COMBINATORIAL MODELS FOR CERTAIN STRUCTURES IN ALGEBRAIC TOPOLOGY AND FORMAL GROUP THEORY

A thesis submitted to the University of Manchester
for the degree of Doctor of Philosophy
in the Faculty of Science

May 1996

By
Cristian Paul Lenart
Department of Mathematics

Contents

Abstract

This work in algebraic combinatorics is concerned with a new, combinatorial approach to the study of certain structures in algebraic topology and formal group theory. Our approach is based on several invariants of combinatorial structures which are associated with a formal group law, and which generalise classical invariants. There are three areas covered by our research, as explained below.

Our first objective is to use the theory of incidence Hopf algebras developed by G.-C. Rota and his school in order to construct and study several *Hopf algebras of set systems* equipped with a group of automorphisms. These algebras are mapped onto certain algebras arising in algebraic topology and formal group theory, such as binomial and divided power Hopf algebras, covariant bialgebras of formal group laws, as well as the Hopf algebroid of cooperations in complex cobordism. We identify the projection maps as certain invariants of set systems, such as the *umbral chromatic polynomial*, which is studied in its own right. Computational applications to formal group theory and algebraic topology are also given.

Secondly, we generalise the *necklace algebra* defined by N. Metropolis and G.-C. Rota, by associating an algebra of this type with every formal group law over a torsion free ring; this algebra is a combinatorial model for the group of Witt vectors associated with the formal group law. The cyclotomic identity is also generalised. We present combinatorial interpretations for certain generalisations of the necklace polynomials, as well as for the actions of the Frobenius operator and of the p-typification idempotent. For an important class of formal group

laws over the integers, we prove that the associated necklace algebras are also defined over the integers; this implies the existence of special ring structures on the corresponding groups of Witt vectors.

Thirdly, we study certain connections between *formal group laws and symmetric functions*, such as those concerning an important map from the Hopf algebra of symmetric functions over a torsion free ring to the covariant bialgebra of a formal group law over the same ring. Applications in this area include: generating function identities for symmetric functions which generalise classical ones, generators for the Lazard ring, and a simplified proof of a classical result concerning Witt vectors.

DECLARATION

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

Statement of Qualifications and Research

The author graduated from the University of Cluj-Napoca in Romania in 1988 with an honours degree in Mathematics. During the year 1992/93 he attended Part III of the Mathematical Tripos at the University of Cambridge, obtaining a Certificate of Advanced Study in Mathematics with distinction. He started his Ph.D. at the University of Manchester in October 1993.

Acknowledgements

First of all I would like to thank my supervisor, Professor Nigel Ray, for introducing me to an extremely rich and beautiful area of research, for his competent guidance throughout my work, as well as for his encouragement and strong support on numerous occasions.

I am grateful to the University of Manchester for offering me a Research Studentship, without which I would not have been able to complete this research.

It is a pleasure to acknowledge the assistance of Peter Cameron, who provided Theorem 1.6.3. I am grateful to William Schmitt for introducing me to the theory of incidence Hopf algebras; the idea of the proof of Proposition 4.1.5 belongs to him. I would also like to thank the members of staff at the University of Manchester who have helped me on various occasions, and in particular Dr. Grant Walker for enjoyable discussions on representation theory, and Dr. David Carlisle for helping me many times with the use of LaTeX.

Introduction

The relevance of Hopf algebras and formal group theory to algebraic topology (and in particular to K-theory and bordism theory) is well-known. The rôle of the Roman-Rota umbral calculus, as an elegant and illuminating framework for computations, became clear through the work of A. Baker, F. Clarke, N. Ray et al. (see [3], [34], [35], [8]). In the last decade combinatorial methods have also been brought to bear on some of the more subtle aspects of algebraic topology. This has been made possible by the important advances in combinatorics in recent years, especially through the establishment of algebraic combinatorics as an independent area of research. One of the main ideas is to find a combinatorial interpretation for the coefficients of various polynomials and formal power series that we investigate (which are in general hard to express and manipulate); thus, we are in a position to apply methods from algebraic combinatorics, which could lead to new insights. Several of the resulting techniques have then been fed back to enrich their combinatorial origins, finding application in areas such as graph theory, modular representations of matrix groups, and symmetric functions. Applications of combinatorial techniques to algebraic topology have been given recently by N. Ray and W. Schmitt; these applications are based on the theory of incidence Hopf algebras, developed by G.-C. Rota and his school (see [19], [46], [47]), which plays a major rôle throughout our work as well.

The aim of this work is to investigate several aspects of the beautiful interplay between algebraic topology, formal group theory, umbral calculus, and combinatorics, by building on the recent results of N. Ray and W. Schmitt. On the one hand, formal group theory and algebraic topology suggest generalisations of certain invariants of combinatorial structures, such as the chromatic polynomial of a graph, the characteristic polynomial of a poset of partitions, the necklace polynomials, and certain symmetric functions. We associate invariants of the type mentioned above with every formal group law, such that the classical invariants all correspond to the multiplicative formal group law $F(X,Y) = X + Y + XY$. Other formal group laws give rise to new invariants, which encode new information about the combinatorial structures. On the other hand, the study of the combinatorial invariants associated with a formal group law leads to a combinatorial framework for investigating certain structures in formal group theory and algebraic topology. Thus, certain Hopf algebras of set systems are combinatorial models for binomial and divided power Hopf algebras, for covariant bialgebras of formal group laws, as well as for the Hopf algebroid of cooperations in complex cobordism. Necklace algebras provide a useful framework for investigating Witt vectors associated with a formal group law, as well as for understanding p-typification. Certain symmetric functions can also be associated with a formal group law, and they prove to be useful new tools in the study of the Lazard ring and Witt vectors, for instance.

In Chapter 1 we have collected a minimum amount of information about the structures and concepts used in this work. The main aim is to establish a notation which is consistent with traditional notation, and to define all the concepts which are not so easily accessible in the literature. We also reformulate some background material (such as that on binomial and divided power Hopf algebras – which have recently come to provide a natural setting for the Roman-Rota umbral calculus, and that on formal group laws) in a way which makes more effective use of the coalgebraic viewpoint. Most of the concepts presented in this chapter also arise in algebraic topology, as explained in §1.4; however, we have preferred to use the classical topological notation only in Chapter 3, where concrete applications to topology are discussed. There are some classical results quoted in this chapter, but in general they are stated only when we need them, or they are just referred to the appropriate source. The main references for this work are:

- Bourbaki [4] for the concepts and notation of graded algebra,
- Sweedler [53] and Nichols and Sweedler [32] for all information concerning Hopf algebras and their applications to umbral calculus,
- Hazewinkel [18] for an encyclopaedic description of the theory of formal groups,
- Adams [1] and Ravenel [33] for all information concerning generalised homology theories,
- Aigner [2] for general combinatorial terminology,
- Schmitt [47] for an up-to-date account on incidence Hopf algebras,
- Macdonald [28] for the theory of symmetric functions.

In Chapter 2 we define and investigate the *umbral chromatic polynomial* of set systems of a fairly general type, which we call *partition systems*. This invariant was first defined for graphs by N. Ray and C. Wright in [41], in which case it encodes the same information about the graph as R. Stanley's *symmetric function generalisation of the chromatic polynomial* [51]. We propose two definitions for a colouring of a partition system, which coincide with the definition due to Wagner [55], in the case of simplicial complexes. These new definitions of colouring enable us to generalise the product formula for the classical chromatic polynomial of a graph, as well as Whitney's formula for expanding this polynomial as the characteristic polynomial of an associated poset. We also present related formulae for our umbral chromatic polynomial of a partition system, such as a deletion-contraction identity. In the last section, automorphism groups of partition systems are considered, and combinatorial interpretations and new formulae are given for the normalised versions of the associated polynomials. One of these formulae generalises the classical formula for expanding the divided conjugate Bell polynomials in terms of divided powers of x. This is the reason for which the results in this chapter are important ingredients for constructing combinatorial models for divided power algebras and covariant bialgebras of formal group laws.

In Chapter 3 we show that *incidence Hopf algebras of partition lattices* provide an efficient combinatorial framework for formal group theory and algebraic topology. We start by showing that the universal Hurwitz group law (respectively universal formal group law) are generating functions for certain leaf-labelled trees (repectively plane trees with coloured leaves). Two formal group law identities are then proved using a combinatorial technique. With reference to p-typical formal group laws, we discuss the way in which the formula for the corresponding characteristic type polynomial of a partition system simplifies; we also discuss the p-typical analogue of Lagrange inversion. As far as applications to algebraic topology are concerned, we illustrate the way in which several computations can be carried out efficiently by using the incidence Hopf algebra framework. Such computations include: expressing certain coactions, computing the images of the coefficients of the universal formal group law under the K-theory Hurewicz homomorphism, proving certain congruences in the complex cobordism ring, and constructing two combinatorial models for the dual of the polynomial part of the modulo p Steenrod algebra, for a given prime p.

In Chapter 4 we construct several *Hopf algebras of set systems*, equipped or not with a group of automorphisms, by using the theory of incidence Hopf algebras. We start by extending the constructions for graphs in [37] to certain cocommutative Hopf algebras of set systems, whose structure is examined. The previously mentioned polynomial invariants of set systems are realised as Hopf algebra maps onto certain binomial and divided power Hopf algebras, as well as onto the covariant bialgebra of a formal group law. An extended version of Stanley's symmetric function generalisation of the chromatic polynomial is also realised as a Hopf algebra map. One of the main themes of this chapter is that passage from a binomial to a divided power algebra corresponds, in the combinatorial setting, to the association of a group of automorphisms with a given set system. Several concepts and properties concerning binomial and divided power Hopf algebras can be lifted to the combinatorial Hopf algebras in a compatible way with the projection maps. Thus, we define delta operators, binomial and divided power sequences, and prove two identities concerning the interaction of a delta operator with the product and the antipode. In the second half of this chapter, we adopt a similar approach for constructing and investigating non-cocommutative Hopf algebras and Hopf algebroids of set systems, equipped or not with a group of automorphisms. These structures project onto such structures as the Faà di Bruno Hopf algebra, the dual of the Landweber-Novikov algebra, and the Hopf algebroid of cooperations in complex cobordism.

In [29] N. Metropolis and G.-C. Rota studied the necklace polynomials, and were lead to define the necklace algebra as a combinatorial model for the classical ring of Witt vectors (which corresponds to the multiplicative formal group law). In Chapter 5 we define and study a *generalised necklace algebra*, which is associated with an arbitrary formal group law $F(X,Y)$ over a torsion free ring A. The map from the ring of Witt vectors associated with $F(X,Y)$ to the necklace algebra is constructed in terms of certain generalisations of the necklace polynomials.

We present a combinatorial interpretation for these polynomials in terms of words on a given alphabet. The actions of the Verschiebung and Frobenius operators, as well as of the p-typification idempotent are described and interpreted combinatorially. A formal group-theoretic generalisation of the cyclotomic identity is also presented. In general, the necklace algebra can only be defined over the rationalisation $A \otimes \mathbb{Q}$. Nevertheless, we show that for an important family of formal group laws over $\mathbb{Z}$, namely $F_q(X, Y) = (X + Y - (1+q)XY)/(1 - qXY)$, $q \in \mathbb{Z}$ (which contains the multiplicative formal group law), the corresponding necklace algebra can be defined over $\mathbb{Z}$; furthermore, the generalised necklace polynomials turn out to be numerical polynomials in the variables x and q (that is they take integer values for integer x and q), and they can be interpreted combinatorially when q is a prime power. These results enable us to define ring structures compatible with the associated maps on the groups of Witt vectors and the groups of curves associated with the formal group laws $F_q(X, Y)$; there are few formal group laws with this property, and these ones are not mentioned in Hazewinkel's book.

In Chapter 6 we investigate several connections between formal group laws and symmetric functions, by using a combined approach, combinatorial and algebraic. We start by studying a certain Hopf algebra map from the Hopf algebra of symmetric functions over a torsion free graded ring to the covariant bialgebra of a formal group law over the same ring. This map has a geometrical interpretation in terms of a generalised homology theory and the determinant map, defined on unitary matrices. The study of the adjoint map provides identities for symmetric functions which generalise classical ones, as well as some Catalan number identities. The images of various symmetric functions under the above map are computed using P. Doubilet's formulae for these functions in terms of Möbius inversion on set partition lattices [11]. As an application of our results so far, we discuss a family of elements in the Lazard ring, with elements of degree $p^k - 1$ being polynomial generators, for every prime p. This family is implicit in the construction of the universal p-typical formal group law, and is well-suited for combinatorial manipulations. In the last section, we associate with every formal group law certain symmetric functions similar to the symmetric functions q_n studied by C. Reutenauer in the recent paper [42]; the latter are associated with the multiplicative formal group law. The symmetric functions which we define are used to give a short proof of the fact that addition of Witt vectors associated with a formal group law over a torsion free ring is determined by polynomials with coefficients in that ring. Finally, we prove a Schur positivity result similar to the one conjectured and partially proved by Reutenauer.

Throughout our research, we have used extensively the computer algebra system *Mathematica* and, occasionally, the symmetric function package of J. Stembridge (for *Maple*). We implemented several procedures for computing polynomial invariants of set systems, and for certain computations in algebraic topology and formal group theory. These procedures assisted us in formulating conjectures and identifying counter-examples, and thus lead us to a better understanding of the structures we were investigating.

Chapter 1

Background

In this chapter we give a brief description of the structures in formal group theory, algebraic topology, and combinatorics which will be used in our work. The main aim is to establish notation, while more detailed information on these structures appears in the references.

1.1 Binomial and Divided Power Hopf Algebras

Throughout §1.1, §1.2, §1.7, and §1.10 we let A_* be a non-negatively graded commutative ring with identity, which we refer to as the ring of *scalars*. We emphasise that A_* is free of additive torsion when, and only when, it embeds in its rationalisation $A\mathbb{Q}_* := A_* \otimes \mathbb{Q}$. All rings and algebras we consider are assumed graded by complex dimension, so that products commute without signs. We let C_* be a graded coalgebra over A_*, with comultiplication δ and counit ε; thus δ invests C_* with the structure of both left and right C_*-comodule. We usually assume that C_* is free, and of finite type, and write C^* for the graded dual $\mathrm{Hom}^*(C_*, A_*)$, which is naturally an A_*-algebra with identity.

We first recall how C^* may be interpreted as a ring of operators on C_*.

Let $\mathrm{Lop}(C_*)$ be the A_*-module consisting of those linear endomorphisms Γ of

C_* which are left C_*-comodule maps, and so satisfy the condition

$$\delta \circ \Gamma = (I \otimes \Gamma) \circ \delta \tag{1.1.1}$$

(where $I = I_{C_*}$ denotes the identity on C_*). We refer to these operators as *left-invariant*, and consider $\mathrm{Lop}(C_*)$ as an algebra under composition. Then C^* and $\mathrm{Lop}(C_*)$ are isomorphic as A_*-algebras under the map which assigns to each $f \in C^*$ the composition

$$C_* \xrightarrow{\delta} C_* \otimes C_* \xrightarrow{I \otimes f} C_* \otimes A_* \cong C_*\,, \tag{1.1.2}$$

which we denote by Γ_f. The inverse map associates to each linear operator Γ satisfying (1.1.1) the linear functional f_Γ defined by $f_\Gamma(z) = \varepsilon(\Gamma z)$, for all $z \in C_*$. We often use (1.1.2) to equate C^* and various of its subalgebras with their images in $\mathrm{Lop}(C_*)$, identifying f with Γ_f and Γ with f_Γ. In consequence, for any $\Delta \in \mathrm{Lop}(C_*)$ we may write

$$\langle \Gamma \cdot \Delta \mid z \rangle = \langle \Gamma \mid \Delta z \rangle\,, \tag{1.1.3}$$

where we follow the standard convention of expressing the duality map as $\langle f|z\rangle :=$ $f(z)$, for any $f \in C^*$ and $z \in C_*$. Indeed, 1.1.3 provides an alternative definition for the action of C^* on C_*.

For any A_*-coalgebra map $p : B_* \to C_*$, we note that

$$\Gamma_f \circ p = p \circ \Gamma_{p^*(f)} \tag{1.1.4}$$

in $\mathrm{Hom}^*(B_*, C_*)$, for all $f \in C^*$. We shall apply this formula in §1.2 and §4.1, for example, where C_* is a Hopf algebra, and p is either the product map or the antipode.

We remark that the algebra of *right*-invariant operators is defined by the obvious modification of (1.1.1), and that whenever C_* is cocommutative, the two concepts coincide. Otherwise, the map corresponding to (1.1.2) is actually an anti-isomorphism.

Given a countable basis c_ω for C_*, we denote the dual pseudobasis for C^* by c^ω. We may then deduce directly from (1.1.2) that the comultiplication is given in terms of the action of c^ω on C_* by

$$\delta(z) = \sum_\omega c^\omega z \otimes c_\omega \,. \tag{1.1.5}$$

Whenever f in C^* is of degree -1 and $f(C_1)$ contains the identity of A_*, we refer to Γ_f as a *delta operator*. We may define the category of coalgebras with delta operator by insisting that the morphisms are coalgebra maps which commute with the delta operators given on source and target, respectively.

By way of example, consider the graded polynomial algebra $A_*[x]$, and the comultiplication, counit, and antipode maps specified by

$$\delta(x) = x \otimes 1 + 1 \otimes x \,, \qquad \varepsilon(x^i) = \delta_{i,0} \,, \qquad \text{and} \qquad \gamma(x) = -x \,,$$

respectively. These maps invest $A_*[x]$ with the structure of a commutative and cocommutative Hopf algebra, which is known as the *binomial* Hopf algebra over A_* (in one variable). The standard basis consists of the powers x^n, for $n \geq 0$.

Note that δ may be rewritten as

$$\delta\colon\, A_*[x] \longrightarrow A_*[x,y] \,,$$

in which guise it is given by $\delta(x) = x + y$, and is known as the *shift* (by y). Then the notions of left and right-invariant coincide, and are traditionally referred to as *shift invariant*. The most basic such operator is the derivative d/dx, which we abbreviate to D. Under the isomorphism of (1.1.2), it corresponds to the A_*-linear functional which annihilates all x^n for $n \neq 1$, and satisfies $\langle D \mid x\rangle = 1$ in A_0. Thus D is a delta operator on $A_*[x]$. In fact, the functionals defined by $\langle D_{(n)} \,|\, x^m\rangle := \delta_{n,m}$ form the pseudobasis dual to the standard basis (so $D_{(1)} = D$); by (1.1.2) they act on $A_*[x]$ such that $D_{(n)}x^m = \binom{m}{n}x^{m-n}$ for all non-negative integers n and m. Whenever A_* embeds in $A\mathbb{Q}_*$, we may rewrite $D_{(n)}$ as $D^n/n!$, and interpret (1.1.5) as the formal Taylor expansion

$$p(x+y) = \sum_n \left(\frac{D^n}{n!} p(x) \right) y^n$$

for any polynomial $p(x)$.

We set $A^{-n} := \mathrm{Hom}^n(A_*, A_*)$ and identify it with A_n in a canonical way. The graded dual of $A_*[x]$ (as a coalgebra) can now be viewed as the graded algebra $A^*\{\{D\}\}$ of formal divided power series (or *Hurwitz series*; see [6]) over A_*. For each n, we shall express the elements of $A^n\{\{D\}\}$ in the form

$$\alpha_{-n} I + \alpha_{1-n} D + \cdots + \alpha_{k-n} D_{(k)} + \cdots ,$$

where $\alpha_i \in A_i$, I is the identity operator, and $D_{(k)} D_{(l)} = \binom{k+l}{k} D_{(k+l)}$ for all $k, l > 0$.

We now select a sequence (or *umbra*) $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)$, where $\alpha_i \in A_i$ and $\alpha_0 = 1$. Then the Hurwitz series

$$\alpha(D) := D + \alpha_1 D_{(2)} + \ldots + \alpha_{i-1} D_{(i)} + \ldots \tag{1.1.6}$$

lies in $A^1\{\{D\}\}$, and acts on $A_*[x]$ as a delta operator; in fact any delta operator on $A_*[x]$ is equal to $u\alpha(D)$ for some umbra α and some invertible element u in A_0. Given a positive integer m and a polynomial $p(x)$ in $A_*[x]$, we define the *umbral substitution* by $m\alpha$ in $p(x)$ as follows:

$$p(m\alpha) := \langle (I + \alpha(D))^m \mid p(x) \rangle .$$

It is not difficult to show that $p(m\alpha)$ can be obtained from $p(x_1 + \ldots + x_m)$ by setting $x_i^j \equiv \alpha_{j-1}$.

The Hurwitz series (1.1.6) form a group under substitution, with identity D. The umbra corresponding to the inverse of $\alpha(D)$ will be denoted by $\overline{\alpha}$, and $\overline{\alpha}(D)$ will be referred to as the *conjugate* delta operator of $\alpha(D)$.

Note that the divided powers $\alpha(D)_{(n)}$ define a new pseudobasis for $A^*\{\{D\}\}$, and that there is a dual basis of polynomials $B_n^\alpha(x)$ in $A_*[x]$. By definition, these polynomials satisfy $\langle \alpha(D)^n \mid B^\alpha_m(x)\rangle = n!\delta_{n,m}$; hence, by using (1.1.3), it follows that $B^\alpha_n(x)$ is monic of degree n (so that $B^\alpha_0(x) = 1$), and that

$$B^\alpha_n(0) = 0 \qquad \text{and} \qquad \alpha(D)\, B^\alpha_n(x) = nB^\alpha_{n-1}(x)\,,$$

for all $n > 0$. The sequence of polynomials $B^\alpha = (1, B^\alpha_1(x), B^\alpha_2(x), \dots)$ is known as the *associated sequence* of $\alpha(D)$. It is not difficult to show that

$$B^\alpha_n(m\alpha) = m(m-1)\dots(m-n+1)\,, \tag{1.1.7}$$

for all $n > 0$. On the other hand, let us note that (1.1.5) immediately provides the formula

$$\delta(B^\alpha_n(x)) = \sum_{i=0}^{n} \binom{n}{i} B^\alpha_i(x) \otimes B^\alpha_{n-i}(x)\,, \tag{1.1.8}$$

which defines B^α to be a *binomial sequence*.

We recall some classic examples of delta operators and their associated sequences.

Examples 1.1.9

1. For any A_*, let θ be the umbra $(1, 0, 0, \dots)$; then $\theta(D) = D$ and $B^\theta_n(x) = x^n$.

2. For scalars $k_* = \mathbb{Z}[u]$, where $u \in k_1$, let κ be the umbra $(1, u, u^2, \dots)$; then $\kappa(D)$ is the discrete derivative operator $(e^{uD} - 1)/u$ and $B^\kappa_n(x) = x(x-u)\dots(x-(n-1)u)$. It follows that $\overline{\kappa}(D) = \ln(1 + uD)/u$ and $B^{\overline{\kappa}}_n(x) = \sum_{i=1}^n u^{n-i} S(n,i) x^i$, where the $S(n,i)$ are Stirling numbers of the second kind. Thus $B^\kappa_n(x)$ and $B^{\overline{\kappa}}_n(x)$ are homogeneous versions of the *falling factorial* and *exponential polynomials*, respectively.

3. For scalars $\Phi_* = \mathbb{Z}[\phi_1, \phi_2, \dots]$, where $\phi_i \in \Phi_i$, let ϕ be the umbra $(\phi_0, \phi_1, \phi_2, \dots)$ with $\phi_0 = 1$. Then $B^\phi_n(x)$ and $B^{\overline{\phi}}_n(x)$ are the *conjugate Bell polynomials* and the *Bell polynomials*, respectively.

4. Given a prime p, we consider the summand Φ_*^p of Φ_* which is the image of the idempotent specified by

$$\overline{\phi}_n \mapsto \begin{cases} \overline{\phi}_n & \text{if } n = p^k - 1 \\ 0 & \text{otherwise}. \end{cases} \tag{1.1.10}$$

We denote by λ_n the image of ϕ_n in Φ_*^p, and by λ the corresponding umbra. It is not difficult to prove that $\lambda_n = 0$ unless n is divisible by $p-1$. We clearly have $\Phi_*^p = \mathbb{Z}[\lambda_{p-1}, \lambda_{p^2-1}, \ldots] = \mathbb{Z}[\overline{\lambda}_{p-1}, \overline{\lambda}_{p^2-1}, \ldots]$. The relevance of this example will be discussed in §1.2 and §1.4.

For basic information concerning the Bell polynomials we refer to [9]. The following simple property is taken from [34].

Proposition 1.1.11 *For any binomial Hopf algebra with delta operator* $(A_*[x], \alpha(D))$*, there is a unique ring homomorphism* $g^\alpha\colon \Phi_* \to A_*$*, specified by* $\phi_n \mapsto \alpha_n$*, which induces a map* $g^\alpha\colon (\Phi_*[x], \phi(D)) \to (A_*[x], \alpha(D))$ *of graded Hopf algebras with delta operator. Thus* $(\Phi_*[x], \phi(D))$ *is the universal example, and* B^ϕ *is the universal binomial sequence.*

Proposition 1.1.11 justifies our first important notational convention, to which we shall adhere throughout this work. Given an element $P^\phi(x)$ in $\Phi_*[x]$, we denote $g^\alpha(P^\phi(x))$ in $A_*[x]$ by $P^\alpha(x)$. If we substitute 1 for u in $P^\kappa(x)$, then we obtain a polynomial in $\mathbb{Z}[x]$, which we denote by $P(x)$. Whenever $P^\phi(x)$ is homogeneous, then $P^\kappa(x)$ is a homogenised version of $P(x)$. If f^ϕ is now a function from a set X to Φ_*, we denote the function $g^\alpha \circ f^\phi\colon X \to A_*$ by f^α.

In tandem with $A_*[x]$, and to cope with the situation when the scalars contain torsion, we must also consider the divided power algebra $R_*\{x\}$, where R_* is a similar ring of scalars. This is the free graded R_*-algebra with standard basis the divided powers $x_{(n)} \in R_n\{x\}$ for $n \geq 0$, where $x_{(0)} = 1$, $x_{(1)} = x$, and $x_{(k)}x_{(l)} = \binom{k+l}{k} x_{(k+l)}$. Whenever R_* is torsion free, then $x_{(n)}$ becomes identified with $x^n/n!$ in $R\mathbb{Q}_*\{x\}$. We may specify comultiplication, counit, and antipode maps by

$$\delta(x_{(n)}) = \sum_{i=0}^{n} x_{(i)} \otimes x_{(n-i)}\,, \qquad \varepsilon(x_{(n)}) = \delta_{n,0}\,, \qquad \text{and} \qquad \gamma(x_{(n)}) = (-1)^n x_{(n)}\,,$$

respectively. These maps invest $R_*\{x\}$ with the structure of Hopf algebra. The R_*-linear map $j\colon R_*[x] \to R_*\{x\}$, defined by $x^n \mapsto n!x_{(n)}$ for all $n \geq 0$, is a Hopf algebra map, and is monic whenever R_* is torsion free. The graded dual of $R_*\{x\}$ (as a coalgebra) is the graded algebra $R^*[[D]]$ of formal power series in the variable D, and the duality is expressed by $\langle D^n \mid x_{(m)} \rangle = \delta_{n,m}$. The action of D on $R_*\{x\}$ given by (1.1.2) is differentiation with respect to x, as before, whilst the dual map $j^*\colon R^*[[D]] \to R^*\{\{D\}\}$ is prescribed by $D^n \mapsto n!D_{(n)}$.

For any sequence $r = (1, r_1, r_2, \dots)$ with $r_i \in R_i$, the corresponding delta operator on $R_*\{x\}$ is

$$r(D) := D + r_1 D^2 + r_2 D^3 + \dots\,,$$

which lies in $R^1[[D]]$. As before, there is a conjugate delta operator $\overline{r}(D)$.

We have now established our second important notational convention: a formal power series associated with an umbra denoted by a Greek or upper case Roman letter (or a lower case Roman letter) will be Hurwitz (or standard) respectively.

In $R_*\{x\}$ there is a basis of polynomials $\beta_n^r(x)$ dual to the alternative pseudobasis $r(D)^n$ of $R^*[[D]]$, where $n \geq 0$. These polynomials form a *divided power sequence*, in the sense that

$$\delta(\beta_n^r(x)) = \sum_{i=0}^{n} \beta_i^r(x) \otimes \beta_{n-i}^r(x)\,; \tag{1.1.12}$$

we refer to this sequence as β^r.

In order to construct the universal example, and relate it to Proposition 1.1.11, we take our inspiration from [1] and choose as scalars the polynomial algebra

$H_* := \mathbb{Z}[b_1, b_2, \ldots]$, where b_n has grading n. The element $\overline{b}_n$ will be denoted by m_n, as it is traditionally done in algebraic topology (see §1.4). Clearly, m_i, $i \geq 1$, are also polynomial generators for H_*. Let b be the sequence $(b_0, b_1, b_2, \ldots)$ with $b_0 = 1$, and observe that we may compatibly identify Φ_* as a subalgebra of H_* by means of $\phi_n \mapsto (n+1)!b_n$. Thus there is a canonical inclusion $e\colon (\Phi_*[x], \phi(D)) \to (H_*\{x\}, b(D))$ of Hopf algebras with delta operator, with respect to which $B_n^\phi(x) = n!\,\beta_n^b(x)$. The following analogue of Proposition 1.1.11 then holds.

Proposition 1.1.13 *For any divided power Hopf algebra with delta operator $(R_*\{x\}, r(D))$, there is a unique ring homomorphism $g^r\colon H_* \to R_*$, specified by $b_n \mapsto r_n$, which induces a map $g^r\colon (H_*\{x\}, b(D)) \to (R_*\{x\}, r(D))$ of graded Hopf algebras with delta operator. Thus $(H_*\{x\}, b(D))$ is the universal example, and β^b is the universal divided power sequence.*

This proposition justifies a similar notational convention to the one following Proposition 1.1.11.

1.2 Formal Group Laws

Definition 1.2.1 *A (one-dimensional, commutative)* formal group law *over a commutative ring R is a formal power series $F(X,Y)$ in $R[[X,Y]]$ with the following properties:*

1. $F(X,0) = F(0,X) = X$;
2. $F(X,Y) = F(Y,X)$;
3. $F(X, F(Y,Z)) = F(F(X,Y), Z)$.

The formal power series $[-1]_F(X)$ in $R[[X]]$ defined by $F(X,[-1]_F(X)) = 0$ is called the *formal inverse*. We will use the following standard notation:

$$X +_F Y := F(X,Y)\,, \qquad X -_F Y := F(X,[-1]_F(Y))\,. \tag{1.2.2}$$

The third condition in the definition of a formal group law allows us to iterate the above notation, e.g. $X +_F Y +_F Z := F(F(X,Y),Z)$. It also makes sense to denote by $\sum^F(\;\;)$ the formal sum of the indicated elements. For integers n, we define

$$[n]_F(X) := \begin{cases} F(X,[n-1]_F(X)) & \text{if } n \geq 0 \\ [-1]_F\,([n]_F(X)) & \text{otherwise}\,. \end{cases} \tag{1.2.3}$$

If the ring R is torsion free, the formal group law $F(X,Y)$ has a *log series* $\log_F(X)$, that is a formal power series in $R\mathbb{Q}[[X]]$ satisfying $\log_F(F(X,Y)) = \log_F(X) + \log_F(Y)$. The substitutional inverse of $\log_F(X)$ is called the *exp series* of $F(X,Y)$, and is denoted by $\exp_F(X)$. Given a prime p and assuming R to be torsion free, we call $F(X,Y)$ *p-typical* if the only powers of X in $\log_F(X)$ are of the form X^{p^k}.

We now turn to the product structure on $A_*[x]$, and in particular to the question of expressing each $B_i^\alpha(x)B_j^\alpha(x)$ as a linear combination of the $B_n^\alpha(x)$. Inevitably, this is closely linked with the product structure on $A_*\{x\}$, and the expression of $\beta_i^a(x)\beta_j^a(x)$ in terms of the $\beta_n^a(x)$. The problem displays remarkable combinatorial complexity, and its investigation and interpretation are recurring themes below.

Let us begin with $A_*[x]$. The transpose of the product is a *formal comultiplication*

$$\delta\colon A^*\{\{D\}\} \longrightarrow A^*\{\{D\}\}\widehat{\otimes}A^*\{\{D\}\}\,,$$

in which we use a suitably completed tensor product $\widehat{\otimes}$. Equivalently, and more naturally, we may interpret δ as the A^*-algebra map

$$\delta\colon A^*\{\{X\}\} \to A^*\{\{X,Y\}\} \tag{1.2.4}$$

specified by $\delta(X) = X + Y$. This is tantamount to writing X and Y for the respective shift invariant operators $\partial/\partial x$ and $\partial/\partial y$. Clearly δ is coassociative, cocommutative, and an algebra map, whilst $X \mapsto 0$ defines a counit.

To address our problem we first consider its dual, which asks for a description of $\delta(\alpha(X)) = \alpha(X+Y)$ in terms of $\alpha(X)$ and $\alpha(Y)$.

Proposition 1.2.5 *There are Hurwitz series*

$$F^\alpha(X,Y) \in A^1\{\{X,Y\}\} \qquad \textit{and} \qquad [-1]_\alpha(X) \in A^1\{\{X\}\}$$

such that

$$F^\alpha(\alpha(X),\alpha(Y)) = \alpha(X+Y) \qquad \textit{and} \qquad F^\alpha(X,[-1]_\alpha(X)) = 0\,.$$

Proof. Choose $F^\alpha(X,Y) := \alpha(\overline{\alpha}(X) + \overline{\alpha}(Y))$ and $[-1]_\alpha(X) := \alpha(-\overline{\alpha}(X))$. □

It is important to observe that the transpose of the antipode γ of $A_*[x]$ is the algebra endomorphism induced by $X \mapsto -X$, which may equally well be described by $\alpha(X) \mapsto [-1]_\alpha(\alpha(X))$.

For each positive integer l, we shall write

$$F^\alpha(X,Y)_{(l)} = \sum_{i,j} F^{\alpha,l}_{i,j} X_{(i)} Y_{(j)} \qquad \text{and} \qquad ([-1]_\alpha(X))_{(l)} = \sum_k \iota^{\alpha,l}_k X_{(k)}$$

(omitting the superscript l whenever it takes the value 1). We shall make regular use of the abbreviations

$$X +_\alpha Y := F^\alpha(X,Y) \qquad \text{and} \qquad -_\alpha X := [-1]_\alpha(X)\,;$$

these are extremely convenient, and suitably graphic. The first notation may be iterated, whence it makes sense to write $\sum^\alpha(\ )$ for the formal sum of the indicated elements.

So long as A_* is torsion free, we may reinterpret $X +_\alpha Y$ as the formal group law

$$\sum_{i,j} F^\alpha_{i,j}/(i!j!)\, X^iY^j \tag{1.2.6}$$

over $A\mathbb{Q}_*$. This interpretation holds, more generally, over any extension $A_* \subseteq {}^+\!A_*$ which contains the appropriately divided coefficients; the minimal such extension was referred to in [34] as the *Leibnitz extension* of A_*, and was denoted by ${}^L\!A_*$. We therefore refer to $X +_\alpha Y$ as a *Hurwitz group law* over A_*.

A crucial example of such an extension for arbitrary A_* occurs when there is an embedding $e\colon A_* \to {}^\#\!A_*$ in some ring ${}^\#\!A_*$ which contains a sequence a such that $\alpha_n = (n+1)!a_n$ for all $n \geq 0$. Then ${}^\#\!A_*$ does indeed contain elements $f^a_{i,j}$ for which $F^\alpha_{i,j} = i!j!f^a_{i,j}$, as required, and we denote the formal group law $\sum_{i,j} f^a_{i,j}\, X^iY^j$ in ${}^\#\!A^*[[X,Y]]$ by $f^a(X,Y)$. By analogy with the universal case, e extends to an embedding $e\colon (A_*[x], \alpha(D)) \to ({}^\#\!A_*\{x\}, a(D))$ of Hopf algebras with delta operator, with respect to which $B^\alpha_n(x) = n!\,\beta^a_n(x)$.

Of course, the product on ${}^\#\!A_*\{x\}$ dualises to the formal comultiplication (1.2.4) on ${}^\#\!A^*[[X]]$. Then $X +_\alpha Y$ is identified with the formal group law $f^a(X,Y)$, which we also denote by $X +_a Y$; in addition, we may rewrite 1.2.5 as

$$f^a(a(X), a(Y)) = a(X+Y) \qquad \text{and} \qquad f^a(X, [-1]_a(X)) = 0\,.$$

This is compatible with the definitions at the beginning of this section, up to abbreviating $X +_{f^a} Y$ to $X +_a Y$, and $[-1]_{f^a}(X)$ to $[-1]_a(X)$. Thus the formal group law $X +_a Y$ has exp series $a(X)$, log series $\overline{a}(X)$, and formal inverse $[-1]_a(X) = \sum_k i^a_k X^k$, where e provides the identification $\iota^\alpha_k = k!\, i^a_k$ in ${}^\#\!A_*$.

The *covariant bialgebra* $U(f^a)_*$ of the formal group law $f^a(X,Y)$ over ${}^+\!A_*$ (see [18]) lies in the chain of Hopf algebra maps

$$A_*[x] \longrightarrow U(f^a)_* \longrightarrow {}^\#\!A_*\{x\}\,, \tag{1.2.7}$$

and its underlying module is the free ${}^{+}\!A_*$-module spanned by the polynomials $\beta_i^a(x)$, where $i \geq 0$; we denote this module by ${}^{+}\!A_*\langle\beta_i^a(x)\rangle$. The *contravariant bialgebra* $R(f^a)^*$ of $f^a(X,Y)$ is just ${}^{+}\!A^*[[a(D)]]$.

With reference to our examples 1.1.9, both $F^\theta(X,Y)$ and $f^\theta(X,Y)$ are the *additive* group law $X+Y$, whilst both $F^\kappa(X,Y)$ and $f^\kappa(X,Y)$ are the *multiplicative* group law $X+Y+uXY$. Moreover, $F^\phi(X,Y)$ is the *universal* Hurwitz group law, by virtue of Proposition 1.1.11. The minimal ring with an embedding of the form $e\colon \Phi_* \to {}^{\#}\Phi_*$ is clearly H_*, whilst ${}^{L}\Phi_*$ is the Lazard ring L_*. Clearly, L_* is a subalgebra of H_*, and $L\mathbb{Q}_* = \mathbb{Q}[b_1, b_2, \ldots] = \mathbb{Q}[m_1, m_2, \ldots]$. Furthermore, by Lazard's theorem, L_* is a polynomial algebra over $\mathbb{Z}$. The formal group law $f^b(X,Y)$ over L_* is also universal, and its covariant bialgebra $U(f^b)_*$ is the free L_*-algebra spanned by $\beta_i^b(x)$, where $i \geq 0$. One of the main thrusts of our work in Chapter 4 is to provide combinatorial models for the universal examples of 1.2.7, and for related Hopf algebras.

It can be shown that the idempotent of $(\Phi\mathbb{Q})_*$ specified by (1.1.10) restricts to an idempotent of L_*; its image is precisely the ring ${}^{L}\Phi_*^p$, which we denote by V_*. The above idempotent maps the coefficients of $f^b(X,Y)$ to the coefficients of the formal group law over V_* obtained by reinterpreting the Hurwitz group law $F^\lambda(X,Y)$ over Φ_*^p as in (1.2.6). The exp series of this new formal group law corresponds to an umbra in H_* which we denote by b^p; we clearly have $\lambda_n = (n+1)!\, b_n^p$ in H_*. The formal group law $f^{b^p}(X,Y)$ over V_* is the universal p-typical formal group law. It is isomorphic to $f^b(X,Y)$, when both formal group laws are considered over $L_* \otimes \mathbb{Z}_{(p)}$; here $\mathbb{Z}_{(p)}$ denotes, as usual, the ring of integers localised at a prime p, that is $\{l/n \in \mathbb{Q} \,:\, (n,p)=1\}$. We have $V\mathbb{Q}_* = \mathbb{Q}[m_{(1)}, m_{(2)}, \ldots]$, where $m_{(n)} := m_{p^n-1}$. Furthermore, by an analogue of Lazard's theorem, the ring V_* is a polynomial algebra over $\mathbb{Z}$ with polynomial generators of degree $p^n - 1$. There are some nice choices for the generators, such as *Hazewinkel's generators*

v_n, $n \geq 1$; we also have *Araki's generators* w_n, $n \geq 0$ for $V_* \otimes \mathbb{Z}_{(p)}$ (see [33]). These generators are defined recursively in terms of $m_{(n)}$ by

$$pm_{(n)} = \sum_{i=0}^{n-1} m_{(i)}\, v_{n-i}^{p^i} \qquad \text{and} \qquad pm_{(n)} = \sum_{i=0}^{n} m_{(i)}\, w_{n-i}^{p^i}\,, \tag{1.2.8}$$

where $w_0 = p$.

As a final example, we refer to an important family of p-typical formal group laws indexed by positive integers q. We consider the formal group law $f^{k^{p,q}}(X,Y)$ with logarithm

$$\overline{k^{p,q}}(X) := X + \frac{u^{p^q-1}}{p} X^{p^q} + \frac{u^{p^{2q}-1}}{p^2} X^{p^{2q}} + \ldots \quad \text{in } k\mathbb{Q}^1[[X]]\,.$$

It is easy to check, by using the defining relations (1.2.8), that the ring homomorphism from V_* to $k\mathbb{Q}_*$ mapping the coefficients of the universal p-typical formal group law to those of $f^{k^{p,q}}(X,Y)$ sends v_q to u^{p^q-1}, and the rest of Hazewinkel's generators to 0. Hence $f^{k^{p,q}}(X,Y)$ is a p-typical formal group law over the summand of k_* generated by u^{p^q-1}, which we denote by $k(q)_*$.

Let us now return to the Hurwitz group law $F^\alpha(X,Y)$. Since $F^\alpha(X,Y)$ and $[-1]_\alpha(X)$ respectively encode the action of δ and the antipode in terms of the pseudobasis $\alpha(X)_{(l)}$, we may immediately dualise to obtain

$$B_i^\alpha(x) B_j^\alpha(x) = \sum_{l=1}^{i+j} F_{i,j}^{\alpha,l} B_l^\alpha(x) \qquad \text{and} \qquad B_k^\alpha(-x) = \sum_{l=1}^{k} \iota_k^{\alpha,l} B_l^\alpha(x)\,. \tag{1.2.9}$$

These formulae answer our original question in terms of the Hurwitz group law; they may neatly be summarised as

$$B(X)B(Y) = B(X +_\alpha Y) \qquad \text{and} \qquad \gamma(B(X)) = B(-_\alpha X) \tag{1.2.10}$$

in $A_*[x]\{\{X,Y\}\}$, where $B(X) = \sum_i B_i^\alpha(x) X_{(i)}$. By iterating the former, we conclude that $\prod B(X_i) = B(\sum^\alpha X_i)$ for any finite sequence of variables X_i; we write $F^{\alpha,l}_{n_1,\ldots,n_k}$ for the coefficient of $\prod_{i=1}^k (X_i)_{(n_i)}$ in $(\sum^\alpha X_i)_{(l)}$.

Applying (1.1.4) to the product map and the antipode of $A_*[x]$ respectively, we further deduce that

$$\alpha(D)_{(l)}\,(p(x)q(x)) = \sum_{i,j\geq 0} F_{i,j}^{\alpha,l}\left(\alpha(D)_{(i)}\,p(x)\right)\left(\alpha(D)_{(j)}\,q(x)\right) \qquad \text{and} \tag{1.2.11}$$

$$\alpha(D)_{(l)}\,\gamma(p(x)) = \sum_{k\geq 1} \iota_k^{\alpha,l}\,\gamma\big(\alpha(D)_{(k)}\,p(x)\big)\,, \tag{1.2.12}$$

for arbitrary $p(x)$ and $q(x)$ in $A_*[x]$. We refer to (1.2.11) as the *Leibnitz rule* for $\alpha(D)$.

Returning to our examples 1.1.9, we note that, in the case of $\theta(D)$, the formulae (1.2.9) are trivial, and the Leibnitz formula is the standard one for D. In the case of $\kappa(D)$, the product formula (1.2.9) becomes the *Vandermonde convolution*, and the Leibnitz formula reduces to the well-known action of the discrete derivative on a product. The universal case $\phi(D)$ is considerably more mysterious, and is discussed in later sections.

Formulae (1.2.9), (1.2.10), (1.2.11), and (1.2.12) may easily be rewritten in terms of $a(D)$, in which context the first two are well-known. Note that

$$l!F_{n_1,\dots,n_k}^{\alpha,l} = n_1!\dots n_k!f_{n_1,\dots,n_k}^{a,l} \tag{1.2.13}$$

in ${}^{+}\!A_*$, under the identification provided by e.

1.3 Hopf Algebroids

Definition 1.3.1 *A* Hopf algebroid *over a commutative ring R is a pair (A, Γ) of commutative R-algebras with structure maps*

$\eta_L, \eta_R\colon A \to \Gamma$	*left and right units,*
$\delta\colon \Gamma \to \Gamma \otimes_R \Gamma$	*comultiplication,*
$\varepsilon\colon \Gamma \to A$	*counit,*
$\gamma\colon \Gamma \to \Gamma$	*conjugation,*

satisfying the following conditions:

1. $\varepsilon \circ \eta_L = \varepsilon \circ \eta_R = I_A$;

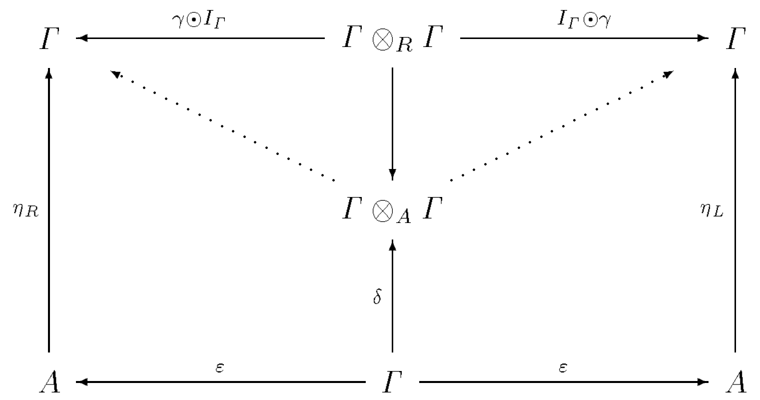
2. $(I_\Gamma \otimes \varepsilon) \circ \delta = (\varepsilon \otimes I_\Gamma) \circ \delta = I_\Gamma$;

3. $(I_\Gamma \otimes \delta) \circ \delta = (\delta \otimes I_\Gamma) \circ \delta$;

4. $\gamma \circ \eta_R = \eta_L$ *and* $\gamma \circ \eta_L = \eta_R$;

5. $\gamma \circ \gamma = I_\Gamma$;

6. *maps exist which make the following diagram commute*



where $(\gamma \odot I_\Gamma)(z_1 \otimes z_2) := \gamma(z_1)\, z_2$ *and* $(I_\Gamma \odot \gamma)(z_1 \otimes z_2) := z_1\, \gamma(z_2)$.

Here Γ is a left A-module via η_L, and a right A-module via η_R; on the other hand, $\Gamma \otimes_A \Gamma$ is the usual tensor product of bimodules, and δ and ε are A-bimodule maps. A graded Hopf algebroid (A_*, Γ_*) is called *connected* if the right and left sub-A_*-modules generated by Γ_0 are both isomorphic to A_*.

We now define Hopf algebroid structures on the algebras $\Phi_* \otimes \Phi_*$, $H_* \otimes H_*$, and $L_* \otimes H_*$, which will be identified with the following polynomial algebras:

$\Phi_* \otimes \Phi_* \cong \Phi_*[\psi_1, \psi_2, \ldots]$	via $\phi_n \otimes 1 \mapsto \phi_n$ and $1 \otimes \phi_n \mapsto \psi_n$,
$H_* \otimes H_* \cong H_*[c_1, c_2, \ldots]$	via $b_n \otimes 1 \mapsto b_n$ and $1 \otimes b_n \mapsto c_n$,
$L_* \otimes H_* \cong L_*[c_1, c_2, \ldots]$	via the restriction of the above isomorphism.

Note that we may compatibly identify $\Phi_* \otimes \Phi_*$ as a subalgebra of $H_* \otimes H_*$ by means of $\psi_n \mapsto (n+1)! c_n$. The structure maps of the Hopf algebroid $(H_*, H_* \otimes H_*)$ are defined as follows:

- $\varepsilon \colon H_* \otimes H_* \to H_*$ is specified by $\varepsilon(c_n) = 0$;
- $\eta_L \colon H_* \to H_* \otimes H_*$ is the standard inclusion;
- $\eta_R \colon H_* \to H_* \otimes H_*$ is given by
$$\sum_{i \geq 0} \eta_R(b_i) = \sum_{i \geq 0} c_i \left(\sum_{j \geq 0} b_j \right)^{i+1} ,$$
where $c_0 = 1$;
- $\delta \colon H_* \otimes H_* \to (H_* \otimes H_*) \otimes_{H_*} (H_* \otimes H_*)$ is specified by
$$\delta(c_n) = \sum_{k \geq 1} \sum_{\substack{n_1 + n_2 + \ldots + n_k = n+1 \\ n_i \geq 1}} \left(\prod_{i=1}^{k} c_{n_i - 1} \right) \otimes c_{k-1} ;$$
- $\gamma \colon H_* \otimes H_* \to H_* \otimes H_*$ is determined by
$$\gamma(b_n) = \eta_R(b_n) \qquad \text{and} \qquad \sum_{i \geq 0} \gamma(c_i) \left(\sum_{j \geq 0} c_j \right)^{i+1} = 1 .$$

The Hopf algebroids $(\Phi_*, \Phi_* \otimes \Phi_*)$ and $(L_*, L_* \otimes H_*)$ are defined by restricting the structure maps of $(H_*, H_* \otimes H_*)$. We abbreviate $\eta_R(\phi_n)$ to ϕ_n^R, and $\eta_R(b_n)$ to b_n^R; we also denote by ϕ^R, b^R, ψ and c the corresponding umbras. For a proof of the fact that the above structures are indeed Hopf algebroids, we refer to [33] Theorem A2.1.16.

1.4 Connections with Algebraic Topology

Interesting instances of the structures discussed in §1.1, §1.2, and §1.3 occur when ${}^L A_*$ is the coefficient ring $E_* := \pi_*(E)$ of a complex oriented cohomology theory

$E^*(\cdot)$ (note that we always have ${}^L E_* = E_*$). All the examples considered in 1.1.9 are of this type.

All homology and cohomology theories referred to in this work are assumed to be unreduced.

Let $E^*(\cdot)$ be a multiplicative cohomology theory with complex orientation Z in $E^2(\mathbb{C}P^\infty)$. The ring of coefficients E_* is assumed to be torsion free. We have $E^*(\mathbb{C}P^\infty) \cong E^*[[Z]]$ and $E_*(\mathbb{C}P^\infty) \cong E_*\langle \beta_1, \beta_2, \ldots \rangle$, that is the free E_*-module generated by $\beta_1, \beta_2, \ldots$ The standard map $\mu\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ classifying the tensor product of the two line bundles over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ determines the multiplicative structure of $E_*(\mathbb{C}P^\infty)$. The diagonal map $\mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ induces the comultiplication $\delta\colon E_*(\mathbb{C}P^\infty) \to E_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E_*(\mathbb{C}P^\infty) \otimes E_*(\mathbb{C}P^\infty)$ specified by

$$\delta(\beta_n) = \sum_{i=0}^{n} \beta_i \otimes \beta_{n-i}\,,$$

which turns $E_*(\mathbb{C}P^\infty)$ into a Hopf algebra. The map μ induces a map

$$\mu^*\colon E^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*(\mathbb{C}P^\infty)\widehat{\otimes} E^*(\mathbb{C}P^\infty)\,.$$

Letting $\mu^*(Z) := F(Z \otimes 1, 1 \otimes Z)$, it is easy to show that $F(X,Y)$ is a formal group law, and that $E^*(\mathbb{C}P^\infty)$ is its contravariant bialgebra, while $E_*(\mathbb{C}P^\infty)$ is its covariant bialgebra.

Let $D \in H^2(\mathbb{C}P^\infty)$ be the first Chern class of the Hopf bundle over $\mathbb{C}P^\infty$, and let $x \in H_2(\mathbb{C}P^\infty)$ be the standard spherical generator. In [35] it is shown that the Boardman map

$$E^*(\mathbb{C}P^\infty) \to H_*(E\mathbb{Q})\widehat{\otimes} H^*(\mathbb{C}P^\infty) \cong E\mathbb{Q}^*[[D]]$$

is a monomorphism, which maps Z to the exp series $\exp_F(D)$ of the formal group law $F(X,Y)$; we let

$$\exp_F(D) = a(D) = D + a_1 D^2 + a_2 D^3 + \ldots \quad \text{in } E\mathbb{Q}^2[[D]]\,.$$

It is also shown that the Hurewicz homomorphism

$$E_*(\mathbb{C}P^\infty) \to H_*(E\mathbb{Q}) \otimes H_*(\mathbb{C}P^\infty) \cong E\mathbb{Q}_*[x]$$

is a monomorphism, which maps β_n to $\beta_n^a(x)$.

Let us now consider the space ΩS^3 of loops on the 3-sphere. Let f_1 be the suspension of the inclusion $S^2 \hookrightarrow \Omega S^3$ and j_1 the evaluation map $\Sigma\Omega S^3 \to S^3$. We define

$$x \in \pi_2^S(\Omega S^3) \qquad \text{and} \qquad D \in \pi_S^2(\Omega S^3)$$

as the classes represented by f_1 and j_1, respectively. The unit $S^0 \to E$ induces elements $x \in E_2(\Omega S^3)$ and $D \in E^2(\Omega S^3)$. In [36] it is shown that $E_*(\Omega S^3) \cong E_*[x]$, and that $E^*(\Omega S^3) \cong E^*\{\{D\}\}$ as Hopf algebras. Let $h\colon \Omega S^3 \to \mathbb{C}P^\infty$ represent the integral cohomology class $D \in H^2(\Omega S^3)$. The use of the notation D for an element in $E\mathbb{Q}^*(\mathbb{C}P^\infty)$ and another one in $E^*(\Omega S^3)$ is now justified, since h^* maps the first element to the second one; the same is true about the notation x. Hence h_* and h^* can be interpreted (via the corresponding isomorphisms) as the embeddings $E_*[x] \hookrightarrow E_*\langle\beta_i^a(x)\rangle$ and $E^*[[a(D)]] \hookrightarrow E^*\{\{D\}\}$, respectively.

