, [2].

In type C_n Kashiwara-Nakashima or KN tableaux will be one of our primary objects of interest. We begin by defining columns:

A column is a Young diagram C of column shape filled with letters from $[\overline{n}]$ strictly increasing from top to bottom.

Here the set $[\overline{n}]$ has the following order:

$$1 < 2 < \cdots < n < \overline{n} < \cdots < \overline{2} < \overline{1} \tag{2.1}$$

Note that this is not the standard order.

The ith entry of a column C shall be denoted $C(i)$.

For a column C let $I = \{z_1 > ... > z_r\}$ be the set of unbarred letters z such that the pair $(z, \overline{z})$ occurs in C. The column C is said to split when there exists a set of r unbarred letters $J = \{t_1 > ... > t_r\} \subset [\overline{n}]$ such that:

(1) t_1 is the greatest letter of $\overline{n}$ satisfying: $t_1 < z_1, t_1 \notin C$ and $\overline{t}_1 \notin C$
(2) for $i = 2, ..., r$, t_i is the greatest letter of $\overline{n}$ satisfying: $t_i < min(t_{i-1}, z_i), t_i \notin C$ and $\overline{t}_i \notin C$.

In the case where the column C may be split we write:

rC for the column obtained by changing in C, $\overline{z}_i$ into $\overline{t}_i$ for each letter $z_i \in I$ and by reordering if necessary,

lC for the column obtained by changing in C, z_i into t_i for each letter $z_i \in I$ and by reordering if necessary.

A column is KN-admissible if and only if it can be split.

Example 2.2.

$$C = \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 5 \\ \hline \overline{5} \\ \hline \overline{2} \\ \hline \end{array}$$

Then $I = \{5, 2\}$ *and* $J = \{4, 1\}$ *and*

$$lC = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \overline{5} \\ \hline \overline{2} \\ \hline \end{array}, rC = \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 5 \\ \hline \overline{4} \\ \hline \overline{1} \\ \hline \end{array}.$$

We place a partial order on columns of length k by giving them the order given by pairwise comparison of entries. For a pair of columns C and D of length k, $C \leq D$ provided that $C(i) \leq D(i)$ for all i such that $1 \leq i \leq k$.

Let σ be a permutation of length $2n$ in type C_n and let C be a column of a tableau of this type of length l for some $1 \leq l \leq n$ such that $\sigma[l] = C$. Here we define $\sigma[k]$ to be the restriction of σ to it's first k entries. Similarly define $\sigma[i, j]$ to be the restriction of σ to positions i through j.

Start with a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n > 0)$ and the conjugate partition $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq ... \geq \lambda'_m > 0)$. Then a KN tableau T of shape λ will be a tableau with n rows where for $1 \leq i \leq n$ the length of row i is λ_i, or equivalently T is a tableau with columns such that for $1 \le j \le m$ the column j has length λ'_j. In order for T to be a KN tableau the column j which we shall denote C_j must be a KN-admissible column, i.e. the column C_j must split.

It will be convenient for us to view T as having $2m$ columns rather than m columns. The way this is done is by replacing each column C_j with the pair of columns lC_j and rC_j. We shall refer to such a tableau as the doubled tableau T which we shall also refer to as T as there is a clear correspondence between tableaux and doubled tableaux.

2.2. **The Alcove Path Model.** In this section, we recall the alcove path model in the representation theory of semisimple Lie algebras, closely following [9, 10].

Let G be a connected, simply connected, simple complex Lie group. Fix a Borel subgroup B and a maximal torus T such that $G \supset B \supset T$. As usual, we denote by B^- be the opposite Borel subgroup, while N and N^- are the unipotent radicals of B and B^-, respectively. Let $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{n}$, and $\mathfrak{n}^-$ be the complex Lie algebras of G, T, N, and N^-, respectively. Let r be the rank of the Cartan subalgebra $\mathfrak{h}$. Let $\Phi \subset \mathfrak{h}^*$ be the corresponding irreducible *root system*, and let $\mathfrak{h}^*_{\mathbb{R}} \subset \mathfrak{h}^*$ be the real span of the roots. Let $\Phi^+ \subset \Phi$ be the set of positive roots corresponding to our choice of B. Then Φ is the disjoint union of Φ^+ and $\Phi^- = -\Phi^+$. We write $\alpha > 0$ (respectively, $\alpha < 0$) for $\alpha \in \Phi^+$ (respectively, $\alpha \in \Phi^-$), and we define $\mathrm{sgn}(\alpha)$ to be 1 (respectively, -1). We also use the notation $|\alpha| := \mathrm{sgn}(\alpha)\alpha$. Let $\alpha_1, \ldots, \alpha_r \in \Phi^+$ be the corresponding *simple roots*, which form a basis of $\mathfrak{h}^*_{\mathbb{R}}$. Let $\langle \cdot\,, \cdot \rangle$ denote the nondegenerate scalar product on $\mathfrak{h}^*_{\mathbb{R}}$ induced by the Killing form. Given a root α, the corresponding *coroot* is $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$. The collection of coroots $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$ forms the *dual root system.*

The *Weyl group* $W \subset \mathrm{Aut}(\mathfrak{h}^*_{\mathbb{R}})$ of the Lie group G is generated by the reflections $s_\alpha : \mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}$, for $\alpha \in \Phi$, given by

$$s_\alpha : \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \, \alpha.$$

In fact, the Weyl group W is generated by the *simple reflections* $s_1, \ldots, s_r$ corresponding to the simple roots $s_i := s_{\alpha_i}$, subject to the *Coxeter relations:*

$$(s_i)^2 = 1 \quad \text{and} \quad (s_i s_j)^{m_{ij}} = 1 \quad \text{for any } i, j \in \{1, \ldots, r\},$$

where m_{ij} is half of the order of the dihedral subgroup generated by s_i and s_j. An expression of a Weyl group element w as a product of generators $w = s_{i_1} \cdots s_{i_l}$ which has minimal length is called a *reduced decomposition* for w; its length $\ell(w) = l$ is called the *length* of w. The Weyl group contains a unique *longest element* $w_\circ$ with maximal length $\ell(w_\circ) = \#\Phi^+$. For $u, w \in W$, we say that u *covers* w, and write $u \gtrdot w$, if $w = u s_\beta$, for some $\beta \in \Phi^+$, and $\ell(u) = \ell(w) + 1$. The transitive closure "$>$" of the relation "$\gtrdot$" is called the *Bruhat order* on W.

The *weight lattice* Λ is given by

$$\Lambda := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for any } \alpha \in \Phi\}. \tag{2.3}$$

The weight lattice Λ is generated by the *fundamental weights* $\omega_1, \ldots, \omega_r$, which are defined as the elements of the dual basis to the basis of simple coroots, i.e., $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. The set Λ^+ of *dominant weights* is given by

$$\Lambda^+ := \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for any } \alpha \in \Phi^+\}.$$

Let $\rho := \omega_1 + \cdots + \omega_r = \frac{1}{2}\sum_{\beta \in \Phi^+} \beta$. The *height* of a coroot $\alpha^\vee \in \Phi^\vee$ is $\langle \rho, \alpha^\vee \rangle = c_1 + \cdots + c_r$ if $\alpha^\vee = c_1\alpha_1^\vee + \cdots + c_r\alpha_r^\vee$. Since we assumed that Φ is irreducible, there is a unique *highest coroot* $\theta^\vee \in \Phi^\vee$ that has maximal height. (In other words, $\theta^\vee$ is the highest root of the dual root system $\Phi^\vee$. It should not be confused with the coroot of the highest root of Φ.) We will also use the *Coxeter number*, that can be defined as $h := \langle \rho, \theta^\vee \rangle + 1$.

Let W_{aff} be the *affine Weyl group* for the Langlands dual group $G^\vee$. The affine Weyl group W_{aff} is generated by the affine reflections $s_{\alpha,k} : \mathfrak{h}_\mathbb{R}^* \to \mathfrak{h}_\mathbb{R}^*$, for $\alpha \in \Phi$ and $k \in \mathbb{Z}$, that reflect the space $\mathfrak{h}_\mathbb{R}^*$ with respect to the affine hyperplanes

$$H_{\alpha,k} := \{\lambda \in \mathfrak{h}_\mathbb{R}^* \mid \langle \lambda, \alpha^\vee \rangle = k\}. \tag{2.4}$$

Explicitly, the affine reflection $s_{\alpha,k}$ is given by

$$s_{\alpha,k} : \lambda \mapsto s_\alpha(\lambda) + k\,\alpha = \lambda - (\langle \lambda, \alpha^\vee \rangle - k)\,\alpha.$$

The hyperplanes $H_{\alpha,k}$ divide the real vector space $\mathfrak{h}_\mathbb{R}^*$ into open regions, called *alcoves.* Each alcove A is given by inequalities of the form

$$A := \{\lambda \in \mathfrak{h}_\mathbb{R}^* \mid m_\alpha < \langle \lambda, \alpha^\vee \rangle < m_\alpha + 1 \text{ for all } \alpha \in \Phi^+\},$$

where $m_\alpha = m_\alpha(A)$, $\alpha \in \Phi^+$, are some integers.