We now turn to the examples considered in 1.1.9, each of which corresponds to a certain complex oriented cohomology theory. Recall that given a torsion free ring A_* and an umbra α in A_*, we have defined a formal group law $f^a(X, Y)$ over ${}^LA_*$. We summarise the examples in 1.1.9 in the following table, where H_* stands for the ring $\mathbb{Z}[b_1, b_2, \ldots]$ (as defined in §1.1), as well as for singular homology, depending on the context. The orientations for the cohomology theories we mention are the usual ones (see [1] or [33]).

${}^{L}A_*$	$f^a(X,Y)$	Corresponding cohomology theory
$\mathbb{Z}$	$X+Y$	$H^*(\cdot)$ (singular cohomology)
k_*	$X+Y+uXY$	$k^*(\cdot)$ (connected K-theory)
L_*	$f^b(X,Y)$	$MU^*(\cdot)$ (complex cobordism)
H_*	$f^b(X,Y)$	$(H \wedge MU)^*(\cdot)$
$V_* \otimes \mathbb{Z}_{(p)}$	$f^{b^p}(X,Y)$	$BP^*(\cdot)$ (Brown-Peterson cohomology)
$k(q)_* \otimes \mathbb{Z}_{(p)}$	$f^{k^{p,q}}(X,Y)$	$g(q)^*(\cdot)$ (connected Morava-type K-theory)

The notation $g(q)^*(\cdot)$ is non-standard; it is motivated by the fact that the case $q=1$ corresponds to the connective version of the Adams summand of K-theory, which is usually denoted by $g^*(\cdot)$. The fact that the complex cobordism ring MU_* is isomorphic to the Lazard ring L_* is a remarkable result due to Quillen. It is also known that $H_*(MU)$ is isomorphic to the ring $H_* = \mathbb{Z}[b_1, b_2, \ldots]$. Furthermore, the Hurewicz homomorphism $MU_* \to H_*(MU)$, which is known to be a monomorphism, can be interpreted (via the above isomorphisms) as the embedding $L_* \hookrightarrow H_*$ discussed in §1.2. Based on these remarks, we shall henceforth identify MU_* with L_* and $H_*(MU)$ with H_*.

Finally, we discuss Hopf algebroids. J. F. Adams showed that under certain conditions on the cohomology theory $E^*(\cdot)$, there is a Hopf algebroid structure on $E_*(E)$, with structure maps defined by topological maps. He also determined the Hopf algebroid structure of $MU_*(MU)$ (see [1] or [33]). It is known that $MU_*(MU) \cong MU_*[b_1^{MU}, b_2^{MU}, \ldots]$ as algebras, whence $MU_*(MU) \cong L_* \otimes H_*$ via $b_n^{MU} \mapsto c_n$. This is actually an isomorphism of Hopf algebroids, since the structure of $L_* \otimes H_*$ defined in §1.3 is precisely the one described by Adams for $MU_*(MU)$. The induced Hopf algebra structure on H_* is known to topologists as the dual of the Landweber-Novikov algebra; we will refer to it once again in Example 1.7.4, from a combinatorial point of view. As far as the Hopf algebroid $H_* \otimes H_*$ is concerned, we note that it is isomorphic to $(H \wedge MU)_*(MU)$.

1.5 Set Systems

We shall always write $|V|$ for the cardinality of a given set V, and $k+[n]$ for the set of integers $\{k+1, k+2, \ldots, k+n\}$.

Given any finite set V of *vertices* (possibly empty), we refer to a collection of subsets $\mathcal{S} \subseteq 2^V$ as a *set system* if $\emptyset \in \mathcal{S}$ and $V = \bigcup_{W \in \mathcal{S}} W$; since $\mathcal{S}$ uniquely determines the vertices, we denote V by $V(\mathcal{S})$ whenever $\mathcal{S}$ is in doubt. Similarly, given a partition π of the set V, we denote the latter by $V(\pi)$. The set systems $\mathcal{S}_1$ and $\mathcal{S}_2$ are *isomorphic* if there is a bijection $f\colon V(\mathcal{S}_1) \to V(\mathcal{S}_2)$ such that $\{f(U) : U \in \mathcal{S}_1\} = \mathcal{S}_2$.

Instead of considering arbitrary set systems, in this work we concentrate on so-called *partition systems*, which we now define, since they provide the most appropriate framework for our constructions. Throughout this work, we employ the non-standard convention that the empty set has the unique partition $\{\emptyset\}$. Given a partition π of the finite set V, we denote by $\mathrm{Bool}(\pi)$ the Boolean algebra of subsets of V consisting of arbitrary unions of blocks of π. A set system $\mathcal{P}$ satisfying $\pi \subseteq \mathcal{P} \subseteq \mathrm{Bool}(\pi)$ for an arbitrary partition π of V will be called a *partition system*. The blocks of π are the atoms of the poset $(\mathcal{P}, \subseteq)$; we will refer to them as the *atoms* of $\mathcal{P}$. Since π is uniquely determined by $\mathcal{P}$, it is often convenient to denote π by $\mathrm{At}(\mathcal{P})$, and $\mathrm{Bool}(\pi)$ by $\mathrm{Bool}(\mathcal{P})$. The sets belonging to $\mathrm{Non}(\mathcal{P}) := \mathcal{P} \setminus \{\emptyset\} \setminus \mathrm{At}(\mathcal{P})$ will be called *non-atoms*.

Any set system which contains every vertex as a singleton is obviously a partition system, with singletons as atoms. Amongst such examples, we shall regularly consider simplicial complexes (or down closed set systems) such as the independence complex $\mathcal{I}(H)$ of a graph H, and

$$\mathcal{N}_V := \{\{x\} : x \in V\} \cup \{\emptyset\}, \qquad \mathcal{K}_V := 2^V, \qquad \mathcal{K}^\pi := \bigcup_{B \in \pi} 2^B,$$

where π is a partition of V. If $V = [n]$, we denote $\mathcal{N}_V$ by $\mathcal{N}_n$ and $\mathcal{K}_V$ by $\mathcal{K}_n$; if π is the partition of $[n_1+\ldots+n_k]$ with blocks $[n_1]$, $n_1+[n_2]$, $\ldots$, $n_1+\ldots+n_{k-1}+[n_k]$, we denote $\mathcal{K}^\pi$ by $\mathcal{K}_{n_1,\ldots,n_k}$. We shall also consider the set systems $\mathcal{I}_n$ consisting of those sets in $\mathcal{K}_n$ which are intervals (in $\mathbb{Z}$).

Given a partition system $\mathcal{P}$ and $U \in \mathrm{Non}(\mathcal{P})$, we define the *deletion* of U to be the partition system $\mathcal{P} \setminus \{U\}$, abbreviated to $\mathcal{P} \setminus U$. We also define the *strong deletion* as

$$\mathcal{P} \setminus\!\setminus U := \mathcal{P} \setminus \{W \in \mathcal{P} \,:\, U \subseteq W\}\,.$$

Now let $\pi \subseteq \mathrm{Bool}(\mathcal{P})$ be a partition of a set $U \subseteq V(\mathcal{P})$ (so that U necessarily lies in $\mathrm{Bool}(\mathcal{P})$). We define the partition system $\mathcal{P}|\pi$ to be

$$\{W \in \mathcal{P} \,:\, W \subseteq B \text{ for some } B \in \pi\}\,,$$

and call it the *restriction* of $\mathcal{P}$ to π. We also define the partition system $\mathcal{P}/\pi$ to be

$$\{W \in \mathcal{P} \,:\, B \subseteq W \text{ or } B \cap W = \emptyset\,, \text{ for all } B \in \pi\} \cup \pi\,,$$

and call it the *contraction* of $\mathcal{P}$ through π. Note that even if all the atoms of a partition system are singletons, not all the atoms of a contraction of it are (except for the trivial case, when we are contracting through singletons). We can transform an arbitrary partition system $\mathcal{P}$ into one which has only singleton atoms by defining

$$\mathrm{Sing}(\mathcal{P}) := \{\mathrm{At}(\mathcal{P}|U) \,:\, U \in \mathcal{P}\}\,.$$

We may then define the *strong contraction* of $\mathcal{P}$ through π as $\mathcal{P}/\!/\pi := \mathrm{Sing}\,(\mathcal{P}/\pi)$. We abbreviate $\mathcal{P}|\{U\}$, $\mathcal{P}/\{U\}$, and $\mathcal{P}/\!/\{U\}$ to $\mathcal{P}|U$, $\mathcal{P}/U$, and $\mathcal{P}/\!/U$, respectively. For instance,

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,2,3\}\}/\!/\{2,3\} = \{\emptyset, \{\overline{1}\}, \{\overline{2}\}, \{\overline{1}, \overline{2}\}\}\,,$$

where $\overline{1} := \{1\}$ and $\overline{2} := \{2,3\}$.

The restriction and the contraction of a partition σ through a set $U \in \mathrm{Bool}(\sigma)$ is defined in a similar way to the restriction and the contraction of a partition system, namely:

$$\sigma|U := \{B \in \sigma \,:\, B \subseteq U\}\,, \qquad \sigma/U := \{B \in \sigma \,:\, B \cap U = \emptyset\} \cup \{U\}\,.$$

Given two partitions π and σ of V satisfying $\pi \leq \sigma$, where the order is refinement (that is: every block of π is a subset of some block of σ), we recall that the *induced partition* σ/π on the blocks of π is the partition of π whose blocks are the sets $\{B \in \pi \,:\, B \subseteq C\}$ for C in σ.

Given two partition systems $\mathcal{P}$ and $\mathcal{Q}$ with the same vertices and atoms, we define the *complement* of $\mathcal{Q}$ in $\mathcal{P}$ to be the partition system

$$\complement_{\mathcal{P}}\mathcal{Q} := \mathcal{P} \setminus \mathrm{Non}(\mathcal{Q})\,.$$

The complement of $\mathcal{P}$ in $\mathrm{Bool}(\mathcal{P})$ will be denoted by $\overline{\mathcal{P}}$, and called, simply, the complement of $\mathcal{P}$. Given partition systems $\mathcal{P}_1$ and $\mathcal{P}_2$, we shall write their *disjoint union* as $\mathcal{P}_1 \cdot \mathcal{P}_2$, and define their *join* by

$$\mathcal{P}_1 \vee \mathcal{P}_2 := \{U_1 \sqcup U_2 \,:\, U_1 \in \mathcal{P}_1,\, U_2 \in \mathcal{P}_2\}\,,$$

where $\sqcup$ denotes disjoint union of sets. Note that the join of partition systems corresponds to disjoint union of graphs, when we identify a graph with its independence complex. It is useful to define the following operations, as well:

$$\mathcal{P}_1 \odot \mathcal{P}_2 := \overline{\overline{\mathcal{P}_1} \cdot \overline{\mathcal{P}_2}}\,, \qquad \mathcal{P}_1 \oplus \mathcal{P}_2 := \overline{\overline{\mathcal{P}_1} \vee \overline{\mathcal{P}_2}}\,.$$

We say that a partition system is *connected* if it differs from $\{\emptyset\}$ and is not isomorphic to a non-trivial disjoint union. Similarly, we call a partition system *join-connected* if it differs from $\{\emptyset\}$ and is not isomorphic to a non-trivial join of partition systems. We write $\mathcal{P}_c$ for the partition system consisting of those sets $U \in \overline{\mathcal{P}}$ for which $\mathcal{P}|U$ is join-connected. Given a graph H, it is not difficult to see that $\mathcal{I}(H)_c$ consists of those sets of vertices U for which the restriction of H to U is a connected graph. In general, finding an alternative description of $\mathcal{P}_c$, which is easier to grasp than the one given above, amounts to finding such a description for join-connectivity; however, we have not been able to do this.

Given a partition system $\mathcal{P}$, we refer to any partition σ of its vertices which satisfies $\sigma \subseteq \mathcal{P}$ as a *division* by $\mathcal{P}$, and denote the set of such divisions by $\Pi(\mathcal{P})$. This is partially ordered, as usual, by refinement, and the partition of $V(\mathcal{P})$ into the atoms of $\mathcal{P}$ is the minimum element $\widehat{0}$ (or $\widehat{0}_{\Pi(\mathcal{P})}$ if the context is unclear). In particular, $\Pi(\mathcal{K}_V)$ is the lattice of all partitions of V, and is usually denoted by $\Pi(V)$, or Π_n when $V = [n]$. The poset $\Pi(\mathcal{I}_n)$, which will be denoted by $\widetilde{\Pi}_n$, is isomorphic to the Boolean algebra $(\mathcal{K}_{n-1}, \subseteq)$. We write $\Pi_{n,k}$ and $\widetilde{\Pi}_{n,k}$ for the subsets of Π_n and $\widetilde{\Pi}_n$ consisting of partitions with k blocks. The set $\bigcup \Pi(\mathcal{P}|U)$, where U ranges over $\mathrm{Bool}(\mathcal{P})$, consists of all divisions of appropriate *subsets* of the vertices by elements of $\mathcal{P}$; this set will also be useful below, and we label it $\widehat{\Pi}(\mathcal{P})$.

Recall that a preferential arrangement of a finite set V is a pair (σ, ω), where σ is a partition of V, and ω is a bijection from $[|\sigma|]$ to σ, inducing a linear order on σ. Given a partition system $\mathcal{P}$, we denote by $A(\mathcal{P})$ the set of preferential arrangements (σ, ω) of $V(\mathcal{P})$ with $\sigma \in \Pi(\mathcal{P})$. This set can be partially ordered by setting $(\pi, \omega') \leq (\sigma, \omega)$ if $\pi \leq \sigma$, and only adjacent blocks are amalgamated in order to obtain σ from π. Following Wagner [55], we define a *colouring* of $\mathcal{P}$ with colours C to be a map $f\colon V(\mathcal{P}) \to C$ whose kernel is a division by $\mathcal{P}$; we denote the set of such colourings by $\Xi_C(\mathcal{P})$. If $C = \mathbb{N}$ or $C = [n]$, we simply call them colourings and colourings with at most n colours, respectively.

For each partition σ of a given set, we define its *type* $\tau^{\phi}(\sigma)$ to be the monomial $\phi_1^{k_1}\phi_2^{k_2}\ldots$ in Φ_*, where k_i is the number of blocks of σ with $i+1$ elements. The type of a colouring is the type of its kernel.

We call two partition systems with singleton atoms $\mathcal{S}_1$ and $\mathcal{S}_2$ *weakly isomorphic* if there are integers $k_1, k_2 \geq 0$ and a set system $\mathcal{S}$ such that $\mathcal{S}_i$ is isomorphic to $\mathcal{S} \cdot \mathcal{N}_{k_i}$ for $i = 1, 2$. Throughout this work, we shall not attempt to distinguish notationally between a set system and its isomorphism class, respectively weak isomorphism class, since in those cases where it matters, we have taken care to ensure that the context is clear. We denote by $\mathfrak{S}$ the set of isomorphism classes of partition systems with singleton atoms; we also denote by $\widehat{\mathfrak{S}}$ the set of weak isomorphism classes of partition systems $\mathcal{S}$ with singleton atoms for which $\Pi(\mathcal{S})$ has a unique maximal element. We write $\mathfrak{S}_\circ$ and $\widehat{\mathfrak{S}}_\circ$ for the subsets of $\mathfrak{S}$ and $\widehat{\mathfrak{S}}$ consisting of isomorphism classes, respectively weak isomorphism classes, of connected set systems. Complementation, as well as all the binary operations discussed above can be defined on $\mathfrak{S}$, while disjoint union can even be defined on $\widehat{\mathfrak{S}}$; thus, we obtain monoid structures on $\mathfrak{S}$ and $\widehat{\mathfrak{S}}$, in each case the unit being $\{\emptyset\}$.

1.6 Set Systems with Automorphism Group

Given a group G acting on a set X, we follow convention by writing the set of orbits under the action of G by X/G, the orbit of $x \in X$ by $G(x)$, and the stabiliser of x by G_x.

In this section, by set system we always mean a partition system with singleton atoms. We refer to a pair $(\mathcal{S}, G)$ consisting of a set system $\mathcal{S}$ and a group G of automorphisms of $\mathcal{S}$ as a *set system with automorphism group*. Obvious examples are $(\mathcal{K}_V, \Sigma_V)$ and $(\mathcal{K}_n, \Sigma_n)$, where Σ_V and Σ_n denote the symmetric groups on the sets V and $[n]$, respectively.

Given a set system with automorphism group $(\mathcal{S}, G)$, the group G acts in an obvious way on $\widehat{\Pi}(\mathcal{S})$, and on the set of colourings $\Xi_A(\mathcal{S})$ via the map $(g, f) \mapsto f \circ g^{-1}$, where $g \in G$ and f is a colouring. It also acts on the following posets by preserving the corresponding order: $(\mathcal{K}_{V(\mathcal{S})}, \subseteq)$, $\Pi(\mathcal{S})$ and $A(\mathcal{S})$; hence, there are induced poset structures on the sets of orbits: $\mathcal{K}_{V(\mathcal{S})}/G$, $\Pi(\mathcal{S})/G$ and $A(\mathcal{S})/G$. It is not difficult to see, and we will use the fact that the poset $A(\mathcal{K}_n)/\Sigma_n$ is isomorphic to $\widetilde{\Pi}_n$.

Consider a pair $(\mathcal{S}, G)$ as before, and a partition π of a subset of $V(\mathcal{S})$. We let $G|\pi$ denote the image of the group $\bigcap_{B\in\pi} G_B$ under the projection $\Sigma_{V(\pi)} \times \Sigma_{V(\mathcal{S})\setminus V(\pi)} \to \Sigma_{V(\pi)}$. Since $G|\pi$ is an automorphism group of $\mathcal{S}|\pi$, it makes sense to define the restriction of $(\mathcal{S}, G)$ to π by $(\mathcal{S}, G)|\pi := (\mathcal{S}|\pi, G|\pi)$. Restriction to $\{U\}$ is abbreviated as before. Now assume that π is a partition of $V(\mathcal{S})$. The stabiliser G_π of the partition π (under the action of G on $\Pi_{V(\mathcal{S})}$) permutes the blocks of π, whence we have a group homomorphism from G_π to Σ_π. The image of this homomorphism is an automorphism group of $\mathcal{S}/\!/\pi$, which we denote by G/π. Hence it makes sense to define $(\mathcal{S}, G)/\pi := (\mathcal{S}/\!/\pi, G/\pi)$, and call it the contraction of $(\mathcal{S}, G)$ through π. Let us note that the kernel of the above homomorphism is precisely $G|\pi$, whence $G_\pi/(G|\pi)$ is isomorphic to G/π; in particular, we have

$$|G_\pi| = |G|\pi|\,|G/\pi|\,. \tag{1.6.1}$$

We define complement and disjoint union of set systems with automorphism group by

$$\overline{(\mathcal{S}, G)} := (\overline{\mathcal{S}}, G) \qquad \text{and} \qquad (\mathcal{S}_1, G_1)\cdot(\mathcal{S}_2, G_2) := (\mathcal{S}_1\cdot\mathcal{S}_2, G_1\times G_2)\,, \tag{1.6.2}$$

respectively. Join and $\odot$ are defined analogously.

We decree that two set systems with automorphism group $(\mathcal{S}_1, G_1)$ and $(\mathcal{S}_2, G_2)$ are isomorphic if there is an isomorphism $f\colon V(\mathcal{S}_1) \to V(\mathcal{S}_2)$ of $\mathcal{S}_1$ and $\mathcal{S}_2$ such that G_1 is isomorphic to G_2 via the map $g \mapsto f\circ g\circ f^{-1}$. We call $(\mathcal{S}_1, G_1)$ and $(\mathcal{S}_2, G_2)$ weakly isomorphic if there are integers $k_1, k_2 \geq 0$ and a set system with automorphism group $(\mathcal{S}, G)$ such that $(\mathcal{S}_i, G_i)$ is isomorphic to $(\mathcal{S}, G) \cdot (\mathcal{N}_{k_i}, \{1\})$ for $i = 1, 2$. Throughout this work, we shall not attempt to distinguish notationally between a set system with automorphism group and its isomorphism class, respectively weak isomorphism class, since in those cases where it matters, we have taken care to ensure that the context is clear.

We now consider the following sets of isomorphism classes, respectively weak isomorphism classes, of set systems with automorphism group:

1. the set $\mathfrak{A}$ of isomorphism classes of all set systems with automorphism group;

2. the set $\widehat{\mathfrak{A}}$ of weak isomorphism classes of set systems with automorphism group $(\mathcal{S}, G)$ for which $\Pi(\mathcal{S})$ has a unique maximal element, and the following condition is satisfied: $(\mathcal{S}, G)$ and $(\mathcal{S}, G)/\sigma$ are disjoint unions of $(\mathcal{S}', G')$ with $\mathcal{S}'$ connected, for every σ in $\Pi(\mathcal{S})$;

3. the set $\mathfrak{C}$ of isomorphism classes of set systems with automorphism group $(\mathcal{S}, G)$ for which G has the property that every cycle of an element of G is also in G;

4. the set $\mathfrak{P}$ of isomorphism classes of set systems with automorphism group $(\mathcal{S}, G)$ for which there is a partition $\pi \in \Pi(\mathcal{S})$ such that $\mathcal{K}^\pi \subseteq \mathcal{S} \subseteq \mathcal{K}^\pi \cup \mathrm{Bool}(\pi)$, and G is the direct product of symmetric groups acting on the blocks of a partition $\sigma \leq \pi$;

5. the set $\widehat{\mathfrak{P}}$ of weak isomorphism classes of set systems with automorphism group $(\mathcal{S}, G)$ satisfying the condition in (4) plus the condition that $\Pi(\mathcal{S})$ has a unique maximal element.

Every set system may be considered as a set system with automorphism group, with respect to the trivial automorphism group $\{1\}$; hence we have the inclusions

$$\mathfrak{S} \subset \mathfrak{P} \subset \mathfrak{C} \subset \mathfrak{A} \qquad \text{and} \qquad \widehat{\mathfrak{S}} \subset \widehat{\mathfrak{P}} \subset \widehat{\mathfrak{A}}\,.$$

Let us also note that the operations disjoint union, join, and $\odot$ define monoid structures on $\mathfrak{P}$, $\mathfrak{C}$, and $\mathfrak{A}$. Disjoint union also defines a monoid structure on $\widehat{\mathfrak{P}}$ and $\widehat{\mathfrak{A}}$.

Peter Cameron has determined all permutation groups G with the property that every cycle of an element of G is also in G. He called such permutation groups *cycle-closed.*

Theorem 1.6.3 ([5]) *A permutation group is cycle-closed if and only if it is the direct product of its transitive constituents, each of which is a symmetric group or a cyclic group of prime order.*

1.7 Incidence Hopf Algebras

In [47], W. Schmitt associated a Hopf algebra, called the *incidence Hopf algebra*, with a family of posets satisfying certain conditions. We review these conditions here, and present some classical examples from this point of view; these examples will be used throughout this work. Other constructions based on the general method in [47] appear in Chapter 4.

Let $\mathbb{P}$ be a non-empty family of posets which are intervals (this means that they have a unique minimal and a unique maximal element). We assume that this family has the property that for all posets P in $\mathbb{P}$ and elements $x \leq y$ in P, the interval $[x, y] := \{z \in P \,:\, x \leq z \leq y\}$ is also in $\mathbb{P}$; such a family is called *interval closed.* We consider an equivalence relation $\sim$ on $\mathbb{P}$ such that, whenever $P \sim Q$ in $\mathbb{P}$, there exists a bijection $\xi\colon P \to Q$ such that $[\widehat{0}_P, x] \sim [\widehat{0}_Q, \xi(x)]$ and $[x, \widehat{1}_P] \sim [\xi(x), \widehat{1}_Q]$, for all x in P; such a relation is called *order compatible*, and the map ξ is called an *order compatible bijection.* Given the above setup and a commutative ring with identity R, there is a natural coalgebra structure on the free R-module $H(\mathbb{P})$ generated by the quotient set $\widetilde{\mathbb{P}} := \mathbb{P}/\sim$ (which is called the *set of types*). The comultiplication δ and the counit ε are defined by

$$\delta[P] := \sum_{x \in P} [\widehat{0}_P, x] \otimes [x, \widehat{1}_P] \qquad \text{and} \qquad \varepsilon[P] := \begin{cases} 1 & \text{if } |P| = 1 \\ 0 & \text{otherwise}. \end{cases}$$

Now let us assume that the interval closed family $\mathbb{P}$ is also closed under formation of direct products; such a family is called *hereditary*. It follows that $\mathbb{P}$ is a semigroup under direct product, generated by the set of indecomposables, which is denoted by $\mathbb{P}_\circ$. Let us assume that the order compatible relation $\sim$ considered above is also a *reduced* semigroup congruence, that is a semigroup congruence satisfying $P \times Q \sim Q \times P \sim P$ for every $P, Q \in \mathbb{P}$ with $|Q| = 1$; such an equivalence is called a *Hopf relation*. Poset isomorphism is an obvious example of a Hopf relation. Under the above conditions, the set of types $\widetilde{\mathbb{P}}$ is a monoid, with identity element equal to the type of any one point interval. Furthermore, the coalgebra structure of $H(\mathbb{P})$ can be enriched to that of a Hopf algebra, by linearly extending the multiplication in the monoid of types; this is called the *incidence Hopf algebra* of $\mathbb{P}$. The antipode of this Hopf algebra is given by the Schmitt formula:

$$\gamma[P] = \sum_{k \geq 0} \sum_{\substack{x_0 < \cdots < x_k \\ x_0 = \widehat{0}_P,\, x_k = \widehat{1}_P}} (-1)^k \prod_{i=1}^{k} [x_{i-1}, x_i]\,. \tag{1.7.1}$$

The dual H^* of $H := H(\mathbb{P})$ is called the *incidence algebra* of $\mathbb{P}$ (reduced modulo $\sim$). This algebra can be identified with the set of all maps from $\mathbb{P}$ to R. The multiplication in H^*, which is dual to the comultiplication in H, is called *convolution* and is given explicitly by

$$\langle f * g \mid [P] \rangle = \sum_{x \in P} \langle f \mid [\widehat{0}_P, x] \rangle \langle g \mid [x, \widehat{1}_P] \rangle\,,$$

for all $[P]$ in $\widetilde{\mathbb{P}}$. The subset $\mathrm{Alg}(H, R)$ of H^* consisting of all algebra maps from H to R can be identified with the set of all maps from $\widetilde{\mathbb{P}}_\circ$ to R. This subset is a group under convolution, called the group of multiplicative functions on H. The inverse of any f in $\mathrm{Alg}(H,R)$ is given by the composition $f \circ \gamma$, where γ is the antipode of H. Given f and g in $\mathrm{Alg}(H,R)$, the map $(f \otimes g) \circ \delta$ in $\mathrm{Alg}(H, R \otimes R)$ will be denoted by $f \circledast g$. The correspondence $R \mapsto \mathrm{Alg}(H,R)$ is a covariant functor from the category of rings to the category of groups. There are situations, such as those mentioned in §3.3, when a ring homomorphism $\xi\colon R \to T$ is uniquely determined by the image of $\mathrm{Alg}(\xi)$ on some function in $\mathrm{Alg}(H,R)$.

We now present three important examples. Recall the graded commutative rings with identity A_* and R_* considered in §1.1, and the corresponding umbras α and r.

Example 1.7.2 The Faà di Bruno Hopf algebra. Consider the family of posets which are isomorphic to a finite product of lattices Π_n. The incidence Hopf algebra (over the integers) of this family modulo isomorphism of posets is a polynomial algebra in infinitely many variables. Hence it can be realised on the set Φ_* by identifying the isomorphism class of the lattice Π_{n+1} with ϕ_n. Note that the isomorphism class of an interval $[\pi,\sigma]$ in Π_{n+1} becomes identified with the type $\tau^{\phi}(\sigma/\pi)$ (as defined in §1.5) of the induced partition σ/π. The comultiplication and counit are specified by

$$\delta(\phi_n) = \sum_{\sigma \in \Pi_{n+1}} \tau^{\phi}(\sigma) \otimes \phi_{|\sigma|-1}\,, \qquad \varepsilon(\phi_n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise}\,, \end{cases} \tag{1.7.3}$$

for all $n \geq 0$. This Hopf algebra is known as the Faà di Bruno Hopf algebra. Now consider the group $\mathrm{Alg}^*(\Phi_*, A_*)$ of *graded* multiplicative functions under convolution. We denote by ζ^{α} the multiplicative function specified by $\phi_i \mapsto \alpha_i$, and by μ^{α} its convolution inverse. In this context, ζ^{ϕ} is the identity map, and μ^{ϕ} is the antipode of the Hopf algebra Φ_*.

Example 1.7.4 The dual of the Landweber-Novikov algebra. Now consider the family of posets which are isomorphic to a finite product of Boolean algebras $\widetilde{\Pi}_n$. The incidence Hopf algebra (over the integers) of this family modulo isomorphism of posets is a polynomial algebra in infinitely many variables. Hence it can be realised on the set H_* by identifying the isomorphism class of the Boolean algebra $\widetilde{\Pi}_{n+1}$ with b_n. Note that the isomorphism class of an interval $[\pi, \sigma]$ in $\widetilde{\Pi}_{n+1}$ becomes identified with $\tau^b(\sigma/\pi)$. The comultiplication and counit are specified by

$$\delta(b_n) = \sum_{\sigma \in \widetilde{\Pi}_{n+1}} \tau^b(\sigma) \otimes b_{|\sigma|-1} = \sum_{k\geq 1} \sum_{\substack{n_1+n_2+\dots+n_k=n+1 \\ n_i \geq 1}} \left(\prod_{i=1}^{k} b_{n_i-1} \right) \otimes b_{k-1}\,, \tag{1.7.5}$$

$$\varepsilon(b_n) = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{otherwise}\,, \end{cases}$$

for all $n \geq 0$. This Hopf algebra is precisely the dual of the Landweber-Novikov algebra, which was mentioned in §1.4. It is not difficult to check that Φ_* can be identified with a sub-Hopf algebra of H_* via the monomorphism $\phi_n \mapsto (n+1)!b_n$; we will actually present a purely combinatorial way of understanding this fact in §4.4. Now consider the group $\mathrm{Alg}^*(H_*, R_*)$ of graded multiplicative functions under convolution. We denote by ζ^r the multiplicative function specified by $b_i \mapsto r_i$, and by μ^r its convolution inverse. In this context, ζ^b is the identity map, and μ^b is the antipode of the Hopf algebra H_*.

The notation ζ^α, μ^α, ζ^r, and μ^r conforms with our conventions. We now present a well-known result (see [12], Theorem 5.1 and [47], Examples 14.1 and 14.2), which has a fundamental rôle in translating problems related to substitution of formal power series into the combinatorial language of incidence Hopf algebras.

Theorem 1.7.6

1. *The group* $\mathrm{Alg}^*(\Phi_*, A_*)$ *is anti-isomorphic to the group under substitution*

of Hurwitz series *in* $A^*\{\{X\}\}$ *of the form*

$$\alpha(X) := \sum_{i\geq 1} \alpha_{i-1} X_{(i)}\,,$$

where α *is an umbra in* A_* *(if* A_* *has no torsion, then we can identify* $X_{(i)}$ *with* $X^i/i!$*); the anti-isomorphism is specified by* $\zeta^\alpha \mapsto \alpha(X)$*.*

2. *The group* $\mathrm{Alg}^*(H_*, R_*)$ *is anti-isomorphic to the group under substitution of formal power series in* $R^*[[X]]$ *of the form*

$$r(X) := \sum_{i\geq 1} r_{i-1} X^i\,,$$

where r *is an umbra in* R_**; the anti-isomorphism is specified by* $\zeta^r \mapsto r(X)$*.*

According to the theorem, we have

$$\mu^\alpha(\phi_n) = \overline{\alpha}_n \qquad \text{and} \qquad \mu^r(b_n) = \overline{r}_n\,. \tag{1.7.7}$$

Let us now recall the binomial Hopf algebra $\varPhi_*[x]$, and the divided power Hopf algebra $H_*\{x\}$. It is well-known that conjugate Bell polynomials and the Bell polynomials in $\varPhi_*[x]$ can be expressed combinatorially as follows:

$$B_n^\phi(x) = x\,(\mu^\phi * \zeta^\xi)(\phi_{n-1}) \quad \text{and} \quad B_n^{\overline{\phi}}(x) = x\,(\zeta^\phi * \zeta^\xi)(\phi_{n-1}) \quad \text{for } n\geq 1\,, \tag{1.7.8}$$

where ξ is the umbra $(1, x, x^2, \dots)$ in $\varPhi_*[x]$. Let us write

$$B_n^\phi(x) = \sum_{k=1}^n s^\phi(n,k)x^k \qquad \text{and} \qquad B_n^{\overline{\phi}}(x) = \sum_{k=1}^n S^\phi(n,k)x^k\,.$$

The coefficients $s^\phi(n,k)$ and $S^\phi(n,k)$ in $\varPhi_{n-k}$ are known as *$\varPhi_*$-Stirling numbers* of the first and second kind, respectively; $S^\phi(n,k)$ are also known as *partial Bell polynomials*. The standard notation $s(n,k)$ and $S(n,k)$ for the classical Stirling numbers of the first and second kind is consistent with our conventions.

Let us now recall the divided conjugate Bell polynomials $\beta_n^b(x)$ in $H_*\{x\}$, which we write as

$$\beta_n^b(x) = \sum_{k=1}^{n} \beta_{n,k}^b \frac{x^k}{k!}\,;$$

clearly, we have $n!\,\beta_{n,k}^b = k!\,s^\phi(n,k)$ in H_*. Considering the umbra $x = (1, x/2, x^2/3!, \dots)$ in $(\Phi\mathbb{Q})_*[x]$, it is now easy to show that the divided conjugate Bell polynomials and the divided Bell polynomials can be expressed combinatorially as follows:

$$\beta_n^b(x) = x\,(\mu^b * \zeta^x)(b_{n-1}) \quad \text{and} \quad \beta_n^{\overline{b}}(x) = x\,(\zeta^b * \zeta^x)(b_{n-1}) \quad \text{for } n \geq 1\,. \tag{1.7.9}$$

Indeed, we know that Φ_* is a sub-Hopf algebra of H_* via the inclusion $\phi_n \mapsto (n+1)!b_n$, which we denote by i. Hence the transpose i^*: $\mathrm{Alg}^*(H_*, (\Phi\mathbb{Q})_*[x]) \to \mathrm{Alg}^*(\Phi_*, (\Phi\mathbb{Q})_*[x])$ of i is a group homomorphism specified by $f \mapsto f \circ i$. More explicitly, we have that $i^*(\zeta^b) = \zeta^\phi$ and $i^*(\zeta^x) = \zeta^\xi$, whence $i^*(\mu^b) = \mu^\phi$ by taking inverses. Finally, we have the relation $i^*(\mu^b * \zeta^x) = \mu^\phi * \zeta^\xi$, which implies (1.7.9) when applied to ϕ_{n-1} and combined with (1.7.8). The second formula in (1.7.9) follows in a similar way. Passing from (1.7.8) to (1.7.9) turns out to be a special case of a more general phenomenon, which is investigated in §2.5 in terms of set systems and their automorphism groups.

Example 1.7.10 The incidence algebra of a poset. Let R be a commutative ring with identity, and P a *locally finite poset*, that is a poset whose intervals are all finite. We denote by $\mathrm{Int}(P)$ the set of intervals of P. The *incidence algebra* $R(P)$ of P is the free R-module generated by all functions from $\mathrm{Int}(P)$ to R, with pointwise addition, scalar multiplication, and product (or *convolution*) of f and g in $R(P)$ defined by

$$(f * g)(x, y) := \sum_{x \leq z \leq y} f(x, z)\, g(z, y)\,,$$

for all $x \leq y$ in P. Note that we have followed convention in abbreviating $f([x,y])$ to $f(x,y)$, and continue to do so. The identity of $R(P)$ is the function e which is defined, using the Kronecker delta, by $e(x,y) := \delta_{x,y}$. We now explain the connection with incidence Hopf algebras. Let $\mathbb{P}$ be the hereditary family consisting of arbitrary products of posets in $\mathrm{Int}(P)$, and let $\sim$ be the smallest (with respect to inclusion) Hopf relation on $\mathbb{P}$ for which the monoid $\mathbb{P}/\sim$ is commutative. We consider the incidence Hopf algebra (over R) of the family $\mathbb{P}$ modulo $\sim$, and denote it by H. As discussed above, the algebra $\mathrm{Alg}(H,R)$ is isomorphic to the subalgebra of $R(P)$ consisting of all functions f taking the value 1 on every one-point interval in P. According to the Schmitt formula (1.7.1), the convolution inverse of such a function exists, and can be expressed as follows:

$$f^{-1}(x,y) = \sum_{k\geq 0} \sum_{x=x_0<\ldots<x_k=y} (-1)^k f(x_0,x_1)\ldots f(x_{k-1},x_k)\,. \tag{1.7.11}$$

We can also express f^{-1} recursively, as follows:

$$f^{-1}(x,y) = -\sum_{x\leq z<y} f^{-1}(x,z)\, f(z,y)\,. \tag{1.7.12}$$

In fact, the convolution inverse of a function g in $R(P)$ exists if and only if $g(x,x)$ is an invertible element of R for every x in P. The general formula for the convolution inverse can be deduced from (1.7.11).

We now present some important special cases of the concepts related to the incidence algebra of a poset; these examples will play an important rôle throughout our work, while other examples will appear in §2.5 and §5.1. The first example is the *Möbius function* of P, which is just the convolution inverse of the *zeta function* ζ, taking the value 1 on all intervals of P. The Möbius function is traditionally denoted by μ, or μ_P if the context is unclear.

Now let A_* be the ring considered in §1.1, α the corresponding umbra, V a finite set, and P a subposet of $\Pi(V)$, ordered by refinement. Consider the function ζ^α in $A_*(P)$ defined by $\zeta^\alpha(\pi,\sigma) := \tau^\alpha(\sigma/\pi)$, which we call the *zeta type function*. Its convolution inverse clearly exists, and is denoted by μ_P^α (or just μ^α if the context is clear, a common situation of this kind being $P = \Pi(V)$). In order to express the convolution inverse, we define the type $\tau^\alpha(\gamma)$ of a chain $\gamma = \{\sigma_1 < \sigma_2 < \ldots < \sigma_k\}$ of length $l(\gamma) := k-1$ by

$$\tau^\alpha(\gamma) := (-1)^{l(\gamma)-1}\zeta^\alpha(\sigma_1,\sigma_2)\,\ldots\,\zeta^\alpha(\sigma_{k-1},\sigma_k)\,,$$

if $k > 1$, and by $\tau^\alpha(\gamma) := 1$, otherwise. According to (1.7.11), we have

$$\mu_P^\alpha(\pi,\sigma) = -\sum_{\gamma\in C(\pi,\sigma)} \tau^\alpha(\gamma)\,; \tag{1.7.13}$$

here, and throughout this work, $C(Q)$ denotes the set of chains between a minimal and a maximal element of the poset Q. The function μ_P^ϕ in $\Phi_*(P)$ is called the *Möbius type function* of P. Observe that both $\zeta^\alpha(\pi,\sigma)$ and $\mu_P^\alpha(\pi,\sigma)$ lie in $A_{|\pi|-|\sigma|}$.

It makes no difference if we replace the ring A_* and umbra α with the ring R_* and umbra r in the above paragraph. Let us also note that the functions $\zeta^\alpha, \mu^\alpha \in A_*(\Pi_n)$ and $\zeta^r, \mu^r \in R_*(\widetilde{\Pi}_n)$ are essentially the same as certain restrictions of the functions in $\mathrm{Alg}^*(\Phi_*, A_*)$ and $\mathrm{Alg}^*(H_*, R_*)$ for which we have used the same notation. The only difference is that the latter are defined on isomorphism classes of intervals rather than intervals; more precisely, we have $\mu^\alpha(\tau^\alpha(\sigma/\pi)) = \mu^\alpha_{\Pi(V)}(\pi,\sigma)$ for $\pi \le \sigma$ in $\Pi(V)$, and similar relations for the other functions. The context will always determine which kind of functions we are using; in fact, functions in $\mathrm{Alg}^*(\Phi_*, A_*)$ and $\mathrm{Alg}^*(H_*, R_*)$ are only used in Chapter 3.

The notation ζ^α and μ_P^α conforms with our convention. In particular, the classical Möbius function $\mu_P(\pi,\sigma)$ is obtained from $\mu_P^\phi(\pi,\sigma)$ by setting each ϕ_i to 1. This suggests that we might generalise certain standard properties of $\mu_P(\pi,\sigma)$ to $\mu_P^\alpha(\pi,\sigma)$. Thus we may establish the following two results by straightforward adaptation of the proofs in [50].

Proposition 1.7.14 *Given the subposets P and Q of $\Pi(V)$ and $\Pi(W)$, where $W \cap V = \emptyset$, we identify the pair $(\sigma, \sigma') \in P \times Q$ with $\sigma \cup \sigma'$; then we have*

$$\mu^{\alpha}_{P\times Q}(\pi \cup \pi', \sigma \cup \sigma') = \mu^{\alpha}_{P}(\pi, \sigma)\, \mu^{\alpha}_{Q}(\pi', \sigma') \qquad \text{in} \qquad A_* \,.$$

Proposition 1.7.15 *If Q is a subposet of the interval $[\pi, \sigma]$ in $P \subseteq \Pi(V)$ which contains both π and σ, then*

$$\mu^{\alpha}_{Q}(\pi, \sigma) = \sum (-1)^k \mu^{\alpha}_{P}(\pi, \pi_1) \,\ldots\, \mu^{\alpha}_{P}(\pi_k, \sigma)\,,$$

where the summation ranges over all chains $\{\pi < \pi_1 < \ldots < \pi_k < \sigma\}$ in $C(\pi, \sigma)$ for which $\pi_i \notin Q$.

1.8 Invariants of Partition Systems

We now associate several polynomials and symmetric functions with a partition system $\mathcal{P}$ and a set system with automorphism group $(\mathcal{S}, G)$ (recall that $\mathcal{S}$ must be a partition system with singleton atoms). We denote by $Sym^{\mathbb{Z}}_*(x)$ the graded $\mathbb{Z}$-algebra of symmetric polynomials over the infinite set of indeterminates $x = \{x_1, x_2, \ldots\}$ (cf. §1.10). Assuming that $V(\mathcal{P}) = V(\mathcal{S}) = V$, we define

$$\begin{aligned}
\rho^{\phi}(\mathcal{P}; x) &:= \textstyle\sum_{\sigma \in \Pi(\mathcal{P})} \tau^{\phi}(\sigma)\, x^{|\sigma|} && \text{in } \Phi_{|V|}[x]\,,\\
c^{\phi}(\mathcal{P}; x) &:= \textstyle\sum_{\sigma \in \Pi(\mathcal{P})} \mu^{\phi}_{\Pi(\mathcal{P})}(\widehat{0}, \sigma)\, x^{|\sigma|} && \text{in } \Phi_{|\mathrm{At}(\mathcal{P})|}[x]\,,\\
\widetilde{\chi}^{\phi}(\mathcal{P}; x) &:= \textstyle\sum_{\sigma \in \Pi(\mathcal{P})} \tau^{\phi}(\sigma)\, B^{\phi}_{|\sigma|}(x) && \text{in } \Phi_{|V|}[x]\,,\\
X(\mathcal{P}; x) &:= \textstyle\sum_{f \in \Xi_{\mathbb{N}}(\mathcal{P})} x^f && \text{in } Sym^{\mathbb{Z}}_{|V|}(x)\,,\\
X(\mathcal{S}, G; x) &:= \textstyle\sum_{f \in \mathcal{T}} x^f && \text{in } Sym^{\mathbb{Z}}_{|V|}(x)\,,
\end{aligned} \tag{1.8.1}$$

where $x^f := \prod_{v \in V} x_{f(v)}$, and $\mathcal{T}$ is an arbitrary transversal of the orbits of G on $\Xi_{\mathbb{N}}(\mathcal{S})$. We call $\rho^{\phi}(\mathcal{P}; x)$ the *partition type polynomial* of $\mathcal{P}$, and $c^{\phi}(\mathcal{P}; x)$ the *characteristic type polynomial* of $\mathcal{P}$. Of course, the first three polynomial invariants in $\Phi_*[x]$ are mapped to polynomial invariants in $A_*[x]$ by the map g^{α}

in Proposition 1.1.11, for every ring A_* of the type considered in §1.1 and every umbra α in A_*.

Special cases of these invariants are well-known. For instance, the partition polynomial of $\mathcal{P}$ investigated in [55] can be retrieved by substituting ϕ_i with 1 in $\rho^\phi(\mathcal{S};x)$. The characteristic type polynomial of a poset of partitions of a finite set appears in [40]; note that we have associated this polynomial with the partition system $\mathcal{P}$ rather than the poset $\Pi(\mathcal{P})$. If $\mathcal{P}$ is a simplicial complex and all the maximal partitions of $\Pi(\mathcal{P})$ have cardinality m, then the substitution $\phi_i \mapsto 1$ maps $c^\phi(\mathcal{P};x)$ to the characteristic polynomial of $\Pi(\mathcal{P})$, up to a factor x^m. The polynomial $\widetilde{\chi}^\phi(\mathcal{P};x)$ was referred to in [40] as the umbral chromatic polynomial of $\mathcal{P}$, since after umbral substitution by $m\phi$, it enumerates by type the colourings of $\mathcal{P}$ with at most m colours. Whenever $\mathcal{P}$ is the independence complex of a graph H, the polynomial $\widetilde{\chi}^\phi(\mathcal{P};x)$ reduces to the umbral chromatic polynomial $\chi^\phi(H;x)$ introduced in [41]. Note that our conventions dictate that we write $\chi(H;x)$ for the classical chromatic polynomial of H, and $\chi^\kappa(H;x)$ for its homogenised version. Considering the latter is natural from the point of view of graded algebras; the combinatorial significance is the following: after umbral substitution by $m\kappa$ (which is a special case of umbral substitution by $m\phi$), $\chi^\kappa(H;x)$ enumerates colourings of H with at most m colours by the number of colours. In Chapter 2 we associate with $\mathcal{P}$ a new polynomial $\chi^\phi(\mathcal{P};x)$, and it is this one that we label the *umbral chromatic polynomial* of $\mathcal{P}$. It turns out that whenever $\mathcal{P}$ is a simplicial complex (in particular, the independence complex of a graph), we have $\chi^\phi(\mathcal{P};x) = \widetilde{\chi}^\phi(\mathcal{P};x)$. There are essentially two reasons for concentrating on the more complex polynomial $\chi^\phi(\mathcal{P};x)$, rather than $\widetilde{\chi}^\phi(\mathcal{P};x)$, whose definition and combinatorial interpretation are straightforward. The first reason is that we are able to generalise Whitney's original formula (in Proposition 2.4.1) for expanding the chromatic polynomial as the characteristic polynomial of an associated poset:

$$\chi(H;x) = x^{n(H)}\, c(L_H;x); \tag{1.8.2}$$

here $n(H)$ denotes the number of connected components, and L_H the *lattice of contractions* (or *bond lattice*) of H, that is the set of all connected partitions of H, partially ordered by refinement. The second reason is that we are able to derive two product formulae for $\chi^\phi(\mathcal{P};x)$ (in Propositions 2.4.9 and 2.4.15) which generalise the classic

$$\chi(H_1 \sqcup H_2;x) = \chi(H_1;x)\,\chi(H_2;x)\,. \tag{1.8.3}$$

The second formula depends on the context of partition systems for its very existence; in terms of application to formal group laws, this formula may best be interpreted in the context of Hopf algebras, as in Chapter 4.

Finally, the symmetric functions $X(\mathcal{P};x)$ and $X(\mathcal{S},G;x)$ are natural extensions of Stanley's recently introduced symmetric function generalisation X_H of the chromatic polynomial of a graph H (see [51]); indeed $X(\mathcal{I}(H);x) = X_H$. As Stanley points out and we discuss in detail in §2.2, the symmetric function X_H encodes the same information as the umbral chromatic polynomial of H.

By way of simple examples, of which the last two are just restating (1.7.8), we remark that

$$\rho^\phi(\mathcal{N}_n;x) = c^\phi(\mathcal{N}_n;x) = x^n\,, \quad \rho^\phi(\mathcal{K}_n;x) = B_n^{\overline{\phi}}(x)\,, \quad c^\phi(\mathcal{K}_n;x) = B_n^\phi(x)\,. \tag{1.8.4}$$

If we combine the standard isomorphism $\Pi(\mathcal{S}_1\cdot\mathcal{S}_2) \cong \Pi(\mathcal{S}_1)\times\Pi(\mathcal{S}_2)$ with Proposition 1.7.14, we may immediately deduce the following result.

Proposition 1.8.5 *For any set systems $\mathcal{S}_1$ and $\mathcal{S}_2$, we have*

$$\rho^\phi(\mathcal{S}_1\cdot\mathcal{S}_2;x) = \rho^\phi(\mathcal{S}_1;x)\,\rho^\phi(\mathcal{S}_2;x)\,, \qquad c^\phi(\mathcal{S}_1\cdot\mathcal{S}_2;x) = c^\phi(\mathcal{S}_1;x)\,c^\phi(\mathcal{S}_2;x)\,.$$

We also record a simple property of $X(\mathcal{S}, G\,; x)$, directly from the definition.

Proposition 1.8.6 *Given set systems with automorphism group* $(\mathcal{S}_1, G_1)$ *and* $(\mathcal{S}_2,G_2)$*, then*

$$X(\mathcal{S}_1 \vee \mathcal{S}_2, G_1 \times G_2\,; x) = X(\mathcal{S}_1, G_1\,; x)\, X(\mathcal{S}_2, G_2\,; x)\,;$$

in particular, $X(\mathcal{S}_1 \vee \mathcal{S}_2\,; x) = X(\mathcal{S}_1\,; x)\, X(\mathcal{S}_2\,; x)$.

1.9 The Classical Necklace Algebra and Ring of Witt Vectors

In [29], Metropolis and Rota studied the properties of the so-called *necklace polynomials*, which are defined for every n in $\mathbb{N}$ by

$$M(x,n) := \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) x^d \qquad \text{in } \mathbb{Q}[x]\,;$$

here μ denotes the number-theoretic Möbius function, which is related to the Möbius function of the lattice $D(n)$ of divisors of n by $\mu(n/d) := \mu(d,n) = \mu(1,n/d)$. For every m in $\mathbb{N}$, $M(m,n)$ represents the number of *primitive necklaces* (that is asymmetric under rotation) with n coloured beads, where the colours are chosen from a set of size m. Hence, $M(x,n)$ are *numerical polynomials* (that is they take integer values for integer x). Metropolis and Rota were lead to define for every torsion free commutative ring A with identity the *necklace algebra* $Nr(A)$ (over A). This algebra is the set A^∞ of infinite sequences of elements of A with componentwise addition, and multiplication defined by

$$(\alpha \cdot \beta)_n := \sum_{[i,j]=n} (i,j)\, \alpha_i\, \beta_j\,;$$

here $[i,j]$ and (i,j) denote, as usual, the least common multiple and greatest common divisor of i and j, respectively. Note the convention of writing α for an element $(\alpha_1, \alpha_2, \dots)$ in A^∞; similarly, if h is a map from a set X to A^∞, we write $h(x) = (h_1(x), h_2(x), \dots)$. Following [29], we define a map $M\colon A\mathbb{Q} \to A\mathbb{Q}^\infty$ by $M_n(b) := M(b, n)$.

The algebra $Nr(A)$ has two remarkable operators for every r in $\mathbb{N}$, namely the *Verschiebung operator* V_r, and the *Frobenius operator* f_r; the former is defined by

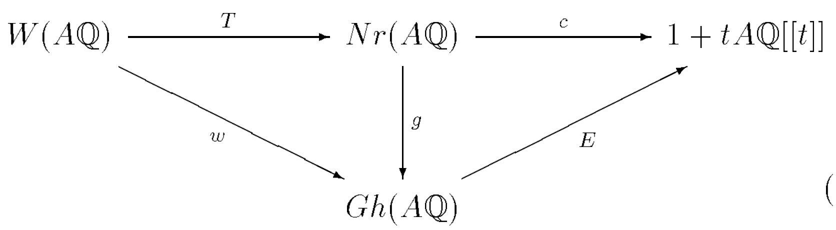
$$\mathrm{V}_{r,n}(\alpha) := \begin{cases} \alpha_i & \text{if } n = ri \\ 0 & \text{otherwise}. \end{cases} \tag{1.9.1}$$

The algebra $Nr(A)$ is closely related to the ring of *Witt vectors* $W(A)$ (see e.g. [21] pages 233–234), and the ring of unital formal power series $1 + tA[[t]]$ under *cyclic sum* and *cyclic product* (see [29]). To explain these relationships, we introduce the *ghost ring* $Gh(A)$, which is just A^∞ with addition and multiplication defined componentwise. We also define the following maps:

$$\begin{aligned} &T\colon W(A\mathbb{Q}) \to Nr(A\mathbb{Q}), && T(\alpha) := \textstyle\sum_{n\geq 1} \mathrm{V}_n M(\alpha_n), \\ &w\colon W(A\mathbb{Q}) \to Gh(A\mathbb{Q}), && w_n(\alpha) := \textstyle\sum_{d|n} d\alpha_d^{n/d}, \\ &g\colon Nr(A\mathbb{Q}) \to Gh(A\mathbb{Q}), && g_n(\alpha) := \textstyle\sum_{d|n} d\alpha_d, \\ &c\colon Nr(A\mathbb{Q}) \to 1 + tA\mathbb{Q}[[t]], && c(\alpha) := \textstyle\prod_{n\geq 1} \left(\frac{1}{1-t^n}\right)^{\alpha_n}, \\ &E\colon Gh(A\mathbb{Q}) \to 1 + tA\mathbb{Q}[[t]], && E(\alpha) := \exp\left(\textstyle\sum_{n\geq 1} \frac{\alpha_n}{n} t^n\right). \end{aligned}$$

Theorem 1.9.2 *(cf. [29], [14], [54])*

1. *All the above maps are ring isomorphisms, and the following diagram is commutative.*



(1.9.3)

2. *The image of $W(A)$ in $1 + tA\mathbb{Q}[[t]]$ is precisely $1 + tA[[t]]$. We also have that $T(W(A)) = Nr(A)$ for $A = \mathbb{Z}$, but not in general.*

3. We have that

$$(c \circ T)(\alpha) = \prod_{n \geq 1} \frac{1}{1 - \alpha_n t^n} .$$

The following generalisation of the cyclotomic identity *(due to V. Strehl [52]) holds:*

$$\prod_{n \geq 1} \left(\frac{1}{1 - kt^n} \right)^{M(m,n)} = \prod_{n \geq 1} \left(\frac{1}{1 - mt^n} \right)^{M(k,n)} \qquad \text{in } 1 + t\mathbb{Z}[[t]] \, , \tag{1.9.4}$$

where $k, m \in \mathbb{Z}$.