Definition 2.5. *A λ-chain of roots is a sequence of positive roots $(\beta_1, \ldots, \beta_n)$ which is determined as indicated below by a reduced decomposition $v_{-\lambda} = s_{i_1} \cdots s_{i_n}$ of $v_{-\lambda}$ as a product of generators of W_{aff}:*

$$\beta_1 = \alpha_{i_1},\ \beta_2 = \bar{s}_{i_1}(\alpha_{i_2}),\ \beta_3 = \bar{s}_{i_1}\bar{s}_{i_2}(\alpha_{i_3}), \ldots,\ \beta_n = \bar{s}_{i_1} \cdots \bar{s}_{i_{n-1}}(\alpha_{i_n})\,.$$

When the context allows, we will abbreviate "λ-chain of roots" to "λ-chain". The λ-chain of reflections associated with the above λ-chain of roots is the sequence $(\widehat{r}_1, \ldots, \widehat{r}_n)$ of affine reflections in W_{aff} given by

$$\widehat{r}_1 = s_{i_1},\ \widehat{r}_2 = s_{i_1}s_{i_2}s_{i_1},\ \widehat{r}_3 = s_{i_1}s_{i_2}s_{i_3}s_{i_2}s_{i_1},\ \ldots,\ \widehat{r}_n = s_{i_1} \cdots s_{i_n} \cdots s_{i_1}.$$

We will present two equivalent definitions of a λ-chain of roots.

Definition 2.6. *An alcove path is a sequence of alcoves $(A_0, A_1, \ldots, A_n)$ such that A_{i-1} and A_i are adjacent, for $i = 1, \ldots, n$. We say that an alcove path is reduced if it has minimal length among all alcove paths from A_0 to A_n.*

Given a finite sequence of roots $\Gamma = (\beta_1, \ldots, \beta_n)$, we define the sequence of integers $(l_1^\emptyset, \ldots, l_n^\emptyset)$ by $l_i^\emptyset := \#\{j < i \mid \beta_j = \beta_i\}$, for $i = 1, \ldots, n$. We also need the following two conditions on Γ.

- (R1) The number of occurrences of any positive root α in Γ is $\langle \lambda, \alpha^\vee \rangle$.
- (R2) For each triple of positive roots (α, β, γ) with $\gamma^\vee = \alpha^\vee + \beta^\vee$, the subsequence of Γ consisting of α, β, γ is a concatenation of pairs (α, γ) and (β, γ) (in any order).

Definition 2.7. *An* admissible subset *is a subset of* $[n] := \{1, \dots, n\}$ *(possibly empty), that is,* $J = \{j_1 < j_2 < \dots < j_s\}$*, such that we have the following saturated chain in the Bruhat order on* W*:*

$$1 \lessdot r_{j_1} \lessdot r_{j_1} r_{j_2} \lessdot \dots \lessdot r_{j_1} r_{j_2} \dots r_{j_s} .$$

We denote by $\mathcal{A} = \mathcal{A}(\Gamma)$ *the collection of all admissible subsets corresponding to our fixed* λ*-chain* Γ*. Given an admissible subset* J*, we use the notation*

$$\mu(J) := -\widehat{r}_{j_1} \dots \widehat{r}_{j_s}(-\lambda) , \qquad w(J) := r_{j_1} \dots r_{j_s} .$$

We call $\mu(J)$ *the* weight *of the admissible subset* J*.*

We have the following character formula in terms of admissible subsets:

$$ch(V_\lambda) = \sum_{J \in \mathcal{A}} e^{\mu(J)}.$$

Let $U(\mathfrak{g})$ be the enveloping algebra of the Lie algebra $\mathfrak{g}$, which is generated by E_i, F_i, H_i, for $i = 1, \dots, r$, subject to the Serre relations. Let $\mathcal{B}$ be the *canonical basis* of $U(\mathfrak{n}^-)$, and let $\mathcal{B}_\lambda := \mathcal{B} \cap V_\lambda$ be the canonical basis of the irreducible representation V_λ with highest weight λ. Let v_λ and v_λ^{low} be the highest and lowest weight vectors in $\mathcal{B}_\lambda$, respectively. Let $\widetilde{E}_i$, $\widetilde{F}_i$, for $i = 1, \dots, r$, be Kashiwara's operators; these are also known as raising and lowering operators, respectively. The *crystal graph* of V_λ is the directed colored graph on $\mathcal{B}_\lambda$ defined by arrows $x \to y$ colored i for each $\widetilde{F}_i(x) = y$.

We now define partial operators known as *root operators* on the collection $\mathcal{A}(\Gamma)$ of admissible subsets corresponding to our fixed λ-chain. They are associated with a fixed simple root α_p, and are traditionally denoted by F_p (also called a lowering operator) and E_p (also called a raising operator).

Let J be a fixed admissible subset, and let

$$\gamma(J) = (F_0, A_0, F_1, \dots, F_n, A_n, F_\infty) , \qquad \Gamma(J) = ((\gamma_1, \gamma_1'), \dots, (\gamma_n, \gamma_n'), \gamma_\infty) .$$

Let us also fix a simple root α_p. We associate with J the sequence of integers $L(J) = (l_1, \dots, l_n)$ defined by $F_i \subset H_{-|\gamma_i|, l_i}$ for $i = 1, \dots, n$. We also define $l_\infty^p := \langle \mu(J), \alpha_p^\vee \rangle$, which means that $F_\infty \subset H_{-\alpha_p, l_\infty^p}$. Finally, we let

$$I(J,p) := \{i \mid \delta_i = -\alpha_p\} , \quad L(J,p) := (\{l_i\}_{i \in I(J,p)}, l_\infty^p) , \quad M(J,p) := \max L(J,p) . \tag{2.8}$$

We first consider F_p on the admissible subset J. This is defined whenever $M(J,p) > 0$. Let $m = m_F(J,p)$ be defined by

$$m_F(J,p) := \begin{cases} \min \{i \in I(J,p) \mid l_i = M(J,p)\} & \text{if this set is nonempty} \\ \infty & \text{otherwise} . \end{cases}$$

Let $k = k_F(J,p)$ be the predecessor of m in $I(J,p) \cup \{\infty\}$, which always exists. It turns out that $m \in J$ if $m \neq \infty$, but $k \notin J$. Finally, we set

$$F_p(J) := (J \setminus \{m\}) \cup \{k\} . \tag{2.9}$$

Let us now define a partial inverse E_p to F_p. The operator E_p is defined on the admissible subset J whenever $M(J,p) > \langle \mu(J), \alpha_p^\vee \rangle$. Let $k = k_E(J,p)$ be defined by

$$k_E(J,p) := \max\{i \in I(J,p) \mid l_i = M(J,p)\};$$

the above set turns out to be always nonempty. Let $m = m_E(J,p)$ be the successor of k in $I(J,p) \cup \{\infty\}$. It turns out that $k \in J$ but $m \notin J$. Finally, we set

$$E_p(J) := (J \setminus \{k\}) \cup (\{m\} \setminus \{\infty\}). \qquad (2.10)$$

2.3. **Root Systems and Weyl Groups in types A_n and C_n.** In type A_{n-1} the Weyl group is S_n. Consider the $n-1$ dimensional subspace of $\mathbb{R}^n$ orthogonal to the vector $e_1 + ... + e_n$ where e_i for $i \in [n]$ are the basis vectors of $\mathbb{R}^n$. Then the root system $\Phi = \{e_i - e_j, i \neq j, i \text{ and } j \in [n]\}$. We shall also need the following notions for the Bruhat order on S_n:

Let t_{ab} be the transposition sending (a,b) to (b,a). The covering relations in the Bruhat order are $v \lessdot w = v \cdot t_{ab}$, where $\ell(w) = \ell(v) + 1$. We denote this by

$$v \xrightarrow{t_{ab}} w\,.$$

A permutation v admits a cover $v \lessdot v \cdot t_{ab}$ with $a < b$ and $v(a) < v(b)$ if and only if whenever $a < c < b$, then either $v(c) < v(a)$ or else $v(b) < v(c)$. Call this the *cover condition*. We will use the following order on pairs of positive integers to compare covers in a k-Bruhat order: $(a,b) \prec (c,d)$ if and only if $(a > c)$ or $(a = c$ and $b < d)$

The fundamental weights in type A_{n-1} are $\omega_i = e_1 + \ldots + e_i$, for $i = 1, \ldots, n-1$.

We order the letters in type C_n as follows: $[\overline{n}] := 1 < 2 < ... < n-1 < n < \overline{n} < \overline{n-1} < ... < \overline{2} < \overline{1}$ (one should note that this is not the standard order). The group C_n is the group of signed permutations. An element of C_n is such that $\sigma(\overline{i}) = \overline{\sigma(i)}$ for all $i \in [n]$, here we use the convention that $\overline{\overline{i}} = i$. It is therefore sufficient to write only the first n entries of any permutation. We write an element $\sigma \in C_n$ as $\sigma = (\sigma(1)...\sigma(n))$.

Consider the space $\mathbb{R}^n$ with basis vectors e_i for $i \in [n]$. Then the root system for type C_n is given by $\Phi = \{\pm e_i \pm e_j, i \text{ and } j \in [n]\}$.