We conclude this section by recalling that Dress and Siebeneicher interpreted the necklace algebra $Nr(\mathbb{Z})$ as the *Burnside-Grothendieck ring* of *almost finite cyclic sets* [14]. They also interpreted the map T in this context, and were lead to a combinatorial interpretation of the ring structure of $W(\mathbb{Z})$. This enabled them to give a surprising generalisation of the ring of Witt vectors $W(A)$ in [13], namely the *Witt-Burnside ring* $W_G(A)$ associated with a profinite group G.

1.10 Symmetric Functions

Let A_* be the ring considered in §1.1. In this section we give a brief description of the graded A_*-algebra $Sym_*^A := Sym_*^A(X)$ of symmetric polynomials over an infinite set of indeterminates $X = \{X_1, X_2, \ldots\}$. Its graded dual is the A^*-algebra $Sym_A^* := Sym_A^*(X)$ of symmetric formal series over the same alphabet X.

There are several remarkable bases for Sym_*^A, which are indexed by partitions of positive integers. A partition of n is a sequence $I = (i_1, \ldots, i_l)$ with $i_1 \geq i_2 \geq \ldots \geq i_l > 0$ and $i_1 + \ldots + i_l = n$. If k occurs r_k times in I for $1 \leq k \leq n$, we write $I = (1^{r_1}, \ldots, n^{r_n})$. We use the notations

$$l(I) := l \, , \quad |I| := i_1 + \ldots + i_l \, , \quad I! := i_1! \ldots i_l! \, , \quad \|I\| := r_1! \ldots r_n! \, ;$$

$l(I)$ is known as the length of I, and $|I|$ as the weight of I. Given a partition σ of a set V, we denote by $I(\sigma)$ the partition of $|V|$ whose parts are the sizes of the blocks of σ. We define a partial order $\leq$ on the set of partitions of $|V|$ by identifying this set (in the obvious way) with the poset Π_n/Σ_n of orbits of the symmetric group Σ_n on Π_n. This partial order will be used in §3.2.

We use the notation of Lascoux and Schützenberger [22] for symmetric functions, which has the advantage of being compatible with the modern interpretation of symmetric functions as operators on λ-rings and polynomial functors. Thus, we denote the *complete symmetric function* corresponding to the partition I by $S^I := S^I(X)$, the *Schur function* by $S_I := S_I(X)$, the *elementary symmetric function* by $\Lambda^I := \Lambda^I(X)$, the *power sum symmetric function* by $\Psi^I := \Psi^I(X)$, and the *monomial symmetric functions* by $\Psi_I := \Psi_I(X)$. We will also use the augmented monomial symmetric functions, which are defined by $\widetilde{\Psi}_I := \|I\|\,\Psi_I$. It is well-known that S_n, $n \geq 1$, on the one hand, and Λ_n, $n \geq 1$, on the other hand, are polynomial generators for Sym_*^A (over A_*), while Ψ_n, $n \geq 1$, are polynomial generators for $Sym_*^{A\mathbb{Q}}$ (over $A\mathbb{Q}_*$). The Schur functions and the monomial symmetric functions form additive bases of Sym_*^A. It is also known that the basis of Schur functions is self-dual, and that the bases of complete and monomial symmetric functions are dual bases (with respect to the Hall inner product). Let us also recall the *forgotten symmetric functions* (see [28]), which form the dual basis to the basis of elementary symmetric functions (with respect to the Hall inner product); alternatively, the forgotten symmetric functions can be defined as images of the monomial symmetric functions under the standard involution on Sym_*^A (see [28]). Using the λ-ring formalism, the forgotten symmetric function corresponding to the partition I can be written as $(-1)^{|I|}\Psi_I(-X)$.

We can define a comultiplication on Sym_*^A by

$$\delta(P) = \xi^{-1}(P(X;Y))\,,$$

where ξ is the canonical isomorphism between $Sym_*^A(X) \otimes Sym_*^A(X)$ and the algebra $Sym_*^A(X;Y)$ of symmetric polynomials over the disjoint union of the alphabets X and Y. The adjoint of this comultiplication is just the ordinary product in Sym_A^*. This comultiplication defines a Hopf algebra structure on Sym_*^A, which was first investigated in [15]. It was shown there that S_n, $n \geq 1$, and Λ_n, $n \geq 1$, are divided power sequences, while Ψ_n, $n \geq 1$, are primitive; this means that

$$\delta(S_n) = \sum_{i=0}^{n} S_i \otimes S_{n-i}\,, \quad \delta(\Lambda_n) = \sum_{i=0}^{n} \Lambda_i \otimes \Lambda_{n-i}\,, \quad \delta(\Psi_n) = \Psi_n \otimes 1 + 1 \otimes \Psi_n\,. \tag{1.10.1}$$

As far as the antipode γ is concerned, we have $\gamma(S_n) = (-1)^n \Lambda_n$.

Chapter 2

Chromatic Polynomials of Partition Systems

In this chapter we define the umbral chromatic polynomial of a partition system and study some of its properties, such as the relation to the characteristic type polynomial defined in (1.8.1). Our results generalise some classical results for the chromatic polynomial of a graph. Apart from the purely combinatorial significance, some of these results will play an important rôle in Chapters 3 and 4, in the construction of combinatorial models for certain Hopf algebras in algebraic topology. Throughout this chapter, we let $\mathcal{P}$ be a fixed partition system, and we make extensive use of the concepts and notation related to partition systems, as presented in §1.5, §1.6, and §1.8.

2.1 Colourings of Partition Systems

In this section we define the concepts of colouring needed for the definition of the umbral chromatic polynomial of a partition system.

Definition 2.1.1

1. *A* factorised colouring *of $\mathcal{P}$ with at most m colours is a pair (γ, f) consisting of a chain $\gamma = \{\widehat{0}_{\Pi(\mathcal{P})} = \sigma_1 < \sigma_2 < \ldots < \sigma_k\}$ of partitions of $V(\mathcal{P})$ and a colouring f of $\mathcal{P}$ with at most m colours, such that the following conditions are satisfied:*

   (a) *$\sigma_i \in \Pi(\overline{\mathcal{P}})$, for $1 \leq i < k$;*

   (b) *the kernel of f is σ_k.*

2. *A* colouring forest *of $\mathcal{P}$ with at most m colours is a pair (δ, f) consisting of a set δ with $\mathrm{At}(\mathcal{P}) \subseteq \delta \subseteq \mathrm{Bool}(\mathcal{P}) \setminus \{\emptyset\}$ and a colouring f of $\mathcal{P}$ with at most m colours, such that in the poset $(\delta, \subseteq)$ we have:*

   (a) *the set of elements covered by U is a division by $\overline{\mathcal{P}}|U$ for all non-atoms $U \in \delta$;*

   (b) *the set $\max(\delta)$ of maximal elements of δ is a partition of $V(\mathcal{P})$, and the kernel of f is $\max(\delta)$.*

The name of the first concept comes from viewing the pair (γ, f) as a factorised function

$$\sigma_0 \xrightarrow{f_1} \sigma_1 \xrightarrow{f_2} \sigma_2 \ldots \xrightarrow{f_k} \sigma_k \xrightarrow{f_{k+1}} [m],$$

where σ_0 is the partition of $V(\mathcal{P})$ into singletons, every function f_i with $1 \leq i \leq k$ sends a block of σ_{i-1} to the block of σ_i containing it, and the composite $f_{k+1} \circ f_k \circ \ldots \circ f_1$ coincides with the colouring f (the last condition assumes that we identify the partition of $V(\mathcal{P})$ into singletons with $V(\mathcal{P})$). The name of the second concept is motivated by the fact that the Hasse diagram of the poset $(\delta, \subseteq)$ is a forest, since every element which is not maximal has a unique cover. Moreover, a colouring forest of $\mathcal{P}$ can be viewed as a forest of rooted trees with coloured roots and leaves labelled with symbols corresponding to the atoms of $\mathcal{P}$. The ordinary colouring f with kernel $\ker(f)$ can be viewed as the factorised colouring $(\{\widehat{0}_{\Pi(\mathcal{P})} \leq \ker(f)\}, f)$, and as the colouring forest $(\mathrm{At}(\mathcal{P}) \cup \ker(f), f)$. It is easy to see that all the factorised colourings and all the colouring forests of a simplicial complex are, in fact, ordinary colourings. We denote by $\Delta(\mathcal{P})$ the collection of sets δ with $\mathrm{At}(\mathcal{P}) \subseteq \delta \subseteq \mathrm{Bool}(\mathcal{P}) \setminus \{\emptyset\}$ satisfying the first condition in the definition of a colouring forest of $\mathcal{P}$ and $\max(\delta) \in \Pi(\mathcal{P})$. Given a chain γ as in the definition of a factorised colouring of $\mathcal{P}$, we associate with it the union of all partitions in γ; this collection of sets lies in $\Delta(\mathcal{P})$, and will be denoted by $\Delta(\gamma)$. We also associate with every factorised colouring (γ, f) the colouring forest $(\Delta(\gamma), f)$. These correspondences are surjective but not injective, as Example 2.1.3 shows.

In [31], Mullin and Rota defined a *reluctant function* from S to X to be a function from S to the disjoint union $S \sqcup X$, such that only a finite number of terms of the sequence $s, f(s), f(f(s)), \ldots$ are defined. Given a factorised colouring (γ, f) as in Definition 2.1.1 (1), we can identify it with the reluctant function $\widehat{f}$ from the disjoint union $\bigsqcup_{i=1}^{k} \sigma_i$ to $[m]$, specified by insisting that the restriction of $\widehat{f}$ to σ_i coincide with the function f_{i+1} discussed above. On the other hand, a colouring forest (δ, f) can be identified with the reluctant function $\overline{f}$ from δ to $[m]$ sending a set B to its cover in $(\delta, \subseteq)$ if B is not maximal, and to $f(x)$ for some x in B otherwise (this is a good definition because of the second condition in Definition 2.1.1 (2)).

There are several reasons for which we concentrate on the enumeration of factorised colourings and colouring forests of partition systems, rather than just ordinary colourings. To justify our choice, let us observe first that the definition of an ordinary colouring of a partition system requires that the maximal monochromatic blocks lie in the partition system. For simplicial complexes $\mathcal{C}$, this is equivalent to *all* monochromatic blocks lying in $\mathcal{C}$. However, for non-simplicial complexes $\mathcal{P}$, there might be monochromatic blocks which do not lie in $\mathcal{P}$; it is precisely these blocks that are taken into account by the two concepts of colouring defined above. Apart from this intuitive reason, we must emphasise, as we have done in §1.8, that certain properties of the classical chromatic polynomial of a graph (such as Whitney's formula and the product formula) could only be generalised using the new concepts of colouring; indeed, Whitney's formula cannot be generalised even in the classical case (corresponding to the umbra κ) if we use ordinary colourings of partition systems (in other words, the polynomial $\widetilde{\chi}^{\kappa}(\mathcal{P};x)$), as Corollary 2.4.2 shows.

We now define the type of the objects considered above. For every δ in $\Delta(\mathcal{P})$, we define its type $\tau^{\phi}(\delta) := (-1)^{s(\delta)}\phi_1^{k_1}\phi_2^{k_2}\ldots$, where $s(\delta)$ is $|\delta \setminus \mathrm{At}(\mathcal{P}) \setminus \max(\delta)|$, and k_i is the number of elements of $(\delta, \subseteq)$ which cover precisely $i+1$ elements. Let us note that

$$\tau^{\phi}(\Delta(\gamma)) = \pm\tau^{\phi}(\gamma)\,. \tag{2.1.2}$$

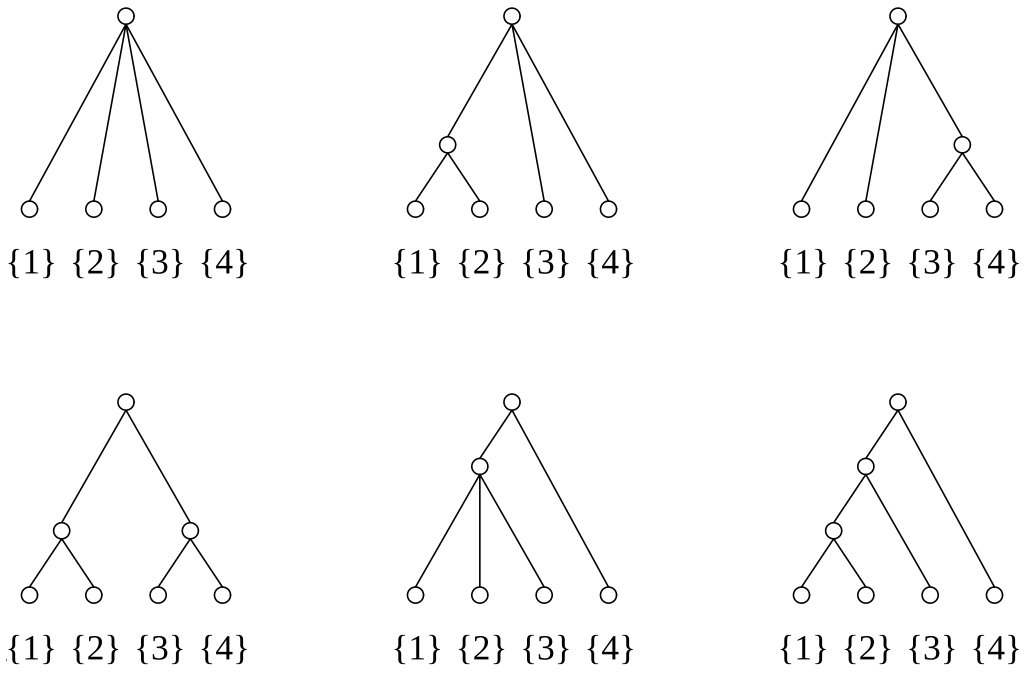
We define the type of a factorised colouring and of a colouring forest by $\tau^{\phi}(\gamma, f) := \tau^{\phi}(\gamma)$ and $\tau^{\phi}(\delta, f) := \tau^{\phi}(\delta)$. Both of these definitions are compatible with the definition of the type of an ordinary colouring.

Example 2.1.3 Let us consider vertices $[4]$, and the partition systems

$$\mathcal{Q} := \{\,\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{3,4\}, \{1,2,3\}\,\} \qquad \text{and } \mathcal{P} := \overline{\mathcal{Q}}\,.$$

There are six *trees* in $\Delta(\mathcal{P})$. They are listed below, and their types are ϕ_3, $-\phi_1\phi_2$, $-\phi_1\phi_2$, ϕ_1^3, $-\phi_1\phi_2$, and ϕ_1^3, respectively. All these trees have only one chain of partitions associated with them, except for the fourth one, which has the following three such chains with types ϕ_1^3, ϕ_1^3, and $-\phi_1^3$, respectively:

$$\widehat{0} < \{\,\{1,2\},\{3\},\{4\}\,\} < \{\,\{1,2\},\{3,4\}\,\} < \{\,\{1,2,3,4\}\,\}\,,$$
$$\widehat{0} < \{\,\{1\},\{2\},\{3,4\}\,\} < \{\,\{1,2\},\{3,4\}\,\} < \{\,\{1,2,3,4\}\,\}\,,$$
$$\widehat{0} < \{\,\{1,2\},\{3,4\}\,\} < \{\,\{1,2,3,4\}\,\}\,.$$



Clearly, the trees and chains discussed above can only be paired with monochromatic colourings of $\mathcal{P}$ in order to obtain colouring forests and factorised colourings.

Proposition 2.1.4 *Given a forest δ in $\Delta(\mathcal{P})$, the sum of types of all chains γ with $\Delta(\gamma) = \delta$ coincides with the type of δ.*

The proposition follows from (2.1.2) and slightly modified versions of two lemmas in [17]. For the sake of completeness, we state these lemmas here, and define the concepts involved. For any finite poset P, a *filtration* F of P is a chain $\{\emptyset = I_0 \subset I_1 \subset \ldots \subset I_k = P\}$ of lower order ideals such that $I_j \setminus I_{j-1}$ is an antichain for all $1 \leq j \leq k$. The number k is the *length* of the filtration, and is denoted by $l(F)$. We now assume that the forest δ is such that $\max(\delta) \neq \widehat{0}_{\Pi(\mathcal{P})}$, and denote by C the set of chains γ associated with δ.

Lemma 2.1.5 *There is a length-preserving bijection between C and the set of filtrations of the poset $(\delta \setminus \mathrm{At}(\mathcal{P}) \setminus \max(\delta)) \cup \{V(\mathcal{P})\}$, ordered by inclusion.*

Lemma 2.1.6 *For any finite poset P, we have that*

$$\sum_F (-1)^{l(F)-1} = (-1)^{|P|-1}\,,$$

where the summation is over all filtrations F of P.

2.2 The Umbral Chromatic Polynomial

In this section we define the umbral chromatic polynomial of the partition system $\mathcal{P}$ as a polynomial enumerating factorised colourings and colouring forests by type.

Let $\mathcal{Q}$ be another partition system with $V(\mathcal{Q}) = V(\mathcal{P})$ and $\mathrm{At}(\mathcal{Q}) = \mathrm{At}(\mathcal{P})$. Given a partition $\sigma \in \widehat{\Pi}(\mathcal{P})$, we define its *Möbius type* $\nu^{\phi}_{\mathcal{Q}}(\sigma)$ with respect to $\mathcal{Q}$ by

$$\sum_{\pi} \mu^{\phi}_{\Pi(\mathcal{Q}|\sigma)}(\widehat{0}, \pi)\, \zeta^{\phi}(\pi, \sigma)\,,$$

where the summation ranges over $\Pi(\mathcal{Q}|\sigma)$. We remark that whenever $\sigma \not\subseteq \mathcal{Q}$, then $\nu^{\phi}_{\mathcal{Q}}(\sigma) = -\mu^{\phi}_{\Pi(\mathcal{Q}|\sigma)\cup\{\sigma\}}(\widehat{0}, \sigma)$; otherwise, $\nu^{\phi}_{\mathcal{Q}}(\sigma)$ is 1 if σ is contained in $\mathrm{At}(\mathcal{Q})$, and 0 if it is not.

In fact, the Möbius type can be viewed as a map from $\widehat{\Pi}(\mathcal{P})$ to the ring Φ_*. We shall refer to any such map w into a commutative ring as a *weight*, and say that w is multiplicative if $w(\sigma_1 \cup \sigma_2) = w(\sigma_1)\, w(\sigma_2)$ for any $\sigma_1, \sigma_2 \in \widehat{\Pi}(\mathcal{P})$ with disjoint vertex sets.

We may now define the umbral chromatic polynomial.

Proposition 2.2.1 *There exists a polynomial in $\Phi_{|\mathrm{At}(\mathcal{P})|}[x]$ with the property that after umbral substitution by $m\phi$, it enumerates by type the factorised colourings* *and the colouring forests of $\mathcal{P}$ with at most m colours. This polynomial can be expressed in interpolated form as follows:*

$$\chi^{\phi}(\mathcal{P};x) := \sum_{\sigma} \nu^{\phi}_{\overline{\mathcal{P}}}(\sigma)\, B^{\phi}_{|\sigma|}(x)\,,$$

where the summation ranges over the poset $\Pi(\mathcal{P})$ of divisions by $\mathcal{P}$.

Proof. Let f be a colouring of $\mathcal{P}$ whose kernel $\ker(f)$ has n blocks. According to (1.7.13),

$$\sum_{\gamma} \tau^{\phi}(\gamma, f) = \nu^{\phi}_{\overline{\mathcal{P}}}(\ker(f))\,,$$

where the summation ranges over all chains γ for which (γ, f) is a factorised colouring. Since there are $m(m-1)\dots(m-n+1)$ colourings of $\mathcal{P}$ with at most m colours having the same kernel as f, the proposition follows by using (1.1.7). The fact that the polynomial $\chi^{\phi}(\mathcal{P};x)$ also enumerates colouring forests of $\mathcal{P}$ by type follows from Proposition 2.1.4. □

We call $\chi^{\phi}(\mathcal{P};x)$ the *umbral chromatic polynomial* of $\mathcal{P}$. Clearly, if $\mathcal{C}$ is a simplicial complex then $\nu^{\phi}_{\overline{\mathcal{C}}}(\sigma) = \tau^{\phi}(\sigma)$, so $\chi^{\phi}(\mathcal{C};x)$ coincides with the polynomial $\widetilde{\chi}^{\phi}(\mathcal{C};x)$ defined in (1.8.1). On the other hand, we can obtain various types of chromatic polynomials of the partition system $\mathcal{P}$ from $\chi^{\phi}(\mathcal{P};x)$ by replacing the umbra ϕ with another umbra. In particular, we obtain $\chi(\mathcal{P};x)$ and $\chi^{\kappa}(\mathcal{P};x)$, which we will call the classical chromatic polynomial of $\mathcal{P}$, and its homogenised version. After appropriate evaluation, the latter enumerates factorised colourings and colouring forests of $\mathcal{P}$ by the type τ^{κ}.

We now present a computational result on Möbius types. Further information on Möbius types will be given in Theorem 2.3.5 below.

Proposition 2.2.2

1. *The weight $\nu^{\phi}_{\mathcal{Q}}$ is multiplicative.*

2. *If $\sigma \in \widehat{\Pi}(\mathcal{P})$, and $\Pi(\mathcal{Q}|\sigma)$ has a unique maximal element $\widehat{1}$ and is non-trivial, then $\nu^{\kappa}_{\mathcal{Q}}(\sigma) = 0$.*

3. *If $\sigma \in \widehat{\Pi}(\mathcal{P})$, then $\nu^{\kappa}_{\overline{\mathcal{P}}}(\sigma) = \nu^{\kappa}_{\mathcal{P}_c}(\sigma)$.*

PROOF. (1) Let $\sigma_1, \sigma_2 \in \widehat{\Pi}(\mathcal{P})$ such that $V(\sigma_1) \cap V(\sigma_2) = \emptyset$, and denote $\sigma_1 \cup \sigma_2$ by σ. Write D_1 for $\Pi(\mathcal{Q}|\sigma_1)$, and D_2 for $\Pi(\mathcal{Q}|\sigma_2)$. Clearly, the poset $\Pi(\mathcal{Q}|\sigma)$ is isomorphic to $D_1 \times D_2$. Using Proposition 1.7.14, we deduce

$$\begin{aligned}
\nu^{\phi}_{\mathcal{Q}}(\sigma) &= \sum_{(\pi_1,\pi_2)\in D_1\times D_2} \mu^{\phi}_{D_1\times D_2}(\widehat{0}_{D_1} \cup \widehat{0}_{D_2}, \pi_1 \cup \pi_2)\, \zeta^{\phi}(\pi_1 \cup \pi_2, \sigma_1 \cup \sigma_2) \\
&= \left(\sum_{\pi_1 \in D_1} \mu^{\phi}_{D_1}(\widehat{0}_{D_1}, \pi_1)\, \zeta^{\phi}(\pi_1, \sigma_1) \right) \left(\sum_{\pi_2 \in D_2} \mu^{\phi}_{D_2}(\widehat{0}_{D_2}, \pi_2)\, \zeta^{\phi}(\pi_2, \sigma_2) \right) \\
&= \nu^{\phi}_{\mathcal{Q}}(\sigma_1)\, \nu^{\phi}_{\mathcal{Q}}(\sigma_2)\,.
\end{aligned}$$

(2) We may assume that $\sigma \notin \widehat{\Pi}(\mathcal{Q})$, since otherwise $\nu^{\phi}_{\mathcal{Q}}(\sigma) = 0$. We can pair each chain $\{\widehat{0} < \sigma_1 < \ldots < \sigma_n < \sigma\}$ in $\Pi(\mathcal{Q}|\sigma) \cup \{\sigma\}$ for which $\sigma_n \neq \widehat{1}$ with the chain $\{\widehat{0} < \sigma_1 < \ldots < \sigma_n < \widehat{1} < \sigma\}$. The contribution to $\nu^{\kappa}_{\mathcal{Q}}(\sigma)$ of each pair is 0, whence $\nu^{\kappa}_{\mathcal{Q}}(\sigma) = 0$.

(3) We need the concept of *coclosure operator* on a poset P (see e.g. [38]), which is a function $x \mapsto \overline{x}$ from P into itself such that: (1) $\overline{x} \leq x$, (2) $\overline{\overline{x}} = \overline{x}$, and (3) $x \leq y$ implies $\overline{x} \leq \overline{y}$, for all $x, y \in P$. An element x of P is *closed* if $\overline{x} = x$. Consider the poset $P := \Pi(\overline{\mathcal{P}}|\sigma) \cup \{\sigma\}$, and define the coclosure operator $\pi \mapsto \overline{\pi}$ by $\overline{\sigma} = \sigma$, and by letting the partition $\overline{\pi}$ be obtained from π by splitting every block B into the sets of vertices of the join-connected components of $\mathcal{P}|B$ whenever $\pi \neq \sigma$. Obviously, the subposet of closed elements is $Q = \Pi(\mathcal{P}_c|\sigma) \cup \{\sigma\}$. According to [44], we have that $\mu^{\kappa}_{Q}(\widehat{0}, \sigma) = \sum \mu^{\kappa}_{P}(\pi, \sigma)$, where the summation is over all π such that $\overline{\pi} = \widehat{0}$ (we have used the fact that $\mu^{\kappa}_{P}(\pi, \sigma) = u^{|\pi|-|\sigma|}\, \mu_P(\pi, \sigma)$). But whenever $\overline{\pi} = \widehat{0}$ and $U \in \pi$, we have $U \in \overline{\mathcal{P}}$ and $\mathcal{P}|U = \mathrm{Bool}(\mathrm{At}(\mathcal{P}|U))$. Thus $U \in \mathcal{P}$, which is possible if and only if $U \in \mathrm{At}(\mathcal{P})$; hence $\pi = \widehat{0}$, which means that $\mu^{\kappa}_{Q}(\widehat{0}, \sigma) = \mu^{\kappa}_{P}(\widehat{0}, \sigma)$. $\square$

According to Proposition 2.2.2 (3), we can define the classical chromatic polynomial of $\mathcal{P}$ and its homogenised version in terms of $\mathcal{P}_c$ only. This is useful because $\overline{\mathcal{P}}$ may be much larger than $\mathcal{P}_c$, whence a summation ranging over $\Pi(\overline{\mathcal{P}})$ may contain many more terms than one ranging over $\Pi(\mathcal{P}_c)$. Moreover, we shall see in §2.4 that it is possible to state the analogue of Whitney's result (1.8.2) for $\chi^\kappa(\mathcal{P};x)$ in terms of $\mathcal{P}_c$, although we were only able to state it in terms of $\overline{\mathcal{P}}$ for $\chi^\phi(\mathcal{P};x)$.

Let $\mathcal{C}$ be a simplicial complex. We conclude this section with a reference to the symmetric function $X(\mathcal{C};x)$ defined in (1.8.1), which reduces to Stanley's symmetric function generalisation of the chromatic polynomial of a graph H when $\mathcal{C}=\mathcal{I}(H)$ (see [51]). Following Stanley, we observe that

$$X(\mathcal{C};x)=\sum_{\sigma\in\Pi(\mathcal{C})}\widetilde{\Psi}_{I(\sigma)}\,. \tag{2.2.3}$$

Let us consider the $\mathbb{Q}$-linear map from the space $Sym_*^{\mathbb{Q}}(x)$ of symmetric functions with rational coefficients to $(\Phi\mathbb{Q})_*[x]$ specified by

$$\widetilde{\Psi}_{I(\sigma)}\mapsto\tau^\phi(\sigma)\,B^\phi_{|\sigma|}(x)\,, \tag{2.2.4}$$

where σ is a partition of $[n]$. Clearly, this map preserves the gradings, is injective, but it is neither surjective, nor an algebra map. Comparing (2.2.3) with the formula for the umbral chromatic polynomial in Proposition 2.2.1, we easily see that the above map sends $X(\mathcal{C};x)$ to $\chi^\phi(\mathcal{C};x)$.

2.3 The Characteristic Type Polynomial

In §2.4 we shall study the properties of the umbral chromatic polynomial by relating it to the characteristic type polynomial defined in (1.8.1); so in this section we study the latter.

We start by presenting a deletion/contraction identity which enables us to compute $c^\phi(\mathcal{P};x)$ recursively. We use it as our main tool in proofs by induction, such as those of Theorem 2.3.5 and Theorem 2.3.6.

Theorem 2.3.1 *If* $U \in \mathrm{Non}(\mathcal{P})$ *is arbitrary, then*

$$c^\phi(\mathcal{P};x) = c^\phi(\mathcal{P} \setminus U;x) + \mu^\phi_{\Pi(\mathcal{P}|U)}(\widehat{0},\{U\})\, c^\phi(\mathcal{P}/U;x)\,;$$

moreover, the identity still holds if we replace contraction by strong contraction.

Proof. For simplicity, we write P, Q, R, and S for $\Pi(\mathcal{P})$, $\Pi(\mathcal{P}\setminus U)$, $\Pi(\mathcal{P}/U)$, and $\Pi(\mathcal{P}|U)$ respectively.

Consider an arbitrary partition $\sigma \in P$, for which three possible cases arise. Firstly, if no block of σ contains U, then $\mu^\phi_P(\widehat{0}_P,\sigma) = \mu^\phi_Q(\widehat{0}_P,\sigma)$. Secondly, if one block of σ is U itself, then

$$\mu^\phi_P(\widehat{0}_P,\sigma) = \mu^\phi_S(\widehat{0}_S,\widehat{1}_S)\,\mu^\phi_P(\widehat{0}_R,\sigma) = \mu^\phi_P(\widehat{0}_P,\widehat{0}_R)\,\mu^\phi_P(\widehat{0}_R,\sigma)\,, \tag{2.3.2}$$

as follows from the poset isomorphisms $[\widehat{0}_P,\sigma] \cong [\widehat{0}_S,\widehat{1}_S] \times [\widehat{0}_R,\sigma] \cong [\widehat{0}_P,\widehat{0}_R] \times [\widehat{0}_R,\sigma]$ by using Proposition 1.7.14. Thirdly, if one block of σ strictly contains U, then by Proposition 1.7.15

$$\mu^\phi_Q(\widehat{0}_P,\sigma) = \sum (-1)^k \mu^\phi_P(\widehat{0}_P,\sigma_1)\,\ldots\,\mu^\phi_P(\sigma_k,\sigma)\,, \tag{2.3.3}$$

where $\sigma_1 < \sigma_2 < \ldots < \sigma_k$ all have U as a block, and $\sigma_k < \sigma$. Using (2.3.2), we deduce that the terms of the form $\mu^\phi_P(\widehat{0}_P,\sigma_1)\,\mu^\phi_P(\sigma_1,\sigma)$ with $\sigma_1 \neq \widehat{0}_R$ cancel with terms $\mu^\phi_P(\widehat{0}_P,\widehat{0}_R)\,\mu^\phi_P(\widehat{0}_R,\sigma_1)\,\mu^\phi_P(\sigma_1,\sigma)$, and so on. Hence, (2.3.3) becomes

$$\mu^\phi_Q(\widehat{0}_P,\sigma) = \mu^\phi_P(\widehat{0}_P,\sigma) - \mu^\phi_S(\widehat{0}_S,\widehat{1}_S)\,\mu^\phi_P(\widehat{0}_R,\sigma)\,. \tag{2.3.4}$$

The required formula follows by considering each of these cases in turn.

The last statement of the theorem follows by noting that

$$c^\phi(\mathcal{P};x) = c^\phi(\mathrm{Sing}(\mathcal{P});x)\,.$$

□

We may now give another method for computing $\nu^{\phi}_{\overline{\mathcal{P}}}(\sigma)$, where $\sigma \in \Pi(\mathcal{P})$, which may also be regarded as a complementation formula for the Möbius type function. It expresses $\nu^{\phi}_{\overline{\mathcal{P}}}(\sigma) = -\mu^{\phi}_{\Pi(\overline{\mathcal{P}}|\sigma)\cup\{\sigma\}}(\widehat{0},\sigma)$ in terms of the Möbius type function of the poset $\Pi(\mathcal{P}|\sigma)$, and so is valuable when this poset is smaller than $\Pi(\overline{\mathcal{P}}|\sigma)\cup\{\sigma\}$.

Theorem 2.3.5 *Let $\mathcal{P}$ be an arbitrary partition system; then for every partition $\sigma \in \Pi(\mathcal{P})$, we have that*

$$\nu^{\phi}_{\overline{\mathcal{P}}}(\sigma) = \mu^{\overline{\phi}}_{\Pi(\mathcal{P})}(\widehat{0},\sigma)\,.$$

Proof. Write V for $V(\mathcal{P})$. According to Proposition 1.7.14 and Proposition 2.2.2 (1), we may assume that $\sigma = \{V\}$, which means that $V \in \mathcal{P}$. We use induction with respect to $|\mathrm{Non}(\mathcal{P})|$, which starts at 1 (we assume that $|\mathrm{At}(\mathcal{P})| > 1$). If $|\mathrm{Non}(\mathcal{P})| > 1$, we choose $U \in \mathrm{Non}(\mathcal{P}) \setminus \{V\}$, set $\mathcal{Q} := \overline{\mathcal{P}} \cup \{U, V\}$, and observe that $\overline{\mathcal{P}\setminus U} = \overline{\mathcal{P}} \cup \{U\}$, that $\overline{\mathcal{P}|U} = \overline{\mathcal{P}}|U$, and that $\overline{\mathcal{P}/U} = (\overline{\mathcal{P}} \cup \{U\})/U$. We then employ the inductive hypothesis, applying (2.3.4) twice in the process; for clarity, it helps to set $\omega^{\phi}_{\mathcal{P}}(\sigma) := \mu^{\phi}_{\Pi(\mathcal{P})}(\widehat{0},\sigma)$ for any $\sigma \in \Pi(\mathcal{P})$, thereby yielding

$$\begin{aligned}
\omega^{\overline{\phi}}_{\mathcal{P}}(\{V\}) &= \omega^{\overline{\phi}}_{\mathcal{P}\setminus U}(\{V\}) + \omega^{\overline{\phi}}_{\mathcal{P}|U}(\{U\})\,\omega^{\overline{\phi}}_{\mathcal{P}/U}(\{V\}) \\
&= \nu^{\phi}_{\overline{\mathcal{P}\setminus U}}(\{V\}) + \nu^{\phi}_{\overline{\mathcal{P}|U}}(\{U\})\,\nu^{\phi}_{\overline{\mathcal{P}/U}}(\{V\}) \\
&= -\omega^{\phi}_{\mathcal{Q}}(\{V\}) + \omega^{\phi}_{\mathcal{Q}|U}(\{U\})\,\omega^{\phi}_{\mathcal{Q}/U}(\{V\}) \\
&= -\omega^{\phi}_{\overline{\mathcal{P}}\cup\{V\}}(\{V\}) = \nu^{\phi}_{\overline{\mathcal{P}}}(\{V\})\,,
\end{aligned}$$

as sought. □

We now apply Theorem 2.3.1 to prove a complementation formula for the characteristic type polynomial, remarking that similar such formulae were obtained in a very general context for the matching polynomial in [56]. Our formula expresses the characteristic type polynomial of a partition systems $\mathcal{Q}$ in terms of divisions by the complement of $\mathcal{Q}$ in a partition system containing it.

Theorem 2.3.6 *If $\mathcal{P}$ and $\mathcal{Q}$ are partition systems such that $V(\mathcal{P}) = V(\mathcal{Q})$, $\mathrm{At}(\mathcal{P}) = \mathrm{At}(\mathcal{Q})$, and $\mathcal{Q} \subseteq \mathcal{P}$, then*

$$c^{\phi}(\mathcal{Q};x) = \sum_{\sigma} \nu^{\phi}_{\mathcal{Q}}(\sigma)\, c^{\phi}(\mathcal{P}/\sigma;x)\,,$$

where the summation is taken over the poset of divisions by $\complement_{\mathcal{P}}\mathcal{Q}$.

Proof. Let us denote $\complement_{\mathcal{P}}\mathcal{Q}$ by $\mathcal{R}$. We will prove our theorem by induction on $|\mathrm{Non}(\mathcal{R})|$, noting that the induction starts at 0 because then $\mathcal{Q} = \mathcal{P}$, whilst $\Pi(\mathcal{R}) = \{\mathrm{At}(\mathcal{P})\}$, $\nu^{\phi}_{\mathcal{P}}(\mathrm{At}(\mathcal{P})) = 1$, and $\mathcal{P}/\mathrm{At}(\mathcal{P}) = \mathcal{P}$.

If $\mathrm{Non}(\mathcal{R}) \neq \emptyset$, we choose $W \in \mathrm{Non}(\mathcal{R})$ to be minimal with respect to inclusion, and let $\overline{W} := V(\mathcal{P}) \setminus W$. Then the inductive hypothesis yields

$$c^{\phi}(\mathcal{Q};x) = \sum_{\sigma} \nu^{\phi}_{\mathcal{Q}}(\sigma)\, c^{\phi}((\mathcal{P}\setminus W)/\sigma;x)\,, \tag{2.3.7}$$

where the summation is taken over $\Pi(\mathcal{R}\setminus W)$. Given $\sigma \in \Pi(\mathcal{R}\setminus W)$, we have

$$(\mathcal{P}\setminus W)/\sigma = \begin{cases} (\mathcal{P}/\sigma)\setminus W & \text{if} \quad \sigma \leq \{W,\overline{W}\} \\ \mathcal{P}/\sigma & \text{otherwise}\,. \end{cases}$$

Consider a partition $\sigma \in \Pi(\mathcal{R}\setminus W)$ satisfying $\sigma \leq \{W,\overline{W}\}$. The choice of W implies that $\sigma|W = \mathrm{At}(\mathcal{P})|W$; hence $(\mathcal{P}/\sigma)|W = \mathcal{P}|W$. Combining this fact with Theorem 2.3.1, we deduce

$$c^{\phi}((\mathcal{P}/\sigma)\setminus W;x) = c^{\phi}(\mathcal{P}/\sigma;x) - \mu^{\phi}_{\Pi(\mathcal{P}|W)}(\widehat{0},\{W\})\, c^{\phi}((\mathcal{P}/\sigma)/W;x)\,. \tag{2.3.8}$$

Now $\sigma/W \in \Pi(\mathcal{R})$ and $(\mathcal{P}/\sigma)/W = \mathcal{P}/(\sigma/W)$. Recalling the choice of W again, we observe that $\nu^{\phi}_{\mathcal{Q}}(\sigma|W) = 1$ and $\mu^{\phi}_{\Pi(\mathcal{P}|W)}(\widehat{0},\{W\}) = \mu^{\phi}_{\Pi((\mathcal{Q}|W)\cup\{W\})}(\widehat{0},\{W\}) = -\nu^{\phi}_{\mathcal{Q}}(\{W\})$. Using these facts, and applying Proposition 2.2.2 (1) twice, we obtain

$$\nu^{\phi}_{\mathcal{Q}}(\sigma)\, \mu^{\phi}_{\Pi(\mathcal{P}|W)}(\widehat{0},\{W\}) = -\nu^{\phi}_{\mathcal{Q}}(\sigma|W)\, \nu^{\phi}_{\mathcal{Q}}(\sigma|\overline{W})\, \nu^{\phi}_{\mathcal{Q}}(\{W\}) = -\nu^{\phi}_{\mathcal{Q}}(\sigma/W)\,. \tag{2.3.9}$$

According to (2.3.8) and (2.3.9), we may replace each term in the right-hand side of (2.3.7) corresponding to a partition $\sigma \leq \{W, \overline{W}\}$ with

$$\nu^\phi_{\mathcal{Q}}(\sigma)\, c^\phi(\mathcal{P}/\sigma; x) + \nu^\phi_{\mathcal{Q}}(\sigma/W)\, c^\phi(\mathcal{P}/(\sigma/W); x)\,.$$

It remains only to define a bijection between the sets $\{\sigma \in \Pi(\mathcal{R} \setminus W) : \sigma \leq \{W, \overline{W}\}\}$ and $\{\sigma \in \Pi(\mathcal{R}) : W \in \sigma\}$; we take $\sigma \mapsto \sigma/W$, with inverse $\pi \mapsto (\mathrm{At}(\mathcal{P})|W) \cup (\pi|\overline{W})$, thereby concluding the induction. □

2.4 Properties of the Umbral Chromatic Polynomial

In this section, we begin by establishing our promised relation between the umbral chromatic polynomial and the characteristic type polynomial, generalising the main result of [40] in passing, and enabling us to investigate further properties of the former. All the results in this section follow directly from those in the previous one, and mainly from Theorem 2.3.6.

Proposition 2.4.1 *For any partition system $\mathcal{P}$, we have*

$$\chi^\phi(\mathcal{P}; x) = c^\phi(\overline{\mathcal{P}}; x)\,.$$

Proof. This follows from Theorem 2.3.6 by considering the partition systems $\overline{\mathcal{P}} \subseteq \mathcal{K}_{V(\mathcal{P})}$, and noting that $c^\phi(\mathcal{K}_{V(\mathcal{P})}/\sigma; x) = c^\phi(\mathcal{K}_{|\sigma|}; x) = B^\phi_{|\sigma|}(x)$, according to (1.8.4). The right-hand side of the identity in Theorem 2.3.6 is precisely $\chi^\phi(\mathcal{P}; x)$. □

Proposition 2.4.1 reduces to Whitney's formula (1.8.2) after replacing the umbra ϕ with κ, as implied by the proof of Corollary 2.4.2. In fact, it reduces to a generalisation of (1.8.2) to a formula for the *classical* chromatic polynomial of a partition system (or its homogenised version).

Corollary 2.4.2 *For any partition system $\mathcal{P}$, we have*

$$\chi^\kappa(\mathcal{P};x) = c^\kappa(\mathcal{P}_c;x)\,.$$

PROOF. Let us define the coclosure operator (cf. the proof of Proposition 2.2.2 (3)) $\sigma \mapsto \overline{\sigma}$ on $\Pi(\overline{\mathcal{P}})$, where the partition $\overline{\sigma}$ is obtained from σ by splitting every block B into the sets of vertices of the join-connected components of $\mathcal{P}|B$. Using a similar argument to the one in the proof of Proposition 2.2.2 (3), we obtain

$$\mu^\kappa_{\Pi(\overline{\mathcal{P}})}(\widehat{0},\sigma) = \begin{cases} \mu^\kappa_{\Pi(\mathcal{P}_c)}(\widehat{0},\sigma) & \text{if } \sigma \in \mathcal{P}_c \\ 0 & \text{otherwise}\,. \end{cases}$$

Hence $c^\kappa(\overline{\mathcal{P}};x) = c^\kappa(\mathcal{P}_c;x)$, so we may now apply Proposition 2.4.1. □

We cannot use the map specified by (2.2.4) to transform the formula in Proposition 2.4.1 into a formula for the symmetric function $X(\mathcal{C};x)$ associated with the simplicial complex $\mathcal{C}$, because this map is not surjective. However, there is an analogue of Proposition 2.4.1 for $X(\mathcal{C};x)$, which generalises Theorem 2.6 in [51], and which provides a different generalisation of Whitney's formula. We present these results below.

Proposition 2.4.3 *Given a simplicial complex $\mathcal{C}$, we have*

$$X(\mathcal{C};x) = \sum_{\sigma\in\Pi(\mathcal{C}_c)} \mu_{\Pi(\mathcal{C}_c)}(\widehat{0},\sigma)\,\Psi^{I(\sigma)}\,.$$

PROOF. We only have to adapt Stanley's proof to the context of simplicial complexes. Given σ in $\Pi(\mathcal{C}_c)$, we define

$$X_\sigma := \sum_f x^f\,, \tag{2.4.4}$$

where the summation ranges over all functions f from $V(\mathcal{C})$ to $\mathbb{N}$ satisfying $\overline{\ker(f)} \le \sigma \le \ker(f)$; here we have used the same coclosure operator as in the proof of Corollary 2.4.2. Given *any* $f\colon V(\mathcal{C}) \to \mathbb{N}$, there is a unique σ in $\Pi(\mathcal{C}_c)$, namely $\overline{\ker(f)}$, such that f is one of the maps appearing in the sum (2.4.4). It follows that for any π in $\Pi(\mathcal{C}_c)$, we have

$$\Psi^{I(\pi)} = \sum_{\pi \leq \sigma \in \Pi(\mathcal{C}_c)} X_\sigma \,.$$

By Möbius inversion,

$$X_\pi = \sum_{\pi \leq \sigma \in \Pi(\mathcal{C}_c)} \Psi^{I(\sigma)} \, \mu_{\Pi(\mathcal{C}_c)}(\pi, \sigma) \,.$$

But $X_{\widehat{0}} = X(\mathcal{C}; x)$ (note that it is essential for $\mathcal{C}$ to be a simplicial complex), and the proof follows. □

Proposition 2.4.5 *We have*

$$\chi^\phi(\mathcal{C}; x) = \sum_{\sigma \in \Pi(\mathcal{C}_c)} \mu_{\Pi(\mathcal{C}_c)}(\widehat{0}, \sigma) \, P^\phi_{I(\sigma)}(x) \,,$$

where

$$P^\phi_{I(\sigma)}(x) := \sum_{\pi \geq \sigma} \tau^\phi(\sigma, \pi) \, x^{|\pi|} \,, \qquad \tau^\phi(\sigma, \pi) := \sum_{\sigma \leq \rho \leq \pi} \zeta^\phi(\widehat{0}, \rho) \, \mu^\phi_{\Pi(V)}(\rho, \pi) \,,$$

and $V := V(\mathcal{C})$. In particular, the umbral chromatic polynomial of a graph can be expressed in terms of the lattice of contractions of that graph.

Proof. We apply the map specified by (2.2.4) to the formula in Proposition 2.4.3. To this end, we compute the images of the power sum symmetric functions under this map. According to Theorem 1 in [12], we have

$$\Psi^{I(\sigma)} = \sum_{\pi \geq \sigma} \widetilde{\Psi}_{I(\pi)} \,.$$

Combining this result with (1.8.4), we find that the map specified by (2.2.4) sends $\Psi^{I(\sigma)}$ to

$$\begin{aligned} \sum_{\rho \geq \sigma} \tau^\phi(\rho) \, B^\phi_{|\rho|}(x) &= \sum_{\rho \geq \sigma} \zeta^\phi(\widehat{0}, \rho) \left(\sum_{\pi \geq \rho} \mu^\phi_{\Pi(V)}(\rho, \pi) \, x^{|\pi|} \right) \\ &= \sum_{\pi \geq \sigma} x^{|\pi|} \left(\sum_{\sigma \leq \rho \leq \pi} \zeta^\phi(\widehat{0}, \rho) \, \mu^\phi_{\Pi(V)}(\rho, \pi) \right) = P^\phi_{I(\sigma)}(x) \,. \end{aligned}$$

□

Let us note that $\tau^{\phi}(\sigma,\sigma)=\tau^{\phi}(\sigma)$ and $\tau^{\kappa}(\sigma,\pi)=0$ unless $\sigma=\pi$, because we can pair the chains from σ to π contributing to $\tau^{\kappa}(\sigma,\pi)$ such that the contribution of each pair is 0. Hence, after replacing ϕ by κ, Proposition 2.4.5 reduces to a special case of Corollary 2.4.2, and to Whitney's formula if $\mathcal{C}$ is the independence complex of a graph.

We can immediately deduce from Theorem 2.3.5 another formula relating the umbral chromatic polynomial to the characteristic type polynomial of a partition system. To state our formula in a nice way, we recall from [31] the *umbral notation*, according to which we write $p(B^{\phi}(x))$ for the image of the polynomial $p(x)$ under the Φ_*-linear operator on $\Phi_*[x]$ mapping x^n to $B_n^{\phi}(x)$.

Proposition 2.4.6 *For any partition system $\mathcal{P}$, we have*

$$\chi^{\phi}(\mathcal{P};x)=c^{\overline{\phi}}(\mathcal{P}\,;B^{\phi}(x))\,.$$

A deletion/contraction procedure for the umbral chromatic polynomial follows easily from Proposition 2.4.1.

Proposition 2.4.7 *Given any set $U\in\mathrm{Non}(\mathcal{P})$, we have*

$$\chi^{\phi}(\mathcal{P};x)=\chi^{\phi}(\mathcal{P}\setminus U;x)+\nu^{\phi}_{\overline{\mathcal{P}}}(\{U\})\,\chi^{\phi}(\mathcal{P}/U;x)\,;$$

moreover, the identity still holds if we replace contraction by strong contraction.

Proof. Apply Proposition 2.4.1, Theorem 2.3.1, and the fact that $\overline{\mathcal{P}\setminus U}=\overline{\mathcal{P}}\cup\{U\}$ and $\overline{\mathcal{P}/U}=(\overline{\mathcal{P}}\cup\{U\})/U$. □

Proposition 2.4.7 provides an analogue of the well-known addition-contraction procedure for graphs (see [37]). There is no known deletion-contraction formula for the umbral chromatic polynomial of a graph H, which we could use to obtain a similar formula for Stanley's symmetric function X_H. Not even $X(\mathcal{C};x)$ with $\mathcal{C}$ a simplicial complex has an obvious deletion-contraction formula which would only involve simplicial complexes (deducible from Proposition 2.4.7, for instance). One way around this problem is to allow arbitrary partition systems, in which case (2.2.3) still holds, with summation now ranging over $\Pi(\mathcal{P})$. Since $\Pi(\mathcal{P})$ is the disjoint union of $\Pi(\mathcal{P}\setminus\setminus U)$ and $\Pi(\mathcal{P}/U)$ for arbitrary U in $\mathrm{Non}(\mathcal{P})$, we have

$$X(\mathcal{P};x) = X(\mathcal{P}\setminus\setminus U;x) + X(\mathcal{P}/U;x)\,. \tag{2.4.8}$$

As was observed in [41], formula (1.8.3) does not extend in a straightforward way to the umbral chromatic polynomial of a graph, although such a generalisation was attempted in [37]. We offer here a superior version, as a special case of formula (2.4.10) for the umbral chromatic polynomial of a join of two partition systems (which corresponds to the disjoint union of graphs, after identification via the independence complex). Simultaneously, we replace ϕ by κ, and show that (1.8.3) *does* generalise to the *classical* chromatic polynomial for partition systems (in homogeneous form).

Proposition 2.4.9 *For arbitrary partition systems $\mathcal{P}_1$ and $\mathcal{P}_2$, we have*

$$\chi^\phi(\mathcal{P}_1\vee\mathcal{P}_2;x) = \chi^\phi(\mathcal{P}_1;x)\chi^\phi(\mathcal{P}_2;x) - \sum_\sigma \nu^\phi_{\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2}}(\sigma)\,\chi^\phi((\mathcal{P}_1\vee\mathcal{P}_2)/\sigma;x)\,, \tag{2.4.10}$$

where the summation is taken over non-$\widehat{0}$ divisions by the complement of the disjoint union $\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2}$ in $\overline{\mathcal{P}_1}\oplus\overline{\mathcal{P}_2}$. After replacing ϕ by κ, formula (2.4.10) reduces to

$$\chi^\kappa(\mathcal{P}_1\vee\mathcal{P}_2;x) = \chi^\kappa(\mathcal{P}_1;x)\,\chi^\kappa(\mathcal{P}_2;x)\,. \tag{2.4.11}$$

Proof. According to Proposition 2.4.1 and (1.8.5), we have

$$\chi^\phi(\mathcal{P}_1\vee\mathcal{P}_2;x) = c^\phi(\overline{\mathcal{P}_1}\oplus\overline{\mathcal{P}_2};x) \text{ and} \tag{2.4.12}$$

$$\chi^\phi(\mathcal{P}_1;x)\,\chi^\phi(\mathcal{P}_2;x) = c^\phi(\overline{\mathcal{P}_1};x)\,c^\phi(\overline{\mathcal{P}_2};x) = c^\phi(\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2};x)\,. \tag{2.4.13}$$

Substituting $\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2}$ for $\mathcal{Q}$ and $\overline{\mathcal{P}_1}\oplus\overline{\mathcal{P}_2}$ for $\mathcal{P}$ in Theorem 2.3.6, we get

$$c^{\phi}(\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2};x)=\sum_{\sigma}\nu^{\phi}_{\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2}}(\sigma)\,c^{\phi}((\overline{\mathcal{P}_1}\oplus\overline{\mathcal{P}_2})/\sigma;x)\,,$$

where the summation is taken over divisions by the complement of $\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2}$ in $\overline{\mathcal{P}_1}\oplus\overline{\mathcal{P}_2}$. Identity (2.4.10) follows from the above relations by noting that $\overline{(\overline{\mathcal{P}_1}\oplus\overline{\mathcal{P}_2})/\sigma}=(\mathcal{P}_1\vee\mathcal{P}_2)/\sigma$, and then applying Proposition 2.4.1 once more.