For $i < j$ we shall make the following identifications:

- (i,j) with $e_i - e_j$ and $s_{e_i - e_j} = t_{ij} t_{\overline{i}\overline{j}}$,
- $(i,\overline{j})$ with $e_i + e_j$ and $s_{e_i + e_j} = t_{i\overline{j}} t_{\overline{i}j}$,
- $(i,\overline{i})$ with $2e_i$ and $s_{2e_i} = t_{i\overline{i}}$.

Let $\sigma \in C_n$ be a permutation.

Theorem 2.11. *π covers σ in the strong Bruhat order on C_n if and only if there exist $i, j \in [1, \overline{1}]$ such that:*

1. *$\sigma(i) < \sigma(j)$*
2. *if $j > n$ then either $\sigma(j) \leq n$ or $\sigma(i) \geq \overline{n}$*
3. *$\pi = \sigma(i,j)$ with $i \leq n$, $i < j \leq \overline{1}$*
4. *There is no $i < l < j$ such that $\sigma(i) < \sigma(l) < \sigma(j)$.*

Let $\ell(\sigma)$ be used to denote the length of the element $\sigma \in C_n$. This is then given by the formula

$$\ell(\sigma) = \#\{i < j \leq n : \sigma(i) > \sigma(j)\} + \sum_{i \leq n : v(i) \geq \overline{n}} (n + 1 - \overline{v(i)}) \tag{2.12}$$

The Hasse diagram of the Bruhat order for C_n will be the graph whose vertices are the elements of C_n and whose labels are given by $\sigma \stackrel{(i,j)}{\to} \rho$ provided that $\rho = \sigma(i,j)$ and $\ell(\rho) = \ell(\sigma) + 1$, i.e. ρ covers σ.

3. Type A

3.1. **Specializing the Alcove Path Model to A_n.** We define a k-increasing chain(of permutations) to be a saturated increasing chain in the k-Bruhat order which is increasing according to the above ordering. We shall also denote such a chain by

$$\pi_0 \xrightarrow{(a_1 b_1)} \pi_1 \xrightarrow{(a_2 b_2)} \ldots \xrightarrow{(a_p b_p)} \pi_p$$

For $\lambda' = (\lambda'_1, ..., \lambda'_m)$ we define a λ-increasing chain(of permutations) γ to be a concatenation of k-increasing chains $\gamma = \gamma_1 ... \gamma_m$ where each γ_i is a λ'_i-increasing chain.

Similarly define $\Gamma(k)$ as having the following form:

$$\begin{array}{llll} (\,(k,k+1), & (k,k+2), & \ldots, & (k,n), \\ \;\;(k-1,k+1), & (k-1,k+2), & \ldots, & (k-1,n), \\ \;\;\vdots & \vdots & & \vdots \\ \;\;(1,k+1), & (1,k+2), & \ldots, & (1,n)\;)\,. \end{array}$$

A λ-increasing chain may then be represented by using the following construction: $\Gamma(\lambda) = (a_1, b_1)...(a_N, b_N)$ where $a_i < b_i$. Here $N = \sum_i \lambda'_i(n - \lambda'_i)$. The λ-increasing chain γ is then viewed as a subset $J := \{j_1 < ... < j_s\}$ of $[N]$. So $\gamma = (a_{j_1}, b_{j_1})...(a_{j_s}, b_{j_s})$.

We may then associate to a tableaux T of shape λ, $\Gamma(\lambda) = \Gamma(\lambda'_1)...\Gamma(\lambda'_m)$. It is then immediate that a λ-increasing chain, γ, is an increasing sequence of transpositions extracted $\Gamma(\lambda)$. We may then think of the transpositions contained in the λ-increasing chain as being marked or underlined positions in $\Gamma(\lambda)$, so $\gamma = (a_1, b_1)...\underline{(a_{j_1}, b_{j_1})}...\underline{(a_{j_s}, b_{j_s})}...(a_N, b_N)$.

Before continuing we shall also need some background with regard to the root systems of Lie algebras of type A_{n-1}. The root system is given by $\Phi = \{e_i - e_j, i \neq j\}$ where e_i are the standard basis vectors in $\mathbb{R}^n$. Here we note that S_n acts on Φ via permutation of indices. We then let the hyperplane H_α be the hyperplane through the origin orthogonal to the root $\alpha \in \Phi$(α not necessarily a simple root), that is the set $H_\alpha = \{v \in \mathbb{R}^n \mid \langle v, \alpha \rangle = 0\}$ where $\langle v, w \rangle$ is the standard inner product on $\mathbb{R}^n$. Similarly we define $H_{\alpha,k} = \{v \in \mathbb{R}^n \mid \langle v, \alpha \rangle = k\}$.

Let (a,b) be the transposition given by sending the root $e_a - e_b$ to its negative. Let $(a,b;l)$ be the corresponding affine transposition. Here we shall also think of $(a,b;l)$ as a map $(a,b;l) : \mathbb{Z}^n \to \mathbb{Z}^n$ given by $(a,b;l)(\lambda) = (\lambda_1, ..., \lambda_b + l, ..., \lambda_a - l, ..., \lambda_n)$ for $\lambda \in \mathbb{Z}^n$. We see that we may write the chain γ as a product of transpositions $\gamma = (a_{j_1}, b_{j_1})...(a_{j_s}, b_{j_s})$. We attach a notion of level to the chain, we define the level l_i of (a_i, b_i) to be $l_i = \left|\{k < j \mid \lambda'_k < b_i\}\right|$. Note that l_i is merely the number of columns of length less than b_i, that is

$$l_i = |\{j < i \mid (a_j, b_j) = (a_i, b_i)\}| \ .$$

Letting α_i be the root associated to (a_i, b_i), allows for viewing γ as a chain of roots $\alpha_{j_1}...\alpha_{j_s}$.

We define β_i to be the root given by $\beta_i = (a_{j_1}, b_{j_1})...(a_{j_k}, b_{j_k})(\alpha_i)$ where k is the largest such that $j_k < i$. The root β_i will be the root sent to it's negative by (c_i, d_i). Let $s_i = (c_i, d_i; m_i)$. We define $s_{j_i} = (a_{j_1}, b_{j_1}; l_{j_1})...(a_{j_i}, b_{j_i}; l_{j_i})...(a_{j_1}, b_{j_1}; l_{j_1})$. We wish to define an analog to l_i for tableaux which shall be called m_i as above. The object to be considered is the hyperplane $(a_{j_1}, b_{j_1}; l_{j_1})...(a_{j_k}, b_{j_k}; l_{j_k})(H_{\alpha_i, l_i}) = s_{j_k}...s_{j_1}(H_{\alpha_i, l_i}) = H_{\beta_i, m_i}$. We shall wish to observe the effect of each s_{j_i} on the previous hyperplane. In a later lemma we shall find that m_i may be found easily by a straightforward counting at the tableaux level.

Let us also define T_i to be the tableaux corresponding (via the bijection above) to the first i terms of the chain associated to T having been applied, so T_0 would be the tableau with the entries in each row being the number of that row and $(a_i, b_i)(T_i)$ is the result of replacing the entries a_i with the entry b_i in tableau T_{i-1} from column q onward, where (a_i, b_i) is in the column or block q of the λ-increasing chain γ associated to tableau T.

The weight of a chain γ, $\mu(\gamma)$, is defined to be $\mu(\gamma) = (a_1, b_1)...(a_s, b_s)(\lambda)$.

3.2. **The bijection between chains and tableaux.** Given a partition λ, we define a map from λ-increasing chains γ in S_n to semi-standard Young tableaux of shape λ and entries $1, \dots, n$ as follows. If $\gamma = \gamma_1 * \dots * \gamma_m$, then the i-th column of the associated tableau is given by the first λ'_i entries of the permutation (written in one-line notation) with which the subchain γ_i ends.

Theorem 3.1. *The above map is a bijection between semi-standard Young tableaux of shape λ with entries in $[n]$ and λ-increasing chains in S_n.*

We shall first need the following lemma.

Lemma 3.2. *For $i \le k < j \le n$, $\pi(j) = b$ and $\pi(l) > b$ for $i < l \le k$, there exists a unique sequence $k < j_1 < ... < j_p = j$ such that*

$$\ell(\pi(i, j_1)...(i, j_r)) = \ell(\pi(i, j_1)...(i, j_{r-1})) + 1 \quad for \ \ 0 < r \le p \tag{3.3}$$

Proof. We shall prove this by giving an algorithm that produces the unique sequence of transpositions of the given form. We begin by showing existence.Define $j_0 = i$. Let $j_1 > i$ be the first position such that $\pi(j_0) < \pi(j_1) \le b$. We see that $\ell(\pi t_{i,j_1}) = \ell(\pi) + 1$ since by construction either $\pi(l) < \pi(j_0)$ or $\pi(l) > b$ for $i < l < j_1$. We then repeat this process with $\pi(i, j_1)$ finding j_2 using j_1 instead of j_0 and then repeat as necessary, and since $j - k$

is finite this process will terminate. It is not hard to see that the sequence of j_l's will be increasing. To show uniqueness, we need to show that if $k < l_1 < ... < l_q = j$ is another such sequence then $j_1 = l_1$(here we shall let $l_0 = i$ as well). Once this has been shown the same argument may be iterated so that we obtain $p = q$ and $j_m = l_m$ for $1 \leq m \leq p$. Thus we examine the situation for j_1 and assume that $l_1 \neq j_1$. We first note that $l_1 \geq j_1$, otherwise length would decrease. There exists a position m such that $\pi(l_{m-1}) < \pi(j_1)$ and $\pi(l_m) > \pi(j_1)$. We then observe that $\ell(\pi(i, j_1)...(i, j_m)) > \ell(\pi(i, j_1)...(i, j_{m-1})) + 1$. But this violates the condition that the length increase by exactly one.

The algorithm used here is as follows:

Algorithm 3.4. *set* $\rho = \pi$*;*
set $i = k$*;*
while $i \geq 1$ *do*
  set $j = k + 1$*;*
  while $j < n$ *do*
   if ($\rho(j) > \rho(i)$ *and* $\rho(j) \leq C'(i)$*)*
    return $\rho = \rho(i, j)$*;*
   end if
   set $j = j + 1$*;*
  end while
  set $i = i - 1$*;*
end while

□

Proof. (of Theorem3.1) We now find a unique chain associated to a tableaux T by giving an explicit algorithm, resulting in an explicit inverse to the above defined map from chains to tableaux. Here we assume T has entries in the set $[n]$. We shall begin by observing the shape of the conjugate tableaux $\widetilde{T}$, that is by noting the length of each column given by $(\lambda'_1, ..., \lambda'_m)$. Begin with the first column of length λ_1 and with the sequence $s = (1, 2, ..., n)$ and using the above lemma find the unique chain that replaces the λ_1^{st} entry of s with the associated value of that box in the tableaux T. We then repeat this for the $\lambda_1 - 1$ entry. It suffices to check that this does not effect the entries further down the column, which is clear since transpositions of the type would violate the above lemma, in particular the length condition would not be preserved. Then repeat this throughout the column until the appropriate entry is in each box of the column, giving a chain γ_1. Then repeat this procedure on the remaining columns, resulting in a chain $\gamma = \gamma_1...\gamma_m$(we note that it is possible for γ_i to be the empty chain). Due to the uniqueness in each step we have that this in fact a well define inverse, thus establishing the claim that the map from chains to tableaux is in fact a bijection. □

Example 3.5. *Consider* $\lambda = (4, 3, 1)$
$\Gamma(\lambda) = (34)(24)(14)|(23)(24)(13)(14)|(23)(24)(13)(14)|(12)(13)(14)$.

Now consider the λ*-increasing chain* $\gamma = (34)|(24)|(14)|(12)(13)$ *which may be viewed as*
$\underline{(34)}(24)(14)|(23)\underline{(24)}(13)(14)|(23)(24)(13)\underline{(14)}|\underline{(12)(13)}(14)$

$$\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\ \hline 2&2&2\\ \cline{1-3} 3\\ \cline{1-1}\end{array} \overset{(34)}{\to} \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\ \hline 2&2&2\\ \cline{1-3} 4\\ \cline{1-1}\end{array} \overset{(24)}{\to} \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\ \hline 2&3&3\\ \cline{1-3} 4\\ \cline{1-1}\end{array} \overset{(14)}{\to} \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\ \hline 2&3&3\\ \cline{1-3} 4\\ \cline{1-1}\end{array} \overset{(12)}{\to} \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline 2&3&3\\ \cline{1-3} 4\\ \cline{1-1}\end{array} \overset{(13)}{\to} \begin{array}{|c|c|c|c|}\hline 1&1&2&4\\ \hline 2&3&3\\ \cline{1-3} 4\\ \cline{1-1}\end{array}$$

$$1234 \overset{(34)}{\to} 1243 \overset{(24)}{\to} 1342 \overset{(14)}{\to} 2341 \overset{(12)}{\to} 3241 \overset{(13)}{\to} 4231$$

The tableau to which γ maps would then be the rightmost tableau above. It can also be easily seen in this example how the map from tableaux to chains is constructed.

3.3. **The Crystal Graph Structure and root operators.** We shall require the following well known formula for reflections of hyperplanes which we shall reference in the proof of the lemma below:

$$t_{\alpha,k}(H_{\beta,l}) = H_{t_\alpha\beta,l-k\langle\alpha,\beta\rangle};\;\; \langle(a,b),(c,d)\rangle = \langle e_a - e_b, e_c - e_d\rangle, \langle e_i, e_j\rangle = \delta_{ij}. \tag{3.6}$$

Recall the λ-chain Γ and let us write $\Gamma = (\beta_1, \ldots, \beta_m)$. As such, we recall the hyperplanes H_{β_k,l_k} and the corresponding affine reflections $\widehat{r}_k = s_{\beta_k,l_k}$. If $\beta_k = (a,b)$ falls in the segment Γ_p of Γ (upon the factorization $\Gamma = \Gamma_1 \ldots \Gamma_{\lambda_1}$ of the latter), then it is not hard to see that

$$l_k = |\{i \,:\, 1 \le i < p,\; \lambda'_i \ge a\}|\,.$$

An affine reflection $s_{(a,b),l}$ acts on our vector space V by

$$s_{(a,b),l}(\mu_1, \ldots, \mu_a, \ldots, \mu_b, \ldots, \mu_n) := (\mu_1, \ldots, \mu_b + l, \ldots, \mu_a - l, \ldots, \mu_n)\,. \tag{3.7}$$

Now fix a permutation w in S_n and a subset $J = \{j_1 < \ldots < j_s\}$ of $[m]$ (not necessarily w-admissible). Let Π be the alcove path corresponding to Γ, and define the alcove walk Ω by

$$\Omega := \phi_{j_1} \ldots \phi_{j_s}(w(\Pi))\,.$$

Given k in $[m]$, let $i = i(k)$ be the largest index in $[s]$ for which $j_i < k$. Let $\gamma_k := wr_{j_1} \ldots r_{j_i}(\beta_k)$, and let H_{γ_k,m_k} be the hyperplane containing the face F_k of Ω. In other words

$$H_{\gamma_k,m_k} = w\widehat{r}_{j_1} \ldots \widehat{r}_{j_i}(H_{\beta_k,l_k})\,.$$

Our first goal is to describe m_k purely in terms of the filling associated to (w,J).

Let $\widehat{t}_k$ be the affine reflection in the hyperplane H_{γ_k,m_k}. Note that

$$\widehat{t}_k = w\widehat{r}_{j_1} \ldots \widehat{r}_{j_i}\widehat{r}_k\widehat{r}_{j_i} \ldots \widehat{r}_{j_1}w^{-1}\,.$$

Thus, we can see that

$$w\widehat{r}_{j_1} \ldots \widehat{r}_{j_i} = \widehat{t}_{j_i} \ldots \widehat{t}_{j_1}w\,.$$

Let $T = ((a_1,b_1), \ldots, (a_s,b_s))$ be the subsequence of Γ indexed by the positions in J. Let T^i be the initial segment of T with length i, let $w_i := wT^i$, and $\sigma_i := f(w,T^i)$. In particular, σ_0 is the filling with all entries in row i equal to $w(i)$, and $\sigma := \sigma_s = f(w,T)$. The columns of a filling of λ are numbered, as usual, from left to right by λ_1 to 1. Note that, if $\beta_{j_{i+1}} = (a_{i+1}, b_{i+1})$ falls in the segment Γ_p of Γ, then σ_{i+1} is obtained from σ_i by replacing the entry $w_i(a_{i+1})$ with $w_i(b_{i+1})$ in the columns $p, \ldots, \lambda_1$ (and the row a_{i+1}) of σ_i.

Given a fixed k, let $\beta_k = (a, b)$, $c := w_i(a)$, and $d := w_i(b)$, where $i = i(k)$ is defined as above. Then $\gamma := \gamma_k = (c, d)$, where we might have $c > d$. Let Γ_q be the segment of Γ where β_k falls. Given a filling ϕ, we denote by $\phi(p)$ and $\phi[p, q)$ the parts of ϕ consisting of the columns $1, 2, \ldots, p-1$ and $p, p+1, \ldots, q-1$, respectively. We use the notation $N_e(\phi)$ to denote the number of entries e in the filling ϕ.

Proposition 3.8. *With the above notation, we have*

$$m_k = \langle \mathrm{ct}(\sigma(q)), \gamma^\vee \rangle = N_c(\sigma(q)) - N_d(\sigma(q))\,.$$

Proof. We apply induction on i, which starts at $i = 0$. We will now proceed from $j_1 < \ldots < j_i < k$, where $i = s$ or $k \leq j_{i+1}$, to $j_1 < \ldots < j_{i+1} < k$, and we will freely use the notation above. Let

$$\beta_{j_{i+1}} = (a', b')\,, \;\; c' := w_i(a')\,, \;\; d' := w_i(b')\,.$$

Let Γ_p be the segment of Γ where $\beta_{j_{i+1}}$ falls, where $p \geq q$.