We now deduce (2.4.11) from (2.4.10). Given a non-$\widehat{0}$ division σ by the complement of $\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2}$ in $\overline{\mathcal{P}_1}\oplus\overline{\mathcal{P}_2}$, we will show that $\nu^{\kappa}_{\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2}}(\sigma)=0$. According to Proposition 2.2.2 (1), we have

$$\nu^{\kappa}_{\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2}}(\sigma)=\prod_{U\in\sigma}\nu^{\kappa}_{\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2}}(\{U\})\,.$$

Clearly, every block U of σ is either an atom of $\mathcal{P}_1\cdot\mathcal{P}_2$ or intersects both $V(\mathcal{P}_1)$ and $V(\mathcal{P}_2)$. Since $\sigma\neq\widehat{0}$, we may find a block $U=U_1\cup U_2$, where $U_i\in\mathrm{Bool}(\mathcal{P}_i)\setminus\{\emptyset\}$ for $i=1,2$. We cannot have $U_1\in\mathcal{P}_1$ and $U_2\in\mathcal{P}_2$ because $U\in\mathrm{Non}(\overline{\mathcal{P}_1\vee\mathcal{P}_2})$; hence, we may assume that $U_1\in\mathrm{Non}(\overline{\mathcal{P}_1})$, for instance. We now consider all chains contributing to $\nu^{\kappa}_{\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2}}(\{U\})$, and pair every chain

$$\{\widehat{0}<\pi_1<\ldots<\pi_k<\{U\}\}$$

for which $U_1\notin\pi_k$ with the chain

$$\{\widehat{0}<\pi_1<\ldots<\pi_k<\{U_1\}\cup(\pi_k|U_2)<\{U\}\}\,.$$

The contribution of each pair is 0, whence $\nu^{\kappa}_{\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2}}(\{U\})=0$.

□

We may apply Proposition 2.4.9 to recover a more systematic version of a result of [37]. Consider graphs H_1 and H_2, and denote their disjoint union by $H_1\sqcup H_2$ and $\overline{\mathcal{I}(H_i)}$ by $\mathcal{A}(H_i)$. For every non-$\widehat{0}$ division σ by the complement of $\mathcal{A}(H_1)\cdot\mathcal{A}(H_2)$ in $\mathcal{A}(H_1\sqcup H_2)$, construct the graph $M_\sigma(H_1,H_2)$ with vertices the blocks of σ, and with edges joining either a non-singleton block to another block, or two singleton blocks if the corresponding vertices of $H_1\sqcup H_2$ are adjacent.

Corollary 2.4.14 *The umbral chromatic polynomial of* $H_1 \sqcup H_2$ *is given by*

$$\chi^\phi(H_1 \sqcup H_2; x) = \chi^\phi(H_1; x)\, \chi^\phi(H_2; x) - \sum_\sigma \nu^\phi_{\mathcal{A}(H_1)\cdot\mathcal{A}(H_2)}(\sigma)\, \chi^\phi(M_\sigma(H_1, H_2); x)\,,$$

where the summation is taken over all divisions σ *specified in the construction.*

PROOF. The stated formula follows from (2.4.10) by replacing $\mathcal{P}_i$ with $\mathcal{I}(H_i)$, and observing that

$$\mathcal{I}(H_1 \sqcup H_2) = \mathcal{I}(H_1) \vee \mathcal{I}(H_2)\,, \qquad \mathcal{A}(H_1 \sqcup H_2) = \mathcal{A}(H_1) \oplus \mathcal{A}(H_2)\,,$$
$$\text{and} \qquad \mathrm{Sing}\,(\mathcal{I}(H_1 \sqcup H_2)/\sigma) = \mathcal{I}(M_\sigma(H_1, H_2))$$

for any division σ of the stated form. □

We recall that in [37] the divisions by $\mathcal{A}(H)$ were called the *admissible* partitions of $V(H)$, and those by the appropriate σ were labelled as *mixed* partitions of $V(H_1 \sqcup H_2)$.

We have seen that the umbral chromatic polynomial does not behave well with respect to the join of partition systems. However, according to the following result, the umbral chromatic polynomial is multiplicative with respect to the operation (multiplication) $\odot$ defined in §1.5. Note that neither the family of graphs (identified with their independence complexes), nor the family of simplicial complexes are closed with respect to this operation. Let us also note that $\mathcal{P}_1 \vee \mathcal{P}_2 \subseteq \mathcal{P}_1 \odot \mathcal{P}_2$.

Proposition 2.4.15 *We have that*

$$\chi^\phi(\mathcal{P}_1 \odot \mathcal{P}_2; x) = \chi^\phi(\mathcal{P}_1; x)\, \chi^\phi(\mathcal{P}_2; x)\,.$$

PROOF. This result is an immediate consequence of Proposition 2.4.1 and (1.8.5). □

This product formula for the the umbral chromatic polynomial of partition systems clearly reduces to (2.4.11), after replacing the umbra ϕ with κ.

2.5 Polynomial Invariants of Set Systems with Automorphism Group

Throughout this section, we let $(\mathcal{S}, G)$ be a fixed set system with automorphism group (recall that $\mathcal{S}$ must be a partition system with singleton atoms). We intend to give a combinatorial interpretation for the normalised polynomial $\chi^{\phi}(\mathcal{S}; x)/|G|$ in $(\Phi\mathbb{Q})_*[x]$. It will turn out that all such polynomials (of which the divided conjugate Bell polynomials $\beta_n^b(x)$ are a special case) lie in $H_{|V(\mathcal{S})|}\{x\}$. Therefore, these polynomials will be used in Chapter 4 for constructing combinatorial models for divided power Hopf algebras and covariant bialgebras of formal group laws. It is important to understand that whenever the universal ring H_* is replaced by a ring with torsion, we can no longer embed the corresponding divided power algebra in its rationalisation. Thus we must replace the polynomials $B_n^{\phi}(x)/n!$ by purely formal quotients, and the polynomials $\chi^{\phi}(\mathcal{S}; x)/|G|$ by linear combinations of the formal quotients; it is these formal quotients and linear combinations of them that we realise combinatorially by considering set systems equipped with an automorphism group.

For our promised combinatorial interpretation, we need the additional concept of *ordered colouring*; we note that the corresponding concept for graphs differs from the one already appearing in the literature. For us, an ordered colouring of $\mathcal{S}$ is a pair (f, ω), where f is a colouring of $\mathcal{S}$, ω is a bijection from $[|V(\mathcal{S})|]$ to $V(\mathcal{S})$, and $f \circ \omega$ is non-decreasing. We can interpret an ordered colouring as proceeding step-by-step, so that the colours are used in increasing order. An *ordered factorised colouring* of $\mathcal{S}$ is a triple (γ, f, ω), where (f, ω) is an ordered colouring, and (γ, f) is a factorised colouring for which $\omega^{-1}(U)$ is an interval (in $\mathbb{N}$) for any block U of a partition in the chain γ. The type of such a colouring is defined by $\tau^b(\gamma, f, \omega) := \tau^b(\gamma, f) \in H_*$. All the ordered factorised colourings of a simplicial complex are, of course, ordered colourings. We can define ordered colouring forests in a similar way, and we can view them as colourings based on forests of *plane* trees.

For any factorised colouring (γ, f) with $\tau^\phi(\gamma, f) = \pm\phi_1^{k_1}\phi_2^{k_2}\ldots$, there are $(2!)^{k_1}(3!)^{k_2}\ldots$ ways of choosing ω such that (γ, f, ω) is ordered. The types of all these colourings are equal, and their sum is $\tau^\phi(\gamma, f)$. Hence, when we regard $\chi^\phi(\mathcal{S}\,; m\phi)$ as an element of H_*, it enumerates by type the ordered factorised colourings of $\mathcal{S}$ with at most m colours. The group G acts on these colourings by $(g, (\gamma, f, \omega)) \mapsto (g\gamma, f \circ g^{-1}, g \circ \omega)$, and each orbit has precisely $|G|$ elements. Therefore, $\chi^\phi(\mathcal{S}\,; m\phi)/|G| \in H_*$ enumerates by type τ^b the orbits of G on the set of ordered factorised colourings of $\mathcal{S}$ with at most m colours. It also enumerates orbits on the set of ordered colouring forests.

We may give an alternative statement of these facts in terms of the orbits of G simply on the set of factorised colourings. Given such a colouring (γ, f) with $\tau^\phi(\gamma, f) = \pm\phi_1^{k_1}\phi_2^{k_2}\ldots$, there are

$$(2!)^{k_1}(3!)^{k_2}\ldots|G(\gamma, f)|/|G| = (2!)^{k_1}(3!)^{k_2}\ldots/|G_{(\gamma,f)}|$$

orbits of G on the set of ordered factorised colourings which map to $G(\gamma, f)$ via the map $G(\gamma, f, \omega) \mapsto G(\gamma, f)$. Hence, $\chi^\phi(\mathcal{S}\,; m\phi)/|G|$ enumerates the orbits of G on the set of factorised colourings of $\mathcal{S}$ with at most m colours, each orbit $G(\gamma, f)$ giving a contribution of $\tau^\phi(\gamma, f)/|G_{(\gamma,f)}|$. If $\gamma = \{\widehat{0} < \sigma\}$, then $G_{(\gamma,f)} = G|\sigma$; if, in addition, $G|\sigma$ is the direct product of symmetric groups acting on the blocks of σ, then the contribution of the orbit $G(\gamma, f)$ to $\chi^\phi(\mathcal{S}\,; m\phi)/|G|$ is $\tau^b(\gamma, f)$. We remark that since $\chi^\phi(\mathcal{S}\,; m\phi)/|G|$ lies in H_* for all m, the polynomial $\chi^\phi(\mathcal{S}; x)/|G|$ must lie in $H_{|V(\mathcal{S})|}\langle\beta_i^b(x)\rangle$, which is the same as $H_{|V(\mathcal{S})|}\{x\}$.

Example 2.5.1 Let

$$\mathcal{S} := \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,2,3\}\},$$

and $G := \langle (1,2),(2,3)\rangle \cong \Sigma_3$. We have $\chi^\phi(\mathcal{S};x)/|G| = (B_4^\phi(x)+6\phi_1 B_3^\phi(x)+(\phi_2+3\phi_1^2)B_2^\phi(x))/6$. Hence, $\chi^\phi(\mathcal{S};2\phi)/|G| = (2\phi_2+6\phi_1^2)/6 = 2b_2+4b_1^2$. A transversal of the orbits of G on the set of ordered colourings of $\mathcal{S}$ with at most 2 colours is represented by

$$\{(1112,1234),(2221,4123),(1122,1234),(1122,2143),(2211,3412),(2211,4321)\}\,,$$

where we expressed the map ω by the word $\omega(1)\omega(2)\omega(3)\omega(4)$, and f by $f(1)f(2)$ $f(3)f(4)$.

Our next goal is to give a combinatorial interpretation for the coefficients of the polynomial $\chi^\phi(\mathcal{S};x)/|G|$ with respect to the bases $\{\beta_i^b(x) : i \geq 0\}$ and $\{x^i/i! : i \geq 0\}$ of $H_*\{x\}$. Our interpretation uses preferential arrangements of $V(\mathcal{S})$.

Lemma 2.5.2 *Given a sequence of polynomials $p_n(x)$ in $(\Phi\mathbb{Q})_*[x]$, a map w: $\Pi(\mathcal{S}) \to (\Phi\mathbb{Q})_*$, which is constant on the orbits of G on $\Pi(\mathcal{S})$, and an arbitrary transversal $\mathcal{T}$ of $A(\mathcal{S})/G$, we have that*

$$\sum_{(\sigma,\omega)\in\mathcal{T}} \frac{w(\sigma)}{|G|\sigma|}\,\frac{p_{|\sigma|}(x)}{|\sigma|!} = \frac{1}{|G|}\sum_{\sigma\in\Pi(\mathcal{S})} w(\sigma)\,p_{|\sigma|}(x)\,.$$

PROOF. It suffices to observe that

$$\begin{aligned}\sum_{(\sigma,\omega)\in\mathcal{T}} \frac{w(\sigma)}{|G|\sigma|}\,\frac{p_{|\sigma|}(x)}{|\sigma|!} &= \frac{1}{|G|}\sum_{(\sigma,\omega)\in\mathcal{T}} |G(\sigma,\omega)|\,w(\sigma)\,\frac{p_{|\sigma|}(x)}{|\sigma|!}\\ &= \frac{1}{|G|}\sum_{(\sigma,\omega)\in A(\mathcal{S})} w(\sigma)\,\frac{p_{|\sigma|}(x)}{|\sigma|!}\\ &= \frac{1}{|G|}\sum_{\sigma\in\Pi(\mathcal{S})} w(\sigma)\,p_{|\sigma|}(x)\,.\end{aligned}$$

□

Now consider an arbitrary poset P of partitions of V, assume that P contains the partition into singletons, and let G be a permutation group on V which also permutes P (via the obvious action on $\Pi(V)$). Consider the poset $A(P)$ of all preferential arrangements (σ,ω) of V with $\sigma \in P$. Let $\widehat{A}(P) := A(P) \sqcup \{\widehat{0}\}$. Clearly, $A(\Pi(\mathcal{S})) = A(\mathcal{S})$, and we denote $A(\mathcal{S}) \sqcup \{\widehat{0}\}$ by $\widehat{A}(\mathcal{S})$. By insisting that $G(\widehat{0}) = \{\widehat{0}\}$, we obtain a poset action of G on $\widehat{A}(P)$, and hence an induced poset structure on the set of orbits $\widehat{A}(P)/G$.

Consider the incidence algebra over H_* of the poset $\widehat{A}(P)/G$, and the element ζ^b in this algebra which is defined by $\zeta^b(G(\pi,\omega'), G(\sigma,\omega)) := \zeta^b(\pi,\sigma)$, $\zeta^b(\widehat{0},\widehat{0}) := 1$, and

$$\zeta^b(\widehat{0}, G(\sigma,\omega)) := \begin{cases} -1 & \text{if } \sigma = \widehat{0}_P \\ 0 & \text{otherwise}. \end{cases}$$

It is easy to see that the convolution inverse of ζ^b exists; it will be denoted by $\mu^b_{\widehat{A}(P)/G}$, or simply μ^b when the context is clear.

Theorem 2.5.3 *For any $(\sigma,\omega) \in A(P)$, we have that*

$$\frac{\mu_P^\phi(\widehat{0},\sigma)}{|G|\sigma|} = \mu^b_{\widehat{A}(P)/G}(\widehat{0}, G(\sigma,\omega)),$$

so the former lies in H_*.

PROOF. We proceed by induction on the maximum length of chains in $C(\widehat{0},\sigma)$, the case 0 being clear. For $\sigma \neq \widehat{0}$, we have

$$\begin{aligned}
\frac{\mu_P^\phi(\widehat{0},\sigma)}{|G|\sigma|} &= -\frac{1}{|G|\sigma|} \sum_{\widehat{0} \le \pi < \sigma} \mu_P^\phi(\widehat{0},\pi)\, \zeta^\phi(\pi,\sigma) \\
&= -\frac{1}{|G|\sigma|} \sum_{\widehat{0} \le \pi < \sigma} |G|\pi|\, \mu^b(\widehat{0}, G(\pi,\omega_\pi))\, \zeta^\phi(\pi,\sigma) \\
&= -\frac{1}{|G|\sigma|} \sum_{(\pi',\omega') < (\sigma,\omega)} |G|\pi'|\, \mu^b(\widehat{0}, G(\pi',\omega'))\, \zeta^b(G(\pi',\omega'), G(\sigma,\omega)) \\
&= -\sum_{(\pi'',\omega'') \in \mathcal{T}} \mu^b(\widehat{0}, G(\pi'',\omega''))\, \zeta^b(G(\pi'',\omega''), G(\sigma,\omega)) = \mu^b(\widehat{0}, G(\sigma,\omega)),
\end{aligned}$$

where ω_π gives an ordering of the blocks of π such that $(\pi,\omega_\pi) < (\sigma,\omega)$, and $\mathcal{T}$ is a transversal of the set of orbits $\{O \in A(P)/G \,:\, O < G(\sigma,\omega)\}$. The first and the last equality follow from the definition of μ^ϕ and μ^b as convolution inverses. The second follows by induction, and the third is a consequence of the fact that

$$\zeta^\phi(\pi', \sigma) = \sum_{\omega' : (\pi', \omega') < (\sigma, \omega)} \zeta^b(\pi', \sigma) .$$

For each $(\pi', \omega') < (\sigma, \omega)$ we have $|(G|\sigma)(\pi', \omega')| = |G|\sigma|/|G|\pi'|$; therefore, the fourth equality follows from showing that the map from the set of orbits of $G|\sigma$ on $\{(\pi', \omega') : (\pi', \omega') < (\sigma, \omega)\}$ to the set $\{O \in A(P)/G : O < G(\sigma, \omega)\}$, given by $(G|\sigma)(\pi', \omega') \mapsto G(\pi', \omega')$, is a bijection. Injectivity follows from the fact that $(\pi'_1, \omega'_1), (\pi'_2, \omega'_2) < (\sigma, \omega)$ and $g(\pi'_1, \omega'_1) = (\pi'_2, \omega'_2)$ imply $g \in G|\sigma$. Surjectivity follows from the chain of implications: $G(\pi'', \omega'') < G(\sigma, \omega) \Rightarrow g(\pi'', \omega'') < (\sigma, \omega)$ for some $g \in G \Rightarrow (G|\sigma)(g(\pi'', \omega'')) \mapsto G(\pi'', \omega'')$. $\Box$

Given $\sigma \in \Pi(\mathcal{S})$, we define an analogue of the Möbius type by

$$\nu^b_{\overline{\mathcal{S}},G}(\sigma) := -\mu^b_{(\widehat{A}(\overline{\mathcal{S}}) \cup G(\sigma,\omega))/G}(\widehat{0}, G(\sigma, \omega)) ,$$

where ω gives an arbitrary ordering of the blocks of σ. Lemma 2.5.2 and Theorem 2.5.3 immediately imply the following result.

Corollary 2.5.4 *If $\mathcal{T}$ is an arbitrary transversal of $A(\mathcal{S})/G$, then*

$$\frac{\chi^\phi(\mathcal{S}; x)}{|G|} = \sum_{(\sigma,\omega) \in \mathcal{T}} \nu^b_{\overline{\mathcal{S}},G}(\sigma)\, \beta^b_{|\sigma|}(x) \qquad \textit{and} \tag{2.5.5}$$

$$\frac{c^\phi(\mathcal{S}; x)}{|G|} = \sum_{(\sigma,\omega) \in \mathcal{T}} \mu^b_{\widehat{A}(\mathcal{S})/G}(\widehat{0}, G(\sigma, \omega))\, \frac{x^{|\sigma|}}{|\sigma|!} \tag{2.5.6}$$

in $H_{|V(\mathcal{S})|}\{x\}$.

We can now combine (2.5.6) with Proposition 2.4.1 in order to express the polynomial $\chi^\phi(\mathcal{S}; x)/|G|$ in terms of divided powers of x. On the other hand, formula (2.5.6) provides the promised generalisation of (1.7.9). Indeed, the latter can be recovered simply by taking $\mathcal{S} = \mathcal{K}_n$, $G = \Sigma_n$, and using the fact that the poset $A(\mathcal{K}_n)/\Sigma_n$ is isomorphic to $\Pi(\mathcal{I}_n)$. Hence, we have reproved (1.7.9), which we now state in the following more explicit form.

Corollary 2.5.7 *We have*

$$\beta_n^b(x) = \sum_{\sigma \in \Pi(\mathcal{I}_n)} \mu^b_{\Pi(\mathcal{I}_n)}(\widehat{0}, \sigma) \, \frac{x^{|\sigma|}}{|\sigma|!} \, .$$

Since the Boolean algebra $\mathcal{I}_n$, which is isomorphic to $(\mathcal{K}_{n-1}, \subseteq)$, is much smaller than the lattice $\Pi_n = \Pi(\mathcal{K}_n)$ for large n, the above corollary provides an expression for the conjugate Bell polynomials with less terms than (1.8.4).

Chapter 3

Some Applications of Incidence Hopf Algebras to Formal Group Theory and Algebraic Topology

In this chapter we present applications of the combinatorial framework in §1.7 and of our results in Chapter 2 to formal group theory and algebraic topology. It turns out that certain objects in these areas have a rich combinatorial structure, which can be expressed in terms of incidence Hopf algebras of partition lattices. The importance of this new point of view is illustrated with various computational examples. More applications to algebraic topology, which also involve symmetric functions, appear in Chapter 6. Since most of this chapter is devoted to topological applications, we prefer to use here the classical topological notation for those topological structures which have already been introduced in Chapter 1 in another guise; the relevant isomorphisms are explained in §1.4.

3.1 Applications to the Universal Formal Group Law

Let us recall from §1.2 the *universal Hurwitz group law* (over Φ_*) and the *universal formal group law* (over L_*)

$$F^{\phi}(X,Y) \in \Phi^1\{\{X,Y\}\} \quad \text{and} \quad f^b(X,Y) \in L^1[[X,Y]]\,.$$

We start by offering a combinatorial interpretation for the coefficient $F^{\phi,l}_{n_1,\dots,n_k}$ of

$$\prod_{i=1}^{k} \frac{X_i^{n_i}}{n_i!} \quad \text{in} \quad \frac{1}{l!}\left(\sum_{i=1}^{k}{}^{\phi} X_i\right)^l .$$

Here, and throughout this section, n_i are positive integers, $n := n_1 + \ldots + n_k$, and π is the partition of $[n]$ with blocks $[n_1]$, $n_1 + [n_2]$, $\ldots$, $n - n_k + [n_k]$.

Proposition 3.1.1 *The element $F^{\phi,l}_{n_1,\dots,n_k}$ in Φ_* may be expressed as*

$$F^{\phi,l}_{n_1,\dots,n_k} = \sum_{\sigma} \nu^{\phi}_{\mathcal{K}_{n_1,\dots,n_k}}(\sigma)\,, \tag{3.1.2}$$

where the summation ranges over those divisions by the complement of $\mathcal{K}_{n_1,\dots,n_k}$ which have cardinality l. In particular,

$$F^{\phi}_{n,l} = \nu^{\phi}_{\mathcal{K}_{n,l}}(\{[n+l]\})\,, \tag{3.1.3}$$

and

$$F^{\phi}_{n_1,\dots,n_k} = \sum_{\pi \geq \sigma \in \Pi_n} \mu^{\phi}(\widehat{0},\sigma)\,\zeta^{\phi}(\sigma,\widehat{1})\,. \tag{3.1.4}$$

PROOF. Using the iterated version of (1.2.9), (1.8.4), Proposition 1.8.5, and Proposition 2.4.1 successively, we have

$$\begin{aligned} F^{\phi,l}_{n_1,\dots,n_k} &= \langle \phi(D)^l/l! \mid \prod_{i=1}^{k} B^{\phi}_{n_i}(x)\rangle = \langle \phi(D)^l/l! \mid \prod_{i=1}^{k} c^{\phi}(\mathcal{K}_{n_i};x)\rangle \\ &= \langle \phi(D)^l/l! \mid c^{\phi}(\mathcal{K}_{n_1,\dots,n_k};x)\rangle = \langle \phi(D)^l/l! \mid \chi^{\phi}(\overline{\mathcal{K}_{n_1,\dots,n_k}};x)\rangle\,. \end{aligned}$$

This establishes 3.1.2, directly from the definition of $\chi^{\phi}(\overline{\mathcal{K}_{n_1,\dots,n_k}};x)$. □

We are going to present a similar expression (3.1.9) for $f^{b}_{n_1,\dots,n_k}$. Several cancellations occur in (3.1.4) and (3.1.9). However, we are able to give a combinatorial interpretation for the coefficients of the monomials in $F^{\phi}_{n_1,\dots,n_k}$ and $f^{b}_{n_1,\dots,n_k}$ in terms of trees, by using the Haiman-Schmitt form of Lagrange inversion [17]. To do this, we need to choose other polynomial generators for Φ_* and H_*, namely $\overline{\phi}_1,\overline{\phi}_2,\dots$, and $m_1,m_2,\dots$, respectively. We consider two kinds of rooted trees with n leaves, namely rooted trees with leaves labelled $1,2,\dots,n$, and rooted plane trees with a k-colouring of the leaves of type $(n_1,\dots,n_k)$ (that is a colouring with colours $1,\dots,k$ such that exactly n_i leaves are coloured i). We also assume that no vertex has only one descendant. The number of vertices of a tree T is denoted by $|T|$. A vertex of a tree is called *peripheral* if all its descendants are leaves; the set of descendants of a peripheral vertex will be called a *peripheral class*. The type $\tau^{\overline{\phi}}(T')$ of a tree T' of the first kind is defined as $\overline{\phi}_1^{\,i_1}\overline{\phi}_2^{\,i_2}\dots$, where i_j is the number of vertices of T' with $j+1$ descendants. The type $\tau^{m}(T'')$ of a tree T'' of the second kind is defined similarly, as a monomial in $m_1,m_2,\dots$. We can now state the combinatorial interpretation mentioned earlier.

Theorem 3.1.5 *We have that*

$$F^{\phi}_{n_1,\dots,n_k}=\sum_{T'}(-1)^{|T'|-n}\,\tau^{\overline{\phi}}(T')\,, \qquad (3.1.6)$$

$$f^{b}_{n_1,\dots,n_k}=\sum_{T''}(-1)^{|T''|-n}\,\tau^{m}(T'')\,; \qquad (3.1.7)$$

the first sum ranges over those trees T' of the first kind for which none of the sets of labels corresponding to a peripheral class are contained in a block of the partition π; the second sum ranges over those trees of the second kind for which no peripheral class is monochromatic. Furthermore, we have that

$$F^{\phi}_{n_1,\dots,n_k}=\sum_{k\geq 1}(-1)^{k}\,\widetilde{B}_{n+k-1,k}\,; \qquad (3.1.8)$$

here $\widetilde{B}_{i,k} := \sum_{\sigma} \tau^{\overline{\phi}}(\sigma)$, *with summation ranging over those partitions* $\sigma \in \Pi_{i,k}$ *with no singleton blocks, for which none of the blocks are contained in a block of the partition* π.

Proof. We denote by $\mathcal{L}_i$ the set of rooted trees with i leaves labelled $1, 2, \ldots, i$; for $T \in \mathcal{L}_i$, we denote by $p(T)$ the partition of $[i]$ whose blocks are either peripheral classes or singletons. We write $\|\sigma\|$ for the number of non-singleton blocks of the partition σ. We also define a new partial order $\preceq$ on Π_n by insisting that $\pi \preceq \sigma$ if and only if σ is obtained from π by amalgamating only singleton blocks. With these notations, and using the expression for Lagrange inversion in terms of rooted leaf-labelled trees (see [17] Corollary 1), we have:

$$\begin{aligned}
F^{\phi}_{n_1,\ldots,n_k} &= \sum_{\pi \geq \sigma \in \Pi_n} \zeta^{\overline{\phi}}(\widehat{0}, \sigma)\, \mu^{\overline{\phi}}(\sigma, \widehat{1}) \\
&= \sum_{\pi \geq \sigma \in \Pi_n} \zeta^{\overline{\phi}}(\widehat{0}, \sigma) \left(\sum_{T \in \mathcal{L}_{|\sigma|}} (-1)^{|T|-|\sigma|}\, \tau^{\overline{\phi}}(T) \right) \\
&= \sum_{\pi \geq \sigma \in \Pi_n} \; \sum_{T' \in \mathcal{L}_n :\, \sigma \preceq p(T')} (-1)^{|T'|-n-\|\sigma\|}\, \tau^{\overline{\phi}}(T') \\
&= \sum_{T' \in \mathcal{L}_n} (-1)^{|T'|-n}\, \tau^{\overline{\phi}}(T') \left(\sum_{\sigma \leq \pi,\, \sigma \preceq p(T')} (-1)^{\|\sigma\|} \right) .
\end{aligned}$$

To compute the last sum, assume that there are l (possibly $l = 0$) non-singleton blocks of $p(T')$ which are contained in some block of π; then the only partitions σ satisfying $\sigma \leq \pi$ and $\sigma \preceq p(T')$ are those obtained from $p(T')$ by splitting all blocks into singletons, except some of the l blocks mentioned above. Hence the last sum is $\sum_{I \subseteq [l]} (-1)^{|I|} = \delta_{l,0}$, which proves (3.1.6). Formula (3.1.8) now follows by using the bijection between leaf-labelled rooted trees and partitions established in [17], Theorem 5.

To prove (3.1.7), we must first find an analogue of (3.1.4). We do this by using Theorem 2.5.3, which provides an expression for the *Möbius type function* μ^{ϕ}_P of a subposet P of Π_n, when there is a permutation group G on $[n]$ which also permutes P (via the obvious action on Π_n). We choose P to be $[\widehat{0},\pi]\cup\{\widehat{1}_{\Pi_n}\}$, and G to be the direct product of symmetric groups on the blocks of π. In §2.5 we have considered the poset $A(P)$ of *preferential arrangements* (σ,ω) of $[n]$ with $\sigma\in P$, and by $\widehat{A}(P)$ the poset obtained by adjoining a least element $\widehat{0}$ to $A(P)$. We also let $\overline{A}(P):=A(P)\setminus\{\widehat{1}\}$. By insisting that $G(\widehat{0})=\{\widehat{0}\}$, we obtain a poset action of G on $\widehat{A}(P)$, and hence an induced poset structure on the set of orbits $\widehat{A}(P)/G$. Let us also recall from §2.5 the function ζ^b and its convolution inverse $\mu^b_{\widehat{A}(P)/G}$ in the incidence algebra over H_* of the poset $\widehat{A}(P)/G$. According to Theorem 2.5.3, we have

$$\begin{aligned} f^b_{n_1,\ldots,n_k} &= \frac{F^\phi_{n_1,\ldots,n_k}}{n_1!\ldots n_k!} = -\frac{\mu^\phi_P(\widehat{0},\widehat{1})}{|G|} \\ &= -\mu^b_{\widehat{A}(P)/G}(\widehat{0},\widehat{1}) = \sum_{(\sigma,\omega)\in\mathcal{T}} \mu^b_{\widehat{A}(P)/G}(\widehat{0},G(\sigma,\omega))\,\zeta^b(\sigma,\widehat{1})\,, \end{aligned} \tag{3.1.9}$$

where $\mathcal{T}$ is an arbitrary transversal of $\overline{A}(P)/G$. We can view $\overline{A}(P)/G$ as the subposet of $A(\Pi_n)$ consisting of "shuffles" of preferential arrangements of the sets $[n_1]$, $n_1+[n_2]$, ..., $n-n_k+[n_k]$, whose blocks are intervals (in $\mathbb{N}$) ordered in the natural way. If we do this, we can establish a bijection between $\overline{A}(P)/G$ and the set of pairs (ρ,c), where c is a k-colouring of $[n]$ of type $(n_1,\ldots,n_k)$, and ρ is a partition of $[n]$ with monochromatic blocks which are intervals (in $\mathbb{N}$). Indeed, if (σ,ω) is the shuffle $(B_{i_1j_1},B_{i_2j_2},\ldots)$ of $(B_{11},B_{12},\ldots)$, ..., $(B_{k1},B_{k2},\ldots)$, then the associated partition ρ is $\{[|B_{i_1j_1}|],|B_{i_1j_1}|+[|B_{i_2j_2}|],\ldots\}$, and all the elements in the r-th block of ρ are coloured i_r. Finally, we denote by $\mathcal{P}_i$ the set of rooted plane trees with i leaves. If we label the leaves of $T\in\mathcal{P}_i$ with $1,\ldots,i$, we can define $p(T)$ as before. Using (3.1.9) and the expression for Lagrange inversion in terms of rooted plane trees (the analogue of Corollary 1 in [17]), we finally have:

$$\begin{aligned} f^b_{n_1,\ldots,n_k} &= \sum_{(\sigma,\omega)\in\mathcal{T}} \zeta^m(\widehat{0},\sigma)\,\mu^m_{\widetilde{\Pi}_{|\sigma|}}(\widehat{0},\widehat{1}) \\ &= \sum_{(\sigma,\omega)\in\mathcal{T}} \zeta^m(\widehat{0},\sigma)\left(\sum_{T\in\mathcal{P}_{|\sigma|}} (-1)^{|T|-|\sigma|}\,\tau^m(T)\right) \\ &= \sum_{(\rho,c)} \sum_{T''\in\mathcal{P}_n:\,\rho\preceq p(T'')} (-1)^{|T''|-n-\|\rho\|}\,\tau^m(T'') \\ &= \sum_{(T'',c)} (-1)^{|T''|-n}\,\tau^m(T'')\left(\sum_{(\rho,c):\,\rho\preceq p(T'')} (-1)^{\|\rho\|}\right). \end{aligned}$$

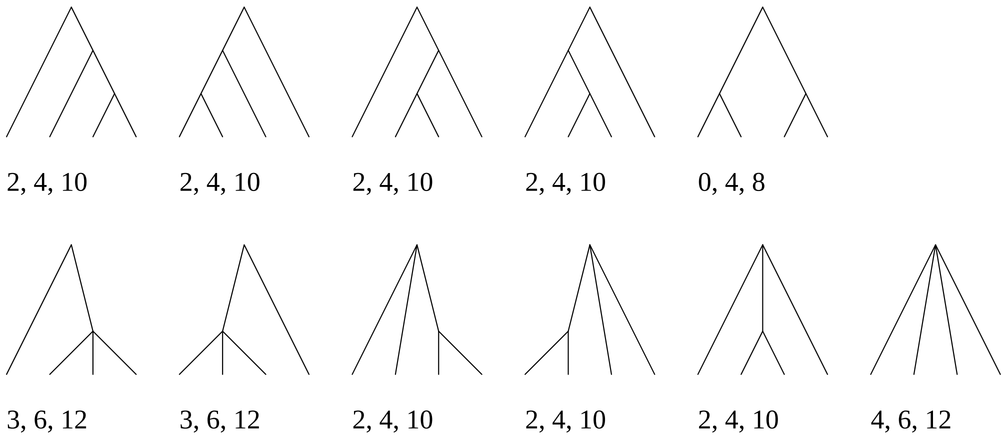
The last sum is computed as before, and (3.1.7) follows. □

Note that for $n_1 = \ldots = n_k = 1$, the theorem is just the Haiman-Schmitt form of Lagrange inversion.

Example 3.1.10 In order to express $f^b_{1,3}$, $f^b_{2,2}$, and $f^b_{1,1,2}$, we consider all the plane trees with 4 leaves, as shown below. The trees on the first line have type m_1^3; those on the second line have type m_1m_2, except for the last one, whose type is m_3. The three numbers corresponding to a tree represent the number of 2-colourings of type $(1,3)$, of 2-colourings of type $(2,2)$, and of 3-colourings of type $(1,1,2)$ for the leaves, which satisfy the condition in Theorem 3.1.5. According to the theorem, we have

$$\begin{aligned} &f^b_{1,3} = -8m_1^3 + 12m_1m_2 - 4m_3\,, \quad f^b_{2,2} = -20m_1^3 + 24m_1m_2 - 6m_3\,, \\ &f^b_{1,1,2} = -48m_1^3 + 54m_1m_2 - 12m_3\,. \end{aligned}$$

We now present applications of Proposition 3.1.1 to combinatorial proofs of formal group law identities. We prove two such identities in Propositions 3.1.11 and 3.1.13. The first identity is a familiar one, and is usually proved by formal power series manipulations (see e.g. [33]).



Proposition 3.1.11 *In $\Phi^*\{\{X\}\}$, we have that*

$$\overline{\phi}'(X) = \left(\frac{\partial F^{\phi}}{\partial Y}(X,0)\right)^{-1},$$

where $\overline{\phi}'(X)$ and $\frac{\partial F^{\phi}}{\partial Y}(X,Y)$ denote the formal derivatives of the corresponding power series, and $(\,\cdot\,)^{-1}$ denotes the multiplicative inverse.

PROOF. The given formula may easily be seen to be equivalent to the set of identities

$$-\,\overline{\phi}_n = \sum_{i=0}^{n-1} \binom{n}{i} \overline{\phi}_i \, F^{\phi}_{1,n-i}\,, \qquad n > 0\,. \tag{3.1.12}$$

Fix $n > 0$. By (1.7.7) and (1.7.13), we have

$$-\overline{\phi}_n = \sum_{\gamma \in C(\Pi_{n+1})} \tau^{\phi}(\gamma)\,,$$

so it suffices to establish that the right-hand side of (3.1.12) also enumerates by type the chains in $C(\Pi_{n+1})$.

We first partition these chains into classes $\mathcal{C}(A)$, with $\{1\} \subseteq A \subsetneqq [n+1]$, by assigning the chain $\{\widehat{0} < \sigma_1 < \ldots < \sigma_r < \widehat{1}\}$ to $\mathcal{C}(A)$ if and only if $A \in \sigma_r$. As $1 \leq |A| \leq n$, and there are $\binom{n}{k-1}$ ways of choosing A of cardinality k, we see that it is enough to prove that if $|A| = k$ then $\sum_{\gamma \in \mathcal{C}(A)} \tau^{\phi}(\gamma) = \overline{\phi}_{k-1} \, F^{\phi}_{1,n-k+1}$. We do this as follows, by using (1.7.13), Proposition 1.7.14 twice, (1.7.7), and (3.1.3);

for clarity, it helps to set $\overline{A} := [n+1] \setminus A$, $\widetilde{A} := \overline{A} \cup \{1\}$, and $\omega := \{\{1\}, \overline{A}\}$, thereby yielding

$$\begin{aligned}\sum_{\gamma \in \mathcal{C}(A)} \tau^{\phi}(\gamma) &= -\sum_{\sigma \in \Pi(\mathcal{K}_{\overline{A}})} \left(\sum_{\gamma \in C(\Pi(\mathcal{K}^{\sigma \cup \{A\}}))} \tau^{\phi}(\gamma) \right) \zeta^{\phi}(\sigma \cup \{A\}, \widehat{1}) \\ &= \sum_{\sigma \in \Pi(\mathcal{K}_{\overline{A}})} \mu^{\phi}(\widehat{0}, \sigma \cup \{A\})\, \zeta^{\phi}(\sigma \cup \{A\}, \widehat{1}) \\ &= \overline{\phi}_{k-1} \sum_{\omega \geq \pi \in \Pi(\widetilde{A})} \mu^{\phi}(\widehat{0}, \pi)\, \zeta^{\phi}(\pi, \widehat{1}) \\ &= \overline{\phi}_{k-1}\, \nu^{\phi}_{\mathcal{K}_1 \cdot \mathcal{K}_{\overline{A}}}(\{\widetilde{A}\}) = \overline{\phi}_{k-1}\, F^{\phi}_{1,n+1-k}\,,\end{aligned}$$

as sought. □

Proposition 3.1.13 *We have that*

$$n m_{n-1} = \sum_{|I|=n} (-1)^{l(I)-1}\, \frac{n(l(I)-1)!}{\|I\|}\, f^{b}_{I}\,.$$

PROOF. According to (3.1.4), we have

$$\begin{aligned}\sum_{\pi \in \Pi_n} F^{\phi}_{I(\pi)}\, \mu(\pi, \widehat{1}) &= \sum_{\pi \in \Pi_n} \left(\sum_{\sigma \leq \pi} \mu^{\phi}(\widehat{0}, \sigma)\, \zeta^{\phi}(\sigma, \widehat{1}) \right) \mu(\pi, \widehat{1}) \\ &= \sum_{\sigma \in \Pi_n} \mu^{\phi}(\widehat{0}, \sigma)\, \zeta^{\phi}(\sigma, \widehat{1}) \left(\sum_{\pi \geq \sigma} \mu(\pi, \widehat{1}) \right) \\ &= \mu^{\phi}(\widehat{0}, \widehat{1}) = \overline{\phi}_{n-1}\,;\end{aligned}$$

according to our conventions, μ denotes here the classical Möbius function of Π_n. It is well-known that $\mu(\pi, \widehat{1}) = (-1)^{|\pi|-1}(|\pi|-1)!$, and that the number of partitions σ of $[n]$ with $I(\sigma) = I$ is $n!/(I!\|I\|)$. Hence

$$n m_{n-1} = \frac{\overline{\phi}_{n-1}}{(n-1)!} = \frac{1}{(n-1)!} \sum_{|I|=n} (-1)^{l(I)-1}\, \frac{n!}{I!\|I\|}\, F^{\phi}_{I}\, (l(I)-1)!\,,$$

which implies the identity to be proved. □

Let us note that the number $n(l(I)-1)!/\|I\|$ is an integer. Indeed, if $I = (1^{r_1}, \ldots, k^{r_k})$, we have that

$$\frac{(r_1+2r_2+\ldots+kr_k)\,(r_1+\ldots+r_k-1)!}{r_1!\ldots r_k!} = \binom{r_1+\ldots+r_k-1}{r_1-1,\ r_2,\ \ldots r_k}$$

$$+2\binom{r_1+\ldots+r_k-1}{r_1,\ r_2-1,\ \ldots r_k} + \ldots + k\binom{r_1+\ldots+r_k-1}{r_1,\ r_2,\ \ldots r_k-1}.$$

Hence, Proposition 3.1.13 provides another expression of nm_{n-1} as an integer linear combination of elements in the Lazard ring.

3.2 Combinatorial Models for p-typical Formal Group Laws

Our results so far, as well as those in Chapter 4, are concerned mainly with algebraic structures and combinatorial invariants associated with the universal formal group law. This section is intended to be a starting point for understanding the combinatorics of p-typical formal group laws, by constructing more appropriate combinatorial models in this case. We concentrate once again on the universal case, by considering the umbra λ in the ring Φ_*^p, where p is a fixed prime.

We start by discussing the way in which the formula defining the characteristic type polynomial $c^\phi(\mathcal{S};x)$ of a partition system with singleton atoms $\mathcal{S}$ simplifies when we consider the image $c^\lambda(\mathcal{S};x)$ of this polynomial under the projection $\Phi_*[x] \to \Phi_*^p[x]$. Let us associate with $\mathcal{S}$ the following partition system:

$$\mathcal{S}^p := \mathcal{S} \setminus \{U \in \mathcal{S} \ : \ |U| \neq p^k,\ \ k \geq 1,\ \ \text{and}\ \ \mathcal{S}|U = \mathcal{K}_U\}.$$

Proposition 3.2.1 *We have that*

$$c^\lambda(\mathcal{S};x) = c^\lambda(\mathcal{S}^p;x).$$

In particular, $\mu^\lambda_{\Pi(\mathcal{S})}(\widehat{0},\sigma) = \mu^\lambda_{\Pi(\mathcal{S}^p)}(\widehat{0},\sigma)$ *for every partition* σ *in* $\Pi(\mathcal{S}^p)$.

PROOF. We use the deletion/contraction identity in Theorem 2.3.1. Since $\mu^{\lambda}_{\Pi(\mathcal{K}_U)}(\widehat{0},\widehat{1}) = \overline{\lambda}_{|U|-1} = 0$ for $|U| \neq p^k$, the characteristic type polynomial is not affected by successively removing from $\mathcal{S}$ all the sets U satisfying $|U| \neq p^k$ and $\mathcal{S}|U = \mathcal{K}_U$. The only thing we have to ensure is that these sets are removed in decreasing order of their cardinalities. The statement about μ^{λ} follows by considering the set system $\mathcal{S}|\sigma$. □

According to the above result, we can express the polynomial $B_n^{\lambda}(x)$ in terms of the poset $\Pi(\mathcal{K}_n^p)$, which is much smaller than $\Pi(\mathcal{K}_n)$ for large n; in other words, we can restrict ourselves to partitions for which the block sizes are powers of p. For instance, for $p = 2$, we have the simpler expression

$$B_4^{\lambda}(x) = x^4 - 6\lambda_1 x^3 + 3\lambda_1^2 x^2 + (-3\lambda_1^3 + 6\lambda_1\lambda_2 - \lambda_3)x\,,$$

instead of the usual formula provided by (1.7.8):

$$B_4^{\lambda}(x) = x^4 - 6\lambda_1 x^3 + (15\lambda_1^2 - 4\lambda_2)x^2 + (-15\lambda_1^3 + 10\lambda_1\lambda_2 - \lambda_3)x\,.$$

But we know that the ring Φ_*^p is polynomial in $\lambda_{p-1}, \lambda_{p^2-1}, \ldots$, so the coefficients of the polynomials $B_n^{\lambda}(x)$ are expressible only in terms of these generators. For $p = 2$ we have $\lambda_2 = 3\lambda_1^2$ (because $\overline{\lambda}_2 = 3\lambda_1^2 - \lambda_2 = 0$), which means that we have in fact

$$B_4^{\lambda}(x) = x^4 - 6\lambda_1 x^3 + 3\lambda_1^2 x^2 + (15\lambda_1^3 - \lambda_3)x\,.$$

Thus we have arrived at the crucial problem of expressing the elements λ_i, for $i \neq p^k - 1$, in terms of the polynomial generators of Φ_*^p. This turns out to be a difficult problem, and is closely related to the open problem of characterising the subgroup of the group of Hurwitz series $X + r_1X^2/2! + r_2X^3/3! + \ldots$ (where r_i lie in some torsion free ring R) under substitution, which is generated by Hurwitz series of the form $X + q_1X^p/p! + q_2X^{p^2}/p^2! + \ldots$. The first problem can in fact be reduced to understanding how Lagrange inversion works for the Hurwitz series $\lambda(X)$ and $\overline{\lambda}(X)$ (which were reinterpreted in §1.2 as the exp and log series of the universal p-typical formal group law). Note that expressing the coefficients of $\lambda(X)$ in terms of those of $\overline{\lambda}(X)$ is just a special case of Lagrange inversion; however, the converse requires more, if we want to express $\overline{\lambda}_{p-1}, \overline{\lambda}_{p^2-1}, \ldots$ only in terms of $\lambda_{p-1}, \lambda_{p^2-1}, \ldots$. Before investigating this problem, we show how it helps us express the elements λ_i, for $i \neq p^k - 1$, in terms of $\lambda_{p-1}, \lambda_{p^2-1}, \ldots$.

Let us start with a few definitions and notations. We say that a partition $I = (i_1, \ldots, i_l)$ of n is *p-typical* if all its parts i_j are of the form $p^k - 1$. All partitions considered from now on in this section are assumed to be p-typical. We set $\lambda_I := \lambda_{i_1} \ldots \lambda_{i_l}$, and denote the coefficient of λ_I in the expression of an element z in Φ_*^p by $c(\lambda_I, z)$; we do the same thing for $\overline{\lambda}_I$, ϕ_I, and $\overline{\phi}_I$, where in the last two cases the element z lies in Φ_*. Note that $c(\phi_I, \overline{\phi}_J) = c(\overline{\phi}_I, \phi_J)$ by Lagrange inversion; however, this equality does not hold when we replace the umbra ϕ with λ, as we have discussed in the previous paragraph. Let us consider the set $P(n)$ of p-typical partitions of n with the partial order defined in §1.10. Note that $c(\phi_I, \overline{\phi}_J)$, $c(\lambda_I, \overline{\lambda}_J)$, and $c(\overline{\lambda}_I, \lambda_J)$ can only be non-zero if $I \leq J$. We have

$$\lambda_n = \sum_{J \in P(n)} c(\overline{\phi}_J, \phi_n)\, \overline{\lambda}_J = \sum_{I \in P(n)} \left(\sum_{I \leq J \in P(n)} c(\lambda_I, \overline{\lambda}_J)\, c(\phi_J, \overline{\phi}_n) \right) \lambda_I\,, \tag{3.2.2}$$

whence

$$c(\lambda_I, \lambda_n) = \sum_{I \leq J \in P(n)} c(\lambda_I, \overline{\lambda}_J)\, c(\phi_J, \overline{\phi}_n)\,. \tag{3.2.3}$$

Formula (3.2.3) answers our original question of expressing λ_n in terms of λ_{p-1}, $\lambda_{p^2-1}, \ldots$, provided that we are able to determine the coefficients $c(\lambda_I, \overline{\lambda}_J)$.

We now fix $n = p^k - 1$ for the rest of this section. We have the following analogue of (3.2.2):

$$\overline{\lambda}_n = \sum_{J \in P(n)} c(\lambda_J, \overline{\lambda}_n)\, \lambda_J = \sum_{I \in P(n)} \left(\sum_{I \leq J \in P(n)} c(\phi_I, \overline{\phi}_J)\, c(\lambda_J, \overline{\lambda}_n) \right) \overline{\lambda}_I\,,$$

which provides

$$\sum_{I\leq J\in P(n)} c(\phi_I,\overline{\phi}_J)\, c(\lambda_J,\overline{\lambda}_n) = \delta_{I,(n)}\,. \tag{3.2.4}$$

We can interpret this formula in terms of the incidence algebra (over $\mathbb{Z}$) of the poset $P(n)$. Indeed, let us define the function f in this algebra by $f(I,J) := c(\phi_I,\overline{\phi}_J)$. Its convolution inverse f^{-1} exists because $f(I,I)=(-1)^{l(I)}$ is invertible, and (3.2.4) tells us that $c(\lambda_I,\overline{\lambda}_n)=f^{-1}(I,(n))$.

Let us consider an example for $p=2$. We have seen above that $\overline{\lambda}_3 = 15\lambda_1^3-\lambda_3$, so let us now compute $\overline{\lambda}_7$. The poset $P(7)$ is totally ordered, and we have $(1^7) < (3,1^4) < (3^2,1) < (7)$. First of all, we have $c(\lambda_1\lambda_3^2,\overline{\lambda}_7) = -c(\phi_1\phi_3^2,\overline{\phi}_7) = 1575$ (this holds in general, for all the monomials λ_I with I a maximal element in $P(n)\setminus\{(n)\}$). Secondly, we have

$$\begin{aligned} c(\lambda_1^4\lambda_3,\overline{\lambda}_7) &= -c(\phi_1^3\phi_3,\overline{\phi}_3^2)\, c(\lambda_1\lambda_3^2,\overline{\lambda}_7) - c(\phi_1^4\phi_3,\overline{\phi}_7)\\ &= -30\cdot 1575 - (-51975) = 4725\,. \end{aligned}$$

Finally, we have

$$\begin{aligned} c(\lambda_1^7,\overline{\lambda}_7) &= c(\phi_1^3,\overline{\phi}_3)\, c(\lambda_1^4\lambda_3,\overline{\lambda}_7) - c(\phi_1^6,\overline{\phi}_3^2)\, c(\lambda_1\lambda_3^2,\overline{\lambda}_7) - c(\phi_1^7,\overline{\phi}_7)\\ &= (-15)\cdot 4725 - 225\cdot 1575 - (-135135) = -290115\,. \end{aligned}$$

We now apply (3.2.3) and obtain

$$\lambda_2 = 3\lambda_1^2\,,\;\; \lambda_4 = -120\lambda_1^4 + 15\lambda_1\lambda_3\,,\;\; \lambda_5 = -2205\lambda_1^5 + 210\lambda_1^2\lambda_3\,,$$
$$\lambda_6 = -28980\lambda_1^6 + 2100\lambda_1^3\lambda_3 + 35\lambda_3^2\,.$$

Returning to the expression of $\overline{\lambda}_7$ in terms of λ_1, λ_3, and λ_7, let us note that the sign of a monomial is no longer determined by the number of its factors, like in Lagrange inversion.

Although it is easy to implement, the above procedure does not offer us any indication about the combinatorial significance of the coefficients $c(\lambda_I,\overline{\lambda}_n)$, as

Lagrange inversion does about the coefficients $c(\phi_I, \overline{\phi}_n)$ (here we use the form of Lagrange inversion in terms of leaf-labelled rooted trees, which was proved in [17]). We now intend to show that finding the convolution inverse of the function f in $\mathbb{Z}(P(n))$ discussed above reduces to several classical Möbius inversions on certain posets of trees. Moreover, we can easily recover the Haiman-Schmitt form of Lagrange inversion.

We need a few more definitions and notations. In what follows, a tree means a rooted tree with $n+1$ leaves labelled $1, 2, \ldots, n+1$, and no vertex having only one descendant (such trees were called "of the first kind" in §3.1). We only consider trees for which the number of descendants of every vertex is a power of p; we call these trees p-typical, and denote their set by $\Omega(n+1)$. We recall the definition of the type $\tau^\lambda(T)$ of a tree T from §3.1. We define a partial order on $\Omega(n+1)$ as follows: $T_1 \leq T_2$ if and only if T_2 is obtained from T_1 by contracting certain internal edges (that is edges not incident to leaves). This poset clearly has a unique maximal element, namely the tree of type λ_n, and several minimal elements, namely the trees of type $\lambda_{p-1}^{n/(p-1)}$. Let us also note that the map from $\Omega(n+1)$ to $P(n)$ specified by $T \mapsto I$, where $\tau^\lambda(T) = \lambda_I$, is order preserving. We are now able to state our main result.

Proposition 3.2.5 *We have that*

$$c(\lambda_I, \overline{\lambda}_n) = -\sum_{\tau^\lambda(T)=\lambda_I} \mu(T, \widehat{1}),$$

where μ denotes the Möbius function of the poset $\Omega(n+1)$.

Proof. We prove this result by induction on $l(I)$, which clearly starts at 1. Now assume that it holds for $l(i) < k$, and consider a p-typical partition I of n with $l(I) = k$. The idea is to sum the relations

$$\mu(T, \widehat{1}) + \sum_{T' > T} \mu(T', \widehat{1}) = 0, \qquad \tau^\lambda(T) = \lambda_I,$$

prove that

$$\sum_{\tau^\lambda(T)=\lambda_I} \sum_{T'>T} \mu(T',\widehat{1}) = (-1)^{l(I)-1} \sum_{J>I} c(\phi_I,\overline{\phi}_J)\, c(\lambda_J,\overline{\lambda}_n)\,, \tag{3.2.6}$$

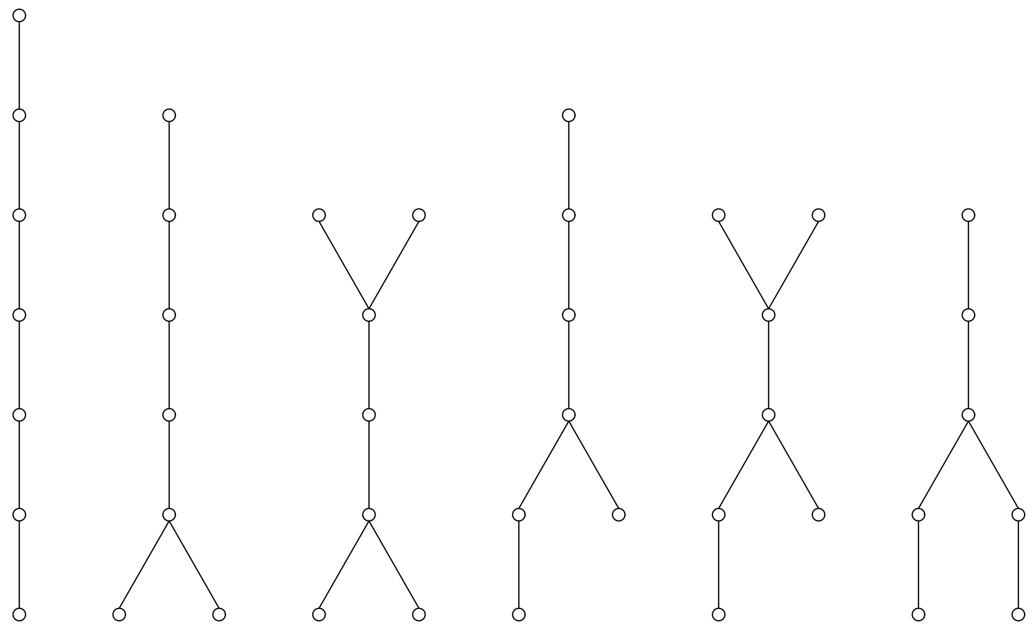
and use (3.2.4). Hence, let us concentrate on (3.2.6). We clearly have

$$\sum_{\tau^\lambda(T)=\lambda_I} \sum_{T'>T} \mu(T',\widehat{1}) = \sum_{J>I} \sum_{\tau^\lambda(T')=\lambda_J} \mu(T',\widehat{1})\, N(I,T')\,,$$

where $N(I,T')$ denotes the number of trees T satisfying $T<T'$ and $\tau^\lambda(T)=\lambda_I$. It is easy to see that this number only depends on the type of T', not on the tree T' itself. Furthermore, by a slightly more general form of the Lagrange inversion formula due to Haiman and Schmitt, $N(I,T')=(-1)^{l(I)}c(\phi_I,\tau^{\overline{\phi}}(T'))$. Thus (3.2.6) follows by induction. □

Let us make a few remarks. The poset $\Omega(n+1)$ is not a lattice, and it seems to admit no obvious closure or coclosure operator. This means that we cannot apply the standard techniques for computing the Möbius function, which makes this problem quite difficult. On the other hand, we notice that if we let our poset of trees contain *all* trees with $n+1$ leaves, then we obtain a Boolean algebra. In consequence, $\mu(T,\widehat{1})=(-1)^{l(I)-1}$ for all trees with $\tau^\phi(T)=\phi_I$, and Proposition 3.2.5 still holds (with λ replaced by ϕ), by reducing to the Haiman-Schmitt form of Lagrange inversion.

We now consider an example, again for $p=2$ and $n=7$. We present below the value of $\mu(T,\widehat{1})$ for several (binary) trees which contribute to the expression of $c(\lambda_1^7,\overline{\lambda}_7)$. For simplicity, we have drawn these trees with edges incident to leaves removed, and without distinguishing the root; we will refer below to the tree obtained in this way as the *skeleton* of the original tree. We notice that the values of the Möbius function are greater than 0, which explains the fact that $-c(\lambda_1^7,\overline{\lambda}_7)$ is more than twice $-c(\phi_1^7,\overline{\phi}_7)$. This leads us to the following conjecture.



1 1 1 3 4 5

Conjecture 3.2.7 *We have* $\mu(T, \widehat{1}) \geq 1$ *for all binary trees* T *with* 2^k *leaves, where* $k \geq 3$.

Note that this is not true for the trees contributing to the expression of $c(\lambda_1^4\lambda_3, \overline{\lambda}_7)$, for instance. We computed $\mu(T, \widehat{1})$ for the trees with 2^k leaves whose skeleton is a path, for $k = 3, 4, 5, 6, 7, 8$.

k	3	4	5	6	7	8
$\mu(T, \widehat{1})$	1	7	769	14678615	$\approx 7741 \cdot 10^{12}$	$\approx 2954 \cdot 10^{30}$
$\log_2(\mu(T, \widehat{1}))$	0	2.80735	9.58684	23.8072	52.7816	111.187

These computations suggest that the value of $\mu(T, \widehat{1})$ for trees with 2^k leaves whose skeleton is a path might be of the order $2^{2^{k-1}}$.

If Conjecture 3.2.7 is true, it confirms the importance of Proposition 3.2.5, and suggests that we might be able to find some objects which are counted by $\mu(T, \widehat{1})$.

3.3 Classical Results in Algebraic Topology Restated

In this section we express several classical coactions in algebraic topology in the language of incidence Hopf algebras. The main instrument for translating topological formulae into this language is Theorem 1.7.6, which establishes the connection between substitution of formal power series and the convolution product in the incidence Hopf algebras Φ_* and H_*. Sometimes we prefer to write formulae using the incidence algebras $\Phi_*(\Pi_n)$ and $H_*(\widetilde{\Pi}_n)$ of the lattices Π_n and $\widetilde{\Pi}_n$. The context will always determine which algebras we are using. The principle on which our formalism is based is the one already discussed in §1.7; namely, that in some cases, such as those considered below, a ring homomorphism $\xi\colon R_* \to T_*$ is uniquely determined by the image of $\mathrm{Alg}(\xi)$ on some function in $\mathrm{Alg}^*(\Phi_*, R_*)$ or $\mathrm{Alg}^*(H_*, R_*)$.

Let us consider the *Hopf algebroid* (see [33]) $MU_*(MU) \cong MU_*[b_1^{MU}, b_2^{MU}, \ldots]$ $\cong MU_* \otimes H_*$, where H_* is the dual of the Landweber-Novikov algebra (see Example 1.7.4). Let η_L, η_R be the left and right units, δ the comultiplication, and γ the conjugation of $MU_*(MU)$. We write ϕ_n^{MU} for $(n+1)!b_n^{MU}$, ϕ_n^R for $\eta_R(\phi_n)$, and consider the umbras $b^{MU} := (1, b_1^{MU}, b_2^{MU}, \ldots)$, $\phi^{MU} := (1, \phi_1^{MU}, \phi_2^{MU}, \ldots)$, and $\phi^R := (1, \phi_1^R, \phi_2^R, \ldots)$. The right unit η_R is usually expressed via the formula

$$\overline{\phi}^R(X) = \overline{\phi}(\overline{\phi}^{MU}(X))\,.$$

By Theorem 1.7.6 (1) this is equivalent to specifying η_R as follows:

$$\eta_R: \qquad \mu^{\phi} \mapsto \mu^{\phi^R} = \mu^{\phi^{MU}} * \mu^{\phi} \qquad \text{or} \qquad \zeta^{\phi} \mapsto \zeta^{\phi^R} = \zeta^{\phi} * \zeta^{\phi^{MU}}, \tag{3.3.1}$$

by taking inverses. If we work in $(H \wedge MU)_*(MU)$, we can also write

$$\eta_R : \qquad \mu^b \mapsto \mu^{b^R} = \mu^{b^{MU}} * \mu^b \qquad \text{or} \qquad \zeta^b \mapsto \zeta^{b^R} = \zeta^b * \zeta^{b^{MU}}, \tag{3.3.2}$$

where $b_n^R := \eta_R(b_n)$, and $b^R := (1, b_1^R, b_2^R, \dots)$. The comultiplication δ can be specified using the notation in §1.7 in either of following ways:

$$\delta : \qquad \zeta^{b^{MU}} \mapsto \zeta^{b^{MU}} \circledast \zeta^{b^{MU}} \qquad \text{or} \qquad \zeta^{\phi^{MU}} \mapsto \zeta^{\phi^{MU}} \circledast \zeta^{\phi^{MU}}. \tag{3.3.3}$$

Similarly, the conjugation γ is specified by

$$\gamma : \qquad \zeta^{b^{MU}} \mapsto \mu^{b^{MU}} \qquad \text{or} \qquad \zeta^{\phi^{MU}} \mapsto \mu^{\phi^{MU}}. \tag{3.3.4}$$

We will present in §3.4 and §3.5 more substantial applications, demonstrating the computational advantages of this new point of view. For now, we consider a few simple applications, demonstrating the advantages in simplifying notation and proofs. For our applications, we need the umbra $\kappa := (1, u, u^2, \dots)$ in $K_* \cong \mathbb{Z}[u, u^{-1}]$ and $\kappa^R := (1, v, v^2, \dots)$ in $K_*(K)$, where $v = \eta_R^K(u)$. Let us recall that the image of $K_*(K)$ in $K_*(K) \otimes \mathbb{Q}$ consists precisely of those finite Laurent series $f(u, v)$ satisfying $f(it, jt) \in \mathbb{Z}[t, t^{-1}, \frac{1}{ij}]$ for all $i, j \in \mathbb{Z} \setminus \{0\}$. We also need the standard map of ring spectra $g \colon MU \to K$ representing the universal Thom class in $K^0(MU)$; the map $g_* \colon MU_* \to K_*$ is the Todd genus, mapping ϕ_n to u^n. Finally, let us recall the fact that $K_*(MU) \cong \mathbb{Z}[u, u^{-1}, b_1^K, b_2^K, \dots]$, and that $g_* \colon MU_*(MU) \to K_*(MU)$ maps b_n^{MU} to b_n^K. We will also need the umbras $b^K := (1, b_1^K, b_2^K, \dots)$ and $\phi^K := (1, \phi_1^K, \phi_2^K, \dots)$, where $\phi_n^K := (n+1)! b_n^K$.

Example 3.3.5 Let us first check that the elements ϕ_n^R are *primitive* in $MU_*(MU)$. Indeed, we have

$$\delta : \ \zeta^{\phi^R} = \zeta^\phi * \zeta^{\phi^{MU}} \mapsto \left(\zeta^\phi * \zeta^{\phi^{MU}}\right) \circledast \zeta^{\phi^{MU}} = \zeta^{\phi^R} \circledast \zeta^{\phi^{MU}} = 1 \otimes \left(\zeta^\phi * \zeta^{\phi^{MU}}\right) = 1 \otimes \zeta^{\phi^R}.$$

Example 3.3.6 It is well-known that the map $f_*\colon MU_*(\mathbb{C}P^\infty) \to MU_*(MU)$ induced by the inclusion $f\colon \mathbb{C}P^\infty \simeq MU(1) \hookrightarrow \Sigma^2 MU$ is specified by $\beta_n^{MU} \mapsto b_{n-1}^{MU}$. Let us identify $MU_*(\mathbb{C}P^\infty)$ with its image in $(H \wedge MU)_*(\mathbb{C}P^\infty)$, and try to determine the map f_*. It is an immediate consequence of (1.7.8), and it is also a well-known fact in umbral calculus that

$$x^n = \sum_{\sigma \in \Pi_n} \zeta^\phi(\widehat{0}, \sigma)\, B_n^\phi(x)\,.$$

Applying the map f_*, we immediately obtain from (3.3.1)

$$f_*(x^n) = (\zeta^\phi * \zeta^{\phi^{MU}})(\phi_{n-1}) = \phi_{n-1}^R\,. \qquad (3.3.7)$$

Hence, the map f_* can be interpreted as *umbral substitution* by ϕ^R.

Formula (3.3.7) and the following commutative diagram show that the elements x^n are primitive under the coaction $MU_*(\mathbb{C}P^\infty) \to MU_*(MU) \otimes_{MU_*} MU_*(\mathbb{C}P^\infty)$.

$$\begin{array}{ccc} MU_*(\mathbb{C}P^\infty) & \longrightarrow & MU_*(MU) \otimes_{MU_*} MU_*(\mathbb{C}P^\infty) \\ \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle I \otimes f_*} \\ MU_*(MU) & \xrightarrow{\ \delta\ } & MU_*(MU) \otimes_{MU_*} MU_*(MU) \end{array} .$$

It is shown in [8] that $K_*(MU)$ is isomorphic to the direct limit

$$MU_*(\mathbb{C}P^\infty) \xrightarrow{\cdot x} MU_*(\mathbb{C}P^\infty) \xrightarrow{\cdot x} MU_*(\mathbb{C}P^\infty) \xrightarrow{\cdot x} \ldots,$$

where the maps are "multiplication by x". This means that there is a $\mathbb{Z}$-linear map $MU_*(\mathbb{C}P^\infty) \to K_*(MU)$ sending x^n to u^{n-1}. We deduce that x^n is not divisible (by integers) in $MU_*(\mathbb{C}P^\infty)$, whence the ring of primitive elements in $MU_*(\mathbb{C}P^\infty)$ is precisely $\mathbb{Z}[x]$. This is a simplified proof of the result which was first proved by D. Segal in [48], Theorem 2.1.

Example 3.3.8 The Hurewicz homomorphism $h_*\colon MU_* \to K_*(MU)$ can be easily determined using the following commutative diagram:

$$\begin{array}{ccc} MU_* & \xrightarrow{\eta_R} & MU_*(MU) \\ & \searrow^{h_*} & \downarrow{\scriptstyle g_*} \\ & & K_*(MU) \end{array} .$$

Hence, we can specify h_* as follows:

$$h_*: \qquad \zeta^\phi \mapsto \zeta^\kappa * \zeta^{\phi^K} \qquad \text{or} \qquad \mu^\phi \mapsto \mu^{\phi^K} * \mu^\kappa ,$$

which means that

$$h_*(\phi_n) = \sum_{k=0}^{n} S(n+1, k+1) \phi_k^K u^{n-k} , \tag{3.3.9}$$

and

$$h_*(\overline{\phi}_n) = \sum_{k=0}^{n} (-1)^k k! \, s^{\phi^K}(n+1, k+1) \, u^k . \tag{3.3.10}$$

Example 3.3.11 We would now like to give a more explicit expression (than the usual one) for the coaction $\delta^K\colon K_*(MU) \to K_*(K) \otimes_{K_*} K_*(MU)$. Let us present first a simple way of computing the image of b_n^{MU} under the map $(g \wedge g)_*\colon MU_*(MU) \to K_*(K)$. Since g is a map of ring spectra, we have the commutative diagram

$$\begin{array}{ccc} MU_* & \xrightarrow{\eta_R} & MU_*(MU) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle (g\wedge g)_*} \\ K_* & \xrightarrow{\eta_R^K} & K_*(K) \end{array} .$$

Hence $(g \wedge g)_*(\phi_n^R) = v^n$. On the other hand, from (3.3.1) we obtain $\zeta^{\phi^{MU}} = \mu^\phi * \zeta^{\phi^R}$. By applying $(g \wedge g)_*$, we deduce

$$(g \wedge g)_* : \qquad \zeta^{\phi^{MU}} \mapsto \mu^\kappa * \zeta^{\kappa^R}, \tag{3.3.12}$$

which means that

$$(g \wedge g)_*(b_n^{MU}) = \frac{B_{n+1}^{\kappa}(\kappa^R)}{(n+1)!} = \frac{(v-u)\dots(v-nu)}{(n+1)!}.$$

Let us now recall the commutative diagram

$$\begin{array}{ccc} MU_*(MU) & \xrightarrow{\delta} & MU_*(MU) \otimes_{MU_*} MU_*(MU) \\ \Big\downarrow{\scriptstyle g_*} & & \Big\downarrow{\scriptstyle (g\wedge g)_*\otimes g_*} \\ K_*(MU) & \xrightarrow{\delta^K} & K_*(K) \otimes_{K_*} K_*(MU) \end{array}.$$

Hence, by combining (3.3.3) and (3.3.12), we have

$$\delta^K: \qquad \zeta^{\phi^K} \mapsto \left(\mu^{\kappa} * \zeta^{\kappa^R}\right) \circledast \zeta^{\phi^K},$$

which means that

$$\begin{aligned}\delta^K(b_n^K) &= \sum_{k=1}^{n+1} \frac{k!}{(n+1)!} \left(\sum_{\pi \leq \sigma \in \Pi_{n+1,k}} \mu^{\kappa}(\widehat{0}, \pi)\, \zeta^{\kappa^R}(\pi, \sigma) \right) \otimes b_{k-1}^K \\ &= \sum_{k=1}^{n+1} \frac{k!}{(n+1)!} \left(\sum_{i=k}^{n+1} \sum_{\pi \in \Pi_{n+1,i}} \mu^{\kappa}(\widehat{0}, \pi) \left(\sum_{\pi \leq \sigma \in \Pi_{n+1,k}} \zeta^{\kappa^R}(\pi, \sigma) \right) \right) \otimes b_{k-1}^K .\end{aligned}$$

Finally, we have the following result:

$$\delta^K(b_n^K) = \sum_{k=1}^{n+1} \frac{k!}{(n+1)!} \left(\sum_{i=k}^{n+1} s(n+1, i)\, S(i,k)\, u^{n+1-i}\, v^{i-k} \right) \otimes b_{k-1}^K . \tag{3.3.13}$$

3.4 The K-theory Hurewicz Homomorphism

In this section, we intend to compute the images of the coefficients $f_{n,l}^b$ of the universal formal group law under the Hurewicz homorphism $h_*\colon MU_* \to K_*(MU)$. According to (1.2.13), these coefficients are related to the coefficients $F_{n,l}^{\phi}$ of the universal Hurwitz group law by $F_{n,l}^{\phi} = n!l!f_{n,l}^b$. An algorithm for this computation appears in [1], but no closed formula is given. All the set partitions considered in this section lie in the lattice Π_{n+l}, with meet denoted by $\wedge$. According to 3.1.4, we have

$$F^{\phi}_{n,l} = \sum_{\sigma \le \sigma_0} \mu^{\phi}(\widehat{0}, \sigma)\, \zeta^{\phi}(\sigma, \widehat{1})\,,$$

where σ_0 is the partition $\{[n], n + [l]\}$ of $[n + l]$. Using (3.3.9) and (3.3.10), we have

$$\begin{aligned} h_*(F^{\phi}_{n,l}) &= \sum_{\pi \le \sigma \le \omega,\ \sigma \le \sigma_0} \mu^{\phi^K}(\widehat{0}, \pi)\, \mu^{\kappa}(\pi, \sigma)\, \zeta^{\kappa}(\sigma, \omega)\, \zeta^{\phi^K}(\omega, \widehat{1}) \\ &= \sum_{\pi \le \sigma \le \sigma_0 \wedge \omega} \mu^{\phi^K}(\widehat{0}, \pi)\, \mu^{\kappa}(\pi, \sigma)\, \zeta^{\kappa}(\sigma, \sigma_0 \wedge \omega)\, \zeta^{\kappa}(\sigma_0 \wedge \omega, \omega)\, \zeta^{\phi^K}(\omega, \widehat{1})\,. \end{aligned}$$

If we sum only over $\pi < \sigma_0 \wedge \omega$ we get 0, since we have the factor $(\mu^{\kappa} * \zeta^{\kappa})(\pi, \sigma_0 \wedge \omega) = 0$. Hence

$$\begin{aligned} h_*(F^{\phi}_{n,l}) &= \sum_{\omega} \mu^{\phi^K}(\widehat{0}, \sigma_0 \wedge \omega)\, \zeta^{\kappa}(\sigma_0 \wedge \omega, \omega)\, \zeta^{\phi^K}(\omega, \widehat{1}) \\ &= \sum_{\pi \le \sigma_0} \mu^{\phi^K}(\widehat{0}, \pi) \left(\sum_{\omega \,:\, \sigma_0 \wedge \omega = \pi} \zeta^{\phi^K}(\omega, \widehat{1})\, u^{|\pi| - |\omega|} \right) \\ &= \sum_{\pi \le \sigma_0} \mu^{\phi^K}(\widehat{0}, \pi) \left(\sum_{k=0}^{\min\{n_\pi, l_\pi\}} k! \binom{n_\pi}{k} \binom{l_\pi}{k} \phi^K_{|\pi|-k-1} u^k \right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{l} s^{\phi^K}(n, i)\, s^{\phi^K}(l, j) \left(\sum_{k=0}^{\min\{i,j\}} k! \binom{i}{k} \binom{j}{k} \phi^K_{i+j-k-1}\, u^k \right)\,, \end{aligned}$$

where n_π and l_π denote the number of blocks of the partition $\pi \le \sigma_0$ contained in $[n]$ and $n + [l]$, respectively; the third equality follows by counting the partitions ω with $\sigma_0 \wedge \omega = \pi$: concentrating on such partitions with precisely k blocks intersecting both $[n]$ and $n + [l]$, there are $\binom{n_\pi}{k}\binom{l_\pi}{k}$ ways of choosing the blocks of π to be amalgamated, and $k!$ ways of matching them. Dividing both sides by $n!l!$, we finally obtain the formula for $h_*(f^b_{n,l})$.

Proposition 3.4.1 *The images of the coefficients of the universal formal group law $f^b_{n,l}$ under the Hurewicz homomorphism h_* are specified by*

$$h_*(f^b_{n,l}) = \sum_{i=1}^{n}\sum_{j=1}^{l} \beta^{bK}_{n,i}\,\beta^{bK}_{l,j} \left(\sum_{k=0}^{\min\{i,j\}} \binom{i+j-k}{k,\; i-k,\; j-k} b^K_{i+j-k-1}\, u^k \right) .$$

3.5 Congruences in MU_*

In this section, we prove some congruences for the Φ_*-Stirling numbers $S^\phi(n,k)$ in $MU_{2(n-k)}$ modulo a prime p. The main tool will be the Hattori-Stong theorem and the periodicity modulo p of the classical Stirling numbers of the second kind, for which there is a nice proof using group actions (see [45]):

$$S(n,k) \equiv S(n-p+1,k) \mod p\,, \quad \text{for } n > p-1\,. \tag{3.5.1}$$

The Hattori-Stong theorem essentially says that the Hurewicz homomorphism $h_*\colon MU_* \to K_*(MU)$ is integrality preserving, that is for all $z \in MU_* \otimes \mathbb{Q}$ we have that $z \in MU_*$ if and only if $(h_* \otimes 1)(z) \in K_*(MU)$. This turns out to be a purely algebraic statement, and such a proof can be found in [8].

Proposition 3.5.2 *We have the following congruences in MU_*:*

$$S^\phi(n,p-1) \equiv \begin{cases} 0 \quad mod\; p & \text{if } n \not\equiv 0 \quad mod\; p-1 \\ \phi_{n-p+1} \quad mod\; p & otherwise, \end{cases} \tag{3.5.3}$$

in $MU_{2(n-p+1)}$;

$$S^\phi(n,p-2) \equiv \begin{cases} 0 \quad mod\; p & \text{if } n \not\equiv 0 \quad mod\; p-1 \\ \phi_{n-p+2} \quad mod\; p & otherwise, \end{cases} \tag{3.5.4}$$

in $MU_{2(n-p+2)}$*, if* $p \geq 3$*;*

$$S^{\phi}(n, p-3) \equiv \begin{cases} 0 \quad mod\ p & if\ n \not\equiv 0, -1, -2 \quad mod\ p-1 \\ \phi_{n-p+3} \quad mod\ p & if\ n \equiv -2 \quad mod\ p-1 \\ 3\phi_{n-p+3} \quad mod\ p & if\ n \equiv -1 \quad mod\ p-1 \\ 3\phi_1\phi_{n-p+2} - \phi_{n-p+3} \quad mod\ p & if\ n \equiv 0 \quad mod\ p-1, \end{cases} \tag{3.5.5}$$

in $MU_{2(n-p+3)}$*, if* $p \geq 5$*.*

Proof. Let us compute the image of $S^{\phi}(n,k)$ under the Hurewicz homomorphism $h_*\colon MU_* \to K_*(MU)$. By (3.3.9) we have

$$\begin{aligned} h_*(S^{\phi}(n,k)) &= \sum_{\pi \leq \sigma \in \Pi_{n,k}} \zeta^{\kappa}(\widehat{0}, \pi)\, \zeta^{\phi^K}(\pi, \sigma) \\ &= \sum_{i=k}^{n} \sum_{\pi \in \Pi_{n,i}} \zeta^{\kappa}(\widehat{0}, \pi) \left(\sum_{\pi \leq \sigma \in \Pi_{n,k}} \zeta^{\phi^K}(\pi, \sigma) \right) \\ &= \sum_{i=k}^{n} S(n,i)\, S^{\phi^K}(i,k)\, u^{n-i}\,. \end{aligned}$$

We will first show that if $i \geq p$ and $k \leq p-1$, then $S^{\phi^K}(i,k) \equiv 0 \mod p$. Consider a partition $\sigma \in \Pi_{i,k}$ with $k \leq p-1$ blocks and type $\tau^{\phi^K}(\sigma) = (\phi_1^K)^{r_2} \ldots (\phi_{j-1}^K)^{r_j}$. If $j \geq p$, then $\tau^{\phi^K}(\sigma)$ is divisible by p in $K_*(MU)$ since ϕ_{p-1}^K is. If $j < p$, then there are

$$\frac{i!}{(2!)^{r_2} \ldots (j!)^{r_j} (k - r_2 - \ldots - r_j)! r_2! \ldots r_j!}$$

partitions in $\Pi_{i,k}$ having the same type as σ; but this number is divisible by p under the above hypothesis.

Now consider the image of ϕ_l under h_* given by (3.3.9). Using (3.5.1), we deduce that

$$h_*(\phi_l) \equiv \begin{cases} u^l \mod p & \text{if } l \equiv 0 \mod p-1 \\ u^l + \phi_1^K u^{l-1} \mod p & \text{if } l \equiv 1 \mod p-1 \\ u^l + 3\phi_1^K u^{l-1} + \phi_2^K u^{l-2} \mod p & \text{if } l \equiv 2 \mod p-1, \end{cases} \tag{3.5.6}$$

in $K_{2l}(MU)$.

According to the above remarks, and using (3.5.1) again, we have the following congruences, implying (3.5.3), (3.5.4), and (3.5.5), respectively:

$$h_*(S^\phi(n,p-1)) \equiv S(n,p-1)u^{n-p+1} \equiv \begin{cases} 0 \mod p & \text{if } n \not\equiv 0 \mod p-1 \\ u^{n-p+1} \mod p & \text{otherwise;} \end{cases}$$

$$\begin{aligned} h_*(S^\phi(n,p-2)) &\equiv S(n,p-2)u^{n-p+2} + S(n,p-1)S(p-1,p-2)\phi_1^K u^{n-p+1} \\ &\equiv \begin{cases} 0 \mod p & \text{if } n \not\equiv 0,-1 \mod p-1 \\ u^{n-p+2} \mod p & \text{if } n \equiv -1 \mod p-1 \\ u^{n-p+2} + \phi_1^K u^{n-p+1} \mod p & \text{if } n \equiv 0 \mod p-1; \end{cases} \end{aligned}$$

$$\begin{aligned} h_*(S^\phi(n,p-3)) &\equiv S(n,p-3)u^{n-p+3} + S(n,p-2)S(p-2,p-3)\phi_1^K u^{n-p+2} \\ &\quad + S(n,p-1)S^{\phi^K}(p-1,p-3)u^{n-p+1} \end{aligned}$$

$$\equiv \begin{cases} 0 \mod p & \text{if } n \not\equiv 0,-1,-2 \\ u^{n-p+3} \mod p & \text{if } n \equiv -2 \\ 3u^{n-p+3} + 3\phi_1^K u^{n-p+2} \mod p & \text{if } n \equiv -1 \\ 2u^{n-p+3} + 3\phi_1^K u^{n-p+2} + (3(\phi_1^K)^2 - \phi_2^K)u^{n-p+1} \mod p & \text{if } n \equiv 0, \end{cases}$$

where all the congruences for n are mod $p-1$. We have used the following facts:

$$S(p-1,p-2) = \binom{p-1}{2} \equiv 1 \mod p\,, \quad S(p-2,p-3) = \binom{p-2}{2} \equiv 3 \mod p,$$

$$S^{\phi^K}(p-1,p-3) = \frac{1}{2}\binom{p-1}{2}\binom{p-3}{2}(\phi_1^K)^2 + \binom{p-1}{3}\phi_2^K \equiv 3(\phi_1^K)^2 - \phi_2^K \mod p$$

in $K_4(MU)$. □

Congruence (3.5.3) was proved in [34] using arguments related to the universal formal group law; it had an essential rôle therein for proving the universal von Staudt theorems. We believe that our technique is more powerful, since it also provides (3.5.4) and (3.5.5), which seem to be new.

3.6 Combinatorial Models for the Dual of the Steenrod Algebra

In this section, we discuss some connections between the Hopf algebras Φ_* and H_*, and the dual of the Steenrod algebra. Some of the results are known; nevertheless, we believe that the combinatorial proofs presented here provide new insights.

Consider a prime p, and let $\phi_{(n)} := \phi_{p^n-1}$. In [39] it is shown that the subalgebra $\mathbb{Z}/(p)[\phi_{(1)}, \phi_{(2)}, \ldots]$ of the mod p Faà di Bruno Hopf algebra $\Phi_* \otimes \mathbb{Z}/(p)$ is actually a sub-Hopf algebra, isomorphic to the polynomial part of the dual of the mod p Steenrod algebra; the proof is based on number-theoretical arguments. We present here an alternative proof, which is purely combinatorial and was inspired from [45].

Consider the cyclic group C_{p^n} acting on $[p^n]$, and hence on the partition lattice Π_{p^n}. We want to determine the partitions fixed by every element of C_{p^n}. If σ is such a partition, then C_{p^n} acts on its blocks. Let g be the cycle $(1, 2, \ldots, p^n)$, let B be the block of σ containing 1, and let $\langle g^{p^k} \rangle$ be the stabiliser of B. From $g^{p^k} B = B$, we deduce that $\{1, 1+p^k, \ldots, 1+(p^{n-k}-1)p^k\} \subseteq B$. Furthermore, the sets $\{i, i+p^k, \ldots, i+(p^{n-k}-1)p^k\}$, for $i = 1, 2, \ldots, p^k$, all lie in different blocks of σ, whence they are precisely the blocks of σ. In consequence, we have $n+1$ partitions fixed by every element of C_{p^n}, namely one for each $k = 0, 1, \ldots, n$. The orbit of every partition which is not of the above type has p^i elements, where $i > 0$. Finally, since all partitions in the same orbit have the same type, (1.7.3) becomes

$$\delta(\phi_{(n)}) = \sum_{k=0}^{n} \phi_{(n-k)}^{p^k} \otimes \phi_{(k)} \qquad \text{in } \mathbb{Z}/(p)[\phi_{(1)}, \phi_{(2)}, \ldots]\,. \tag{3.6.1}$$

Let us now consider the Hopf algebra $P_* := H_* \otimes \mathbb{Z}/(p)$, and the ideal J_* generated by the elements b_i, $i \neq p^n - 1$. Writing $b_{(n)}$ for b_{p^n-1}, we clearly have an isomorphism of algebras $P_*/J_* \cong \mathbb{Z}/(p)[b_{(1)}, b_{(2)}, \ldots]$. We intend to show that J_* is also a coideal (and hence a Hopf ideal), and that in the Hopf algebra $\mathbb{Z}/(p)[b_{(1)}, b_{(2)}, \ldots]$ we have

$$\delta(b_{(n)}) = \sum_{k=0}^{n} b_{(n-k)}^{p^k} \otimes b_{(k)} \,. \tag{3.6.2}$$

We will prove these statements together. Consider arbitrary integers $m, k \geq 1$ such that $m \geq p^k$. A partition in $\widetilde{\Pi}_{m,p^k}$ can be represented by a p^k-tuple $(i_1, \ldots, i_{p^k})$ with $i_j \geq 1$ and $\sum_{j=1}^{p^k} i_j = m$. The cyclic group C_{p^k} acts on $\widetilde{\Pi}_{m,p^k}$ in the obvious way, and there is at most one partition fixed by each element of C_{p^k}, namely the one with equal block sizes, if p^k divides m; the sizes of all the other orbits are non-zero powers of p. This shows that if k_0 is highest power of p dividing m, then

$$\delta(b_{m-1}) \in P_* \otimes J_* + \sum_{k=0}^{k_0} b_{m/p^k-1}^{p^k} \otimes b_{(k)} \,,$$

whence the desired statements follow.

In consequence, we have proved:

Proposition 3.6.3 *The polynomial part of the dual of the mod p Steenrod algebra is isomorphic to the sub-Hopf algebra $\mathbb{Z}/(p)[\phi_{(1)}, \phi_{(2)}, \ldots]$ of the mod p Faà di Bruno Hopf algebra $\Phi_* \otimes \mathbb{Z}/(p)$, as well as to the quotient of $H_* \otimes \mathbb{Z}/(p)$ (that is the dual of the Landweber-Novikov algebra tensored with $\mathbb{Z}/(p)$) by the Hopf ideal J_*.*

Let R_* be an evenly graded commutative ring of characteristic p. Since P_*/J_* is a Hopf algebra, then the set $\mathrm{Alg}^*(P_*/J_*, R_*)$ is a group under convolution. We can now derive an analogue of Theorem 1.7.6 (2).

Proposition 3.6.4 *The set of formal power series in $R^*[[X]]$ of the form*

$$r(X) = \sum_{k\geq 1} r_{p^k-1} X^{p^k}$$

(that is p-typical power series) form a group under substitution. There is an anti-isomorphism from $\mathrm{Alg}^*(\mathbb{Z}/(p)[b_{(1)}, b_{(2)}, \ldots], R_*)$ *to this group, specified by* $\zeta^r \mapsto r(X)$.

In consequence, Lagrange inversion for p-typical power series in $R^*[[X]]$ is equivalent to computing the antipode of $\mathrm{Alg}^*(\mathbb{Z}/(p)[b_{(1)}, b_{(2)}, \ldots], R_*)$, which is much easier than computing the antipode of $\mathrm{Alg}^*(H_*, R_*)$.

Chapter 4

Hopf Algebras of Set Systems

In this chapter we construct and study several Hopf algebras/algebroids of set systems which map onto the Hopf algebras/algebroids presented in Chapter 1 via the polynomial/symmetric function invariants defined in §1.8 and Chapter 2. All our constructions concern the universal cases, namely the rings of scalars Φ_* and H_*, with corresponding umbras ϕ and b. Purely as a matter of algebra, our constructions may be carried over to the setting corresponding to any other ring and umbra of the types discussed in §1.1. A notational consequence also deserves comment; since the rings Φ_* and H_* are both torsion free, we choose to rewrite divided powers such as $\phi(D)_{(n)}$ and $x_{(n)}$ in the more explicit forms $\phi(D)^n/n!$ and $x^n/n!$, respectively.

4.1 Cocommutative Hopf Algebras of Set Systems

Consider the free Φ_*-module $S_* := \Phi_*\langle\mathfrak{S}\rangle$ spanned by the set $\mathfrak{S}$ defined in §1.5. In this section we define several graded Hopf algebra structures on S_*, following the general method in [47] (which was summarised in §1.7) for constructing the *incidence Hopf algebra* of a *hereditary family of posets* with a *Hopf relation*. The resulting Hopf algebras map onto $\Phi_*[x]$ in a variety of ways, and so provide combinatorial generalisations of the algebraic phenomena associated with the universal Hurwitz group law.

As pointed out in §1.5, we shall not attempt to distinguish notationally between a set system and its isomorphism class, since in those cases where it matters, we have taken care to ensure that the context is clear.

Let $\mathbb{P}$ be the hereditary family consisting of all finite products of intervals from the posets $(\mathcal{K}_{V(\mathcal{S})}, \subseteq)$, for arbitrary set systems $\mathcal{S}$. Intervals corresponding to different set systems are considered distinct, even if they consist of identical sets, so we index by $\mathcal{S}$ the elements of $\mathcal{K}_{V(\mathcal{S})}$ determining an interval. We define a map from $\mathbb{P}$ to $\mathfrak{S}$ as follows: given set systems $\mathcal{S}_i$ for $i \in [n]$ and intervals $[U_{\mathcal{S}_i}, W_{\mathcal{S}_i}]$ in $\mathcal{K}_{V(\mathcal{S}_i)}$, we map $[U_{\mathcal{S}_1}, W_{\mathcal{S}_1}] \times \ldots \times [U_{\mathcal{S}_n}, W_{\mathcal{S}_n}]$ to the isomorphism class of the set system $\mathcal{S}_1|(W_{\mathcal{S}_1} \setminus U_{\mathcal{S}_1}) \cdot \ldots \cdot \mathcal{S}_n|(W_{\mathcal{S}_n} \setminus U_{\mathcal{S}_n})$. Let $\sim$ be the kernel of this map. The proof of the order compatibility of $\sim$ is mainly based on the fact that disjoint union interacts with restriction such that $(\mathcal{S}_1 \cdot \mathcal{S}_2)|(U_1 \sqcup U_2) = (\mathcal{S}_1|U_1) \cdot (\mathcal{S}_2|U_2)$, where $U_i \subseteq V(\mathcal{S}_i)$ for $i = 1, 2$. This proof can be divided into two steps, as shown below:

$$[U_{\mathcal{S}_1}, W_{\mathcal{S}_1}] \times [U_{\mathcal{S}_2}, W_{\mathcal{S}_2}] \sim [(U_{\mathcal{S}_1} \sqcup U_{\mathcal{S}_2})_{\mathcal{S}_1 \cdot \mathcal{S}_2}, (W_{\mathcal{S}_1} \sqcup W_{\mathcal{S}_2})_{\mathcal{S}_1 \cdot \mathcal{S}_2}],$$

$$(A_{\mathcal{S}_1}, A_{\mathcal{S}_2}) \mapsto (A_{\mathcal{S}_1} \sqcup A_{\mathcal{S}_2})_{\mathcal{S}_1 \cdot \mathcal{S}_2},$$

$$[U_{\mathcal{S}}, W_{\mathcal{S}}] \sim [\emptyset_{\mathcal{S}|(W_{\mathcal{S}} \setminus U_{\mathcal{S}})}, (W_{\mathcal{S}} \setminus U_{\mathcal{S}})_{\mathcal{S}|(W_{\mathcal{S}} \setminus U_{\mathcal{S}})}], \qquad A_{\mathcal{S}} \mapsto (A_{\mathcal{S}} \setminus U_{\mathcal{S}})_{\mathcal{S}|(W_{\mathcal{S}} \setminus U_{\mathcal{S}})};$$

the two maps indicated are the corresponding order compatible bijections. Since isomorphism of set systems is a congruence with respect to disjoint union, the relation $\sim$ is a Hopf relation.

Let $H(\mathbb{P})$ be the Φ_*-incidence Hopf algebra of the family $\mathbb{P}$ modulo the Hopf relation $\sim$. The bijection from $\mathbb{P}/\sim$ to $\mathfrak{S}$ induced by the map above can be extended by linearity to a bijection from $H(\mathbb{P})$ to S_*. We use this bijection to transfer the Hopf algebra structure of $H(\mathbb{P})$ to S_*. Comultiplication in S_* is specified by

$$\delta(\mathcal{S}) := \sum_{W \subseteq V(\mathcal{S})} \mathcal{S}|W \otimes \mathcal{S}|\overline{W}\,,$$

where $\overline{W} := V(\mathcal{S}) \setminus W$. The counit is determined by

$$\varepsilon(\mathcal{S}) := \begin{cases} 1 & \text{if } \mathcal{S} = \{\emptyset\} \\ 0 & \text{otherwise}\,. \end{cases}$$

Multiplication is disjoint union, and the unit is the map η specified by $\eta(1) := \{\emptyset\}$. The antipode is determined by

$$\gamma(\mathcal{S}) = \sum_{\sigma \in \Pi(V(\mathcal{S}))} (-1)^{|\sigma|}\, |\sigma|!\, \mathcal{S}|\sigma\,. \tag{4.1.1}$$

Clearly, the Hopf algebra S_* is commutative and cocommutative. It is also graded, by setting the degree of $\mathcal{S}$ equal to $|V(\mathcal{S})|$, and it has finite type. The indecomposables in S_* are the isomorphism classes of connected set systems, so that S_* is isomorphic, as an algebra, to the polynomial algebra $\Phi_*[\mathfrak{S}_\circ]$. Since each poset in the family $\mathbb{P}$ is a Boolean algebra, we can apply Theorem 10.2 in [47] to obtain further information about the structure of S_*, as in Theorem 4.1.2 below. For this purpose, we recall from [47] the projection p of S_* onto its primitive elements, specified by

$$p(\mathcal{S}) := \sum_{\sigma \in \Pi(V(\mathcal{S}))} (-1)^{|\sigma|-1}\, (|\sigma|-1)!\, \mathcal{S}|\sigma\,.$$

Theorem 4.1.2 *The Hopf algebra S_* is isomorphic to the polynomial Hopf algebra $\Phi_*[p(\mathfrak{S}_\circ)]$, having primitive indeterminates.*

We can define similar Hopf algebra structures on S_* by basing the multiplication on $\vee$ or $\odot$, rather than disjoint union; this is possible because both operations interact with restriction in similar fashion to disjoint union, and isomorphism of set systems is still a congruence. The resulting Hopf algebras are likewise polynomial, and have primitive indeterminates. Complementation of set systems induces a Hopf algebra isomorphism between $(S_*, \cdot)$ and $(S_*, \odot)$. In what follows, by S_* we always mean $(S_*, \cdot)$.

We now consider the graded dual S^* of the coalgebra S_*, which has pseudobasis $\{D_{\mathcal{S}} : \mathcal{S} \in \mathfrak{S}\}$ dual to $\mathfrak{S}$. As a left-invariant operator, $D_{\mathcal{S}}$ acts on S_* according to the rule

$$D_{\mathcal{S}}\, \mathcal{T} = \sum_{W} \mathcal{T}|\overline{W}\,,$$

where $\overline{W} := V(\mathcal{T}) \setminus W$, and the summation ranges over those subsets of $V(\mathcal{T})$ for which $\mathcal{T}|W = \mathcal{S}$. The multiplication in S^* is given by

$$D_{\mathcal{S}_1}\, D_{\mathcal{S}_2} = \sum_{\mathcal{S} \in \mathfrak{S}} (\mathcal{S}; \mathcal{S}_1, \mathcal{S}_2)\, D_{\mathcal{S}}\,,$$

where $(\mathcal{S}; \mathcal{S}_1, \mathcal{S}_2)$ denotes the coefficient of $\mathcal{S}_1 \otimes \mathcal{S}_2$ in $\delta(\mathcal{S})$. We now utilise Theorem 4.1.2 to view S_* as the polynomial algebra $\Phi_*[p(\mathfrak{S}_\circ)]$ with primitive indeterminates, and write $D_{p(\mathcal{S})}$ for the partial differentiation operator with respect to the variable $p(\mathcal{S})$, where $\mathcal{S}$ lies in $\mathfrak{S}_\circ$. It is now easy to see that S^* is isomorphic to the Φ_*-algebra of Hurwitz series in the variables $D_{p(\mathcal{S})}$.

Perceptive readers may have noticed that, thus far, we could have restricted our scalars to $\mathbb{Z}$. We now impose deletion/contraction relations on S_* which involve the scalars Φ_* in an essential manner. Let R_* be the graded submodule of S_* spanned by all elements of the form

$$\mathcal{S} - \mathcal{S} \setminus U - \phi_{|U|-1}\, \mathcal{S}//U\,, \tag{4.1.3}$$

where U is a maximal element of the poset $(\mathcal{S}, \subseteq)$ with $|U| > 1$, and let Q_* be the graded submodule spanned by all elements of the form

$$\mathcal{S} - \mathcal{S} \setminus U - \mu^{\phi}_{\Pi(\mathcal{S}|U)}(\widehat{0}, \{U\})\, \mathcal{S}//U\,,$$

with U as before. Then writing ρ^ϕ, c^ϕ, and χ^ϕ for the Φ_*-linear maps from S_* to $\Phi_*[x]$ obtained by respectively assigning $\rho^\phi(\mathcal{S};x)$, $c^\phi(\mathcal{S};x)$, and $\chi^\phi(\mathcal{S};x)$ to the isomorphism class of the set system $\mathcal{S}$, it is not difficult to check, using Theorem 2.3.1, that $\ker\rho^\phi = R_*$ and $\ker c^\phi = Q_*$. Our first realisation result for polynomial invariants can now be proven.

Proposition 4.1.4 *The maps ρ^ϕ, $c^\phi\colon S_* \to \Phi_*[x]$ and $\chi^\phi\colon (S_*, \odot) \to \Phi_*[x]$ are surjective maps of graded Hopf algebras.*

Proof. Surjectivity follows from (1.8.4), whilst ρ^ϕ and c^ϕ are algebra maps by Proposition 1.8.5. It therefore suffices to prove that they are coalgebra maps as well, since any bialgebra map of Hopf algebras is a Hopf algebra map. We will show that R_* and Q_* are coideals in the corresponding coalgebras, concentrating on the former.

Obviously, $\varepsilon(R_*) = \{0\}$. Now consider an element of the form (4.1.3), and $W \subseteq V(\mathcal{S})$. If $U \subseteq W$, then $(\mathcal{S}\backslash U)|\overline{W} = \mathcal{S}|\overline{W}$. If $W \subseteq \overline{U}$, then $(\mathcal{S}\backslash U)|W = \mathcal{S}|W$. If none of the above hold, then $(\mathcal{S}\setminus U)|W = \mathcal{S}|W$ and $(\mathcal{S}\setminus U)|\overline{W} = \mathcal{S}|\overline{W}$. Finally, since deletion commutes with restriction, we have

$$\delta(\mathcal{S} - \mathcal{S}\setminus U) = \sum_{W\supseteq U} (\mathcal{S}|W - (\mathcal{S}|W)\setminus U) \otimes \mathcal{S}|\overline{W} + \sum_{W\subseteq\overline{U}} \mathcal{S}|W \otimes (\mathcal{S}|\overline{W} - (\mathcal{S}|\overline{W})\setminus U)\,.$$

Since U is a maximal element of $(\mathcal{S},\subseteq)$, then $\mathcal{S}//U$ is isomorphic to $\mathcal{N}_1 \cdot \mathcal{S}|\overline{U}$ and

$$\begin{aligned}\delta(\mathcal{S}//U) &= (\mathcal{N}_1 \otimes \{\emptyset\} + \{\emptyset\} \otimes \mathcal{N}_1) \cdot \left(\sum_{W\subseteq\overline{U}} \mathcal{S}|W \otimes \mathcal{S}|(\overline{U}\setminus W)\right) \\ &= \sum_{W\supseteq U} (\mathcal{S}|W)//U \otimes \mathcal{S}|\overline{W} + \sum_{W\subseteq\overline{U}} \mathcal{S}|W \otimes (\mathcal{S}|\overline{W})//U\,.\end{aligned}$$

These relations show that $\delta(R_*) \subseteq R_* \otimes S_* + S_* \otimes R_*$, whence R_* is a coideal.

The proof for Q_* is similar, and the result for χ^ϕ follows from that for c^ϕ, using Proposition 2.4.1. □

Now consider the algebra $Sym_*^{\mathbb{Z}}(x)$ of symmetric functions with integer coefficients, with the Hopf algebra structure described in §1.10. We give here an alternative description of the comultiplication (cf. [15]), which will be used in the proofs of Proposition 4.1.5 and Theorem 4.2.4. Given $A \subseteq \mathbb{N}$ and $t(x) \in Sym_*^{\mathbb{Z}}(x)$, let $t(x_A)$ denote the symmetric function obtained from $t(x)$ by substituting 0 for x_i whenever $i \notin A$. If $t(x)$ has degree n, then its image under comultiplication can be computed by examining the image of $t(x_{[2n]})$ under the natural isomorphism $\mathbb{Z}[x_1, \ldots, x_{2n}] \cong \mathbb{Z}[x_1, \ldots, x_n] \otimes \mathbb{Z}[x_{n+1}, \ldots, x_{2n}]$. This observation enables us to establish an analogous result to Proposition 4.1.4 for the $\mathbb{Z}$-linear map $X\colon \mathbb{Z}\langle\mathfrak{S}\rangle \to Sym_*^{\mathbb{Z}}(x)$ specified by $\mathcal{S} \mapsto X(\mathcal{S};x)$, where $X(\mathcal{S};x)$ is defined in §1.8.

Proposition 4.1.5 *The map* $X\colon (\mathbb{Z}\langle\mathfrak{S}\rangle, \vee) \to Sym_*^{\mathbb{Z}}(x)$ *is a map of graded Hopf algebras.*

Proof. According to Proposition 1.8.6, we have only to prove that X is a coalgebra map. Consider a set system $\mathcal{S}$ with d vertices. It suffices to show that

$$X(\mathcal{S};x_{[2d]}) = \sum_{W \subseteq V(\mathcal{S})} X(\mathcal{S}|W;x_{[d]})\, X(\mathcal{S}|\overline{W};x_{d+[d]})\,.$$

Now there is an obvious bijection from $\Xi_{[2d]}(\mathcal{S})$ to

$$\bigcup_{W \subseteq V(\mathcal{S})} \Xi_{[d]}(\mathcal{S}|W) \times \Xi_{d+[d]}(\mathcal{S}|\overline{W})\,,$$

namely $f \mapsto (f', f'')$, where $f' = f|f^{-1}([d])$ and $f'' = f|f^{-1}(d+[d])$; moreover, we clearly have $x^f = x^{f'}x^{f''}$, from which the formula follows. □

Inspection of the comultiplication in S_* reveals that the sequences $(\mathcal{N}_n)$ and $(\mathcal{K}_n)$ are binomial in the sense of (1.1.8), and according to (1.8.4) they map (as they must) to familiar binomial sequences in $\Phi_*[x]$ under ρ^ϕ, c^ϕ, and χ^ϕ. It is therefore of interest to determine *all* binomial sequences in S_*, and especially those whose elements lie entirely within the generating set $\mathfrak{S}$. For this purpose, we define the set system

$$\mathcal{K}_n(A) := \{U \subseteq [n] \,:\, |U| \in A\}$$

for every $n \in \mathbb{N} \cup \{0\}$, and every set $A \subseteq \mathbb{N} \cup \{0\}$ containing 0 and 1. Obviously, $\mathcal{K}_n(\{0,1\}) = \mathcal{N}_n$, and $\mathcal{K}_n(\mathbb{N} \cup \{0\}) = \mathcal{K}_n$.

Proposition 4.1.6 *The only binomial sequences whose elements lie in* $\mathfrak{S}$ *are those of the form* $(\mathcal{K}_n(A))$ *for* $\{0,1\} \subseteq A \subseteq \mathbb{N} \cup \{0\}$.

PROOF. Clearly, all sequences $(\mathcal{K}_n(A))$ are binomial. Now let $(\mathcal{B}_n)$ be an arbitrary binomial sequence whose elements lie in $\mathfrak{S}$. Assume for induction that there is a set $A_n \subseteq [n] \cup \{0\}$ such that $\mathcal{B}_i = \mathcal{K}_i(A_n)$ for all $0 \leq i \leq n$; this certainly holds for $n = 0$. Then define

$$A_{n+1} := \begin{cases} A_n \cup \{n+1\} & \text{if } V(\mathcal{B}_{n+1}) \in \mathcal{B}_{n+1} \\ A_n & \text{otherwise}. \end{cases}$$

Choose an arbitrary subset U of $V(\mathcal{B}_{n+1})$ with $1 \leq |U| \leq n$, and $x \in V(\mathcal{B}_{n+1}) \setminus U$. The binomial property implies that $\mathcal{B}_{n+1}|(V(\mathcal{B}_{n+1}) \setminus \{x\})$ is isomorphic to $\mathcal{B}_n$. It follows by the inductive hypothesis that $U \in \mathcal{B}_{n+1}$ if and only if $|U| \in A_n \subseteq A_{n+1}$. We conclude the proof by setting $A := \bigcup_{n \geq 0} A_n$. □

It is now time to embellish our Hopf algebras with delta operators.

Let $\theta \colon S_* \to \Phi_*[x]$ be a surjective map of graded coalgebras. Then its transpose $\theta^* \colon \Phi^*\{\{D\}\} \to S^*$ is an injective algebra map. Hence θ^* induces an algebra isomorphism from $\Phi^*\{\{D\}\}$ to the subalgebra $\Phi^*\{\{D^\theta\}\}$ of S^*, where $D^\theta := \theta^*(D)$. Therefore, given a delta operator $\alpha(D) \in \Phi^1\{\{D\}\}$, we may write the delta operator $\theta^*(\alpha(D))$ as $\alpha(D^\theta)$.