We need to compute

$$w\widehat{r}_{j_1} \ldots \widehat{r}_{j_{i+1}}(H_{\beta_k, l_k}) = \widehat{t}_{j_{i+1}} \ldots \widehat{t}_{j_1} w(H_{\beta_k, l_k}) = \widehat{t}_{j_{i+1}}(H_{\gamma, m})\,,$$

where $m = \langle \mathrm{ct}(\sigma_i(q)), \gamma^\vee \rangle$, by induction. Note that $\gamma' := \gamma_{j_{i+1}} = (c', d')$, and $\widehat{t}_{j_{i+1}} = s_{\gamma', m'}$, where $m' = \langle \mathrm{ct}(\sigma_i(p)), (\gamma')^\vee \rangle$, by induction. We will use the following formula:

$$s_{\gamma', m'}(H_{\gamma, m}) = H_{s_{\gamma'}(\gamma), m - m' \langle \gamma', \gamma^\vee \rangle}\,.$$

Thus, the proof is reduced to showing that

$$m - m' \langle \gamma', \gamma^\vee \rangle = \langle \mathrm{ct}(\sigma_{i+1}(q)), s_{\gamma'}(\gamma^\vee) \rangle\,.$$

An easy calculation, based on the above information, shows that the latter equality is non-trivial only if $p > q$, in which case it is equivalent to

$$\langle \mathrm{ct}(\sigma_{i+1}[p, q)) - \mathrm{ct}(\sigma_i[p, q)), \gamma^\vee \rangle = \langle \gamma', \gamma^\vee \rangle \, \langle \mathrm{ct}(\sigma_{i+1}[p, q)), (\gamma')^\vee \rangle\,. \tag{3.9}$$

This equality is a consequence of the fact that

$$\mathrm{ct}(\sigma_{i+1}[p, q)) = s_{\gamma'}(\mathrm{ct}(\sigma_i[p, q)))\,,$$

which follows from the construction of σ_{i+1} from σ_i explained above. □

Define $\Gamma(\lambda)^i$ to be the portion of $\Gamma(\lambda)$ consisting of only the transpositions which exchange the values i and $i+1$, that is of the form $(i, i+1)$ or $\underline{(i, i+1)}$ or $(i+1, i)$.

An immediate observation from the above lemma is that the level m is constant on a given column, thus we have the following corollary to the lemma.

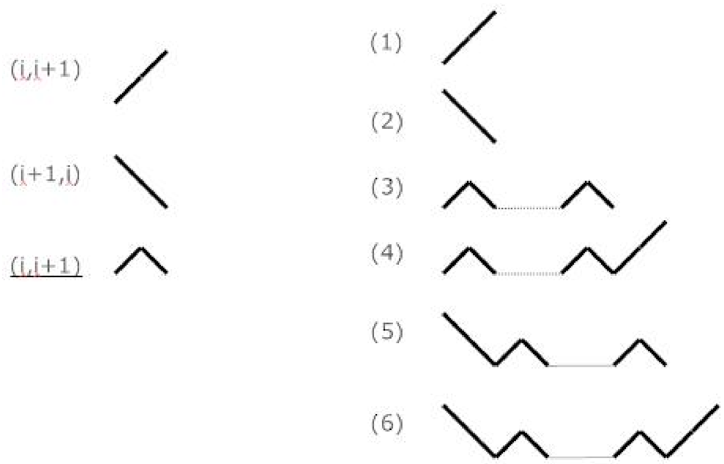
Corollary 3.10. $\Gamma(\lambda)^i$ *in any given column is of one of the following forms:*

(1) $(i, i+1)$
(2) $(i+1, i)$
(3) $\underline{(i, i+1)} \ldots \underline{(i, i+1)}$
(4) $\underline{(i, i+1)} \ldots \underline{(i, i+1)}(i, i+1)$
(5) $(i+1, i)\underline{(i, i+1)} \ldots \underline{(i, i+1)}$
(6) $(i+1, i)\underline{(i, i+1)} \ldots \underline{(i, i+1)}(i, i+1)$

Recall the function $g_{J,p}$ defined in the initial background. We specialize this to the type A case where we shall refer to the graph as $g(i)$. Start at $(0,-1/2)$ as previously. Then $(i,i+1)$, $(i+1,i)$, and $\underline{(i,i+1)}$ are represented as follows:

- $(i,i+1)$ by a linear segment going up by one and to the right by one
- $(i+1,i)$ by a linear segment going down by one and to the right by one
- $\underline{(i,i+1)}$ by a linear segment going up a half, right a half, followed by a linear segment going down a half, to the right by a half.

This yields the following picture in type A



Proof. We prove this be merely observing what roots may occur in succession, or more accurately which cannot occur in succession. The proof follows from the following basic observations:

- An $(i,i+1)$ cannot be followed by an $(i+1,i)$.
- For $(i,i+1)(i,i+1)$ the the second will have level one higher than the previous.
- For $(i+1,i)(i+1,i)$ the second will have level one lower than the first.
- For $(i,i+1)\underline{(i,i+1)}$ the level increases by one.
- For $\underline{(i,i+1)}(i+1)$ the level decreases by one.
- For $\underline{(i,i+1)}\dots\underline{(i,i+1)}$ all have the same level.
- For $(i+1,i)\underline{(i,i+1)}$ the level is the same.
- For $\underline{(i,i+1)}(i,i+1)$ the level is the same for both.

This gives us that an $(i,i+1)$ cannot be followed by anything, an $(i+1,i)$ only by $\overline{(i,i+1)}$, and $\overline{(i,i+1)}$ may be followed either by $\overline{(i,i+1)}$ or by $(i,i+1)$ in which case the chain terminates. Leaving use with the forms above.

A more intuitive way to see this using the graph mentioned above is that the midpoints of each segment all lie on the same horizontal line, which is easily seen for the forms above, it is also easily seen that other forms will violate this condition. ∎

We shall extract some information about the structure of a column based on what $\Gamma(\lambda)^i$ looks like. We can say at the very least that:

- If $\Gamma(\lambda)^i$ is of forms (1), (4) or (6) then there is a single i entry in the given column.
- If $\Gamma(\lambda)^i$ is of forms (2), (5) or (6) then the previous column has an $i+1$ entry.
- If $\Gamma(\lambda)^i$ is of form (2) and the next column has the same level, then either there is both an i and an $i+1$ entry, or there are neither.
- If $\Gamma(\lambda)^i$ is of form (2) and the next column has lower level then there is an $i+1$ entry in the given column.
- If $\Gamma(\lambda)^i$ is of forms (3) or (5) and if the next column has lower level then there is a single $i+1$ entry.
- If $\Gamma(\lambda)^i$ is of forms (3) or (5) and if the next column has the same level then the column has either both an i and an $i+1$ entry or neither.

Another observation to be made from the above lemma is that the first highest level occurs either in the first column, in a column after one with only an i entry (no $i+1$), or as an exceptional case in the last column provided that the last column has only an i entry no $i+1$ (in reality this level may be the same as a previous highest level, but the highest level would actually occur in a column immediately following the last, were there one). We concern ourselves with the first highest level as this plays a role in the definition of F_i.

Lemma 3.11. *Let j be the column in which the highest level occurs (assuming $j>1$), then the graph $g(i)$ restricted to column j is one of the following forms:*

- *empty*
- $\underline{(i,i+1)}\dots\underline{(i,i+1)}$
- $\underline{(i,i+1)}\dots\underline{(i,i+1)}(i,i+1)$

Proof. Merely observe that $\Gamma(\lambda)^i$ in column $j-1$ terminates with a $(i,i+1)$. This together with the fact that an $(i+1,i)$ cannot follow an $(i,i+1)$ establishes the claim. ☐

Write $\Gamma(\lambda)^i=(a_1,b_1)...(a_m,b_m)$ and let $K\subseteq[m]$ be the subset of marked indices $K=\{i_1<...<i_k\}$. So then $\Gamma(\lambda)^i=(a_1,b_1)...\underline{(a_{i_1},b_{i_1})}...\underline{(a_{i_k},b_{i_k})}...(a_m,b_m)$.

Let p be the final position in $[m]\backslash K$ in column $j-1$, that is the last unmarked position in column $j-1$.

Define $F_i(\Gamma(\lambda)^i)=(a_1,b_1)...\underline{(a_{i_1},b_{i_1})}...\underline{(a_p,b_p)}(a_{p+1},b_{p+1})...\underline{(a_{i_k},b_{i_k})}...(a_m,b_m)$, i.e. the position before the highest level becomes marked. In the case where the highest level occurs in the first column define $F_i(\Gamma(\lambda)^i)=\underline{(a_1,b_1)}...(a_{i_1},b_{i_1})...\underline{(a_{i_k},b_{i_k})}...(a_m,b_m)$.

We define E_i similarly, however for our purposes we shall need only F_i.

Our goal at this point will be to show that the root operators E_i and F_i commute with the bijection between chains and tableaux.

Now consider the portion of the column word of T the results from extracting the subsequence of i's and $i+1$'s, then look at the result after r-pairing. This may be of one of the following forms:

(1) $i...i$
(2) $i...i\; i+1...i+1$
(3) $i+1...i+1$

Note that if this is of the third form then this corresponds to having the highest level in the first column in which case the root operator F_i is undefined, so we need not consider this case further.

Theorem 3.12. *The bijection between semi-standard Young tableaux of shape λ with entries in $[n]$ and λ-increasing chains in S_n commutes with the root operators E_i and F_i.*

Proof. It suffices to check that the bijection commutes with F_i. Note that in the column prior to the one with highest level,i.e. the column $j-1$ as in the above, the chain ends with $(i, i+1)$, thus on the level of chains F_i makes this marked. This is precisely the same as replacing the i in that column with an $i+1$, consequently this is the same as the effect at the level of column words. Note that if we are in the case where highest level occurs 'past' the last column that this amounts to marking the last $(i, i+1)$, this occurs in the case where the column word is of the form $i...i$ and the last i is changed to a $i+1$. These are also easily seen to be equivalent. Thus the definition on the level of chains is the same as the definition in terms of column words, thereby showing that F_i commutes with the bijection which is sufficient.

□

We thus have the following immediate corollary to the above theorem:

Corollary 3.13. *The bijection between semi-standard Young tableaux of shape λ with entries in $[n]$ and λ-increasing chains in S_n preserves the crystal graph structure for Young tableaux of shape λ with entries in $[n]$.*

As an immediate corollary to this we have the following:

Corollary 3.14. *The bijection between semi-standard Young tableaux of shape λ with entries in $[n]$ and λ-increasing chains in S_n preserves weight for Young tableaux of shape λ with entries in $[n]$.*

Example 3.15. *Consider the case where $\lambda = (4, 3, 1)$ with tableau as below:*

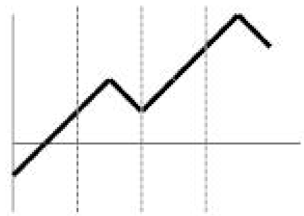
1	1	2	5
3	4	4	
4	5		

In this case the chain after foldings is

$\underline{(3\ 4)}(4\ 5)\underline{(2\ 4)}(2\ 5)(1\ 2)(1\ 5)\|(4\ 2)\underline{(4\ 5)}(3\ 2)\underline{(3\ 4)}(1\ 2)(1\ 4)\|$
$(4\ 5)(4\ 2)(4\ 3)(1\ 5)\underline{(1\ 2)}(2\ 3)\|\underline{(2\ 4)}\underline{(4\ 5)}(5\ 1)(5\ 3)$

$\Gamma(\lambda)^4$ *in this case is* $(4\ 5)\|\underline{(4\ 5)}\|(4\ 5)\|\underline{(4\ 5)}$

The column word in this case is 45445*, when restricted to* 4 *and* 5*, which becomes* 445 *after pairing.*



The levels m_4 are then easily read of of the graph of $g(4)$ above to be, 0 in the first column, 1 in the second column, 1 in the third, and 2 in the fourth.

4. Type C

4.1. **Specializing the Alcove Path Model to C_n.** We shall fix n from this point onward. Define an $\Gamma_i^l(k)$ for $i \leq k$ chain to have the following form:

$$\begin{array}{llll} (\,(i,k+1), & (i,k+2), & \ldots, & (i,n), \\ (i,\overline{i}), & & & \\ (i,\overline{n}), & (i,\overline{n-1}), & \ldots, & (i,\overline{k+1}), \\ (i,\overline{i-1}), & (i,\overline{i-2}), & \ldots, & (i,\overline{1})\,)\,. \end{array}$$

$\Gamma^l(k)$ is then defined as

$$\Gamma^l(k) = \Gamma_k^l(k)\Gamma_{k-1}^l(k)...\Gamma_1^l(k)$$

Define an $\Gamma^r(k)$ chain to have the following form:

$$\begin{array}{llll} (\,(k,\overline{k-1}), & (k,\overline{k-2}), & \ldots, & (k,\overline{1}), \\ \vdots & \vdots & & \\ (3,\overline{2}) & (3,\overline{1}), & & \\ (2,\overline{1})\,)\,. & & & \end{array}$$

The chain $\Gamma(k)$ is then defined as:

$$\Gamma(k) = \Gamma^l(k)\Gamma^r(k). \tag{4.1}$$

The chain $\Gamma(\lambda)$ is then defined as:

$$\Gamma(\lambda) = \Gamma_1(\lambda'_1)\Gamma_2(\lambda'_2)...\Gamma_m(\lambda'_m), \tag{4.2}$$

and breaks down as

$$\Gamma(\lambda) = \Gamma_1^l(\lambda_1')\Gamma_1^r(\lambda_1')\Gamma_2^l(\lambda_2')\Gamma_2^r(\lambda_2')...\Gamma_m^l(\lambda_m')\Gamma_m^r(\lambda_m'). \tag{4.3}$$

Lemma 4.4. *$\Gamma(k)$ is an ω_k-chain.*

Proof. We use the criterion for λ-chains in [10][Definition 4.1, Proposition 4.4], cf. [10][Proposition 10.2]. This criterion says that a chain of roots Γ is a λ-chain if and only if it satisfies the following conditions:

(R1) The number of occurrences of any positive root α in Γ is $\langle \lambda, \alpha^\vee \rangle$.
(R2) For each triple of positive roots (α, β, γ) with $\gamma^\vee = \alpha^\vee + \beta^\vee$, the subsequence of Γ consisting of α, β, γ is a concatenation of pairs (γ, α) and (γ, β) (in any order).

Letting $\lambda = \omega_k = \varepsilon_1 + \ldots + \varepsilon_k$, condition (R1) is easily checked; for instance, a root $(a, \overline{b})$ appears twice in $\Gamma(k)$ if $a < b \le k$, once if $a \le k < b$, and zero times otherwise. For condition (R2), we use a case by case analysis, as follows, where $a < b < c$:

(1) $\alpha = (a, b)$, $\beta = (b, c)$, $\gamma = (a, c)$;
(2) $\alpha = (a, b)$, $\beta = (b, \overline{c})$, $\gamma = (a, \overline{c})$;
(3) $\alpha = (a, c)$, $\beta = (b, \overline{c})$, $\gamma = (a, \overline{b})$;
(4) $\alpha = (b, c)$, $\beta = (a, \overline{c})$, $\gamma = (a, \overline{b})$;
(5) $\alpha = (a, b)$, $\beta = (b, \overline{b})$, $\gamma = (a, \overline{a})$;
(6) $\alpha = (a, \overline{a})$, $\beta = (b, \overline{b})$, $\gamma = (a, \overline{b})$.

Case (1) is the same as in type A. Each of the cases (2)-(4) has the following three sub cases: $k \ge c$, $b \le k < c$, and $a \le k < b$, while each of the cases (5)-(6) has the following two sub cases: $k \ge b$, and $a \le k < b$. For instance, if $b \le k < c$ in Case (3), then the subsequence of $\Gamma(k)$ consisting of α, β, γ is $((a, \overline{b}), (a, c), (a, \overline{b}), (b, \overline{c}))$. □

Definition 4.5. *We then define a σ-right admissible subsequence γ^r to be a subsequence of $\Gamma^r(k)$ such that it is the labels of the covers of a saturated chain in the Bruhat order of C_n starting at σ. We denote the mentioned saturated chain in the Bruhat order by $\sigma \xrightarrow{\gamma^r} \sigma'$, where σ' is the permutation where the chain ends. We shall identify admissible subsequences with corresponding chains called admissible chains. Similarly define a σ-left admissible subsequence γ^l to be a subsequence of $\Gamma^l(k)$ such that it is the labels of the covers of a saturated chain in the Bruhat order of C_n starting at σ.*

Definition 4.6. *Define a σ-admissible subsequence γ to be a subsequence of $\Gamma(\lambda)$ such that it is the labels of the covers of a saturated chain in the Bruhat order of C_n starting at σ. In the particular case where σ is the identity permutation we shall just say admissible subsequence.*

$$\sigma \xrightarrow{\gamma^l} \sigma^l \xrightarrow{\gamma^r} \sigma'. \tag{4.7}$$

Our primary goal at this point shall be to create a bijection from admissible sequences to KN tableaux. We begin by examining the left portion of the chain.

Suppose we have an admissible subsequence which is a subsequence of

$$\Gamma(\lambda) = \Gamma_1^l(\lambda_1')\Gamma_1^r(\lambda_1')\Gamma_2^l(\lambda_2')\Gamma_2^r(\lambda_2')...\Gamma_m^l(\lambda_m')\Gamma_m^r(\lambda_m'). \tag{4.8}$$

Then we may view this as having the following splitting on the corresponding Bruhat chain:

$$id \xrightarrow{\gamma_0^l} \sigma_1^l \xrightarrow{\gamma_1^r} \sigma_1^r \xrightarrow{\gamma_1^l} \sigma_2^l \xrightarrow{\gamma_2^r} ... \xrightarrow{\gamma_m^r} \sigma_m^r. \tag{4.9}$$

Then for $1 \leq i \leq m$ the column $lC_i = \sigma_i^l[\lambda_i']$ and $rC_i = \sigma_i^r[\lambda_i']$. This provides the desired mapping from admissible sequences to doubled KN tableaux which is more than sufficient.

Our goal is to now create an inverse to the above map, i.e. a map from KN tableaux to admissible sequences.

Theorem 4.10. *Given the pair (σ, C'), where $C = \sigma[n] \leq C'$ are columns in C_n, there exists a unique σ-left admissible subsequence γ^l from σ to a unique σ' such that $\sigma'[k] = C'$.*

This is done via an algorithm which explicitly constructs said σ-left admissible subsequence.