Proposition 4.1.7 *The map* $\theta \colon (S_*, \alpha(D^\theta)) \to (\Phi_*[x], \alpha(D))$ *is a map of coalgebras with delta operator.*

PROOF. The fact that $\alpha(D)\circ\theta = \theta\circ\alpha(D^\theta)$ follows immediately from (1.1.4) by setting $p=\theta$, $f=\langle\alpha(D)\mid\cdot\rangle$. □

Clearly, we can take θ to be any one of the maps ρ^ϕ, χ^ϕ or c^ϕ, and combine the results of Proposition 4.1.7 and Proposition 4.1.4. By way of example, we record that

$$D^\rho\mathcal{S} = \sum_{W\in\mathcal{S}\setminus\{\emptyset\}} \phi_{|W|-1}\,\mathcal{S}|\overline{W}\,, \qquad D^c\mathcal{S} = \sum_{W\in\mathcal{S}\setminus\{\emptyset\}} \mu^\phi_{\Pi(\mathcal{S}|W)}(\widehat{0},\{W\})\,\mathcal{S}|\overline{W}\,,$$

$$\text{and} \qquad \phi(D^c)\,\mathcal{S} = \sum_{W\in\overline{\mathcal{S}}\setminus\{\emptyset\}} \nu^\phi_{\mathcal{S}|W}(\{W\})\,\mathcal{S}|\overline{W} \qquad \text{(by Proposition 2.4.1)}$$

in S_*. In particular, we have

$$D^\rho\mathcal{N}_n = D^c\mathcal{N}_n = n\,\mathcal{N}_{n-1} \qquad \text{and} \qquad \phi(D^c)\,\mathcal{K}_n = n\,\mathcal{K}_{n-1}\,. \tag{4.1.8}$$

Let Δ be an arbitrary delta operator on S_*. Let T_* be a subcoalgebra of S_* such that $S_* = T_*\oplus\Phi^*\{\{D_{\mathcal{N}_1}\}\}^\perp$ (see [53]). Then Δ acts on T_* non-trivially, and $\Phi^*\{\{\Delta\}\}$ may be viewed as its dual. In this context, it is consistent to refer to the basis of T_* dual to the pseudobasis $\Delta^n/n!$ as the associated sequence of Δ in T_*; such sequences are obviously binomial. Of special interest are the subcoalgebras $C(A)_*$ spanned by the binomial sequences $(\mathcal{K}_n(A))$ of Proposition 4.1.6. Note that $(\mathcal{K}_n(A))$ is the associated sequence of $D_{\mathcal{N}_1}$ in $C(A)_*$, but that it is also the associated sequence of other delta operators, as exemplified by 4.1.8. There is an isomorphism of coalgebras with delta operator between $(C(A)_*,\Delta)$ and $(\Phi_*[x],\alpha(D))$, specified by $\mathcal{K}_n(A)\mapsto x^n$, where $\alpha_{n-1}=\langle\Delta\mid\mathcal{K}_n(A)\rangle$. The determination of the associated sequences of Δ therefore reduces to the classical case.

We conclude this section by explaining how the identities (1.2.11) and (1.2.12), which hold in $\Phi_*[x]$, can be lifted to S_*.

Proposition 4.1.9 *Let α be an arbitrary umbra in Φ_*, and Δ a delta operator on S_* which is a derivation; then the following identities hold in S_*:*

$$\alpha(\Delta)\,(\mathcal{S}_1\cdot\mathcal{S}_2)=\sum_{i,j\geq 0}F^{\alpha}_{i,j}\,(\alpha(\Delta)^i/i!\,\mathcal{S}_1)\cdot(\alpha(\Delta)^j/j!\,\mathcal{S}_2) \tag{4.1.10}$$

$$\alpha(\Delta)\,\gamma(\mathcal{S})=\sum_{k\geq 1}\iota^{\alpha}_k\,\gamma(\alpha(\Delta)^k/k!\,\mathcal{S})\,. \tag{4.1.11}$$

Given any map $\theta\colon S_*\to\Phi_*[x]$ of graded Hopf algebras, D^θ is a derivation, so the above identities hold for it.

PROOF. Let p denote the multiplication in S_*. We have that $\Delta\circ p=p\circ(\Delta\otimes I+I\otimes\Delta)$, whence, by Proposition 1.2.5,

$$\alpha(\Delta)\circ p=p\circ\alpha(\Delta\otimes I+I\otimes\Delta)=p\circ F^{\alpha}(\Delta\otimes I,I\otimes\Delta)\,.$$

This proves (4.1.10). On the other hand, we know from (1.1.4) that $\Delta\circ\gamma=\gamma\circ\Gamma_{\langle\Delta\circ\gamma\,|\,\cdot\rangle}$. For all $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$ in $\mathfrak{S}$ with $\mathcal{S}_1,\mathcal{S}_2\neq\{\emptyset\}$, we have that

$$\gamma(\mathcal{S})=-\mathcal{S}+\text{decomposables in }S_*\qquad(\text{see (4.1.1)})\,,$$

and

$$\langle\Delta\,|\,\mathcal{S}_1\cdot\mathcal{S}_2\rangle=\langle\Delta\otimes I+I\otimes\Delta\,|\,\mathcal{S}_1\otimes\mathcal{S}_2\rangle=0\,.$$

This implies $\langle\Delta\circ\gamma|\cdot\rangle=\langle-\Delta|\cdot\rangle$, whence the linear operator corresponding $\langle\Delta\circ\gamma|\cdot\rangle$ is $-\Delta$, and $\Delta\circ\gamma=-\gamma\circ\Delta$. Using Proposition 1.2.5 again, we immediately obtain

$$\alpha(\Delta)\circ\gamma=\gamma\circ\alpha(-\Delta)=\gamma\circ\iota^{\alpha}(\alpha(\Delta))\,,$$

which proves (4.1.11).

If θ is the map specified above, then, by using (1.1.4) once again, we have that $D^\theta\circ p=p\circ\Gamma_{\langle D^\theta\circ p\,|\,\cdot\rangle}$. For all $\mathcal{S}_1$ and $\mathcal{S}_2$ in $\mathfrak{S}$, we have

$$\begin{aligned}\langle D^\theta\circ p\,|\,\mathcal{S}_1\otimes\mathcal{S}_2\rangle&=\langle D\,|\,\theta(\mathcal{S}_1\cdot\mathcal{S}_2)\rangle=\langle D\,|\,\theta(\mathcal{S}_1)\,\theta(\mathcal{S}_2)\rangle\\&=\langle D\otimes I+I\otimes D\,|\,\theta(\mathcal{S}_1)\otimes\theta(\mathcal{S}_2)\rangle\\&=\langle D^\theta\otimes I+I\otimes D^\theta\,|\,\mathcal{S}_1\otimes\mathcal{S}_2\rangle\,.\end{aligned}$$

In consequence, the operator corresponding to $\langle D^\theta \circ p \,|\, \cdot\rangle$ is $D^\theta \otimes I + I \otimes D^\theta$, and $D^\theta \circ p = p \circ (D^\theta \otimes I + I \otimes D^\theta)$. □

We may take $\theta = \rho^\phi$ and $\theta = c^\phi$ in Proposition 4.1.9 to obtain identities concerning the interaction of $\alpha(D^\rho)$ and $\alpha(D^c)$ with the multiplication and the antipode in S_*. Analogous identities hold for $\alpha(D^\chi)$ in $(S_*, \odot)$.

4.2 Cocommutative Hopf Algebras of Set Systems with Automorphism Group

In this section, we define cocommutative Hopf algebra structures on the free modules $H_*\langle\mathfrak{C}\rangle$ and $L_*\langle\mathfrak{P}\rangle$ spanned by the sets $\mathfrak{C}$ and $\mathfrak{P}$ defined in §1.6; we also study certain quotients of these Hopf algebras.

As pointed out in §1.6, we shall not attempt to distinguish notationally between a set system with automorphism group and its isomorphism class, since in those cases where it matters, we have taken care to ensure that the context is clear.

The obvious idea for constructing a Hopf algebra structure on $H_*\langle\mathfrak{C}\rangle$ would be to extend the procedure in the previous section. Thus, we would have to consider the hereditary family $\mathbb{P}$ consisting of all finite products of intervals in the posets $(\mathcal{K}_{V(\mathcal{S})}/G, \subseteq)$, for set systems with automorphism group $(\mathcal{S}, G)$ whose isomorphism classes lie in $\mathfrak{C}$. However, the kernel of the map from $\mathbb{P}$ to $\mathfrak{C}$ extending the map defined in the previous section is not an order compatible relation. This is the reason for which we adopt a direct approach, defining the comultiplication in $H_*\langle\mathfrak{C}\rangle$ by

$$\delta(\mathcal{S}, G) = \sum_{W \in \mathcal{T}} (\mathcal{S}, G)|W \otimes (\mathcal{S}, G)|\overline{W}\,,$$

where $\mathcal{T}$ is an arbitrary transversal of $\mathcal{K}_{V(\mathcal{S})}/G$. The counit is specified by

$$\varepsilon(\mathcal{S},G)=\begin{cases}1 & \text{if } (\mathcal{S},G)=(\{\emptyset\},\{1\})\\ 0 & \text{otherwise}\,.\end{cases}$$

Multiplication is induced by disjoint union, as defined by (1.6.2), and the unit is the map η specified by $\eta(1)=(\{\emptyset\},\{1\})$. It is not difficult to check that the above maps define a Hopf algebra structure on $H_*\langle\mathfrak{C}\rangle$. In particular, coassociativity follows by observing that

$$\begin{aligned}|G|\,\delta(\mathcal{S},G)&=\sum_{W\subseteq V(\mathcal{S})}|G_W|\,(\mathcal{S},G)|W\otimes(\mathcal{S},G)|\overline{W} \qquad (4.2.1)\\ &=\sum_{W\subseteq V(\mathcal{S})}\big(|G|W|\,(\mathcal{S},G)|W\big)\otimes\big(|G|\overline{W}|\,(\mathcal{S},G)|\overline{W}\big)\,;\end{aligned}$$

here we have used the standard fact that $|G(W)|=|G|/|G_W|$, as well as the fact that G is cycle-closed, which implies $G_W\cong G|W\times G|\overline{W}$. We can also use (4.2.1) to prove that the antipode of $H_*\langle\mathfrak{C}\rangle$ is specified by

$$|G|\,\gamma(\mathcal{S},G)=\sum_{(\sigma,\omega)\in A(\mathcal{K}_{V(\mathcal{S})})}(-1)^{|\sigma|}\prod_{B\in\sigma}\big(|G|B|\,(\mathcal{S},G)|B\big)\,.$$

Using once again the fact that G is cycle-closed, we deduce that

$$\gamma(\mathcal{S},G)=\sum_{(\sigma,\omega)\in\mathcal{T}}(-1)^{|\sigma|}\prod_{B\in\sigma}(\mathcal{S},G)|B\,,$$

where $\mathcal{T}$ is an arbitrary transversal of $A(\mathcal{K}_{V(\mathcal{S})})/G$. Note that considering cycle-closed automorphism groups is essential; indeed, the obvious extensions of the above maps do not define a Hopf algebra structure on $H_*\langle\mathfrak{A}\rangle$. Clearly, the Hopf algebra $H_*\langle\mathfrak{C}\rangle$ is commutative, cocommutative, graded (in a similar way to S_*), and has finite type. We may define similar Hopf algebra structures on $H_*\langle\mathfrak{C}\rangle$ by basing the multiplication on $\vee$ or $\odot$, rather than disjoint union. We can also replace the ring H_* of scalars with other rings, such as $\mathbb{Z}$.

We would now like to factor the Hopf algebra $H_*\langle\mathfrak{C}\rangle$ by relations of the form $\mathcal{S} - |G|\,(\mathcal{S}, G)$. Let I_* be the graded ideal of $H_*\langle\mathfrak{C}\rangle$ generated by the set

$$\{\mathcal{S} - |G|\,(\mathcal{S}, G) \;:\; (\mathcal{S}, G) \in \mathfrak{C}\}\,. \tag{4.2.2}$$

Proposition 4.2.3 *The ideal I_* is a Hopf ideal.*

PROOF. According to (4.2.1), we have

$$\delta(\mathcal{S} - |G|\,(\mathcal{S}, G)) = \sum_{W \subseteq V(\mathcal{S})} \left(\mathcal{S}|W - |G|W|\,(\mathcal{S}|W, G|W)\right) \otimes \mathcal{S}|\overline{W}$$
$$+ \sum_{W \subseteq V(\mathcal{S})} |G|W|\,(\mathcal{S}, G)|W \otimes \left(\mathcal{S}|\overline{W} - |G|\overline{W}|\,(\mathcal{S}|\overline{W}, G|\overline{W})\right),$$

which lies in $H_*\langle\mathfrak{C}\rangle \otimes I_* + I_* \otimes H_*\langle\mathfrak{C}\rangle$. □

According to the above result, we have the graded Hopf algebra $H_*\langle\mathfrak{C}\rangle/I_*$, which we denote by C_*. We will denote the element $(\mathcal{S}, G) + I_*$ of C_* by $[\mathcal{S}, G]$. Note that $H_*\langle\mathfrak{S}\rangle$ can be regarded as a sub-Hopf algebra of C_*. Let us also note that for a given set system $\mathcal{S}$, the elements $[\mathcal{S}, G]$ in C_* which are not divisible by integers correspond to the maximal cycle-closed subgroups G of the automorphism group of $\mathcal{S}$. In general, there is more than one such subgroup. Indeed, let $\mathcal{S}$ be the set system on 7 vertices corresponding to the projective geometry $PG_2(2)$, with automorphism group $GL_3(2)$; it is not difficult to see that the maximal cycle-closed subgroups of $GL_3(2)$ are its Sylow subgroups.

Let $\rho^b, c^b \colon C_* \to H_*\{x\}$ be the H_*-linear maps specified by

$$[\mathcal{S}, G] \mapsto \rho^\phi(\mathcal{S}; x)/|G| \qquad \text{and} \qquad [\mathcal{S}, G] \mapsto c^\phi(\mathcal{S}; x)/|G|\,,$$

respectively; note that these maps are well-defined, and that we can choose the codomains to be $H_*\{x\}$ by Lemma 2.5.2 and (2.5.6). Similarly, we consider the $\mathbb{Z}$-linear map $X \colon \mathbb{Z}\langle\mathfrak{C}\rangle \to Sym_*^{\mathbb{Z}}(x)$ specified by $(\mathcal{S}, G) \mapsto X(\mathcal{S}, G; x)$.

Theorem 4.2.4 *The maps $\rho^b, c^b \colon C_* \to H_*\{x\}$ and $X \colon (\mathbb{Z}\langle\mathfrak{C}\rangle, \vee) \to Sym_*^{\mathbb{Z}}(x)$ are surjective maps of graded Hopf algebras.*

Proof. Surjectivity follows from (1.8.4) and the fact that $X(\mathcal{N}_n, \Sigma_n; x)$ is the elementary symmetric function Λ_n. According to Propositions 1.8.5 and 1.8.6, it only remains to prove that the given maps preserve comultiplication. For ρ^b and c^b, this follows immediately from Proposition 4.1.4 by regarding $H_*\langle\mathfrak{S}\rangle$ as a sub-Hopf algebra of C_*, and by using the relation $[\mathcal{S}] = |G|\,[\mathcal{S}, G]$, which holds in C_*.

In order to prove that X is a coalgebra map, it suffices to show that

$$X(\mathcal{S}, G; x_{[2d]}) = \sum_{W \in \mathcal{T}} X(\mathcal{S}|W, G|W; x_{[d]})\, X(\mathcal{S}|\overline{W}, G|\overline{W}; x_{d+[d]})\,, \tag{4.2.5}$$

where $\mathcal{T}$ is an arbitrary transversal of $\mathcal{K}_{V(\mathcal{S})}/G$. Let us denote $\Xi_{[d]}(\mathcal{S}|W) \times \Xi_{d+[d]}(\mathcal{S}|\overline{W})$ by Ξ_W, for simplicity. Recall the bijection from $\Xi_{[2d]}(\mathcal{S})$ to $\bigcup_{W\subseteq V(\mathcal{S})} \Xi_W$ constructed in the proof of Proposition 4.1.5. The group G acts on the second set via this bijection, and we have a restricted action of G_W on Ξ_W. There is a second bijection, from $\bigcup_{W\in\mathcal{T}} \Xi_W/G_W$ to $\left(\bigcup_{W\subseteq V(\mathcal{S})} \Xi_W\right)/G$, given by $G_W(f', f'') \mapsto G(f', f'')$. Since G is cycle-closed we have $G_W \cong G|W \times G|\overline{W}$, and hence a third bijection, from Ξ_W/G_W to

$$(\Xi_{[d]}(\mathcal{S}|W)/(G|W)) \times (\Xi_{d+[d]}(\mathcal{S}|\overline{W})/(G|\overline{W}))\,.$$

These three bijections together yield a fourth one, from $\Xi_{[2d]}(\mathcal{S})/G$ to

$$\bigcup_{W\in\mathcal{T}} (\Xi_{[d]}(\mathcal{S}|W)/(G|W)) \times (\Xi_{d+[d]}(\mathcal{S}|\overline{W})/(G|\overline{W}))\,,$$

with the property that if $G(f) \mapsto ((G|W)(f'), (G|\overline{W})(f''))$, then $x^f = x^{f'}\,x^{f''}$. This proves (4.2.5). □

Now let $\theta\colon C_* \to H_*\{x\}$ be a surjective map of graded coalgebras, such as ρ^b or c^b. As in the previous section, we may employ θ^* to associate a delta operator $a(D^\theta)$ on C_* with the delta operator $a(D)$ on $H_*\{x\}$, and check that the map $\theta\colon (C_*, a(D_\theta)) \to (H_*\{x\}, a(D))$ becomes a map of coalgebras with delta operator. This result then yields a suitably strengthened version of Theorem 4.2.4.

As an analogue of (4.1.8), we have

$$D^{\rho}(\mathcal{N}_n, \Sigma_n) = D^{c}(\mathcal{N}_n, \Sigma_n) = (\mathcal{N}_{n-1}, \Sigma_{n-1}), \quad b(D^c)\,(\mathcal{K}_n, \Sigma_n) = (\mathcal{K}_{n-1}, \Sigma_{n-1}).$$

With reference to the binomial sequences $(\mathcal{K}_n(A))$ defined in the previous section, we remark that the sequence $(\mathcal{K}_n(A), \Sigma_n)$ is a divided power sequence, in the sense of (1.1.12); furthermore, every divided power sequence in C_* is of this form.

We also obtain an analogue of Proposition 4.1.9, whose proof is similar.

Proposition 4.2.6 *Let a be an arbitrary umbra in H_*, and Δ a delta operator on C_* which is a derivation; then the following identities hold in C_*:*

$$a(\Delta)\,((\mathcal{S}_1, G_1)\cdot(\mathcal{S}_2, G_2)) = \sum_{i,j\geq 0} f^a_{i,j}\,(a(\Delta)^i\,(\mathcal{S}_1, G_1))\cdot(a(\Delta)^j\,(\mathcal{S}_2, G_2))$$

$$\text{and} \qquad a(\Delta)\,\gamma(\mathcal{S}, G) = \sum_{k\geq 1} i^a_k\,\gamma(a(\Delta)^k\,(\mathcal{S}, G))$$

Given any map $\theta\colon C_* \to H_*\{x\}$ of graded Hopf algebras, D^{θ} is a derivation on C_*, so the above identities hold for it.

All our results for the map c^b may be reformulated by complementation for the map $\chi^b\colon (H_*\langle\mathfrak{C}\rangle/I_*, \odot) \to H_*\{x\}$, specified by $[\mathcal{S}, G] \mapsto \chi^{\phi}(\mathcal{S}; x)/|G|$.

In conclusion, we address the problem of finding a model for the covariant bialgebra $L_*\langle\beta^b_i(x)\rangle$ of the universal formal group law. To this end, we consider the free L_*-module $L_*\langle\mathfrak{P}\rangle$ spanned by the set $\mathfrak{P}$ defined in §1.6; clearly, this module is a sub-Hopf algebra of $L_*\langle\mathfrak{C}\rangle$. If I_* is now the ideal of $L_*\langle\mathfrak{C}\rangle$ generated by the set in (4.2.2), we have the graded Hopf algebra $P_* := L_*\langle\mathfrak{P}\rangle/(I_* \cap L_*\langle\mathfrak{P}\rangle)$, by a similar argument to Proposition 4.2.3. Note that there are inclusions of Hopf algebras

$$L_*\langle\mathfrak{S}\rangle \hookrightarrow P_* \hookrightarrow L_*\langle\mathfrak{C}\rangle/I_*.$$

Theorem 4.2.7 *The restriction of c^b to P_* is a map of graded Hopf algebras onto $L_*\langle\beta^b_i(x)\rangle$.*

PROOF. We establish that $c^b(\mathfrak{P}) \subseteq L_*\langle \beta_i^b(x)\rangle$, from which the result follows by Proposition 4.1.4 and (1.8.4). Consider $(\mathcal{S}, G) \in \mathfrak{P}$, and let π be the associated partition. We will prove that $c^\phi(\mathcal{S}; x)/|G| \in L_*\langle \beta_i^b(x)\rangle$ by induction with respect to $|\mathcal{S} \setminus \mathcal{K}^\pi|$; the induction starts successfully at 0, by the definition of $L_*\langle \beta_i^b(x)\rangle$. If $|\mathcal{S} \setminus \mathcal{K}^\pi| > 0$, choose a set $U \in \mathcal{S} \setminus \mathcal{K}^\pi$ minimal with respect to inclusion, and recall the deletion/contraction formula Theorem 2.3.1. We clearly have $G \cong G|U \times G|\overline{U}$, and $(\mathcal{S} \setminus U, G) \in \mathfrak{P}$. Also, $G|\overline{U}$ can be viewed as an automorphism group of $\mathcal{S}//U$ in the obvious way, whence $(\mathcal{S}//U, G|\overline{U})$ lies in $\mathfrak{P}$ (the corresponding partition is obtained from $\pi|\overline{U}$ by adjoining the block consisting of the singleton $\{U\}$). Using these facts, formula (2.3.1) can be rewritten as

$$\frac{c^\phi(\mathcal{S}; x)}{|G|} = \frac{c^\phi(\mathcal{S} \setminus U; x)}{|G|} + \frac{\mu^\phi_{\Pi(\mathcal{S}|U)}(\widehat{0}, \{U\})}{|G|U|} \, \frac{c^\phi(\mathcal{S}//U; x)}{|G|\overline{U}|} \, .$$

Given the choice of U, we have

$$\frac{\mu^\phi_{\Pi(\mathcal{S}|U)}(\widehat{0}, \{U\})}{|G|U|} = -r \, \frac{F^\phi_{n_1, \ldots, n_k}}{n_1! \ldots n_k!} \quad \text{in } L_* \, ,$$

by (3.1.2) and (1.2.13), where n_i are the sizes of the blocks of $\pi|U$, and r is some positive integer. The induction is now complete. □

4.3 A Non-cocommutative Hopf Algebroid of Set Systems

Recall that we denoted by $\widehat{\mathfrak{S}}$ the set of weak isomorphism classes of set systems for which the poset of divisions has a unique maximal element. In this section we define a non-cocommutative structure on the free Φ_*-module $\widehat{S}_* := \Phi_*\langle \widehat{\mathfrak{S}} \rangle$ spanned by the set $\widehat{\mathfrak{S}}$; this structure is not a Hopf algebra, but a *Hopf algebroid.* However, its construction starts by defining a similar structure on the free $\mathbb{Z}$-module $\mathbb{Z}\langle \widehat{\mathfrak{S}} \rangle$, and this *is* a Hopf algebra. This construction is analogous to the one for S_*, being also based on the general method of constructing incidence Hopf algebras described in [47].

As pointed out in §1.5, we shall not attempt to distinguish notationally between a set system and its weak isomorphism class, since in those cases where it matters, we have taken care to ensure that the context is clear.

Let $\mathbb{P}$ be the hereditary family consisting of all finite products of intervals from the posets $\Pi(\mathcal{S})$ ordered by refinement, for set systems $\mathcal{S}$ with weak isomorphism classes lying in $\widehat{\mathfrak{S}}$. Intervals corresponding to different set systems are considered distinct, even if they consist of identical sets, so we index by $\mathcal{S}$ the elements of $\Pi(\mathcal{S})$ determining an interval. We define a map from $\mathbb{P}$ to $\widehat{\mathfrak{S}}$ as follows: given set systems $\mathcal{S}_i$ for $i \in [n]$ and intervals $[\pi_{\mathcal{S}_i}, \sigma_{\mathcal{S}_i}]$ in $\Pi(\mathcal{S}_i)$, we map $[\pi_{\mathcal{S}_1}, \sigma_{\mathcal{S}_1}] \times \ldots \times [\pi_{\mathcal{S}_n}, \sigma_{\mathcal{S}_n}]$ to the weak isomorphism class of the set system $((\mathcal{S}_1|\sigma_{\mathcal{S}_1})//\pi_{\mathcal{S}_1}) \cdot \ldots \cdot ((\mathcal{S}_n|\sigma_{\mathcal{S}_n})//\pi_{\mathcal{S}_n})$. Let $\sim$ be the kernel of this map. The proof of the order compatibility of $\sim$ is mainly based on the fact that disjoint union interacts with restriction and strong contraction such that

$$((\mathcal{S}_1 \cdot \mathcal{S}_2)|(\sigma_1 \sqcup \sigma_2))//(\pi_1 \sqcup \pi_2) = ((\mathcal{S}_1|\sigma_1)//\pi_1) \cdot ((\mathcal{S}_2|\sigma_2)//\pi_2)\,,$$

where $[\pi_i, \sigma_i]$ are intervals in $\Pi(\mathcal{S}_i)$ for $i = 1, 2$. This proof can be divided into two steps, as shown below:

$$[\pi_{\mathcal{S}_1}, \sigma_{\mathcal{S}_1}] \times [\pi_{\mathcal{S}_2}, \sigma_{\mathcal{S}_2}] \sim [(\pi_{\mathcal{S}_1} \sqcup \pi_{\mathcal{S}_2})_{\mathcal{S}_1 \cdot \mathcal{S}_2}, (\sigma_{\mathcal{S}_1} \sqcup \sigma_{\mathcal{S}_2})_{\mathcal{S}_1 \cdot \mathcal{S}_2}], (\rho_{\mathcal{S}_1}, \rho_{\mathcal{S}_2}) \mapsto (\rho_{\mathcal{S}_1} \sqcup \rho_{\mathcal{S}_2})_{\mathcal{S}_1 \cdot \mathcal{S}_2}$$

$$[\pi_{\mathcal{S}}, \sigma_{\mathcal{S}}] \sim [(\pi_{\mathcal{S}}/\pi_{\mathcal{S}})_{(\mathcal{S}|\sigma_{\mathcal{S}})//\pi_{\mathcal{S}}}, (\sigma_{\mathcal{S}}/\pi_{\mathcal{S}})_{(\mathcal{S}|\sigma_{\mathcal{S}})//\pi_{\mathcal{S}}}]\,, \qquad \rho_{\mathcal{S}} \mapsto (\rho_{\mathcal{S}}/\pi_{\mathcal{S}})_{(\mathcal{S}|\sigma_{\mathcal{S}})//\pi_{\mathcal{S}}}\,;$$

the two maps indicated are the corresponding order compatible bijections. Since isomorphism of set systems is a congruence with respect to disjoint union, and since $\sim$ is a *reduced congruence* (this is the reason for considering weak isomorphism classes of set systems), the relation $\sim$ is a Hopf relation.

Let $H(\mathbb{P})$ be the $\mathbb{Z}$-incidence Hopf algebra of the family $\mathbb{P}$ modulo the Hopf relation $\sim$, as defined in [47]. The bijection from $\mathbb{P}/\sim$ to $\widehat{\mathfrak{S}}$ induced by the map above can be extended by linearity to a bijection from $H(\mathbb{P})$ to $\mathbb{Z}\langle\widehat{\mathfrak{S}}\rangle$. We use this bijection to transfer the Hopf algebra structure of $H(\mathbb{P})$ to $\mathbb{Z}\langle\widehat{\mathfrak{S}}\rangle$. Comultiplication in $\mathbb{Z}\langle\widehat{\mathfrak{S}}\rangle$ is specified by

$$\delta(\mathcal{S}) := \sum_{\sigma\in\Pi(\mathcal{S})} \mathcal{S}|\sigma \otimes \mathcal{S}//\sigma\,.$$

The counit is determined by

$$\varepsilon(\mathcal{S}) := \begin{cases} 1 & \text{if } \mathcal{S} = \{\emptyset\} \\ 0 & \text{otherwise}\,. \end{cases}$$

Multiplication is disjoint union, and the unit is the map η specified by $\eta(1) := \{\emptyset\}$. The antipode can be expressed using the Schmitt formula (1.7.1). Note the rôle of the non-standard convention $\Pi(\{\emptyset\}) := \{\{\emptyset\}\}$; also note the fact that we have no counit if we replace $\widehat{\mathfrak{S}}$ with the set of weak isomorphism classes of all set systems.

Clearly, the Hopf algebra $\mathbb{Z}\langle\widehat{\mathfrak{S}}\rangle$ is commutative and non-cocommutative. It has finite type, and is graded by setting the degree of the weak isomorphism class of $\mathcal{S}$ equal to $|\widehat{0}_{\Pi(\mathcal{S})}|-|\widehat{1}_{\Pi(\mathcal{S})}|$. On the other hand, this Hopf algebra is isomorphic, as an algebra, to the polynomial algebra $\mathbb{Z}[\widehat{\mathfrak{S}}_\circ]$.

Let us denote by $\widehat{\mathfrak{K}}$ the set of weak isomorphism classes of $\mathcal{K}_n$, $n \geq 0$. This set generates a sub-Hopf algebra of $\widehat{S}_*$, which is easily seen to be isomorphic to the Faà di Bruno Hopf algebra discussed in Chapter 1; a slight variation of the above combinatorial model for this algebra appears in [47] Example 14.1, and is based on complete graphs.

We now define the Hopf algebroid $(\Phi_*, \widehat{S}_*)$. The left Φ_*-module structure on $\widehat{S}_*$, which is expressed by the map η_L, is the usual one; we will identify the elements of Φ_* with their images via η_L. The maps δ and ε are defined by the same relations as the corresponding ones for $\mathbb{Z}\langle\widehat{\mathfrak{S}}\rangle$, plus the constraint to be left module maps. The right unit η_R is defined by

$$\eta_R(\phi_n) := \sum_{\sigma \in \Pi_{n+1}} \tau^{\phi}(\sigma)\, \mathcal{K}_{|\sigma|}\,. \tag{4.3.1}$$

The conjugation γ is defined as before for weak isomorphism classes of set systems, while we are constrained to set $\gamma(\phi_n) := \eta_R(\phi_n)$. In order to check that $(\Phi_*, \widehat{S}_*)$ with the above structure maps is indeed a Hopf algebroid, as well as in view of later applications, we define the algebra map $\widehat{c}^{\phi} \colon \widehat{S}_* \to \Phi_* \otimes \Phi_*$ by specifying its image on weak isomorphism classes of connected set systems $\mathcal{S}$:

$$\widehat{c}^{\phi}(\mathcal{S}) := \langle \phi^R(D) \mid c^{\phi}(\mathcal{S}; x) \rangle\,.$$

It is not difficult to see that the map $\widehat{c}^{\phi}$ restricts to an algebra isomorphism between $\Phi_*[\widehat{\mathfrak{K}}]$ and $\Phi_* \otimes \Phi_*$, which commutes with the restrictions of the structure maps defined above. Hence $(\Phi_*, \Phi_*[\widehat{\mathfrak{K}}])$ is a Hopf algebroid isomorphic to $(\Phi_*, \Phi_* \otimes \Phi_*)$. In particular, we have

$$\widehat{c}^{\phi}(\mathcal{K}_{n+1}) = \langle \phi^R(D) \mid B^{\phi}_{n+1}(x) \rangle = \sum_{\sigma \in \Pi_{n+1}} \mu^{\phi}(\widehat{0}, \sigma)\, \zeta^{\phi^R}(\sigma, \widehat{1}) = \psi_n\,, \tag{4.3.2}$$

and

$$\delta(z) = 1 \otimes z \qquad \text{for } \; z \in \eta_R(\Phi_*) \subset \widehat{S}_*\,. \tag{4.3.3}$$

Furthermore, all the axioms of a Hopf algebroid can now be easily checked for $(\Phi_*, \widehat{S}_*)$ either directly, or using the above isomorphism and the fact that $\mathbb{Z}\langle\widehat{\mathfrak{S}}\rangle$ is a Hopf algebra.

In order to write computations in a concise form, it helps to consider the functions ζ^{ϕ}, μ^{ϕ} in $\Phi_*(\Pi_{n+1})$, $\zeta^{\phi^R}, \mu^{\phi^R}$ in $(\Phi_* \otimes \Phi_*)(\Pi_{n+1})$, and $\zeta^{\mathcal{K}}, \mu^{\mathcal{K}}, \zeta^{\eta_R(\phi)}, \mu^{\eta_R(\phi)}$ in $\widehat{S}_*(\Pi_{n+1})$; as the notation suggests, $\mathcal{K}$ is the umbra $(\mathcal{K}_1, \mathcal{K}_2, \dots)$, and $\eta_R(\phi)$ is the umbra $(1, \eta_R(\phi_1), \eta_R(\phi_2), \dots)$. Throughout this section, we let $\widehat{0} := \widehat{0}_{\Pi_{n+1}}$ and $\widehat{1} := \widehat{1}_{\Pi_{n+1}}$, unless there is some other poset in sight. The definition (4.3.1) of η_R can now be rewritten as $\eta_R(\phi_n) := (\zeta^\phi * \zeta^{\mathcal{K}})(\widehat{0},\widehat{1})$, which is equivalent to $\mathcal{K}_{n+1} = (\mu^\phi * \zeta^{\eta_R(\phi)})(\widehat{0},\widehat{1})$.

The Hopf algebroid $(\Phi_*, \widehat{S}_*)$ clearly has the same properties as the Hopf algebra $\mathbb{Z}\langle\widehat{\mathfrak{S}}\rangle$ of being commutative, non-cocommutative, graded, and isomorphic to a polynomial algebra.

Proposition 4.3.4 *The map* $(I, \widehat{c}^\phi)\colon (\Phi_*, \widehat{S}_*) \to (\Phi_*, \Phi_* \otimes \Phi_*)$ *is a surjective map of graded Hopf algebroids.*

PROOF. The map $\widehat{c}^\phi$ clearly preserves gradings, while surjectivity follows from (4.3.2). Now let $\mathcal{S}$ be an arbitrary connected set system with weak isomorphism classes lying in $\widehat{\mathfrak{S}}$. First, we need to show that $(\varepsilon \circ \widehat{c}^\phi)(\mathcal{S}) = 0$. By expressing $c^\phi(\mathcal{S}; x)$ according to Proposition 2.4.1 and by using (4.3.2) once again, we obtain

$$\widehat{c}^\phi(\mathcal{S}) = \sum_{\sigma \in \Pi(\overline{\mathcal{S}})} \nu^\phi_{\mathcal{S}}(\sigma)\, \psi_{|\sigma|-1}\,.$$

Since all partitions in $\Pi(\overline{\mathcal{S}})$ have at least two blocks, we have $\varepsilon(\widehat{c}^\phi(\mathcal{S})) = 0$. The fact that $\widehat{c}^\phi \circ \eta_R = \eta_R$ has been already observed, when we discussed the restriction of $\widehat{c}^\phi$ to $\Phi_*[\widehat{\mathfrak{K}}]$.

The fact that $\delta \circ \widehat{c}^\phi = (\widehat{c}^\phi \otimes \widehat{c}^\phi) \circ \delta$ follows from

$$\begin{aligned}
((\widehat{c}^\phi \otimes \widehat{c}^\phi) \circ \delta)(\mathcal{S}) &= \sum_{\sigma \in \Pi(\mathcal{S})} (\mu^\phi_{\Pi(\mathcal{S})} * \zeta^{\phi^R})(\widehat{0}, \sigma) \otimes (\mu^\phi_{\Pi(\mathcal{S})} * \zeta^{\phi^R})(\sigma, \widehat{1}) \qquad (4.3.5)\\
&= \sum_{\pi \in \Pi(\mathcal{S})} \mu^\phi_{\Pi(\mathcal{S})}(\widehat{0}, \pi) \otimes (\zeta^\phi * (\mu^\phi_{\Pi(\mathcal{S})} * \zeta^{\phi^R}))(\pi, \widehat{1})\\
&= \sum_{\pi \in \Pi(\mathcal{S})} \mu^\phi_{\Pi(\mathcal{S})}(\widehat{0}, \pi) \otimes \zeta^{\phi^R}(\pi, \widehat{1})\\
&= \delta\left(\sum_{\pi \in \Pi(\mathcal{S})} \mu^\phi_{\Pi(\mathcal{S})}(\widehat{0}, \pi)\, \zeta^{\phi^R}(\pi, \widehat{1})\right) = (\delta \circ \widehat{c}^\phi)(\mathcal{S})\,.
\end{aligned}$$

As far as the relation $\widehat{c}^\phi \circ \gamma = \gamma \circ \widehat{c}^\phi$ is concerned, it is easily checked when applied to ϕ_n. Hence, we only need to prove that the above relation holds when applied to $\mathcal{S}$ in $\widehat{\mathfrak{S}}$. We do this by induction on $|V(\mathcal{S})|$, which starts at 1. For $|V(\mathcal{S})| \geq 2$, we use induction and the definition of γ to prove that

$$\begin{aligned} 0 &= \widehat{c}^{\phi}\left(\sum_{\sigma\in\Pi(\mathcal{S})} \mathcal{S}|\sigma \cdot \gamma(\mathcal{S}//\sigma)\right) \\ &= \widehat{c}^{\phi}(\gamma(\mathcal{S})) - \gamma(\widehat{c}^{\phi}(\mathcal{S})) + \sum_{\sigma\in\Pi(\mathcal{S})} \widehat{c}^{\phi}(\mathcal{S}|\sigma)\,\gamma(\widehat{c}^{\phi}(\mathcal{S}//\sigma)) \\ &= \widehat{c}^{\phi}(\gamma(\mathcal{S})) - \gamma(\widehat{c}^{\phi}(\mathcal{S})) + ((I \odot \gamma) \circ (\widehat{c}^{\phi} \otimes \widehat{c}^{\phi}) \circ \delta)(\mathcal{S}) \\ &= \widehat{c}^{\phi}(\gamma(\mathcal{S})) - \gamma(\widehat{c}^{\phi}(\mathcal{S}))\,; \end{aligned}$$

the last equality follows from the axioms of a Hopf algebroid:

$$(I \odot \gamma) \circ (\widehat{c}^{\phi} \otimes \widehat{c}^{\phi}) \circ \delta = (I \odot \gamma) \circ \delta \circ \widehat{c}^{\phi} = \eta_L \circ \varepsilon \circ \widehat{c}^{\phi} = \eta_L \circ \varepsilon\,.$$

□

Note that the map $\widehat{c}^{\phi}$ allows us to define a right $\Phi_* \otimes \Phi_*$-comodule algebra structure on $\widehat{S}_*$ via the map

$$\Delta\colon \widehat{S}_* \to \widehat{S}_* \otimes_{\Phi_*} (\Phi_* \otimes \Phi_*)\,, \qquad \Delta := (I \otimes \widehat{c}^{\phi}) \circ \delta\,.$$

Recall that an element z of this comodule algebra is called *primitive* if $\Delta(z) = z \otimes 1$. The set of primitive elements is a Φ_*-subalgebra of $\widehat{S}_*$, and will be denoted by $P(\widehat{S}_*)$. Our next goal is to apply the following structure theorem to the map $(I, \widehat{c}^{\phi})$.

Theorem 4.3.6 *(cf. [33] Corollary A1.1.19) Let $(I, f)\colon (A_*, \Gamma_*) \to (A_*, \Theta_*)$ be a map of graded connected Hopf algebroids. Suppose*

1. *$f\colon \Gamma_* \to \Theta_*$ is onto, and*

2. *$P(\Gamma_*)$ is an A_*-module and there is a A_*-linear map $p\colon \Gamma_* \to P(\Gamma_*)$ split by the inclusion of $P(\Gamma_*)$ in Γ_*.*

Then there is a map $\xi\colon \Gamma_* \to P(\Gamma_*) \otimes_{A_*} \Theta_*$, defined by $\xi := (p \otimes f) \circ \delta$, which is an isomorphism of $P(\Gamma_*)$-modules and Θ_*-comodules (δ denotes the comultiplication of (A_*, Γ_*)).

To apply the above theorem, we need to construct a projection $p\colon \widehat{S}_* \to P(\widehat{S}_*)$. We believe that it is possible to construct a projection which is an algebra map; thus, the map ξ in the above theorem would be an isomorphism of algebras and $\Phi_* \otimes \Phi_*$-comodules. For the moment, we have only been able to define such a projection on a sub-Hopf algebroid of $(\Phi_*, \widehat{S}_*)$. Let $\widehat{\mathfrak{D}}$ denote the set of those weak isomorphism classes in $\widehat{\mathfrak{S}}$ which correspond to set systems $\mathcal{S}$ with $\mathcal{S} \setminus \max \mathcal{S}$ a simplicial complex (here $\max \mathcal{S}$ denotes the set of maximal elements in the poset $(\mathcal{S}, \subseteq)$). Clearly, the free Φ_*-module $\widehat{D}_* := \Phi_*\langle \widehat{\mathfrak{D}} \rangle$ generated by $\widehat{\mathfrak{D}}$ is a sub-Hopf algebroid and a sub-$\Phi_* \otimes \Phi_*$-comodule algebra of $\widehat{S}_*$. We define the algebra map $p\colon \widehat{D}_* \to \widehat{D}_*$ by specifying its image on weak isomorphism classes of connected set systems $\mathcal{S}$:

$$p(\mathcal{S}) := \mathcal{S} - \sum_{\sigma \in \Pi(\overline{\mathcal{S}})} \nu^{\phi}_{\mathcal{S}}(\sigma)\, \mathcal{K}_{|\sigma|}\,.$$

Theorem 4.3.7 *The algebra map p is a projection onto $P(\widehat{D}_*)$. Hence, the map $\xi\colon \widehat{D}_* \to P(\widehat{D}_*) \otimes_{\Phi_*} (\Phi_* \otimes \Phi_*)$ defined by $\xi := (p \otimes \widehat{c}^{\phi}) \circ \delta$ is an isomorphism of algebras and $\Phi_* \otimes \Phi_*$-comodules.*

PROOF. We have

$$\begin{aligned}
\Delta(\mathcal{S}) - \mathcal{S}\otimes 1 &= \sum_{1\neq\rho\in\Pi(\mathcal{S})} \mathcal{K}^{\rho}\otimes\widehat{c}^{\phi}(\mathcal{S}/\!/\rho)\\
&= \sum_{1\neq\rho\in\Pi(\mathcal{S})} (\mu^{\phi}*\zeta^{\eta_R(\phi)})(\widehat{0},\rho)\otimes(\mu^{\phi}_{\Pi(\mathcal{S})}*\zeta^{\phi^R})(\rho,\widehat{1})\\
&= \sum_{\pi\leq\sigma<\widehat{1};\,\pi,\sigma\in\Pi(\mathcal{S})} \mu^{\phi}_{\Pi(\mathcal{S})}(\widehat{0},\pi)\otimes(\zeta^{\phi}*\mu^{\phi}_{\Pi(\mathcal{S})})(\pi,\sigma)\,\zeta^{\phi^R}(\sigma,\widehat{1})\\
&\qquad + \sum_{\pi\leq\rho<\widehat{1};\,\pi,\rho\in\Pi(\mathcal{S})} \mu^{\phi}_{\Pi(\mathcal{S})}(\widehat{0},\pi)\otimes\zeta^{\phi}(\pi,\rho)\,\mu^{\phi}_{\Pi(\mathcal{S})}(\rho,\widehat{1})\\
&= \sum_{\widehat{1}\neq\sigma\in\Pi(\mathcal{S})} \mu^{\phi}_{\Pi(\mathcal{S})}(\widehat{0},\sigma)\otimes\zeta^{\phi^R}(\sigma,\widehat{1}) - \sum_{\widehat{1}\neq\pi\in\Pi(\mathcal{S})} \mu^{\phi}_{\Pi(\mathcal{S})}(\widehat{0},\pi)\otimes\zeta^{\phi}(\pi,\widehat{1})\\
&= \sum_{\sigma\in\Pi(\mathcal{S})} \mu^{\phi}_{\Pi(\mathcal{S})}(\widehat{0},\sigma)\otimes(\zeta^{\phi^R}(\sigma,\widehat{1})-\zeta^{\phi}(\sigma,\widehat{1}))\,.
\end{aligned}$$

Let $K := (\mu^{\phi}_{\Pi(\mathcal{S})}*\zeta^{\eta_R(\phi)})(\widehat{0},\widehat{1})$. Combining the above result with (4.3.3), we obtain

$$\begin{aligned}
\Delta(K) &= \sum_{\sigma\in\Pi(\mathcal{S})} \mu^{\phi}_{\Pi(\mathcal{S})}(\widehat{0},\sigma)\otimes\zeta^{\phi^R}(\sigma,\widehat{1})\\
&= K\otimes 1 + \sum_{\sigma\in\Pi(\mathcal{S})} \mu^{\phi}_{\Pi(\mathcal{S})}(\widehat{0},\sigma)\otimes(\zeta^{\phi^R}(\sigma,\widehat{1})-\zeta^{\phi}(\sigma,\widehat{1}))\\
&= K\otimes 1 + \Delta(\mathcal{S}) - \mathcal{S}\otimes 1\,;
\end{aligned}$$

hence $\Delta(\mathcal{S}-K) = (\mathcal{S}-K)\otimes 1$. On the other hand, according to Proposition 2.4.1, we have

$$K = \sum_{\sigma\in\Pi(\overline{\mathcal{S}})} \nu^{\phi}_{\mathcal{S}}(\sigma)\,(\mu^{\phi}*\zeta^{\eta^R(\phi)})(\widehat{0}_{\Pi_{|\sigma|}},\widehat{1}_{\Pi_{|\sigma|}}) = \sum_{\sigma\in\Pi(\overline{\mathcal{S}})} \nu^{\phi}_{\mathcal{S}}(\sigma)\,\mathcal{K}_{|\sigma|}\,.$$

We conclude the proof by noting that $p(\mathcal{K}_n)=0$ for $n\geq 1$, whence p is indeed a projection. □

Let us note that $\widehat{D}_*$ is the direct sum of $\Phi_*[\widehat{\mathfrak{K}}]$ and $P(\widehat{D}_*)$; on the other hand, $P(\widehat{D}_*)$ is easily seen to be a polynomial algebra in the set of variables $p(\widehat{\mathfrak{D}}_\circ\setminus\widehat{\mathfrak{K}})$, where $\widehat{\mathfrak{D}}_\circ$ denotes, as expected, the subset of $\widehat{\mathfrak{D}}$ consisting of weak isomorphism classes of connected set systems.

4.4 Non-cocommutative Hopf Algebroids of Set Systems with Automorphism Group

In this section, we define non-cocommutative Hopf algebroid structures on certain quotients of the free modules $H_*\langle\widehat{\mathfrak{A}}\rangle$ and $L_*\langle\widehat{\mathfrak{P}}\rangle$ spanned by the sets $\widehat{\mathfrak{A}}$ and $\widehat{\mathfrak{P}}$ defined in §1.6. We also relate our constructions to the cocommutative Hopf algebras of set systems with automorphism group constructed previously.

As pointed out in §1.6, we shall not attempt to distinguish notationally between a set system with automorphism group and its weak isomorphism class, since in those cases where it matters, we have taken care to ensure that the context is clear.

Let us consider the algebra $\mathbb{Z}\langle\widehat{\mathfrak{A}}\rangle$ with multiplication induced by disjoint union, as defined by (1.6.2), and unit map η specified by $\eta(1) = (\{\emptyset\},\{1\})$. Clearly, this algebra is commutative, graded (in a similar way to $\mathbb{Z}\langle\widehat{\mathfrak{S}}\rangle$), and has finite type. Note that if the weak isomorphism class of $(\mathcal{S},G)$ lies in $\widehat{\mathfrak{A}}$, then the weak isomorphism classes of $(\mathcal{S},G)|\sigma$ and $(\mathcal{S},G)/\sigma$ also lie in $\widehat{\mathfrak{A}}$, for every $\sigma\in\Pi(\mathcal{S})$; later we will need the fact that $\widehat{\mathfrak{P}}$ is also closed with respect to restriction and contraction (in the sense mentioned above), which is again easy to check. Hence, we may define the following comultiplication:

$$\delta(\mathcal{S},G) = \sum_{\sigma\in\mathcal{T}} (\mathcal{S},G)|\sigma \otimes (\mathcal{S},G)/\sigma\,,$$

where $\mathcal{T}$ is an arbitrary transversal of $\Pi(\mathcal{S})/G$. This comultiplication is not coassociative (indeed, $((I\otimes\delta)\circ\delta)(\mathcal{K}_4,\Sigma_4) \neq ((\delta\otimes I)\circ\delta)(\mathcal{K}_4,\Sigma_4)$), but it has a counit determined by

$$\varepsilon(\mathcal{S},G) = \begin{cases} 1 & \text{if } (\mathcal{S},G) = (\{\emptyset\},\{1\}) \\ 0 & \text{otherwise}\,. \end{cases}$$

We would now like to factor the algebra $\mathbb{Z}\langle\widehat{\mathfrak{A}}\rangle$ by the graded ideal J_* generated by the set

$$\{\mathcal{S} - |G|\,(\mathcal{S}, G) \;:\; (\mathcal{S}, G) \in \widehat{\mathfrak{A}}\}\,. \tag{4.4.1}$$

Proposition 4.4.2 *The ideal J_* is also a coideal.*

PROOF. Consider $(\mathcal{S}, G)$ with weak isomorphism class in $\widehat{\mathfrak{A}}$. Using the fact that $|G(\sigma)| = |G|/|G_\sigma|$, we obtain

$$|G|\,\delta(\mathcal{S}, G) = \sum_{\sigma\in\Pi(\mathcal{S})} |G_\sigma|\,(\mathcal{S}, G)|\sigma \otimes (\mathcal{S}, G)/\sigma\,.$$

Using (1.6.1), we have

$$\begin{aligned}\delta(\mathcal{S} - |G|\,(\mathcal{S}, G)) = &\sum_{\sigma\in\Pi(\mathcal{S})} \big(\mathcal{S}|\sigma - |G|\sigma|\,(\mathcal{S}|\sigma, G|\sigma)\big) \otimes \mathcal{S}//\sigma \\ &+ \sum_{\sigma\in\Pi(\mathcal{S})} |G|\sigma|\,(\mathcal{S}, G)|\sigma \otimes \big(\mathcal{S}//\sigma - |G/\sigma|\,(\mathcal{S}//\sigma, G/\sigma)\big)\,,\end{aligned}$$

which lies in $\mathbb{Z}\langle\widehat{\mathfrak{A}}\rangle \otimes J_* + J_* \otimes \mathbb{Z}\langle\widehat{\mathfrak{A}}\rangle$. □

According to the above result, we can define the comultiplication δ on the graded quotient $\mathbb{Z}\langle\widehat{\mathfrak{A}}\rangle/J_*$. The element $(\mathcal{S}, G) + J_*$ of this quotient will be denoted by $[\mathcal{S}, G]$. Note that $\mathbb{Z}\langle\widehat{\mathfrak{S}}\rangle$ can be regarded as a sub-Hopf algebra of $\mathbb{Z}\langle\widehat{\mathfrak{A}}\rangle/J_*$. Furthermore, we can use the fact that $\mathbb{Z}\langle\widehat{\mathfrak{S}}\rangle$ is a Hopf algebra to prove that $\mathbb{Z}\langle\widehat{\mathfrak{A}}\rangle/J_*$ is also a Hopf algebra. Let us also note that for a given set system $\mathcal{S}$, there is a unique element $[\mathcal{S}, G]$ in $\mathbb{Z}\langle\widehat{\mathfrak{A}}\rangle/J_*$ which is not divisible by integers, and this corresponds to G being the automorphism group of $\mathcal{S}$.

The elements $[\mathcal{K}_n, \Sigma_n]$, $n \geq 1$, generate a sub-Hopf algebra of $\mathbb{Z}\langle\widehat{\mathfrak{A}}\rangle/J_*$. This is isomorphic to the dual of the Landweber-Novikov algebra, which was discussed in Example 1.7.4. Indeed, we have

$$\begin{aligned} n!\,\delta([\mathcal{K}_n,\Sigma_n]) &= \sum_{(\sigma,\omega)\in A(\mathcal{K}_n)} \frac{|\Sigma_{n,\sigma}|}{|\sigma|!}\,[\mathcal{K}_n|\sigma,\Sigma_n|\sigma]\otimes[\mathcal{K}_n/\!/\sigma,\Sigma_n/\sigma] \\ &= \sum_{(\sigma,\omega)\in A(\mathcal{K}_n)} |\Sigma_n|\sigma|\,[\mathcal{K}_n|\sigma,\Sigma_n|\sigma]\otimes[\mathcal{K}_{|\sigma|},\Sigma_{|\sigma|}] \\ &= n!\sum_{\sigma\in\Pi(\mathcal{I}_n)}\left(\prod_{U\in\sigma}[\mathcal{K}_{|U|},\Sigma_{|U|}]\right)\otimes[\mathcal{K}_{|\sigma|},\Sigma_{|\sigma|}]\,; \end{aligned}$$

here we have used the following facts:

$$|G_\sigma| = |G|/|G(\sigma)|\,,\quad |\Sigma_n|\sigma| = |\Sigma_{n,\sigma}|/|\Sigma_n/\sigma|\,,\quad\text{and}\quad A(\mathcal{K}_n)/\Sigma_n\cong\Pi(\mathcal{I}_n)\,.$$

Let us now consider the embedding $\mathcal{K}_n\mapsto[\mathcal{K}_n]=n!\,[\mathcal{K}_n,\Sigma_n]$ of the Hopf algebra $\mathbb{Z}\langle\widehat{\mathfrak{K}}\rangle$ in the sub-Hopf algebra of $\mathbb{Z}\langle\widehat{\mathfrak{A}}\rangle/J_*$ generated by $[\mathcal{K}_n,\Sigma_n]$, $n\geq 1$. This is a purely combinatorial way of understanding the fact that the Faà di Bruno Hopf algebra Φ_* embeds in the dual of the Landweber-Novikov algebra H_* via $\phi_{n-1}\mapsto n!b_{n-1}$ (see Example 1.7.4). Another combinatorial model for the dual of the Landweber-Novikov algebra appears in [47] Example 14.2, and is based on paths; however, this model does not help us understand the embedding $\Phi_*\hookrightarrow H_*$.