Algorithm 4.11. *set $\pi = \sigma$;*
set $i = k$;
while $i \geq 1$ do
exchange($k+1, n$);

  if ($\pi(\bar{i}) > \pi(i)$ and $\pi(\bar{i}) \leq C'(i)$)
   return $\pi = \pi(i, \bar{i})$;
  end if

exchange($\overline{n}, \overline{k+1}$);

exchange($\overline{i-1}, \overline{1}$);

  set $i = i - 1$;
end while

exchange(a, b)
  set $j = a$;
  while $j < b$ do

if $(\pi(j) > \pi(i)$ *and* $\pi(j) \leq C'(i))$
return $\pi = \pi(i, j)$;
end if
set $j = j + 1$;
end while

In order to prove that the above algorithm produces the desired bijection we shall first need a few lemmas similar to those used in the case of type A.

Lemma 4.12. *For* $i \leq k < j \leq n$, $\pi(i) = a$, $\pi(j) = b$ *and* $\pi(l) \notin [a, b]$ *for* $i < l \leq k$, *there exists a unique sequence* $k < j_1 < ... < j_p = j$ *such that*

$$\ell(\pi(i, j_1)...(i, j_r)) = \ell(\pi(i, j_1)...(i, j_{r-1})) + 1 \quad for \;\; 1 \leq r \leq p \tag{4.13}$$

Proof. This particular lemma is nearly identical to that in type A and the proof is the same. □

Lemma 4.14. *For* $i \leq k < j \leq n$, $\pi(i) = a$, $\pi(j) = b$ *and* $\pi(l) \notin [a, b]$ *for* $i < l \leq n$, *there exists a unique sequence* $\overline{n} < \overline{j}_1 < ... < \overline{j}_p = \overline{j}$ *such that*

$$\ell(\pi(i, \overline{j}_1)...(i, \overline{j}_r)) = \ell(\pi(i, \overline{j}_1)...(i, \overline{j}_{r-1})) + 1 \quad for \;\; 1 \leq r \leq p \tag{4.15}$$

Lemma 4.16. *For* $i \leq k \leq n, j < k$, $\pi(i) = a$, $\pi(j) = b$ *and* $\pi(l) \notin [a, b]$ *for* $i < l \leq \overline{i}$, *there exists a unique sequence* $i > j_1 > ... > j_p = j$ *such that*

$$\ell(\pi(i, \overline{j}_1)...(i, \overline{j}_r)) = \ell(\pi(i, \overline{j}_1)...(i, \overline{j}_{r-1})) + 1 \quad for \;\; 1 \leq r \leq p \tag{4.17}$$

Theorem 4.18. *Given the pair* (σ, C'), *where* $C = \sigma[n] \leq C'$ *are columns in* C_n, *there exists a unique* σ*-left admissible subsequence* γ^l *from* σ *to a unique* σ' *such that* $\sigma'[k] = C'$.

Proof. We shall show in particular that the above algorithm produces the desired result. Consider $\Gamma_i^l(k)$ for a particular $i \leq k$ We begin exactly as in type A using the first lemma for the first call of the exchange function from the algorithm. The permutation returned at the end of the first call of the exchange function shall be called σ'. If $\sigma'(\overline{i}) < C'(i)$ then positions i and $\overline{i}$ will need to be swapped. We need to ensure that for no j in the interval $[i, \overline{i}]$ is the relation $\sigma'(i) < \sigma'(j) < \sigma'(\overline{i})$ satisfied. We know that $\sigma'(j) >\notin [\sigma'(i), C'(i)]$

for $j \in [k+1, n]$ consequently $\sigma'(j) >\notin [\sigma'(i), C'(i)]$ for $j \in [\overline{n}, \overline{k+1}]$ by symmetry. Thus the cover condition is satisfied in this case. We now check that the condition of the second lemma are satisfied prior to the second call of the exchange function, however we already know this as $\sigma'(i) > \sigma'(j)$ and $\sigma'(\bar{i}) > \sigma'(j)$ for $j \in [k+1, n]$. We shall the call the permutation returned at the end of the second call of the exchange function σ''. It will then suffice to check that σ'' satisfies the condition of the third lemma prior to the call of the third instance of the exchange function. This follows immediately as $\sigma''(j) \notin [\sigma''(i), C'(i)]$ for $j \in [i, \bar{i}]$. This holds for all i, thus the algorithm does in fact produce the desired subsequence.

□

The right chain shall next be examined. The goals here are firstly, to construct the chain from a left column to a right column and secondly, to show that this chains existence is equivalent to a column being admissible (i.e. splitting).

Let $D \leq E$ be KN-columns such that $\ell(D) = \ell(E) = k$, $k \leq n$, $D(i) \neq \overline{D(j)}$ for $i, j \in [n]$, and likewise for E. We shall refer to a column D where the condition $D(i) \neq \overline{D(j)}$ for $i, j \in [n]$ is satisfied as not having repetition of entries.

Theorem 4.19. *Assume that columns $D \leq E$ are KN-columns without repetition of entries. Then the following statements are equivalent:*

(1) *$\exists!$ σ_D-right admissible subsequence ending at E, where σ_D is the permutation corresponding to the column D.*
(2) *$D = lC$ and $E = rC$ for some KN-column C.*

Proof. (2) → (1): We first give an algorithm that produces the desired chain from D to E.

Algorithm 4.20. *set $\pi = \sigma_I$;*
set $i = k$;
while $i \geq 2$ do
  set $j = i - 1$;
  while $j \geq 1$ do
   if ($\pi(\bar{j}) > \pi(i)$ and $\pi(\bar{j}) \leq C'(i)$)
    $\pi = \pi(i, \bar{j})$;
   end if
   set $j = j - 1$;
  end while
  set $i = i - 1$;
end while

We are then required to check that the cover condition is not violated as the algorithm is executed. We first check that $\sigma(i) \notin [\bar{z}_j, \bar{t}_j]$ for $1 \leq j \leq r$ and $i \in [k+1, \overline{k+1}]$ as otherwise the cover condition would be violated. However this is immediate by the construction of t_j as defined in the splitting, as t_j is the largest possible entry such that t_j and $\bar{t}_j$ are not already in $C[1,k]$. Also note that $\sigma(i)\langle\overline{z}_j$ for $i\rangle p_{\overline{z}_j}\langle k$ where $p_{\overline{z}_j}$ is the position of $\overline{z}_j$. These together along with the symmetry of the permutation show the algorithm used above does not violate the cover condition.

(1) $\to$ (2): Here we take the columns D and E and explicitly construct C. Define p_D to be the position of the first barred entry in column D, note that $p_D = p_C$. Then C is the column such that $C[1, p_D - 1] = D[1, p_D - 1]$ and $C[p_D, n] = E[p_D, n]$. It is then easily seen that $lC = D$ and $rC = E$ by the splitting construction given in the introduction.

□

(Insert Example Here)

4.2. **The Crystal Graph Structure and root operators.** The arguments in this section will be analogous to those in the case of Type A.

Recall the λ-chain Γ. Let us write $\Gamma = (\beta_1, \dots, \beta_m)$. As such, we recall the hyperplanes H_{β_k, l_k} and the corresponding affine reflections $\widehat{r}_k = s_{\beta_k, l_k} = s_{\beta_k} + l_k\beta_k$.

Now fix a signed permutation w in C_n and a subset $J = \{j_1 < \dots < j_s\}$ of $[m]$ (not necessarily w-admissible). Let Π be the alcove path corresponding to Γ, and define the alcove walk Ω by

$$\Omega := \phi_{j_1} \dots \phi_{j_s}(w(\Pi))\,.$$

Given k in $[m]$, let $i = i(k)$ be the largest index in $[s]$ for which $j_i < k$, and let $\gamma_k := wr_{j_1} \dots r_{j_i}(\beta_k)$. Then the hyperplane containing the face F_k of Ω is of the form H_{γ_k, m_k}; in other words

$$H_{\gamma_k, m_k} = w\widehat{r}_{j_1} \dots \widehat{r}_{j_i}(H_{\beta_k, l_k})\,.$$

Our first goal is to describe m_k purely in terms of the filling associated to (w, J).

Let $\widehat{t}_k$ be the affine reflection in the hyperplane H_{γ_k, m_k}. Note that

$$\widehat{t}_k = w\widehat{r}_{j_1} \dots \widehat{r}_{j_i}\widehat{r}_k\widehat{r}_{j_i} \dots \widehat{r}_{j_1}w^{-1}\,.$$

Thus, we can see that

$$w\widehat{r}_{j_1} \dots \widehat{r}_{j_i} = \widehat{t}_{j_i} \dots \widehat{t}_{j_1} w\,.$$

Let $T = ((a_1, b_1), \dots, (a_s, b_s))$ be the subsequence of Γ indexed by the positions in J, cf. Section **??**. Let T^i be the initial segment of T with length i, let $w_i := wT^i$, and $\sigma_i := \overline{f(w, T^i)}$, see (4.22). In particular, σ_0 is the filling with all entries in row i equal to $w(i)$, and $\sigma := \sigma_s = \overline{f(w, T)}$. The columns of a filling of 2λ are numbered left to right by 1 to $2\lambda_1$. If $\beta_{j_{i+1}} = (a_{i+1}, b_{i+1}) = (a, b)$ falls in the segment of Γ corresponding to column p of 2λ, then σ_{i+1} is obtained from σ_i by replacing the entry $w_i(a)$ with $w_i(b)$ in the columns $1, \dots, p-1$ of σ_i, as well as, possibly, the entry $w_i(\overline{b})$ with $w_i(\overline{a})$ in the same columns.