Now let $\widehat{A}_*$ be the quotient of the algebra $H_*\langle\widehat{\mathfrak{A}}\rangle$ by the graded ideal generated by the set in (4.4.1). We can extend the map $\widehat{c}^\phi$ defined in the previous section to an algebra map $\widehat{c}^b$ from $\widehat{A}_*$ to $H_*\otimes H_*$ as follows: $[\mathcal{S},G]\mapsto\widehat{c}^\phi(\mathcal{S})/|G|$. Note that $\widehat{c}^b$ is well-defined; furthermore, $\widehat{c}^b$ does indeed take values in $H_*\otimes H_*$ since, assuming $\mathcal{S}$ to be connected, we know from (2.5.6) that $c^\phi(\mathcal{S};x)/|G|$ lies in $H_*\{x\}$, and that $\langle\phi^R(D)\mid x^n/n!\rangle = b^R_{n-1}$. Let us also note that $\widehat{c}^b$ sends $[\mathcal{K}_n,\Sigma_n]$ to c_{n-1}. Using a similar approach to the one in the previous section, we define the Hopf algebroid $(H_*,\widehat{A}_*)$, and prove that $(I,\widehat{c}^b)$ is a surjective map of graded Hopf algebroids. All the definitions and proofs are easily adapted to the new context. The only slight difference appears in the definition of the right unit, which becomes

$$\eta_R(b_n) := \sum_{\sigma\in\Pi(\mathcal{I}_{n+1})} \tau^b(\sigma)\,[\mathcal{K}_{|\sigma|}, \Sigma_{|\sigma|}]\,.$$

This definition is actually equivalent to the condition $\widehat{c}^b \circ \eta_R = \eta_R$.

We now address the problem of finding a model for the Hopf algebroid $(L_*, L_* \otimes H_*)$. To this end, we consider the free L_*-module $L_*\langle\widehat{\mathfrak{P}}\rangle$ spanned by the set $\widehat{\mathfrak{P}}$ defined in §1.6. Clearly, $(L_*, L_*\langle\widehat{\mathfrak{P}}\rangle)$ is a sub-Hopf algebroid of $(L_*, L_*\langle\widehat{\mathfrak{A}}\rangle)$. Now let J_* be the ideal of $L_*\langle\widehat{\mathfrak{A}}\rangle$ generated by the set in (4.4.1). By a similar argument to Proposition 4.4.2, we have the graded Hopf algebroid $(L_*, \widehat{P}_*)$, where $\widehat{P}_* := L_*\langle\widehat{\mathfrak{P}}\rangle/(J_* \cap L_*\langle\widehat{\mathfrak{P}}\rangle)$. Note that there are inclusions of Hopf algebroids

$$(L_*, L_*\langle\widehat{\mathfrak{S}}\rangle) \hookrightarrow (L_*, \widehat{P}_*) \hookrightarrow (L_*, L_*\langle\widehat{\mathfrak{A}}\rangle/J_*)\,.$$

Theorem 4.4.3 *The restriction of $(I, \widehat{c}^b)$ to $(L_*, \widehat{P}_*)$ is a surjective map of graded Hopf algebroids onto $(L_*, L_* \otimes H_*)$.*

Proof. This is an immediate consequence of Theorem 4.2.7, and the fact that

$$\widehat{c}^b([\mathcal{K}_n, \Sigma_n]) = \langle\phi^R(D) \mid B_n^\phi(x)/n!\rangle = c_{n-1}\,.$$

□

In conclusion, we present a combinatorial model for the map $\langle\phi^R(D)\,|\,\cdot\rangle$ from the covariant bialgebra of the universal formal group law $L_*\langle\beta_i^b(x)\rangle$, which is isomorphic to $MU_*(\mathbb{C}P^\infty)$, to $L_* \otimes H_*$, which is isomorphic to $MU_*(MU)$. Recall from Example 3.3.6 that the above map is induced by the topological inclusion $\mathbb{C}P^\infty \simeq MU(1) \hookrightarrow \Sigma^2 MU$. The map $\langle\phi^R(D) \mid \cdot\rangle$ is neither an algebra nor a coalgebra map. We lift this map to a purely combinatorial map s from a certain sub-Hopf algebra P'_* of P_* to $\widehat{P}_*$. Let us first introduce the following notation:

$$\widehat{\mathcal{S}} := \mathcal{S} \cup \{V(\mathcal{S})\}\,, \qquad P'_* := L_*\langle\mathfrak{P}'\rangle/(I_* \cap L_*\langle\mathfrak{P}'\rangle)\,;$$

here $\mathfrak{P}'$ denotes the subset of $\mathfrak{P}$ consisting of isomorphism classes of set systems with automorphism group $(\mathcal{S}, G)$ for which $\Pi(\mathcal{S})$ has a unique maximal element, and I_* is the ideal of $L_*\langle\mathfrak{C}\rangle$ generated by the set in (4.2.2). We now define the map $s\colon P'_* \to \widehat{P}_*$ as follows:

$$s([\mathcal{S},G]) := \begin{cases} [\mathcal{S},G] & \text{if } \mathcal{S}\in\mathfrak{S}_\circ \\ \left[\widehat{\mathcal{S}},G\right] - \mu^{\phi}_{\Pi(\widehat{\mathcal{S}})}(\widehat{0},\widehat{1})/|G| & \text{otherwise}. \end{cases}$$

This map is well-defined as long as we show that $\mu^{\phi}_{\Pi(\widehat{\mathcal{S}})}(\widehat{0},\widehat{1})/|G|$ lies in L_*. This is indeed the case, since

$$\frac{\mu^{\phi}_{\Pi(\widehat{\mathcal{S}})}(\widehat{0},\widehat{1})}{|G|}\,x = \frac{c^{\phi}(\widehat{\mathcal{S}};x)}{|G|} - \frac{c^{\phi}(\mathcal{S};x)}{|G|}, \tag{4.4.4}$$

and we know from Theorem 4.2.7 that both of these polynomials lie in $L_*\langle\beta_i^b(x)\rangle$. By using (4.4.4) once again, we prove our final result, which states that s is a lifting of $\langle\phi^R(D)\mid\cdot\rangle$.

Proposition 4.4.5 *The following diagram is commutative:*

$$\begin{array}{ccc} P'_* & \xrightarrow{\quad s \quad} & \widehat{P}_* \\ {\scriptstyle c^b}\downarrow & & \downarrow{\scriptstyle \widehat{c}^b} \\ L_*\langle\beta_i^b(x)\rangle & \xrightarrow{\langle\phi^R(D)\mid\cdot\rangle} & L_*\otimes H_* \end{array}$$

The above diagram captures the essence of the results contained in this chapter, namely that the universal objects $L_*\langle\beta_i^b(x)\rangle$ and $L_*\otimes H_*$ are images of some combinatorial structures via maps of Hopf algebras/algebroids, which are compatible with the embedding $L_*\langle\beta_i^b(x)\rangle \hookrightarrow L_*\otimes H_*$.

Chapter 5

Necklace Algebras and Witt Vectors Associated with Formal Group Laws

In this chapter, we generalise the constructions in §1.9 in the context of formal group laws. Thus, we obtain combinatorial models for Witt vectors associated with a formal group law. We shall see that the classical necklace algebra of Metropolis and Rota corresponds to the multiplicative formal group law. Other special cases are also investigated, including a family of formal group laws not mentioned in [18] for which there are ring structures on the associated Witt vectors and curves.

5.1 Constructing the Generalised Necklace Algebra

We start this section with a brief survey of Witt vectors associated with formal group laws (cf. [18] Chapter 3).

Throughout this chapter, we let $F(X,Y)$ be a formal group law over a torsion free ring A with

$$\log_F(X) = \sum_{n\geq 1} a_n X^n\,, \quad a_1 = 1\,, \quad a_n \in A\mathbb{Q}\,.$$

Unlike the rest of this work, here we do not require A to be graded. In particular, for every integer q, we consider the formal group law

$$F_q(X,Y) := \frac{X+Y-(q+1)XY}{1-qXY} \qquad \text{in } \mathbb{Z}[[X,Y]] \tag{5.1.1}$$

with logarithm

$$\log_q(X) = \sum_{n\geq 1} \frac{[n]_q}{n} X^n \qquad \text{in } \mathbb{Q}[[X]]\,,$$

where $[n]_q := 1+q+\ldots+q^{n-1}$. Note that we have written $\log_q(X)$ instead of $\log_{F_q}(X)$, for simplicity; actually, throughout this chapter, we replace every subscript or superscript F_q by q. Let us also note that $F_0(X,Y)$ is the multiplicative formal group law, while $F_{-1}(X,Y)$ gives the addition formula for the hyperbolic tangent. It is worth mentioning that the formal group law $F_q(X,Y)$ is relevant to algebraic topology in the following sense: the ring homomorphism from the Lazard ring, which we identify with MU_*, to $\mathbb{Z}$ mapping the coefficients of the universal formal group law to the coefficients of $F_q(X,Y)$ is precisely the *Euler characteristic* for $q=1$, the *Todd genus* for $q=0$, and the *L-genus* for $q=-1$ (see e.g. [30]).

Recall that in §1.9 we have defined A^∞ to be the set of infinite sequences of elements of A, as well as the ghost ring $Gh(A)$. Given the formal group law $F(X,Y)$ over A, we follow [18] by defining the map

$$w^F\colon A\mathbb{Q}^\infty \to Gh(A\mathbb{Q})\,, \qquad w_n^F(\alpha) := \sum_{d|n} a_{n/d}\, \alpha_d^{n/d}\,.$$

The group of Witt vectors $W^F(A\mathbb{Q})$ has underlying set $A\mathbb{Q}^\infty$, and is defined by insisting that w^F be a group homomorphism. Let $\mathcal{C}(F,A)$ denote the group of curves in the formal group law $F(X,Y)$, that is the group $tA[[t]]$ with addition specified by

$$\alpha(t) +_F \beta(t) := F(\alpha(t), \beta(t)) \qquad \text{(cf. (1.2.2))}.$$

We define the map

$$E^F \colon Gh(A\mathbb{Q}) \to \mathcal{C}(F, A\mathbb{Q})\,, \qquad E^F(\alpha) := \exp_F(\alpha(t))\,,$$

where $\alpha(t) := \sum_{n\geq 1} \alpha_n t^n$. The map $H^F \colon W^F(A\mathbb{Q}) \to \mathcal{C}(F, A\mathbb{Q})$ defined by $H^F := E^F \circ w^F$ is known as an *Artin-Hasse type exponential map* associated with the formal group law $F(X,Y)$. It is easy to check that

$$H^F(\alpha) = \sum_{n\geq 1}^{F} \alpha_n t^n\,.$$

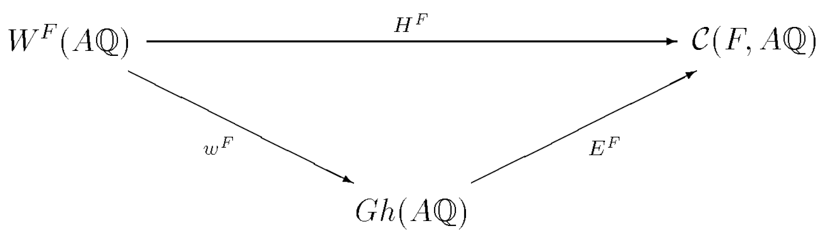
For every positive integer r, the Verschiebung operator V_r is defined on $W^F(A\mathbb{Q})$ and on $Gh(A\mathbb{Q})$ as in (1.9.1), and on $\mathcal{C}(F, A\mathbb{Q})$ by

$$\mathrm{V}_r\, \alpha(t) = \alpha(t^r)\,. \tag{5.1.2}$$

The Frobenius operator f_r is defined on $Gh(A\mathbb{Q})$ and $\mathcal{C}(F, A\mathbb{Q})$ by

$$\mathrm{f}_{r,n}\, \alpha = r\alpha_{rn} \qquad \text{and} \qquad \mathrm{f}_r\, \alpha(t) = \alpha(\rho t^{1/r}) +_F \alpha(\rho^2 t^{1/r}) +_F \ldots +_F \alpha(\rho^r t^{1/r})\,, \tag{5.1.3}$$

respectively, where ρ is a primitive r-th root of unity (see [18]). The Frobenius operator is also defined on $W^F(A\mathbb{Q})$ such that it commutes with H^F. Clearly, V_r acts on $W^F(A)$, $Gh(A)$ and $\mathcal{C}(F, A)$, while f_r acts on $Gh(A)$ and $\mathcal{C}(F, A)$.



(5.1.4)

Theorem 5.1.5 *(cf. [18] §25.1 and Theorem 6.5.8)*

1. *Addition in $W^F(A\mathbb{Q})$ is defined by polynomials with coefficients in A, which means that A^∞ is a subgroup of $W^F(A\mathbb{Q})$ (this is the group of Witt vectors $W^F(A)$).*

2. *The maps w^F, E^F, and H^F are isomorphisms of abelian groups.*

3. *The image of $W^F(A)$ in $\mathcal{C}(F, A\mathbb{Q})$ is precisely $\mathcal{C}(F, A)$.*

4. *The Frobenius operator f_r acts on $W^F(A)$. The maps w^F, E^F, and H^F commute with the actions of the operators V_r and f_r.*

Note that if $F(X,Y)$ is the multiplicative formal group law $F_0(X,Y)$ over A, then $W^F(A)$ coincides with the additive group of $W(A)$. As pointed out in [18], it is quite remarkable that in this case we are able to define a multiplicative structure on $W^F(A)$ as well, such that $\nu \circ g^F$ is a ring homomorphism, for some map $\nu\colon Gh(A\mathbb{Q}) \to Gh(A\mathbb{Q})$ of the form $\nu_n(\alpha) = k_n\alpha_n$ with $k_n \in \mathbb{Q}$ (for $F_0(X,Y)$ we have $\nu_n(\alpha) = n\alpha_n$, as discussed at the end of this section). In §5.4 we prove that this actually happens for every formal group law $F_q(X,Y)$.

We now define and study the necklace algebra associated with the formal group law $F(X,Y)$. In general, we are only able to define it over $A\mathbb{Q}$, so we will denote it by $Nr^F(A\mathbb{Q})$. The module structure of $Nr^F(A\mathbb{Q})$ is the same as that of $Nr(A\mathbb{Q})$. In order to define the multiplicative structure and to relate $Nr^F(A\mathbb{Q})$ to the other structures in diagram 5.1.4, we need to associate with $F(X,Y)$ generalised necklace polynomials. Let us consider the incidence algebra over $A\mathbb{Q}$ of the lattice $D(n)$ of divisors of n. Let ζ^F be the element of this algebra defined by

$$\zeta^F(d_1, d_2) := a_{d_2/d_1}\,,$$

for every $d_1, d_2 \in D(n)$ with $d_1|d_2$. Since $a_1 = 1$, the element ζ^F has a convolution inverse, which will be denoted by μ^F. It is easy to see that $\mu^0(d_1, d_2) =$

$d_1/d_2\,\mu(d_1,d_2) = d_1/d_2\,\mu(d_2/d_1)$, and that $\mu^1(d_1,d_2) = \mu(d_1,d_2) = \mu(d_2/d_1)$. We now define the polynomials

$$M^F(x,n) := \sum_{d|n} \mu^F(d,n)\, a_d\, x^d \qquad \text{in } A\mathbb{Q}[x]\,.$$

Clearly, $M^0(x,n) = M(x,n)$, and $M^F(1,n) = 0$ for $n > 1$. In order to give a combinatorial interpretation for the polynomials $M^F(x,n)$, we recall from [29] the polynomials $S(x,n) := nM(x,n)$. Let us also recall that n in $\mathbb{N}$ is a *period* of the word w (on a given alphabet), if there is a word u such that $w = u^{|w|/n}$, where $|w|$ denotes the length of w; the smallest period is called the *primitive period.* A word with primitive period equal to its length is called *aperiodic.* It is not difficult to prove, via Möbius inversion, that $S(m,n)$ represents the number of aperiodic words of length n on an alphabet with m letters. Necklaces can be defined as equivalence classes of words under the conjugacy relation (that is $w \sim w'$ if and only if there are words u, v such that $w = uv$ and $w' = vu$); moreover, primitive necklaces can be defined as equivalence classes of aperiodic words.

Proposition 5.1.6 *The polynomials $M^F(x,n)$ can be expressed in the basis $\{S(x,i)\}$ of the $A\mathbb{Q}$-module $A\mathbb{Q}[x]$ by the following formula:*

$$M^F(x,n) = \sum_{d|n} \tau^F\left(\frac{n}{d}, n\right)\, S(x,d)\,,$$

where $\tau^F(i,n) := \sum_{j|i} \mu^F(1,j)\,\zeta^F(j,n)\,.$

It turns out that this proposition is a special case of Theorem 5.2.3, so we postpone the proof until then. Let us note that $\tau^F(n,n) = 0$ for $n > 1$, and that $\tau^0(i,n) = \tau^1(i,n) = 0$ unless $i = 1$; indeed, we can pair the chains in $D(n)$ contributing to $\tau^0(i,n)$ such that each pair consists of a chain containing i, and the same chain with i removed. We now explain the combinatorial significance of the above formula in terms of a combinatorial object which we call a *factorised* *word*. This is a word w (on a given alphabet), together with an expression of the following form:

$$w = (\ldots((w_0^{i_1})^{i_2})\ldots)^{i_k}\,.$$

Clearly, $|w_0| = |w|/(i_1 \ldots i_k)$. The word w_0 will be called the *root* of the factorised word. We define the *type* of the factorised word to be the element $(-1)^k a_{|w_0|} a_{i_1} \ldots a_{i_k}$ in $A\mathbb{Q}$. In this section, as well as in §5.2 and §5.3, we usually think of the formal group law $F(X, Y)$ as being the universal one; then the type of a factorised word is a signed monomial in the polynomial generators m_i of $L\mathbb{Q}$. With these definitions, we can now state the following corollary of Proposition 5.1.6.

Corollary 5.1.7 *For all m, n in $\mathbb{N}$, $M^F(m, n)$ in $A\mathbb{Q}$ enumerates by type the factorised words of length n on an alphabet with m letters.*

Proof. A factorised word $w = (\ldots((w_0^{i_1})^{i_2})\ldots)^{i_k}$ of length n on an alphabet with m letters is uniquely determined by the primitive period u of w (and w_0) and the chain $\{1 = d_0|d_1|\ldots|d_k|n\}$ in $D(n)$ with d_k dividing $n/|u|$. Indeed, we set $w_0 := u^{n/(|u|d_k)}$, and $i_j := d_{k-j+1}/d_{k-j}$ for $1 \leq j \leq k$. We can choose u of length d dividing n in $S(m, d)$ ways. According to the above remark, the sum of types of the factorised words of length n and primitive period u is

$$\sum (-1)^k \,\zeta^F(d_0, d_1) \ldots \zeta^F(d_k, n) = \sum_{d_k|n/d} \mu^F(1, d_k)\,\zeta^F(d_k, n) = \tau^F\left(\frac{n}{d}, n\right)\,,$$

where the first summation ranges over all chains which can be associated with u as above; the first equality follows from the generalisation of the formula for the Möbius function of a poset. □

We now relate $Nr^F(A\mathbb{Q})$ to the other groups in diagram 5.1.4, by defining the following maps:

$$T^F\colon W^F(A\mathbb{Q}) \to Nr^F(A\mathbb{Q})\,, \qquad T^F(\alpha) := \sum_{n\geq 1} \mathrm{V}_n M^F(\alpha_n)\,,$$

$$g^F\colon Nr^F(A\mathbb{Q}) \to Gh(A\mathbb{Q})\,, \qquad g_n^F(\alpha) := \sum_{d|n} a_{n/d}\,\alpha_d\,,$$

$$c^F\colon Nr^F(A\mathbb{Q}) \to \mathcal{C}(F, A\mathbb{Q})\,, \qquad c^F(\alpha) := \sum_{n\geq 1}{}^F\, [\alpha_n]_F\, t^n\,;$$

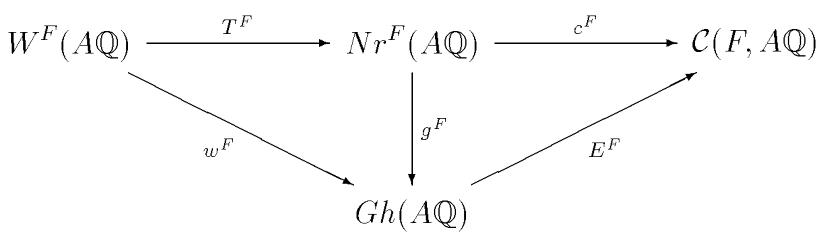
here $M_n^F(b) := M^F(b,n)$, the Verschiebung operator V_r on $Nr^F(A\mathbb{Q})$ is defined as in (1.9.1), and

$$[b]_F(\alpha(t)) := \exp_F(b\,\log_F(\alpha(t))) \qquad \text{for } b \in A\mathbb{Q}\,.$$

Note that the last definition is compatible with (1.2.3).

For every map $\nu\colon Gh(A\mathbb{Q}) \to Gh(A\mathbb{Q})$ of the form $\nu_n(\alpha) = k_n\alpha_n$ with $k_n \in \mathbb{Q}$, we define a multiplication in $Nr^F(A\mathbb{Q})$ by insisting that $\nu \circ g^F$ be a ring homomorphism. For $F(X,Y) = F_0(X,Y)$ and $k_n = n$, we obtain the necklace algebra defined by Metropolis and Rota. The ring structure of $Nr^F(A\mathbb{Q})$ will only be important in §5.4; until then, we regard $Nr^F(A\mathbb{Q})$ only as an abelian group.

Proposition 5.1.8 *All the above maps are isomorphisms of abelian groups, commuting with the action of the Verschiebung operator, and the following diagram is commutative.*



(5.1.9)

Proof. We note first that g^F is invertible, and that its inverse is defined by

$$(g^F)_n^{-1}(\alpha) = \sum_{d|n} \mu^F(d,n)\,\alpha_d\,. \tag{5.1.10}$$

We now have

$$((g^F)^{-1} \circ w^F)_n(\alpha) = \sum_{i|n} \mu^F(i,n) \left(\sum_{j|i} a_j \, \alpha_{i/j}^j \right) ,$$

$$T_n^F(\alpha) = \sum_{i|n} M^F\left(\alpha_i, \frac{n}{i}\right) = \sum_{i|n} \sum_{j|n/i} \mu^F\left(j, \frac{n}{i}\right) \, a_j \, \alpha_i^j$$

$$= \sum_{i|n} \sum_{ij|n} \mu^F(ij,n) \, a_j \, \alpha_i^j = \sum_{k|n} \sum_{j|k} \mu^F(k,n) \, a_j \, \alpha_{k/j}^j \, ;$$

the last equality follows by setting $k := ij$. Hence $g^F \circ T^F = w^F$. To check the commutativity of the second triangle, we note that

$$(\log_F \circ E^F \circ g^F)(\alpha) = (g^F(\alpha))(t) \qquad \text{and} \qquad (\log_F \circ c^F)(\alpha) = \sum_{i \geq 1} \alpha_i \, \log_F(t^i) \, .$$

The coefficients of t^n in both power series above are equal to $\sum_{i|n} a_{n/i} \, \alpha_i$, whence $E^F \circ g^F = c^F$.

The map g^F is clearly an isomorphism of abelian groups, and commutes with the Verschiebung operator. By using the commutativity of diagram 5.1.9 and Theorem 5.1.5, we deduce that T^F and c^F have the same properties as g^F. Note that the inverse of T^F can be found by using an algorithm similar to the *clearing algorithm* in [29]. □

Diagram 5.1.9 for $F(X,Y) = F_0(X,Y)$ is not exactly the same as diagram 1.9.3. In order to explain the relation between them, we define the following homomorphisms:

$$\omega \colon Gh(A\mathbb{Q}) \to Gh(A\mathbb{Q}) \, , \qquad \omega_n(\alpha) := n\alpha_n \, , \tag{5.1.11}$$

$$\iota \colon \mathcal{C}(F_0, A\mathbb{Q}) \to 1 + tA\mathbb{Q}[[t]] \, , \qquad \iota(\alpha(t)) := \frac{1}{1-\alpha(t)} \, .$$

We can easily check that

$$w = \omega \circ w^0 \, , \;\; g = \omega \circ g^0 \, , \;\; c = \iota \circ c^0 \, , \;\; E = \iota \circ E^0 \circ \omega^{-1} \, .$$

A first result which validates our constructions is a formal group-theoretic generalisation of the cyclotomic identity; in some cases, we are able to derive from it nice explicit identities.

Proposition 5.1.12 *The following formal group-theoretic generalisation of the cyclotomic identity (V. Strehl's form) holds:*

$$\sum_{n\geq 1}^{F} [M^F(u,n)]_F(vt^n) = \sum_{n\geq 1}^{F} [M^F(v,n)]_F(ut^n) \qquad \text{in } \mathcal{C}(F, A\mathbb{Q})\,, \tag{5.1.13}$$

where $u, v \in A\mathbb{Q}$. *In particular, for* $F_0(X,Y)$ *we obtain (1.9.4), and for* $F_{-1}(X,Y)$ *we obtain*

$$\gamma(t;\, k,\, m) = \gamma(t;\, m,\, k)\,, \tag{5.1.14}$$

where $k, m \in \mathbb{Z}$, *and*

$$\gamma(t;\, i,\, j) := \frac{\prod_{n\geq 1}(1+it^{2n-1})^{M(j,2n-1)} - \prod_{n\geq 1}(1-it^{2n-1})^{M(j,2n-1)}}{\prod_{n\geq 1}(1+it^{2n-1})^{M(j,2n-1)} + \prod_{n\geq 1}(1-it^{2n-1})^{M(j,2n-1)}} \qquad \text{in } \mathbb{Z}[[t]]\,.$$

Proof. For the first part, we use the following identity which holds in $Nr^F(A\mathbb{Q})$:

$$\sum_{n\geq 1} M^F(u,n)\, \mathrm{V}_n\, M^F(v) = \sum_{n\geq 1} M^F(v,n)\, \mathrm{V}_n\, M^F(u)\,;$$

indeed, the n-th term in both sequences is

$$\sum_{d|n} M^F(u,d)\, M^F\left(v, \frac{n}{d}\right)\,.$$

We apply c^F to this identity, using the following facts:

$$c^F(\mathrm{V}_n\, M^F(u)) = \mathrm{V}_n\, c^F(M^F(u)) = \mathrm{V}_n\, H^F(u,0,0,\dots) = ut^n \quad \text{and}$$
$$c^F(u\alpha) = \sum_{n\geq 1}^{F} [u\alpha_n]_F(t^n) = \sum_{n\geq 1}^{F} [u]_F([\alpha_n]_F(t^n)) = [u]_F(c^F(\alpha))\,;$$

here $u \in A\mathbb{Q}$ and $\alpha \in A\mathbb{Q}^\infty$.

In order to derive (5.1.14) from (5.1.13), we note first that we have

$$\sum_{m\geq 1}^{-1} X_m = \frac{\prod_{m\geq 1}(1+X_m) - \prod_{m\geq 1}(1-X_m)}{\prod_{m\geq 1}(1+X_m) + \prod_{m\geq 1}(1-X_m)}\,. \tag{5.1.15}$$

This is easy to prove by induction when we have a finite sum on the left-hand side; we then take the limit in the filtration topology of $\mathbb{Z}[[X_1, X_2, \ldots]]$. Since

$$[k]_F(X) = \sum_{1\le i\le |k|}^{F} [\mathrm{sign}(k)]_F(X)$$

for any formal group law F and integer k, and since

$$[-1]_{-1}(X) = -X = \frac{(1+X)^{-1} - (1-X)^{-1}}{(1+X)^{-1} + (1-X)^{-1}}\,,$$

formula (5.1.15) generalises to

$$\sum_{m\ge 1}^{-1}[k_m]_{-1}(X_m) = \frac{\prod_{m\ge 1}(1+X_m)^{k_m} - \prod_{m\ge 1}(1-X_m)^{k_m}}{\prod_{m\ge 1}(1+X_m)^{k_m} + \prod_{m\ge 1}(1-X_m)^{k_m}}\,, \quad k_m \in \mathbb{Z}\,.$$

Finally, we note that

$$M^{-1}(i,m) = \begin{cases} M(i,m) & \text{if } m \text{ odd} \\ 0 & \text{otherwise}\,, \end{cases}$$

since $\log_{-1}(X) = X + X^3/3 + X^5/5 + \ldots$. $\square$

5.2 Verschiebung and Frobenius Operators

In the previous section, we have defined for all positive integers r the Verschiebung operator V_r and the Frobenius operator f_r on $Gh(R)$, $W^F(R)$, and $\mathcal{C}(F,R)$, where R is one of the rings $A\mathbb{Q}$ or A. We have also defined V_r on $Nr^F(A\mathbb{Q})$ and $Nr^F(A)$. We have seen that the isomorphisms in diagram 5.1.9 commute with the actions of these operators. It is natural to define f_r on $Nr^F(A\mathbb{Q})$ in a compatible way with the isomorphisms mentioned above. It turns out that, in general, f_r is not an operator on $Nr^F(A)$. Let us recall the well-known identities concerning the interaction of the Verschiebung and Frobenius operators on any of the rings on which they act (see [18], [29], [14], [54]):

$$\begin{aligned} &\mathrm{V}_r \mathrm{V}_s = \mathrm{V}_{rs}\,, \qquad \mathrm{f}_r \mathrm{f}_s = \mathrm{f}_{rs}\,, \\ &\mathrm{f}_r \mathrm{V}_r = r\, I\,, \\ &\mathrm{f}_r \mathrm{V}_s = (r,s)\, \mathrm{f}_{r/(r,s)} \mathrm{V}_{s/(r,s)} = (r,s)\, \mathrm{V}_{s/(r,s)} \mathrm{f}_{r/(r,s)}\,; \end{aligned} \tag{5.2.1}$$

these identities are most easily checked in $Gh(A\mathbb{Q})$. In this section, we intend to express and interpret combinatorially the action of the Frobenius operator on $Nr^F(A\mathbb{Q})$.

Theorem 5.2.2 *The Frobenius operator* f_r *acts on* $Nr^F(A\mathbb{Q})$ *as follows:*

$$\mathrm{f}_{r,n}\, \alpha = r \sum_{d|rn} \tau^F \left(\frac{rn}{[r,d]}, \frac{rn}{d} \right) \alpha_d\,.$$

Proof. By (5.1.10), we have

$$\begin{aligned} \mathrm{f}_{r,n}\, \alpha &= (g^F)_n^{-1} (\mathrm{f}_r\, g^F(\alpha)) = r \sum_{i|n} \mu^F(i,n) \left(\sum_{d|ri} a_{ri/d}\, \alpha_d \right) \\ &= r \sum_{d|rn} \left(\sum_{d|ri,\, i|n} \mu^F\left(1, \frac{n}{i}\right) \zeta^F \left(\frac{n}{i}, \frac{rn}{d} \right) \right) \alpha_d \\ &= r \sum_{d|rn} \left(\sum_{j|rn/[r,d]} \mu^F(1,j)\, \zeta^F \left(j, \frac{rn}{d} \right) \right) \alpha_d = r \sum_{d|rn} \tau^F \left(\frac{rn}{[r,d]}, \frac{rn}{d} \right) \alpha_d\,. \end{aligned}$$

The fourth equality follows by setting $j := n/i$ and noting that the conditions $d|rn/j$ and $j|n$ are equivalent to $j|rn/[r,d]$. □

Note that if $r|d$ and $d \neq rn$, then $\tau^F(rn/[r,d], rn/d) = 0$. On the other hand, according to the observations about μ^0 and τ^0 in §5.1, we have that $\tau^0(rn/[r,d], rn/d) = 0$ unless $[r,d] = rn$, in which case it is equal to d/rn; hence, we recover the formula in [29] for the action of f_r on $Nr(A)$, namely

$$\mathrm{f}_{r,n}\, \alpha = \sum_d \frac{d}{n}\, \alpha_d\,,$$

where the summation ranges over the set $\{d \,:\, [r,d]=rn\}$.

We now interpret combinatorially the action of f_r on $Nr^F(A\mathbb{Q})$ by computing $\mathrm{f}_{r,n}\, M^F(m)$ for $m,n\in\mathbb{N}$.

Theorem 5.2.3 *We have that*

$$\mathrm{f}_{r,n}\, M^F(x) = r\sum_{d|n}\mu^F(d,n)\, a_{rd}\, x^{rd}\,. \tag{5.2.4}$$

The above polynomial can be expressed in the basis $\{S(x,i)\}$ of the $A\mathbb{Q}$-module $A\mathbb{Q}[x]$ by the following formula

$$\mathrm{f}_{r,n}\, M^F(x) = r\sum_{d|n}\tau^F\left(\frac{n}{d}, rn\right)\, S(x^r,d)\,. \tag{5.2.5}$$

PROOF. Formula (5.2.4) follows easily:

$$\begin{aligned}\mathrm{f}_{r,n}\, M^F(x) &= (g^F)_n^{-1}(\mathrm{f}_r\, w^F(x,0,0,\dots)) = r\,(g^F)_n^{-1}(a_r x^r, a_{2r}x^{2r},\dots)\\ &= r\sum_{d|n}\mu^F(d,n)\, a_{rd}\, x^{rd}\,.\end{aligned}$$

Formula (5.2.5) follows by rewriting its right-hand side:

$$\begin{aligned}\sum_{d|n}\tau^F\left(\frac{n}{d},rn\right)\, S(x^r,d) &= \sum_{d|n}\left(\sum_{i|n/d}\mu^F(1,i)\,\zeta^F(i,rn)\right)\left(\sum_{j|d}\mu(j,d)\,x^{rj}\right)\\ &= \sum_{j|n}\sum_{i|n/j}\mu^F(1,i)\,\zeta^F(i,rn)\,x^{rj}\left(\sum_{j|d|n/i}\mu(j,d)\right)\\ &= \sum_{j|n}\mu^F\left(1,\frac{n}{j}\right)\,\zeta^F\left(\frac{n}{j},rn\right)\,x^{rj}\\ &= \sum_{j|n}\mu^F(j,n)\,a_{rj}\,x^{rj} = \mathrm{f}_{r,n}\, M^F(x)\,.\end{aligned}$$

□

Let us note that $\mathrm{f}_r\mathrm{V}_s\, M^F(x)$ can be easily computed now, by using (5.2.1). Proposition 5.1.6 follows from (5.2.5) by setting $r:=1$. Let us also note that $\tau^0(n/d,rn)=0$ unless $d=n$, in which case it is equal to $1/(rn)$; hence, (5.2.5) implies Theorem 4 (p. 100) in [29], namely the fact that $\mathrm{f}_{r,n}\, M(x) = M(x^r,n)$.

We now define the *repetition factor* of a word w to be the quotient of $|w|$ by the primitive period of w. With this definition, we can interpret (5.2.5) as follows.

Corollary 5.2.6 *For all $m, n \in \mathbb{N}$, $1/r\, \mathrm{f}_{r,n}\, M^F(m)$ in $A\mathbb{Q}$ enumerates by type those factorised words of length rn on an alphabet with m^r letters, for which r divides the repetition factor of the root.*

PROOF. Let us recall from the proof of Corollary 5.1.7 the correspondence between factorised words $w = (\ldots((w_0^{i_1})^{i_2})\ldots)^{i_k}$ and pairs consisting of an aperiodic word u and a chain $\{1 = d_0|d_1|\ldots|d_k\}$ with d_k dividing $|w|/|u|$. In this case, the alphabet has size m^r, the words w have length rn, the length of their primitive period u divides n, and $d_k = rn/|w_0|$ divides $n/|u|$. The last condition is equivalent to "r divides $|w_0|/|u|$", and this implies the fact that $|u|$ divides n. □

5.3 The p-typification Idempotent

Let $A_{(p)} := A \otimes \mathbb{Z}_{(p)}$. Recall from [18] that a curve $\alpha(t)$ in $\mathcal{C}(F, A)$ is called *p-typical* if $\log_F(\alpha(t))$ is of the form $\sum_{n\geq 0} \beta_n t^{p^n}$. There is a remarkable idempotent ε_p on $\mathcal{C}(F, A_{(p)})$, which is a projection onto the subgroup of p-typical curves; we will call it the *p-typification idempotent*. It is expressed in terms of V_r and f_r as follows:

$$\varepsilon_p = \sum_{(r,p)=1} \frac{1}{r}\mu(r)\, \mathrm{V}_r\, \mathrm{f}_r\,.$$

The p-typification idempotent has an important rôle in formal group theory, since the curve $\varepsilon_p\, t$ is an isomorphism over the localisation of the Lazard ring $L_{(p)}$ between the universal formal group law and the universal p-typical formal group law (see [18] or [33]). We can define ε_p on $Gh(A)$ (not just $Gh(A_{(p)})$), $W^F(A_{(p)})$, and $Nr^F(A\mathbb{Q})$. The action on $Gh(A)$ is very easy to describe, namely:

$$\varepsilon_{p,n}\,\alpha = \begin{cases} \alpha_n & \text{if } n = p^k \\ 0 & \text{otherwise}. \end{cases}$$

In order to describe the action of ε_p on $Nr^F(A\mathbb{Q})$, we need some additional notation. First, we denote by $v_p(n)$ the p-valuation of n (that is the largest integer k such that $p^k|n$). Now assume that $m \neq p^k$, $k > 0$, and consider the poset $D_p(m)$ obtained from the lattice of divisors of m by removing all non-zero powers of p. Let μ_p^F denote the convolution inverse of ζ^F in the incidence algebra (over $A\mathbb{Q}$) of this poset. We will write $\mu_p^F(m)$ for $\mu_p^F(1,m)$ if $m \neq p^k$, $k > 0$; otherwise, we set $\mu_p^F(m) = 0$.

Theorem 5.3.1 *The idempotent ε_p acts on $Nr^F(A\mathbb{Q})$ as follows*

$$\varepsilon_{p,n}\,\alpha = \sum_{k=0}^{v_p(n)} \mu_p^F\left(\frac{n}{p^k}\right)\,\alpha_{p^k}\,. \tag{5.3.2}$$

In particular, the idempotent ε_p acts on $Nr(A_{(p)})$ by

$$\varepsilon_{p,n}\,\alpha = \frac{p^{v_p(n)}}{n}\,\mu\left(\frac{n}{p^{v_p(n)}}\right)\,\alpha_{p^{v_p(n)}}\,. \tag{5.3.3}$$

PROOF. From $g^F(\varepsilon_p\,\alpha) = \varepsilon_p\,g^F(\alpha)$, it follows that

$$\sum_{d|n} a_{n/d}\,\varepsilon_{p,d}\,\alpha = \begin{cases} \sum_{i=0}^{k} a_{p^{k-i}}\,\alpha_{p^i} & \text{if } n = p^k \\ 0 & \text{otherwise}. \end{cases} \tag{5.3.4}$$

An easy induction provides $\varepsilon_{p,p^k}\,\alpha = \alpha_{p^k}$; hence (5.3.2) holds in this case, according to the convention $\mu_p^F(p^k) = 0$ for $k > 0$. We now use induction once more and (5.3.4) to prove (5.3.2) for $n \neq p^k$, $k > 0$:

$$\begin{aligned}
\varepsilon_{p,n}\,\alpha &= -\sum_{d|n,\, d\neq n} a_{n/d}\,\varepsilon_{p,d}\,\alpha = -\sum_{d|n,\, d\neq n} a_{n/d}\left(\sum_{l=0}^{v_p(d)} \mu_p^F\left(\frac{d}{p^l}\right)\alpha_{p^l}\right) \\
&= \sum_{l=0}^{v_p(n)}\left(-\sum_{p^l|d|n,\, d\neq n} a_{n/d}\,\mu_p^F\left(\frac{d}{p^l}\right)\right)\alpha_{p^l} \\
&= \sum_{l=0}^{v_p(n)}\left(-\sum_{i|n/p^l,\, i\neq n/p^l} \mu_p^F(1,i)\,\zeta^F\left(i,\frac{n}{p^l}\right)\right)\alpha_{p^l} \\
&= \sum_{l=0}^{v_p(n)} \mu_p^F\left(\frac{n}{p^l}\right)\alpha_{p^l}\,.
\end{aligned}$$

We now compute $\mu_p(n) := n\mu_p^0(n)$ by using Proposition 1.7.15. Assuming that $n \neq p^k$, $k > 0$, we have

$$\begin{aligned}
\mu_p(n) = \mu_{D_p(n)}(1,n) &= \sum_{l=0}^{v_p(n)} \sum_{0=l_0<l_1<\ldots<l_r=l} (-1)^r\,\mu(1,p^{l_1})\ldots\mu(p^{l_{r-1}},p^l)\,\mu(p^l,n) \\
&= \sum_{l=0}^{v_p(n)} \mu\left(\frac{n}{p^l}\right).
\end{aligned}$$

Here we have used

$$\mu(p^k) = \begin{cases} 0 & \text{if } k > 1 \\ -1 & \text{if } k = 1\,. \end{cases}$$

Recalling the additional fact that $\mu(rs) = \mu(r)\,\mu(s)$ if $(r,s) = 1$, we finally have

$$\mu_p(n) = \begin{cases} 0 & \text{if } p|n \\ \mu(n) & \text{otherwise}\,. \end{cases}$$

This is clearly true for $n = p^k$ as well, whence (5.3.3) holds. □

Finally, we interpret combinatorially the action of ε_p on $Nr^F(A\mathbb{Q})$ by computing $\varepsilon_{p,n}\,M^F(m)$ for $m, n \in \mathbb{N}$.

Theorem 5.3.5 *We have that*

$$\varepsilon_{p,n}\,M^F(x) = \sum_{k=0}^{v_p(n)} \mu^F(p^k,n)\,a_{p^k}\,x^{p^k} \qquad \textit{in } A\mathbb{Q}[x]\,. \tag{5.3.6}$$

The above polynomial can be expressed in the basis $\{S(x,i)\}$ of the $A\mathbb{Q}$-module $A\mathbb{Q}[x]$ by the following formula

$$\varepsilon_{p,n}\, M^F(x) = \sum_{k=0}^{v_p(n)} \left(\sum_{i=0}^{v_p(n)-k} \mu^F\left(1, \frac{n}{p^{i+k}}\right) a_{p^{i+k}} \right) S(x, p^k)\,. \tag{5.3.7}$$

Proof. Formula (5.3.6) follows easily:

$$\begin{aligned}
\varepsilon_{p,n}\, M^F(x) &= (g_n^F)^{-1}(\varepsilon_p\, w^F(x,0,0,\dots)) \\
&= (g_n^F)^{-1}(x, 0, \dots, 0, a_p x^p, 0, \dots, 0, a_{p^2} x^{p^2}, 0, \dots) \\
&= \sum_{k=0}^{v_p(n)} \mu^F(p^k, n)\, a_{p^k}\, x^{p^k}\,.
\end{aligned}$$

Formula (5.3.7) follows by noting that its right-hand side can be written as

$$\sum_{i=0}^{v_p(n)} \mu^F\left(1, \frac{n}{p^i}\right) a_{p^i}\, x + \sum_{k=1}^{v_p(n)} \left(\sum_{i=0}^{v_p(n)-k} \mu^F\left(1, \frac{n}{p^{i+k}}\right) a_{p^{i+k}} \right) (x^{p^k} - x^{p^{k-1}})\,.$$

Hence the coefficient of x^{p^k} (with $0 \le k \le v_p(n)$) in the right-hand side is equal to

$$\sum_{i=0}^{v_p(n)-k} \mu^F\left(1, \frac{n}{p^{i+k}}\right) a_{p^{i+k}} - \sum_{i=0}^{v_p(n)-k-1} \mu^F\left(1, \frac{n}{p^{i+k+1}}\right) a_{p^{i+k+1}} = \mu^F(p^k, n)\, a_{p^k}\,;$$

here the second sum does not appear if $k = v_p(n)$. □

We can interpret (5.3.7) combinatorially as follows.

Corollary 5.3.8 *For all $m, n \in \mathbb{N}$, $\varepsilon_{p,n}\, M^F(m)$ in $A\mathbb{Q}$ enumerates by type those factorised words of length n on an alphabet with m letters for which the root length is a power of p.*

Proof. We recall once again from the proof of Corollary 5.1.7 the correspondence between factorised words $w = (\dots((w_0^{i_1})^{i_2})\dots)^{i_l}$ and pairs consisting of an aperiodic word u and a chain $\{1 = d_0|d_1|\dots|d_l\}$ with d_l dividing $|w|/|u|$. In this case the alphabet has size m, the words w have length n, their primitive period

u has length p^k for some $0 \leq k \leq v_p(n)$, and $n/|w_0| = d_l = n/p^{i+k}$ for some $0 \leq i \leq v_p(n) - k$. The last two conditions are equivalent to $|w_0|$ being a power of p. $\square$

5.4 Special Cases

The main special case which we consider is the family of formal group laws $F_q(X,Y)$, $q \in \mathbb{Z}$, over $\mathbb{Z}$ defined in (5.1.1). Recall that the classical ring of Witt vectors and the necklace algebra of Metropolis and Rota correspond to $q = 0$ (in other words, to the multiplicative formal group law). According to the general constructions, we have the group of Witt vectors $W^q(\mathbb{Z})$ and the necklace algebra $Nr^q(\mathbb{Q})$, where the multiplicative structure of the latter depends on the choice of a map $\nu\colon Gh(\mathbb{Q}) \to Gh(\mathbb{Q})$ of the form $\nu_n(\alpha) = k_n \alpha_n$ with $k_n \in \mathbb{Q}$; more precisely, this structure is defined by insisting that $\nu \circ g^q$ be an algebra map.

Let us consider first the case $q = 1$ and $\nu = I$. We have that

$$g_n^1(\alpha) = \sum_{d|n} \alpha_d \,.$$

Hence, according to [54], $\mathbb{Z}^\infty$ is a subring of $Nr^1(\mathbb{Q})$, and this is precisely the *aperiodic ring* $Ap(\mathbb{Z})$. Multiplication in $Ap(\mathbb{Z})$ is defined by

$$(\alpha \cdot \beta)_n = \sum_{[i,j]=n} \alpha_i \beta_j \,.$$

From now on, we let $\nu := \omega$, where ω was defined in (5.1.11). In order to simplify notation, we set $\widetilde{g}^q := \nu \circ g^q$ and $\widetilde{\tau}^q(d,n) := n\tau^q(d,n)$. Theorem 5.4.8 represents the main result of this section, generalising the classical necklace algebra construction (which can be recovered for $q = 0$); its proof is based on the following two lemmas.

Lemma 5.4.1

1. *If $q \equiv 1 \mod p$ for a given prime p, then $[p^l m]_q$ is divisible by p^l for any positive integers l, m.*

2. *The polynomials $n/d\,\tau^q(d,n)$ in $\mathbb{Q}[q]$ are numerical polynomials for all positive integers d, n with $d|n$.*

PROOF. We fix q in $\mathbb{Z}$, and divide the proof into three steps.

Step 1. Clearly, it is enough to consider $m = 1$, and $q = pr + 1$ with $r \neq 0$. In this case we have

$$[p^l]_q = \frac{(pr+1)^{p^l} - 1}{pr} = \sum_{i=1}^{p^l} \binom{p^l}{i} p^{i-1} r^{i-1} .$$

It would be enough to show that p^l divides $\binom{p^l}{i} p^{i-1}$ for all $i = 1, \dots, p^l$. Let $i = p^k j$ with $(p, j) = 1$ be such a number, and denote $\binom{p^l}{p^k j} p^{p^k j - 1}$ by N. We use the formula $v_p(n!) = \sum_{s=1}^{\infty} [n/p^s]$, where $[\alpha]$ denotes the greatest integer which is less or equal to α. The crucial observation is that

$$[p^l/p^s] \geq [p^k j/p^s] + [(p^l - p^k j)/p^s] ,$$

and that this inequality is strict for all integers s with $k + 1 \leq s \leq l$. Hence

$$v_p(N) \geq l - k + p^k - 1 \geq l .$$

Step 2. We now show that p^k divides $\widetilde{\tau}^q(p^k, p^k n)$, where p is a prime and n a positive integer. If $q \equiv 1 \mod p$, then this is clearly true by step 1, since every term of $\widetilde{\tau}^q(p^k, p^k n)$ is of the form $\pm [p^{l_1}]_q \dots [p^{l_s}]_q [p^{l_{s+1}} n]_q$ with $l_1 + \ldots + l_{s+1} = k$.

If $q \not\equiv 1 \mod p$, we use induction on k, which obviously starts at 0. Partitioning the terms in $\widetilde{\tau}^q(p^k, p^k n)$ according to the smallest element different from 1 in the chains from 1 to $p^k n$ in $D(p^k n)$ corresponding to them, we obtain

$$\widetilde{\tau}^q(p^k, p^k n) = [p^k n]_q - \sum_{i=1}^{k} [p^i]_q \, \widetilde{\tau}^q(p^{k-i}, p^{k-i} n) . \tag{5.4.2}$$

We know that p^l divides $q^{p^l m} - q^{p^{l-1}m}$ for all positive integers l, m, since $M(x, p^l)$ is a numerical polynomial; hence p^l divides $[p^l m]_q - [p^{l-1}m]_q$, by the assumption on q. Applying this fact and induction to (5.4.2), we deduce that $\widetilde{\tau}^q(p^k, p^k n)$ is congruent modulo p^k to

$$[p^{k-1}n]_q - \widetilde{\tau}^q(p^{k-1}, p^{k-1}n) - \sum_{i=2}^{k} [p^{i-1}]_q \, \widetilde{\tau}^q(p^{k-i}, p^{k-i}n) \, .$$

But this is equal to 0, by using (5.4.2) once again.

Step 3. We prove that d divides $\widetilde{\tau}^q(d, n)$ for every $d|n$. Given a prime p dividing d, we have $d = p^k m$ and $n = p^k m r$ for some positive integers k, m, r with $(p, m) = 1$. We now use the following identity:

$$\widetilde{\tau}^q(p^k m, p^k m r) = \sum (-1)^{s-1} \, \widetilde{\tau}^q(p^{i_1}, p^{i_1} j_1) \ldots \widetilde{\tau}^q(p^{i_s}, p^{i_s} j_s) \, , \tag{5.4.3}$$

where the summation ranges over all $i_t \geq 0$, $j_t > 1$ with $i_1 + i_2 + \ldots + i_s = k$, $j_1 j_2 \ldots j_s = mr$, and $r | j_s$. According to the previous step, we have $p^k | \widetilde{\tau}^q(d, n)$. To prove (5.4.3), we consider the sets $\mathcal{C}(i_1, \ldots, i_s; j_1, \ldots, j_s)$, with i_t, j_t as above, consisting of all chains from 1 to n in $D(n)$ of the form

$$1|p^{i_{11}}| \ldots p^{i_{1l_1}}|p^{i_1} j_1|p^{i_1+i_{21}} j_1| \ldots p^{i_1+i_{2l_2}} j_1|p^{i_1+i_2} j_1 j_2| \ldots p^{i_1+\ldots+i_{s-1}+i_{sl_s}} j_1 \ldots j_{s-1}|n.$$

These sets determine a partition of the chains from 1 to n in $D(n)$ contributing to $\widetilde{\tau}^q(d, n)$. Furthermore, the sum of terms corresponding to the chains in $\mathcal{C}(i_1, \ldots, i_s; j_1, \ldots, j_s)$ is precisely $(-1)^{s-1} \, \widetilde{\tau}^q(p^{i_1}, p^{i_1} j_1) \ldots \widetilde{\tau}^q(p^{i_s}, p^{i_s} j_s)$, whence (5.4.3) is proved. □

Lemma 5.4.4 *For every $q \neq 1$, we have that*

$$e_i \cdot e_j = j \, \mathrm{V}_{[i,j]} \left(\sum_{d|n, \, d \neq 1} \tau^q \left(\frac{n}{d}, \frac{[i,j]}{i} n \right) \frac{S(q^{[i,j]/j}, d)}{q-1} + \tau^q \left(n, \frac{[i,j]}{i} n \right) \left[\frac{[i,j]}{j} \right]_q \right)_{n \geq 1}$$

where $e_{r,s} = \delta_{r,s}$.

PROOF. We base our computation of $e_i \cdot e_j$ on the following formula:

$$e_i \cdot e_j = (\widetilde{g}^q)^{-1}(\widetilde{g}^q(e_i) \cdot \widetilde{g}^q(e_j)) \,. \tag{5.4.5}$$

In order to make the map $\widetilde{g}^q \colon Nr^q(\mathbb{Q}) \to Gh(\mathbb{Q})$ commute with the Verschiebung operator, we have to consider the operator $\mathrm{V}'_r := r\mathrm{V}_r$ on $Gh(\mathbb{Q})$. We have

$$\widetilde{g}^q(e_i) \cdot \widetilde{g}^q(e_j) = (\mathrm{V}'_i \, \widetilde{g}^q(e_1)) \cdot (\mathrm{V}'_j \, \widetilde{g}^q(e_1)) \,,$$

and $\widetilde{g}^q_n(e_1) = [n]_q$, whence

$$\widetilde{g}^q(e_i) \cdot \widetilde{g}^q(e_j) = ij \, \mathrm{V}_{[i,j]} \, \pi = (i,j) \, \mathrm{V}'_{[i,j]} \, \pi \,,$$

where $\pi_n := [kn]_q [ln]_q$, and $k := [i,j]/i$, $l := [i,j]/j$. By (5.4.5), we have

$$e_i \cdot e_j = (i,j) \, \mathrm{V}_{[i,j]} \, (\widetilde{g}^q)^{-1}(\pi) \,. \tag{5.4.6}$$

On the other hand, by (5.1.10) and (5.2.4) we have

$$\begin{aligned}(\widetilde{g}^q)^{-1}_n(\pi) &= \sum_{d|n} \mu^q(d,n) \, \frac{\pi_d}{d} = k \sum_{d|n} \mu^q(d,n) \, \frac{[kd]_q}{kd} \, [ld]_q \\ &= \mathrm{f}_{k,n} \, M^q(x) \,, \quad x^{kr} \equiv [lr]_q \,;\end{aligned}$$

the "umbral notation" $x^{kr} \equiv [lr]_q$ means that x^{kr} is replaced by $[lr]_q$ after collecting powers of x in $\mathrm{f}_{k,n} \, M^q(x)$. Furthermore, combining this result with (5.2.5), we have

$$(\widetilde{g}^q)^{-1}_n(\pi) = k \sum_{d|n} \tau^q\left(\frac{n}{d}, kn\right) \; S(x^k, d) \,, \quad x^{kr} \equiv [lr]_q \,. \tag{5.4.7}$$

Finally, since $S(1,d) = 0$ for $d > 1$, we can rewrite (5.4.7) as

$$(\widetilde{g}^q)^{-1}_n(\pi) = k \sum_{d|n,\, d\neq 1} \tau^q\left(\frac{n}{d}, kn\right) \; \frac{S(q^l, d)}{q-1} + k \, \tau^q(n, kn) \, [l]_q \,.$$

The lemma now follows by combining this result with (5.4.6). □

Theorem 5.4.8

1. *The polynomials $M^q(x,n)$ are numerical polynomials in x and q.*

2. *Multiplication in $Nr^q(\mathbb{Q})$ is defined by numerical polynomials $P_{n,i,j}(q)$ in $\mathbb{Q}[q]$, with $[i,j]$ dividing n, in the sense that*

$$(\alpha \cdot \beta)_n = \sum_{[i,j]|n} (i,j)\, P_{n,i,j}(q)\, \alpha_i\, \beta_j\,.$$

Hence, there is a $\mathbb{Z}$-algebra structure on $Nr^q(\mathbb{Z})$.