Now fix a position k, and consider $i = i(k)$ and the roots β_k, $\gamma := \gamma_k$, as above, where γ_k might be negative. Assume that β_k falls in the segment of Γ corresponding to column q of 2λ. Given a filling ϕ, we denote by $\phi(p)$ and $\phi[p, q)$ the parts of ϕ consisting of the columns $1, \dots, 2p-1$ and $p, \dots, q-1$, respectively.

Proposition 4.21. *With the above notation, we have*
$$m_k = \langle \mathrm{ct}(\sigma[q]), \gamma^\vee \rangle \,.$$

Let us first define the content of a filling. For this purpose, we first associate with a filling σ a "compressed" version of it, namely the filling $\overline{\sigma}$ of the partition 2λ. This is defined as follows:
$$\overline{\sigma} = \overline{\mathcal{C}}^{\lambda_1} \dots \overline{\mathcal{C}}^{1} \,, \quad \text{where } \overline{\mathcal{C}}^{i} := C'_{i2} C_{i1} \,, \tag{4.22}$$
. Now define $\mathrm{ct}(\sigma) = (c_1, \dots, c_n)$, where c_i is half the difference between the number of occurrences of the entries i and $\overline{\imath}$ in $\overline{\sigma}$. Sometimes, this vector is written in terms of the coordinate vectors ε_i, as
$$\mathrm{ct}(\sigma) = c_1\varepsilon_1 + \dots + c_n\varepsilon_n = \frac{1}{2}\sum_{b\in\overline{\sigma}} \varepsilon_{\overline{\sigma}(b)} \,; \tag{4.23}$$
here the last sum is over all boxes b of $\overline{\sigma}$, and we set $\varepsilon_{\overline{\imath}} := -\varepsilon_i$.

Proof. We apply induction on i, which starts at $i = 0$, when the verification is straightforward. We will now proceed from $j_1 < \dots < j_i < k$, where $i = s$ or $k \le j_{i+1}$, to $j_1 < \dots < j_{i+1} < k$, and we will freely use the notation above. Assume that $\beta_{j_{i+1}}$ falls in the segment of Γ corresponding to column p of 2λ, where $p \ge q$.

We need to compute
$$w\widehat{r}_{j_1} \dots \widehat{r}_{j_{i+1}}(H_{\beta_k,l_k}) = \widehat{t}_{j_{i+1}} \dots \widehat{t}_{j_1} w(H_{\beta_k,l_k}) = \widehat{t}_{j_{i+1}}(H_{\gamma,m}) \,,$$
where $m = \langle \mathrm{ct}(\sigma_i[q]), \gamma^\vee \rangle$, by induction. Let $\gamma' := \gamma_{j_{i+1}}$, and $\widehat{t}_{j_{i+1}} = s_{\gamma',m'}$, where $m' = \langle \mathrm{ct}(\sigma_i[p]), (\gamma')^\vee \rangle$, by induction. We will use the following formula:
$$s_{\gamma',m'}(H_{\gamma,m}) = H_{s_{\gamma'}(\gamma), m - m'\langle \gamma', \gamma^\vee \rangle} \,.$$
Thus, the proof is reduced to showing that
$$m - m'\langle \gamma', \gamma^\vee \rangle = \langle \mathrm{ct}(\sigma_{i+1}[q]), s_{\gamma'}(\gamma^\vee) \rangle \,.$$
An easy calculation, based on the above information, shows that the latter equality is non-trivial only if $p > q$, in which case it is equivalent to
$$\langle \mathrm{ct}(\sigma_{i+1}(p,q]) - \mathrm{ct}(\sigma_i(p,q]), \gamma^\vee \rangle = \langle \gamma', \gamma^\vee \rangle \, \langle \mathrm{ct}(\sigma_{i+1}(p,q]), (\gamma')^\vee \rangle \,.$$
This equality is a consequence of the fact that
$$\mathrm{ct}(\sigma_{i+1}(p,q]) = s_{\gamma'}(\mathrm{ct}(\sigma_i(p,q])) \,,$$
which follows from the construction of σ_{i+1} from σ_i explained above. □

From this proposition we see that we do not get as clean of a result as in Type A, where the level did not change in a given column. We do however have that the level m_k does not change in a left column or a right column if we view the doubled tableaux.

Let $(\overline{i+1}, \overline{i})$ be represented by $(i, i+1)$ to simplify notation.

Then a left hand column is of one of the following forms as it was in Type A by the exact same argument, in particular the realization that the level m_i does not change in a left hand column, we have a similar result for right hand columns.

Corollary 4.24. $\Gamma(\lambda)^i$ *in any left or right hand column is of one of the following forms:*

(1) $(i, i+1)$
(2) $(i+1, i)$
(3) $\underline{(i, i+1)} \dots \underline{(i, i+1)}$
(4) $\underline{(i, i+1)} \dots \underline{(i, i+1)}(i, i+1)$
(5) $(i+1, i)\underline{(i, i+1)} \dots \underline{(i, i+1)}$
(6) $(i+1, i)\underline{(i, i+1)} \dots \underline{(i, i+1)}(i, i+1)$

From here we can take most all of the results from type A, with minor exception. The note to be made is that the highest level may occur in either a left or a right column. Where this highest level occurs will follow the same restrictions as in Type A, particularly:

Lemma 4.25. *Let j be the left or right column in which the highest level occurs (assuming $j > 1$), then the graph $g(i)$ restricted to column j is one of the following forms:*

- *empty*
- $\underline{(i, i+1)} \dots \underline{(i, i+1)}$
- $\underline{(i, i+1)} \dots \underline{(i, i+1)}(i, i+1)$

Proof. The proof here is identical to Type A. □

Recall the construction of column words from type A, in particular the word is read starting at the bottom of the column working toward the top of the column, starting with the leftmost column. In type C the same process is used. Call this word w. Then consider the subword of w consisting of the entries of the form, i, $i+1$, $\bar{i}$ or $\overline{i+1}$ and call this word w_i. Denote $\overline{i+1}$ or i by a $+$ and denote $\bar{i}$ or $i+1$ by a $-$. Factors of the form $+-$ may be ignored. This may be repeated until a subword of the form $\rho(w) = -^r +^s$ is reached.

If $r > 0$ $e_i(w)$ is obtained by changing the rightmost $-$ to a $+$ (i.e. changing $i+1$ into i and $\bar{i}$ into $\overline{i+1}$) and all other letters remain unchanged. If $r = 0$ then $e_i(w) = 0$. Then $f_i(w)$ is defined as the inverse.

If $s > 0$ $f_i(w)$ is obtained by changing the rightmost $+$ to a $-$ (i.e. changing i into $i+1$ and $\overline{i+1}$ into $\bar{i}$) and all other letters remain unchanged. If $s = 0$ then $f_i(w) = 0$. Then $e_i(w)$ is defined as the inverse.

Alternately as in type A:

$f_i(\Gamma(\lambda)^i) = (a_1, b_1)...\underline{(a_{i_1}, b_{i_1})}...\underline{(a_p, b_p)}(a_{p+1}, b_{p+1})...\underline{(a_{i_k}, b_{i_k})}...(a_m, b_m)$, i.e. the position before the highest level becomes marked. In the case where the highest level occurs in the first column $f_i(\Gamma(\lambda)^i) = \underline{(a_1, b_1)}...(a_{i_1}, b_{i_1})...\underline{(a_{i_k}, b_{i_k})}...(a_m, b_m)$.

Now consider the portion of the column word of T the results from extracting the subsequence of $+$'s and $-$'s, then look at $\rho(w)$ as defined above. This may be of one of the following forms:

(1) $+...+$
(2) $+...+\ -...-$
(3) $-...-$

Note that if this is of the third form then this corresponds to having the highest level in the first column in which case the root operator f_i is undefined, so we need not consider this case further.

Theorem 4.26. *The bijection between KN-tableaux of shape λ with entries in $[\overline{n}]$ and λ-increasing chains in C_n commutes with the root operators e_i and f_i.*

Proof. It suffices to check that the bijection commutes with f_i. Note that in the column prior to the one with highest level,i.e. the column $j-1$ as in the above, the chain ends with $(i, i+1)$ or $(\overline{i+1}, \bar{i})$, thus on the level of chains f_i makes this marked. This is precisely the same as replacing the i in that column with an $i+1$(or an $\overline{i+1}$ with a $\bar{i}$), consequently this is the same as the effect at the level of column words. Note that if we are in the case where highest level occurs 'past' the last column that this amounts to marking the last $(i, i+1)$ or $(\overline{i+1}, \bar{i})$. Thus the definition on the level of chains is the same as the definition in terms of column words, thereby showing that f_i commutes with the bijection, which is sufficient.

□

We thus have the following immediate corollary to the above theorem:

Corollary 4.27. *The bijection between KN-tableaux of shape λ with entries in $[\overline{n}]$ and λ-increasing chains in C_n preserves the crystal graph structure for KN-tableaux of shape λ with entries in $[\overline{n}]$.*

As an immediate corollary to this we have the following:

Corollary 4.28. *The bijection between KN-tableaux of shape λ with entries in $[\overline{n}]$ and λ-increasing chains in C_n preserves weight for KN-tableaux of shape λ with entries in $[\overline{n}]$.*

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