3. *The Frobenius operator* f_r *acts on $Nr^q(\mathbb{Z})$.*

4. *The map T^q induces a group isomorphism between $W^q(\mathbb{Z})$ and $Nr^q(\mathbb{Z})$.*

Proof. (1) According to Proposition 5.1.6, we have

$$M^q(x,n) = \sum_{d|n} \tau^q\left(\frac{n}{d}, n\right)\, S(x,d)\,.$$

The claim now follows from Lemma 5.4.1 (2), and the fact that $M(x,d) = S(x,d)/d$ are numerical polynomials.

(2) It suffices to show that $e_i \cdot e_j$ is obtained by applying $\mathrm{V}_{[i,j]}$ to a sequence of integers divisible by (i,j). This is clearly true for $q=1$, since formula (5.4.7) still holds. We now fix the integers $q \neq 1$ and $i,j,n>0$, and use Lemma 5.4.4. By Lemma 5.4.1 (2)

$$\frac{j}{(i,j)}\, \tau^q\left(n, \frac{[i,j]}{i} n\right) = \frac{[i,j]}{i}\, \tau^q\left(n, \frac{[i,j]}{i} n\right)$$

is an integer. Hence it suffices to show that the following number is an integer

$$\frac{j}{(i,j)}\, \tau^q\left(\frac{n}{d}, \frac{[i,j]}{i} n\right)\, \frac{S(q^{[i,j]/j},d)}{q-1} = \frac{N}{n}\,,$$

where

$$N := \widetilde{\tau}^q\left(\frac{n}{d}, \frac{[i,j]}{i} n\right)\, \frac{S(q^{[i,j]/j},d)}{q-1}\,,$$

and $d \neq 1$ is a divisor of n. Note that $q-1$ divides $S(q^{[i,j]/j},d)$, since $S(1,d)=0$ for $d \neq 1$. On the other hand, n divides $(q-1)N$, since $M(q^{[i,j]/j},d) = S(q^{[i,j]/j},d)/d$

is a numerical polynomial and n/d divides $\widetilde{\tau}^q(n/d, [i,j]n/i)$ by Lemma 5.4.1 (2). We now show that every prime power p^k dividing n also divides N. If $q \not\equiv 1 \bmod p$, this claim follows from the fact that n divides $(q-1)N$; otherwise, p^k divides $\widetilde{\tau}^q(n/d, [i,j]n/i)$ by Lemma 5.4.1 (1).

(3) This follows from Lemma 5.4.1 (2) and Theorem 5.2.2.

(4) This follows from the fact that $M^q(x, n)$ are integral polynomials, and from the construction of the inverse of T^q via an algorithm similar to the *clearing algorithm* in [29]. □

The main thrust of Theorem 5.4.8 is the existence of necklace algebras $Nr^q(\mathbb{Z})$ for all $q \in \mathbb{Z}$. We now use the maps T^q and H^q to define multiplicative structures on $W^q(\mathbb{Z})$ and $\mathcal{C}(F^q, \mathbb{Z})$.

Corollary 5.4.9 *There are ring structures on $W^q(\mathbb{Z})$ and $\mathcal{C}(F^q, \mathbb{Z})$ such that the restrictions of the maps T^q, H^q and c^q are ring isomorphisms, and the restriction of $\omega \circ w^q$ is a ring homomorphism.*

Thus, we have identified a family of formal group laws not mentioned in [18], for which the corresponding groups of Witt vectors and curves have ring structures compatible with the maps in diagram 5.1.9.

Recall the formula $\mathrm{f}_{r,n}\, M(x) = M(x^r, n)$ in [29], which holds in $Nr(A)$, and which was generalised to $Nr^F(A\mathbb{Q})$ in (5.2.5). We present here a conjecture, which attempts to provide a different generalisation of the original formula of Metropolis and Rota.

Conjecture 5.4.10 *We have that*

$$\mathrm{f}_{r,n}\, M^q(x) = \sum_{d|n} Q_{r,n,d}(q)\, M^q(x^r, d) \qquad \text{in } \mathbb{Q}[x,q]\,,$$

where $Q_{r,n,d}(q)$ in $\mathbb{Q}[q]$ are numerical polynomials.

If q is a prime power p^k, we are able to give a combinatorial interpretation for the polynomials $M^q(x,n)$. Our ingredients are: the field $\mathrm{GF}(q)$, an alphabet Γ with m letters, and the free monoid $(\Gamma \times \mathrm{GF}(q))^*$ generated by $\Gamma \times \mathrm{GF}(q)$. We let $\mathrm{GF}(q) \setminus \{0\}$ act on this monoid by

$$(\theta, (c_1, \rho_1) \dots (c_s, \rho_s)) \mapsto (c_1, \theta\rho_1) \dots (c_s, \theta\rho_s)\,.$$

Note that the equivalence relation determined by the orbits of this action is not a congruence. We define a *q-word* as an orbit in $(\Gamma \times \mathrm{GF}(q))^* \setminus (\Gamma \times \{0\})^*$. We call $s \in \mathbb{N}$ a period of the q-word $[w]$ if there is w_0 in $(\Gamma \times \mathrm{GF}(q))^*$ of length s and $\rho_1, \dots, \rho_t$ in $\mathrm{GF}(q)$ such that $[w] = [(\rho_1 w_0) \dots (\rho_t w_0)]$ (here $0w_0$ is defined in the obvious way). The primitive period of w, aperiodic q-words, q-necklaces, and primitive q-necklaces can now be defined in the usual way. Let us denote $n\, M^q(x,n)$ by $S^q(x,n)$. We claim that these polynomials are uniquely defined by the relations

$$\sum_{d|n} [n/d]_q\, S^q(x,d) = [n]_q\, x^n\,; \tag{5.4.11}$$

indeed, we have that

$$(\omega \circ g^q \circ M^q)_n(x) = (\omega \circ w^q)_n(x, 0, 0, \dots) = [n]_q\, x^n\,.$$

Examining (5.4.11), we obtain the combinatorial interpretation mentioned above.

Proposition 5.4.12 *For every $m, n \in \mathbb{N}$, $S^q(m,n)$ represents the number of aperiodic q-words of length n, and $M^q(m,n)$ represents the number of aperiodic q-necklaces of length n on the given alphabet Γ with m letters.*

We suggest that the constructions of Dress and Siebeneicher [14], [13] could be extended to the above setting.

We conclude this section by briefly investigating the case when $F(X,Y)$ is the universal p-typical formal group law corresponding to the prime p, which we have seen that is defined over a certain summand V of the Lazard ring L. The corresponding group $W^F(V\mathbb{Q})$ is defined as a certain subgroup of the group of Witt vectors (with underlying set $L\mathbb{Q}^\infty$) associated with the universal formal group law; more precisely, it is the subgroup consisting of those infinite sequences α of elements of $V\mathbb{Q}$ for which $\alpha_k = 0$ whenever k is not a power of p. We define $Nr^F(V\mathbb{Q})$ similarly, and abbreviate the sequence $(\alpha_1, 0, \ldots, 0, \alpha_p, 0, \ldots, 0, \alpha_{p^2}, 0, \ldots)$ in $W^F(V\mathbb{Q})$, $Nr^F(V\mathbb{Q})$, or $Gh^F(V\mathbb{Q})$ to $(\alpha_1, \alpha_p, \alpha_{p^2}, \ldots)$.

Recall from (1.2.8) Hazewinkel's generators of V and Araki's generators of $V_{(p)}$, which were defined recursively in terms of the coefficients $m_{(i)}$ of the logarithm of the universal p-typical formal group law ($m_{(i)}$ is the coefficient of X^{p^i}). It turns out that we can express these generators very easily by using the necklace algebra $Nr^F(V\mathbb{Q})$ associated with the universal p-typical formal group law.

Proposition 5.4.13 *We have that*

$$T^F(v_1, v_2, \ldots) = \mathrm{f}_p\,(1, 0, 0, \ldots) \quad \textit{and} \quad T^F(w_0, w_1, w_2, \ldots) = (p, 0, 0, \ldots)\,.$$

Proof. According to the defining relations (1.2.8), we have

$$w^F(v_1, v_2, \ldots) = p\,(m_{(1)}, m_{(2)}, \ldots)\,, \quad w^F(w_0, w_1, w_2, \ldots) = p\,(m_{(0)}, m_{(1)}, \ldots)\,.$$

On the other hand, we have

$$g^F(1, 0, 0, \ldots) = (m_{(0)}, m_{(1)}, \ldots)\,, \quad \mathrm{f}_p\,(m_{(0)}, m_{(1)}, \ldots) = p\,(m_{(1)}, m_{(2)}, \ldots)\,.$$

The propositions now follows from the fact that g^F is an isomorphism and $g^F \circ T^F = w^F$. □

Chapter 6

Formal Group Laws and Symmetric Functions

This chapter is devoted to a brief study of the interaction between formal group theory and the theory of symmetric functions. This interaction is reciprocal, in the sense that we are able to use concepts/results in one of the two areas in order to obtain results in the other area. We rely heavily on the notation and comments in §1.1, §1.2, and §1.10, which we use without further comment.

6.1 A Remarkable Homomorphism and Its Geometrical Interpretation

Consider a ring A_* as in §1.1, and an umbra a in $A\mathbb{Q}_*$ such that the formal group law $f^a(X,Y)$ lies in $A^1[[X,Y]]$. Let us define the map of graded Hopf algebras $d_*\colon Sym_*^A \to U(f^a)_*$ by

$$d_*(S_n) = \beta_n^a(x)\,.$$

This is indeed a Hopf algebra map since we know from (1.1.12) and (1.10.1) that $\{S_n\}$ and $\{\beta_n^a(x)\}$ are divided power sequences. In particular, considering the multiplicative formal group law $f^k(X,Y)$ over k_*, the map $d_*\colon Sym_*^k \to U(f^k)_*$ is defined by

$$d_*(S_n) = \frac{x(x-u)\ldots(x-(n-1)u)}{n!}\,.$$

We now present a geometrical interpretation for the map d_*. This shows that algebraic topology has a great deal to offer in enlightening and guiding our understanding of symmetric functions, as well as of the covariant and contravariant bialgebras of a formal group law.

Let $E^*(\cdot)$ be the multiplicative cohomology theory with complex orientation Z in $E^2(\mathbb{C}P^\infty)$ which was considered in §1.4. It is well-known that $E^*(BU) \cong E^*[[c_1, c_2, \ldots]]$, where c_n are the generalised Chern classes. It is also known that the map

$$E^*(BU(n)) \to E^*(\mathbb{C}P^\infty \times \ldots \times \mathbb{C}P^\infty) \cong E^*(\mathbb{C}P^\infty)\widehat{\otimes}\ldots\widehat{\otimes}E^*(\mathbb{C}P^\infty)$$

induced by the classifying map of the direct product of n copies of the Hopf bundle over $\mathbb{C}P^\infty$ is a monomorphism mapping c_i, with $i \leq n$, to the i-th elementary symmetric function in $X_1 := Z\otimes 1\otimes\ldots\otimes 1$, $X_2 := 1\otimes Z\otimes\ldots\otimes 1$, ..., $X_n := 1\otimes 1\otimes \ldots\otimes Z$. On the other hand, we have that $E_*(BU) \cong E_*[b_1, b_2, \ldots]$, and that c_n is dual to b_1^n with respect to the monomial basis of $E_*[b_1, b_2, \ldots]$. The multiplicative structure of $E_*(BU)$ is determined by the map $BU \times BU \to BU$ classifying the Whitney sum of vector bundles. The diagonal map $BU \to BU \times BU$ induces a comultiplication $\delta\colon E_*(BU) \to E_*(BU \times BU) \cong E_*(BU) \otimes E_*(BU)$ satisfying

$$\delta(b_n) = \sum_{i=0}^{n} b_i \otimes b_{n-i}\,,$$

which turns $E_*(BU)$ into a Hopf algebra. The standard inclusion $\mathbb{C}P^\infty = BU(1) \hookrightarrow BU$ induces a monomorphism $E_*(\mathbb{C}P^\infty) \hookrightarrow E_*(BU)$ mapping β_n to b_n. The determinant map $\det\colon U \to S^1$ defined on unitary matrices gives rise to a map $\mathrm{Bdet}\colon BU \to BS^1 = \mathbb{C}P^\infty$; furthermore, the composite of the inclusion $\mathbb{C}P^\infty \hookrightarrow BU$ with Bdet is the identity on $\mathbb{C}P^\infty$, whence $\mathrm{Bdet}_*\colon E_*(BU) \to$

$E_*(\mathbb{C}P^\infty)$ maps b_n to β_n. Since the determinant map is a group homomorphism, the map Bdet_* is a ring homomorphism; moreover, it is a Hopf algebra map.

It follows from the above topological arguments that we may identify the Hopf algebras $E_*(BU)$ and $E^*(BU)$ with the Hopf algebras Sym_*^E and Sym_E^* respectively, in such a way that b_n is identified with S_n and c_n with Λ_n; furthermore, the map Bdet_* is identified with d_*. Let us now consider the composite

$$\mathbb{C}P^\infty \times \ldots \times \mathbb{C}P^\infty \to BU(n) \xrightarrow{\mathrm{Bdet}} \mathbb{C}P^\infty\,,$$

where the first map classifies the direct product of n copies of the Hopf bundle over $\mathbb{C}P^\infty$. It is easy to see that the composite is precisely the map classifying the tensor product of the n line bundles over $\mathbb{C}P^\infty \times \ldots \times \mathbb{C}P^\infty$; in other words, it is obtained from the map μ considered in §1.4 by iterating it $n-1$ times. Let us also recall from §1.4 that Z in $E^2(\mathbb{C}P^\infty)$ was identified with some formal power series $a(D)$ in $E\mathbb{Q}^1[[D]]$, and that $\mu^*(Z) = f^a(Z \otimes 1, 1 \otimes Z)$. Combining the above remarks, we finally obtain $d^*(a(D)) = \sum_{n\geq 1}^a X_n$; this identity will be proved in Proposition 6.2.1 in a purely algebraic way.

6.2 Identities Related to d_*

Consider the transpose map $d^*\colon R(f^a)^* \to Sym_A^*$, and its extension $\widehat{d^*}\colon A\mathbb{Q}^*[[D]] \to Sym_{\widehat{A}}^*$. We denote $d^*(a(D))$ in Sym_A^* by Δ, and $\widehat{d^*}(D)$ in $Sym_{\widehat{A}}^*$ by Δ_0.

Proposition 6.2.1 *We have that*

$$\Delta = \sum_{n\geq 1}{}^a X_n \qquad \textit{in } Sym_A^*\,, \tag{6.2.2}$$

and

$$\Delta_0 = \sum_{n\geq 1} \overline{a}_{n-1}\,\Psi_n \qquad \textit{in } Sym_{\widehat{A}}^*\,. \tag{6.2.3}$$

Proof. We have

$$\langle d^*(a(D)) \mid S^I\rangle = \langle a(D) \mid d_*(S^I)\rangle = \langle a(D) \mid \beta^a_{i_1}(x)\ldots\beta^a_{i_l}(x)\rangle$$
$$= \langle \delta^{l-1}(a(D)) \mid \beta^a_{i_1}(x)\otimes\ldots\otimes\beta^a_{i_l}(x)\rangle = f^a_{i_1,\ldots,i_l}\,;$$

here δ^{l-1} denotes the comultiplication iterated $l-1$ times, and $I=(i_1,\ldots,i_l)$. Since the dual basis to S^I is Ψ_I, we have

$$d^*(a(D)) = \sum_I f^a_I\,\Psi_I = \sum_{n\geq 1}{}^a X_n\,.$$

On the other hand,

$$\langle \widehat{d^*}(D) \mid S^I\rangle = \langle D \mid \beta^a_{i_1}(x)\ldots\beta^a_{i_l}(x)\rangle = \begin{cases} \overline{a}_{i_1-1} & \text{if } l=1 \\ 0 & \text{otherwise}\,, \end{cases}$$

whence (6.2.3) follows. □

Corollary 6.2.4 *We have*

$$\sum_{n\geq 1}{}^a X_n = a\left(\sum_{n\geq 1}\overline{a}_{n-1}\,\Psi_n\right) \qquad \text{in } Sym^*_{\widehat{A}}\,.$$

We will now consider the important special case corresponding to the umbra t^q in $k\mathbb{Q}_*$ with

$$\overline{t^q} := \left(1, \frac{[2]_q}{2}u, \frac{[3]_q}{3}u^2, \ldots\right)\,;$$

here q is an integer, and $[n]_q := 1+q+\ldots+q^{n-1}$. We have

$$\overline{t^q}(Z) = \frac{1}{(1-q)u}\ln\frac{1-quZ}{1-uZ}\,, \qquad t^q(Z) = \frac{\exp((1-q)uZ)-1}{(\exp((1-q)uZ)-q)\,u}$$

for $q\neq 1$, and

$$\overline{t^1}(Z) = \frac{Z}{1-uZ}\,, \qquad t^1(Z) = \frac{Z}{1+uZ}\,.$$

The formal group law $f^{t^q}(X_1,X_2)$ is given by

$$f^{t^q}(X_1,X_2) = \frac{X_1+X_2-(1+q)uX_1X_2}{1-qu^2X_1X_2}\,,$$

whence $f^{t^q}(X_1,X_2)$ lies in $k^1[[X_1,X_2]]$. Note that this formal group law is actually the graded version of $F_q(X,Y)$ in (5.1.1).

Corollary 6.2.5 *The following identity holds in $Sym^*_{k\mathbb{Q}}$:*

$$t^q\left(\sum_{n\geq 1}\frac{[n]_q}{n}\,\Psi_n\,u^{n-1}\right)=\frac{\sum_{n\geq 1}(-1)^{n-1}\,[n]_q\,\Lambda_n\,u^{n-1}}{1+q\sum_{n\geq 2}(-1)^{n-1}\,[n-1]_q\,\Lambda_n\,u^n}\,.$$

PROOF. The identity follows immediately from Corollary 6.2.4 after proving that

$$\sum_{n\geq 1}{}^{t^q}X_n=\frac{\sum_{n\geq 1}(-1)^{n-1}\,[n]_q\,\Lambda_n\,u^{n-1}}{1+q\sum_{n\geq 2}(-1)^{n-1}\,[n-1]_q\,\Lambda_n\,u^n}\,.$$

It is not difficult to prove a similar identity for which the left-hand side sum is finite, by induction on n. Then, we consider the limit with respect to the filtration topology of $k\mathbb{Q}_*[[X_1,X_2,\ldots]]$. □

Let us note that for $q=0$ we obtain the well-known identity

$$\exp\left(\sum_{n\geq 1}\frac{\Psi_n}{n}\,u^n\right)=\sum_{n\geq 0}S_n\,u^n\,,$$

while for $q=1$ we obtain the identity

$$\frac{1}{\sum_{n\geq 0}\Psi_n\,u^n}=\frac{\sum_{n\geq 0}(-1)^n\,\Lambda_n\,u^n}{\sum_{n\geq 0}(-1)^{n-1}\,(n-1)\,\Lambda_n\,u^n}\,. \tag{6.2.6}$$

The latter appears in a slightly different form in [49], Proposition 2.2, and is attributed to I. Gessel; hence Corollary 6.2.5 represents the q-analogue of (6.2.6).

Other types of combinatorial identities, not necessarily involving symmetric functions, can be derived from Corollary 6.2.4. For instance, let us consider the formal group law $f^b(X_1,X_2)$ over the ring H_*. We view H_* as $\mathbb{Z}[m_1,m_2,\ldots]$, and choose the monomial symmetric function basis in Sym^*_H. The coefficient of Ψ_n in $\sum^b_{i\geq 1}X_i$ is clearly 0, for every $n>1$. This means that we can obtain a family of identities by computing the coefficient of $m_1^{i_1}\ldots m_l^{i_l}\,\Psi_n$ in $b(\sum_{i\geq 1}m_{i-1}\,\Psi_i)$, where $i_1+\ldots+li_l=n-1$. The key ingredient for this computation is Lagrange inversion, namely the fact that the coefficient of $m_1^{i_1}\ldots m_l^{i_l}$ in b_{n-1} is equal to the number of (unlabelled) rooted plane trees with n leaves and outdegree sequence $(2^{i_1},\ldots,(l+1)^{i_l})$ for the internal vertices. For instance, the coefficient of $m_1^{n-1}\,\Psi_n$

provides the identity

$$\sum_{i=\lceil \frac{n}{2} \rceil}^{n} (-1)^{i-1} \, C_i \binom{i}{n-i} = 0 \,, \tag{6.2.7}$$

where $C_i = \frac{1}{i}\binom{2i-2}{i-1}$ is the i-th *Catalan number*. This is a special case of an identity in [43] §4.5 Problem 1(c). More generally, the coefficient of $m_r^s \, \Psi_{rs+1}$ provides the identity

$$\sum_{i=\lceil \frac{s-1}{r+1} \rceil}^{s} (-1)^i \, C_{ri+1}^{r+1} \binom{ri+1}{s-i} = 0 \,, \tag{6.2.8}$$

where $C_{ri+1}^{r+1} = \frac{1}{(r+1)i+1}\binom{(r+1)i+1}{i}$ represents the number of $r+1$-ary rooted plane trees with $ri+1$ leaves (see e.g. [16], or [7] for a bijective proof).

6.3 Computing the Images of Certain Bases of Sym_*^A under the Map d_*

The images of the elementary symmetric functions under the map d_* are easy to compute. Indeed, using the fact that d_* is a Hopf algebra map, and denoting by γ the antipodes of Sym_*^A and $U(f^a)_*$, we have

$$d_*(\Lambda_n) = (-1)^n \, d_*(\gamma(S_n)) = (-1)^n \, \gamma(d_*(S_n)) = (-1)^n \, \gamma(\beta_n^a(x)) = (-1)^n \, \beta_n^a(-x) \,.$$

We now move on to the computation of the images of the power sum symmetric functions under the map d_*. The key ingredients for most of the computations in this section are Doubilet's change of basis formulae for symmetric functions in [11], which use *Möbius inversion* on set partition lattices. We will use the zeta type function ζ^α and the Möbius type function μ^α in the incidence algebra $A_*(\Pi_n)$. According to our conventions, μ denotes the classical Möbius function of the lattice Π_n, as it traditionally does.

Proposition 6.3.1 *For every partition $I = (i_1, \ldots, i_l)$ of n, we have that*

$$d_*(\Psi^I) = \left(\prod_{j=1}^{l} i_j \, \overline{a}_{i_j - 1} \right) x^l .$$

PROOF. *First method.* This is a combinatorial method. Choose σ in Π_n such that $I(\sigma) = I$. Combining Doubilet's formula

$$\Psi^I = \frac{1}{|\mu(\widehat{0}, \sigma)|} \sum_{\pi \leq \sigma} \mu(\pi, \sigma) \, I(\pi)! \, S^{I(\pi)} ,$$

with (1.7.8), we obtain

$$\begin{aligned} d_*(\Psi^I) &= \frac{1}{|\mu(\widehat{0}, \sigma)|} \sum_{\pi \leq \sigma} \mu(\pi, \sigma) \left(\sum_{\omega \leq \pi} \mu^{\alpha}(\widehat{0}, \omega) \, x^{|\omega|} \right) \\ &= \frac{1}{|\mu(\widehat{0}, \sigma)|} \sum_{\omega \leq \sigma} \mu^{\alpha}(\widehat{0}, \omega) \, x^{|\omega|} \left(\sum_{\omega \leq \pi \leq \sigma} \mu(\pi, \sigma) \right) \\ &= \frac{1}{|\mu(\widehat{0}, \sigma)|} \mu^{\alpha}(\widehat{0}, \sigma) \, x^{|\sigma|} = \left(\prod_{j=1}^{l} \frac{\overline{\alpha}_{i_j - 1}}{(i_j - 1)!} \right) x^l . \end{aligned}$$

Second method. Since d_* is an algebra map, it is enough to prove the result for partitions I of length 1. Since d_* is a coalgebra map and Ψ_n is a primitive element of Sym_*^A, we have that $d_*(\Psi_n) = cx$ for some c in A_*. Now

$$c = \langle D \mid d_*(\Psi_n) \rangle = \langle \Delta_0 \mid \Psi_n \rangle = n \overline{a}_{n-1} ;$$

the last equality follows from (6.2.3) and the well-known fact that $\langle \Psi_i \,|\, \Psi_j \rangle = i \delta_{i,j}$.
□

Using other two formulae of Doubilet, we can immediately express the images of the monomial and forgotten symmetric functions.

Corollary 6.3.2 *Let K be a partition of n, and choose σ in Π_n such that $I(\sigma) = K$. We have*

$$d_*(\Psi_K) = \frac{1}{\|K\|} \sum_{\pi \geq \sigma} \mu(\sigma, \pi) \left(\prod_{j=1}^{|\pi|} i_j(\pi) \, \overline{a}_{i_j(\pi) - 1} \right) x^{|\pi|} , \qquad (6.3.3)$$

$$d_*(\Psi_K(-X)) = \frac{(-1)^{l(K)}}{\|K\|} \sum_{\pi \geq \sigma} |\mu(\sigma,\pi)| \left(\prod_{j=1}^{|\pi|} i_j(\pi)\, \overline{a}_{i_j(\pi)-1} \right) x^{|\pi|}, \tag{6.3.4}$$

where $I(\pi) := (i_1(\pi), \ldots, i_{|\pi|}(\pi))$. *Furthermore, given the partition* $R = (r^s)$ *of* $n = rs$, *we have*

$$\begin{aligned} d_*(\Psi_R) &= \sum_{|I|=s} \frac{(-1)^{s-l(I)}}{\|I\|} \left(\prod_{j=1}^{l(I)} \overline{a}_{ri_j-1} \right) (rx)^{l(I)} \qquad (6.3.5) \\ &= \frac{1}{n!} \sum_{\pi \in \Pi_n^{(r)}} (-1)^{s-|\pi|}\, \mu^{\alpha}(\widehat{0},\pi)\, (rx)^{|\pi|}, \end{aligned}$$

$$\begin{aligned} d_*(\Psi_R(-X)) &= (-1)^s \sum_{|I|=s} \frac{1}{\|I\|} \left(\prod_{j=1}^{l(I)} \overline{a}_{ri_j-1} \right) (rx)^{l(I)} \qquad (6.3.6) \\ &= \frac{(-1)^s}{n!} \sum_{\pi \in \Pi_n^{(r)}} \mu^{\alpha}(\widehat{0},\pi)\, (rx)^{|\pi|}, \end{aligned}$$

where $\Pi_n^{(r)}$ *represents the poset of* r-divisible partitions *of* $[n]$ *(that is partitions with all block sizes divisible by* r*).*

PROOF. (6.3.3) and (6.3.4) follow immediately from Proposition 6.3.1 and [11]. To deduce (6.3.5) from (6.3.3), we use the fact that the number of partitions π of $[n]$ with $I(\pi) = I$ is $n!/(I!\,\|I\|)$. Given the partition σ in Π_n with $I(\sigma) = R$, we have

$$\begin{aligned} d_*(\Psi_R) &= \frac{1}{s!} \sum_{|I|=s} (-1)^{s-l(I)} \frac{s!}{I!\,\|I\|} \left(\prod_{j=1}^{l(I)} (i_j - 1)!\,(ri_j)\, \overline{a}_{ri_j-1} \right) x^{l(I)} \\ &= \frac{1}{n!} \sum_{|I|=s} (-1)^{s-l(I)} \frac{n!}{(ri_1)! \ldots (ri_{l(I)})!\,\|I\|} \left(\prod_{j=1}^{l(I)} \overline{\alpha}_{ri_j-1} \right) (rx)^{l(I)} \\ &= \frac{1}{n!} \sum_{\pi \in \Pi_n^{(r)}} (-1)^{s-|\pi|}\, \mu^{\alpha}(\widehat{0},\pi)\, (rx)^{|\pi|}. \end{aligned}$$

(6.3.6) follows similarly. □

Remark 6.3.7 Computing the images of the forgotten symmetric functions is important because we can immediately obtain the coefficients in the expansion of Δ in the elementary symmetric function basis. All we need to do is replace every x^k in (6.3.4) by a_{k-1}, and multiply the result by $(-1)^{|K|}$; indeed, we have $\langle \Delta | \Psi_R(-X) \rangle = \langle a(D) | d_*(\Psi_R(-X)) \rangle$. According to (6.2.2), we obtain, in particular, the coefficient of $E_1^i E_2^j$ in the formal group law $f^a(X,Y)$, where $E_1 := X+Y$ and $E_2 := XY$.

Since Sym_A^* is the dual algebra of the coalgebra Sym_*^A, we have a canonical action of the former on the latter, as discussed as the beginning of §1.1; this action is well-known in the theory of symmetric functions (see [28]). For instance

$$\Delta\, S_n = \sum_{i=1}^{n} \langle \Delta \mid S_i \rangle\, S_{n-i} = S_{n-1}\,, \quad \text{and} \quad \Delta\, \Psi_n = \langle \Delta \mid \Psi_n \rangle = n\overline{a}_{n-1}\,. \tag{6.3.8}$$

The operator Δ is, in fact, a delta operator on Sym_*^A, and the map d_* is a map of Hopf algebras with delta operator; the second claim follows immediately from (1.1.4). The following proposition, which extends the similar results (1.2.11) and (1.2.12) concerning the action of $a(D)$ on $U(f^a)_*$, will enable us to compute the action of Δ on other symmetric functions.

Proposition 6.3.9 *For every P, Q in Sym_*^A, we have*

$$\Delta\, PQ = \sum_{i,j \geq 0} f_{i,j}^a\, (\Delta^i\, P)\, (\Delta^j\, Q)\,, \tag{6.3.10}$$

$$\Delta\, \gamma(P) = \sum_{j \geq 1} i_j^a\, \gamma(\Delta^j\, P)\,. \tag{6.3.11}$$

PROOF. Let p denote the product in Sym_*^A and $g := \langle \Delta \mid \cdot \rangle$. We have

$$\begin{aligned} \langle \Delta \mid PQ \rangle &= \langle \delta(d^*(a(D))) \mid P \otimes Q \rangle = \langle (d^* \otimes d^*)(\delta(a(D))) \mid P \otimes Q \rangle \\ &= \langle (d^* \otimes d^*)(f^a(a(D) \otimes 1, 1 \otimes a(D))) \mid P \otimes Q \rangle \\ &= \left\langle \sum_{i,j \geq 0} f_{i,j}^a\, \Delta^i \otimes \Delta^j \mid P \otimes Q \right\rangle . \end{aligned}$$

Hence $\Gamma_{gop} = \sum_{i,j\geq 0} f^a_{i,j}\, \Delta^i \otimes \Delta^j$, and (6.3.10) follows by recalling (1.1.4). Formula (6.3.11) is proved similarly, using

$$\begin{aligned}\langle \Delta \mid \gamma(P)\rangle &= \langle a(D) \mid d_*(\gamma(P))\rangle = \langle a(D) \mid \gamma(d_*(P))\rangle = \langle a(-D) \mid d_*(P)\rangle \\ &= \left\langle \sum_{j\geq 1} i^a_j\, a(D)^j \mid d_*(P) \right\rangle = \left\langle \sum_{j\geq 1} i^a_j\, \Delta^j \mid P \right\rangle .\end{aligned}$$

□

Clearly, formula (6.3.10) can be iterated in order to express the action of Δ on an arbitrary product. Here are some applications of Proposition 6.3.9 (cf. (6.3.8)):

$$\Delta\, S^I = \sum_{0\leq s_k\leq i_k} f^a_{s_1,\ldots,s_{l(I)}} \prod_{j=1}^{l(I)} S_{i_j - s_j}\,, \tag{6.3.12}$$

$$\Delta\, \Psi^I = \sum_{0\leq s_k\leq 1} \alpha_{s_1+\ldots+s_{l(I)}-1} \prod_{j=1}^{l(I)} \Delta^{s_j}\, \Psi_{i_j}\,, \tag{6.3.13}$$

$$\Delta\, \Lambda_n = \sum_{j=1}^{n} (-1)^j\, i^a_j\, \Lambda_{n-j}\,. \tag{6.3.14}$$

In order to express the action of Δ on the basis of monomial symmetric functions, we need to combine the formula in [15] for the comultiplication corresponding to these functions with (6.3.3), and use (1.1.2); this gives a more complicated formula, which we do not present here.

6.4 Applications Related to the Lazard Ring

In this section, we study a certain family of elements in the Lazard ring L_*, by using the results in the previous section corresponding to the universal formal group law $f^b(X_1, X_2)$. More precisely, given two integers r, s with $r > 1$, and a partition $R := (r^s)$ of $n := rs$, we consider the elements

$$g_{n,r} := (-1)^s \langle \Delta \mid \Psi_R(-X)\rangle = (-1)^s \langle b(D) \mid d_*(\Psi_R(-X))\rangle \quad \text{in } L_{n-1}\,. \tag{6.4.1}$$

In other words, $(-1)^{(r-1)s}\, g_{n,r}$ is the coefficient of Λ_r^s in the expression of Δ (or just $X_1 +_b + \ldots +_b X_r$) in the elementary symmetric function basis. Furthermore, if ρ is a primitive r-th root of unity, then $g_{n,r}$ is the coefficient of Z^n in $\rho Z +_b \ldots +_b \rho^r Z$.

The elements $g_{n,r}$ are implicit in the construction of the universal *p-typical formal group law* (see [33]), to which we will refer below. These elements are manipulated in a purely formal way in the process of p-typification, whence no explicit formula for them is needed. Corollary 6.3.2 enables us to derive several explicit formulae for $g_{n,r}$.

Proposition 6.4.2 *Let σ is a partition of $[n]$ with $I(\sigma) = R$. We have*

$$\begin{aligned} g_{n,r} &= \frac{1}{s!} \sum_{\pi \geq \sigma} |\mu(\sigma,\pi)| \left(\prod_{j=1}^{|\pi|} i_j(\pi)\, m_{i_j(\pi)-1} \right) |\pi|!\, b_{|\pi|-1} \\ &= \sum_{|I|=s} \frac{r^{l(I)}\, l(I)!}{\|I\|} \left(\prod_{j=1}^{l(I)} m_{ri_j-1} \right) b_{l(I)-1} \\ &= \frac{1}{n!} \sum_{\pi \in \Pi_n^{(r)}} r^{|\pi|}\, \mu^{\phi}(\widehat{0},\pi)\, \zeta^{\phi}(\pi,\widehat{1}) \qquad \text{in } L_{n-1}\,, \end{aligned}$$

where $I(\pi) := (i_1(\pi), \ldots, i_{|\pi|}(\pi))$. In particular, $g_{n,n} = n m_{n-1}$.

Recall that the *Milnor genus* of an element z in L_{n-1} is the coefficient of b_{n-1} in the expression of z as a polynomial in the b_i's. According to Proposition 6.4.2, the Milnor genus of $g_{n,r}$ is $-r$. Hence, according to the structure of the Lazard ring, we have the following result.

Proposition 6.4.3 *For every prime p and integer $l \geq 1$, the element $g_{p^l,p}$ is a canonical polynomial generator for the Lazard ring in dimension $p^l - 1$. Furthermore, if p and q are two distinct primes dividing n, and i, j are integers such that $ip + jq = 1$, then $ig_{n,p} + jg_{n,q}$ is a polynomial generator in dimension $n-1$. In consequence, the Lazard ring is generated by the set of elements $g_{n,p}$, where $n \geq 2$ is an integer, and p is a prime dividing n.*

Note that the elements $\langle \Delta \mid \Psi_R \rangle$ have similar properties with $g_{n,r}$.

Let us now recall the universal p-typical formal group law corresponding to the prime p, which is defined over the summand V_* of L_*, and whose exp series was denoted by $b^p(Z)$. We denote by $h_*^{b^p}$ the canonical projection from L_* to V_*. The above remark on the Milnor genus of the elements $g_{n,r}$ implies that the elements $z_l := h_*^{b^p}(g_{p^l,p})$ are polynomial generators of V_*. Proposition 6.4.2 provides similar expressions for z_l, simply by setting $m_i = 0$ for $i \neq p^k - 1$, and by replacing b_i with b_i^p for all i (recall from §1.1 that $b_i^p = 0$ unless i is divisible by $p-1$).

Let ρ be a primitive p-th root of unity. According to the definition of the elements $g_{n,r}$ at the beginning of this section, we have

$$\sum_{s \geq 1} g_{ps,p}\, Z^{ps} = \rho Z +_b \ldots +_b \rho^p Z\,.$$

On the other hand, it was proved in [20] that Hazewinkel's generators v_l satisfy

$$\sum_{l \geq 1}{}^{b^p} v_l\, Z^{p^l} = \rho Z +_{b^p} \ldots +_{b^p} \rho^p Z\,.$$

Projecting the first relation onto V_* and using the second one, we obtain

$$\sum_{s \geq 1} h_*^{b^p}(g_{ps,p})\, Z^{ps} = \sum_{l \geq 1}{}^{b^p} v_l\, Z^{p^l}\,. \tag{6.4.4}$$

Hence, we have a formal expression for z_l in terms of Hazewinkel's generators.

We now intend to derive more explicit information from (6.4.4). To this end, we recall from §1.2 the formal group law $f^{k^{p,q}}(X_1, X_2)$ over the summand $k(q)_*$ of k_*, where q is an integer greater than 1. We denote by $\overline{h}_*^{k^{p,q}}$ the ring homomorphism from V_* to $k(q)_*$ mapping the coefficients of the universal p-typical formal group law to those of $f^{k^{p,q}}(X_1, X_2)$. We have seen that this homomorphism sends v_q to u^{p^q-1}, and the rest of Hazewinkel's generators to 0. By projecting formula (6.4.4) via $\overline{h}_*^{k^{p,q}}$, we obtain a similar result for the generators z_l of V_*. We state this result in terms of the composite $\overline{h}_*^{k^{p,q}} \circ h_*^{b^p}$, which we denote by $h_*^{k^{p,q}}$.

Proposition 6.4.5 *We have that*

$$h_*^{k^{p,q}}(g_{n,r}) = \begin{cases} u^{p^q-1} & \text{if } n = p^q \text{ and } r = p \\ 0 & \text{otherwise}. \end{cases}$$

In particular, the homomorphism $\overline{h}_*^{k^{p,q}}$ *sends* z_q *to* u^{p^q-1}, *and the rest of the generators* z_l *of* V_* *to 0.*

Proposition 6.4.5 can be reformulated by using the interpretation of the elements $g_{n,r}$ at the beginning of this section. On the other hand, we obtain the following corollary as an immediate consequence.

Corollary 6.4.6 *Let* $l > 1$ *be an integer, and let* q *be a divisor of* l.

1. *The coefficient of* $v_q^{(p^l-1)/(p^q-1)}$ *in the expression of* z_l *in terms of the generators* v_i *is* 1 *if* $q = l$, *and* 0 *otherwise. The same holds for the coefficient of* $z_q^{(p^l-1)/(p^q-1)}$ *in the expression of* v_l *in terms of the generators* z_i. *In particular,* $z_1 = v_1$ *and* $z_2 = v_2$.

2. *The coefficients of* $z_q^{(p^l-1)/(p^q-1)}$ *and* $v_q^{(p^l-1)/(p^q-1)}$ *in the expressions of* m_{p^l-1} *in terms of the generators* z_i *and* v_i *of* V_* *are both equal to* $p^{-l/q}$.

We conclude this section with a purely combinatorial proof of Proposition 6.4.5, which does not use Hazewinkel's generators. The main point made here is that the elements $g_{n,r}$ are well-suited for combinatorial manipulations, due to the formulae in Proposition 6.4.2.

PROOF. From Proposition 6.4.2 it follows that $h_*^{k^{p,q}}(g_{n,r}) = 0$ if $r \neq p$, or if $r = p$ and p^q does not divide n. Now let s be an integer divisible by p^{q-1}, and let

$$k'(Z) := \overline{k^{p,q}}(Z^{p^{q-1}}) = Z^{p^{q-1}} + \frac{u^{p^q-1}}{p} Z^{p^{2q-1}} + \frac{u^{p^{2q}-1}}{p^2} Z^{p^{3q-1}} + \dots .$$

We claim that $h_*^{k^{p,q}}(g_{ps,p})$ is equal to $u^{ps(1-p^{-q})}$ times the coefficient of Z^s in $k^{p,q}(k'(Z)) = Z^{p^{q-1}}$.

Let σ be a partition $[ps]$ such that $I(\sigma) = (p^s)$. Given a subset A of $\mathbb{N}$, we denote by Π_n^A the set of those partitions of $[n]$ for which every block size lies in A. Now consider the sets $P := \{p^{qi} \,:\, i \geq 1\}$ and $Q := \{p^{qi-1} \,:\, i \geq 1\}$. According to Proposition 6.4.2, we have

$$\begin{aligned} h_*^{k^{p,q}}(g_{ps,p}) &= \frac{1}{s!} \sum_{\pi \in \Pi_{ps}^P,\, \pi \geq \sigma} |\mu(\sigma,\pi)| \left(\prod_{j=1}^{|\pi|} p^{(q-1)e_j(\pi)} \right) u^{ps-|\pi|}\, |\pi|!\, k_{|\pi|-1}^{p,q} \\ &= \frac{1}{s!} \sum_{\pi \in \Pi_s^Q} \left(\prod_{j=1}^{|\pi|} (p^{qe_j(\pi)-1} - 1)!\, p^{(q-1)e_j(\pi)} \right) u^{ps-|\pi|}\, |\pi|!\, k_{|\pi|-1}^{p,q} \\ &= \frac{1}{s!} \sum_{\pi \in \Pi_s^Q} \left(\prod_{j=1}^{|\pi|} (p^{qe_j(\pi)-1})!\, p^{1-e_j(\pi)} \right) u^{ps-|\pi|}\, |\pi|!\, k_{|\pi|-1}^{p,q}\,; \end{aligned}$$

here $I(\pi) := (p^{qe_1(\pi)}, \ldots, p^{qe_{|\pi|}(\pi)})$ if π lies in Π_{ps}^P, and $I(\pi) := (p^{qe_1(\pi)-1}, \ldots, p^{qe_{|\pi|}(\pi)-1})$ if π lies in Π_s^Q. The above claim now follows by comparing the formula for $h_*^{k^{p,q}}(g_{ps,p})$ with the formula for the coefficient of Z^s in $k^{p,q}(k'(Z))$ given by Theorem 1.7.6 (1). $\square$

6.5 Symmetric Functions and Witt Vectors Associated with a Formal Group Law

In this section we associate certain symmetric functions with a formal group law, and discuss their connection with Witt vectors associated with the same formal group law.

Given the formal group law $f^a(X_1, X_2)$ in $A^1[[X_1, X_2]]$ considered in §6.1, we define symmetric functions $q_n^a := q_n^a(X)$ in Sym_A^1 by

$$\sum_{n\geq 1}{}^{a}\, q_n^a t^n = \sum_{n\geq 1}{}^{a}\, X_n t \quad \text{in } Sym_A^*[[t]]\,. \tag{6.5.1}$$

Since $f^a(X_1, X_2) \equiv X_1 + X_2 \bmod (X_1, X_2)^2$, this is a good definition. Let us note that

$$\sum_{n\geq 1}^{a} q_n^a(X;Y)\, t^n = \sum_{n\geq 1}^{a} q_n^a(X)\, t^n +_a \sum_{n\geq 1}^{a} q_n^a(Y)\, t^n\,,$$

which implies the existence of a polynomial $Q_n^a(x;y) := Q_n^a(x_1,\ldots,x_n;y_1,\ldots,y_n)$ in $A_*[x;y] := A_*[x_1,\ldots,x_n;y_1,\ldots,y_n]$ such that

$$q_n^a(X;Y) = Q_n^a(q^a(X);q^a(Y))\,. \tag{6.5.2}$$

For example, consider the multiplicative formal group law $f^k(X_1,X_2) = X_1 + X_2 + uX_1X_2$, for which we have

$$\sum_{n\geq 1}^{k} X_n = \frac{-1+\prod_{n\geq 1}(1+uX_n)}{u}\,.$$

Hence, (6.5.1) becomes

$$\prod_{n\geq 1}(1+q_n^k u t^n) = \sum_{n\geq 0} \Lambda_n u^n t^n\,. \tag{6.5.3}$$

Now recall the symmetric functions q_n in $Sym_{\mathbb{Z}}^n$ studied by C. Reutenauer in [42], which are defined by

$$\prod_{n\geq 1}\frac{1}{1-q_n t^n} = \sum_{n\geq 0} S_n t^n\,. \tag{6.5.4}$$

It is easy to see by substituting $u := -1$ in (6.5.3), and comparing with (6.5.4), that $q_n^k = (-u)^{n-1}\, q_n$.

It is not difficult to show, by applying the logarithm $\overline{a}(Z)$ of $f^a(X_1,X_2)$ to (6.5.1), that

$$\sum_{d|n} \overline{a}_{n/d-1}\,(q_d^a)^{n/d} = \overline{a}_{n-1}\Psi_n\,. \tag{6.5.5}$$

We write this, using the Witt vector notation introduced in §5.1, as

$$w_n^a(q^a) = \overline{a}_{n-1}\Psi_n\,, \tag{6.5.6}$$

where $q^a := (q_1^a, q_2^a, \ldots)$; note that we have written $w_n^a(\cdot)$ instead of $w_n^{f^a}(\cdot)$, for simplicity. In other words, $\overline{a}_{n-1}\Psi_n$ are the *ghost components* of q_n^a.

Let us now recall from [18] §15.3 the polynomials $\Sigma_n^a(x;y) := \Sigma_n^a(x_1,\ldots,x_n;$ $y_1,\ldots,y_n)$ in $A\mathbb{Q}_*[x;y] := A\mathbb{Q}_*[x_1,\ldots,x_n;y_1,\ldots,y_n]$, which are defined by

$$w_n^a(\Sigma^a(x;y)) = w_n^a(x) + w_n^a(y)\,. \tag{6.5.7}$$

The following is a classical result, whose proof in [18] §25.1 uses the so-called functional equation lemma. The proof presented here is much simpler, and it generalises the proof given in [42] in the case of the multiplicative formal group law. Our proof relies heavily on the symmetric functions we have associated with a formal group law.

Theorem 6.5.8 *(cf. [18] §25.1) The polynomials $\Sigma_n^a(x;y)$ have coefficients in A_*.*

Proof. We concentrate on the universal formal group law $f^b(X_1,X_2)$ in $L^1[[X_1,X_2]]$, with logarithm $m(Z)$ in $L\mathbb{Q}^1[[Z]] = \mathbb{Q}[m_1,m_2,\ldots][[Z]]$. Let us note first that q_n can be obtained from q_n^b by substituting m_{i-1} with $1/i$, for all i. According to (6.5.6), we have

$$w_n^b(q^b(X;Y)) = w_n^b(q^b(X)) + w_n^b(q^b(Y))\,.$$

Combining this relation with (6.5.2) and (6.5.7), and using the fact that q_n^b are algebraically independent (since q_n are), we finally deduce that $\Sigma_n^b(x;y) = Q_n^b(x;y)$. But we have already seen that $Q_n^b(x;y)$ has coefficients in L_*; furthermore, $\Sigma_n^a(x;y)$ is the image of $\Sigma_n^b(x;y)$ under the homomorphism from $L_*[x;y]$ to $A_*[x;y]$ mapping the coefficients of the universal formal group law to those of $f^a(X_1,X_2)$. The theorem now follows. □

The main consequence of this result is the definition of the group of *Witt vectors* associated with the formal group law $f^a(X_1,X_2)$. Addition of Witt vectors is defined by the polynomials $\Sigma_n^a(x;y)$.

C. Reutenauer conjectured in [42] that for $n \geq 2$, the symmetric functions $-q_n$ are *Schur positive*, that is linear combinations of Schur functions with non-negative integer coefficients. He proved this conjecture when n is a power of 2 by explicitly describing the representation of the symmetric group Σ_n whose image under the characteristic map is $-q_n$. Reutenauer's conjecture was proved in general by W. Doran in [10], and independently by T. Scharf and J.-Y. Thibon. Here we formulate a related Schur positivity result for the symmetric functions q_n^b in Sym_L^1. Let

$$q_n^b = \sum_{|I|=n-1} (-1)^{l(I)} \, m_I \, \overline{q}_I \,,$$

where $m_I := m_{i_1} m_{i_2} \ldots$. Clearly, $\overline{q}_I$ are symmetric functions in $Sym_{\mathbb{Z}}^n$; furthermore, from (6.5.6) it follows that $\overline{q}_I = 0$ unless $i_j + 1$ divides n for every part i_j of I. We now state the promised Schur positivity result, and refer to [28] for the classical results used in the proof.

Proposition 6.5.9 *The symmetric functions $\overline{q}_I$ are Schur positive.*

Proof. We can rewrite (6.5.5) as

$$q_n^a = \overline{a}_{n-1}(\Psi_n - (q_1^a)^n) - \sum_{d|n,\, 1\neq d\neq n} \overline{a}_{n/d-1}(q_d^a)^{n/d} \,.$$

We use induction based on the Littlewood-Richardson rule. Since $q_1^a = S_1$, it only remains to prove that $S_1^n - \Psi_n$ is Schur positive. Let us note first that

$$\Psi_n = S_n - S_{(n-1,1)} + S_{(n-2,1^2)} - \ldots \,.$$

This follows from the fact that

$$\Psi_n = \sum_{|I|=n} K_{(n),I}^{(-1)} \, S_I \,,$$

where $K_{(n),I}^{(-1)}$ are entries of the inverse Kostka matrix, which are computed in [28] page 107. On the other hand, the coefficient of $S_{(n-k,1^k)}$ in S_1^n is greater than 0 by Young's rule, which concludes the proof. □

Let us note that our result for $n = 2^k$ implies the Schur positivity of $-q_n$, since the partitions of $2^k - 1$ with parts of the form $2^i - 1$, $i \geq 1$, have odd lengths. It would be interesting to describe the representations of Σ_n corresponding to the symmetric functions $\overline{q}_I$ in this case, and relate them to the representations constructed by Reutenauer.

Bibliography

[1] J. F. Adams. *Stable Homotopy and Generalised Homology.* Chicago University Press, Chicago, 1972.

[2] M. Aigner. *Combinatorial Theory.* Springer-Verlag, Berlin–New York, 1978.

[3] A. Baker. Combinatorial and arithmetic identities based on formal group laws. volume 1298 of *Lecture Notes in Math.*, pages 17–34. Springer-Verlag, Berlin–New York, 1987.

[4] Bourbaki. *Algèbre.* Hermann, Paris, 1970.

[5] P. Cameron. Private communication, October 1994.

[6] L. Carlitz. Some properties of Hurwitz series. *Duke Math. J.*, 16:285–295, 1949.

[7] W. Y. C. Chen. A general bijective algorithm for trees. *Proc. Natl. Acad. Sci. USA*, 87:9635–9639, 1990.

[8] F. Clarke, J. Hunton, and N. Ray. Extensions of umbral calculus II: double delta operators and Hattori-Stong theorems. Preprint, University of Manchester, 1995.

[9] L. Comtet. *Advanced Combinatorics.* Reidel, Dordrecht, 1974.

[10] W. F. Doran. A proof of Reutenauer's $-q_{(n)}$ conjecture. Preprint, California Institute of Technology, 1995.

[11] P. Doubilet. On the foundations of combinatorial theory VII: Symmetric functions through the theory of distribution and occupancy. *Studies Appl. Math.*, LI:377–396, 1972.

[12] P. Doubilet, G.-C. Rota, and R. Stanley. On the foundations of combinatorial theory VI: the idea of generating function. In *Sixth Berkeley Symposium on Mathematical Statistics and Probability*, volume 2, pages 267–318. University of California Press, 1972.

[13] A. W. M. Dress and C. Siebeneicher. The Burnside ring of profinite groups and the Witt vector construction. *Adv. Math.*, 70:87–132, 1988.

[14] A. W. M. Dress and C. Siebeneicher. The Burnside ring of the infinite cyclic group and its relation to the necklace algebra, λ-rings, and the universal ring of Witt vectors. *Adv. Math.*, 78:1–41, 1989.

[15] L. Geissinger. Hopf algebras of symmetric functions and class functions. In A. Dold and B. Eckmann, editors, *Combinatoire et Représentation du Group Symétrique*, volume 579 of *Lecture Notes in Math.*, pages 168–181. Springer-Verlag, Berlin–Heidelberg–New York, 1977.

[16] I. P. Goulden and D. M. Jackson. *Combinatorial Enumeration.* Wiley Intersci. Ser. in Discrete Math. John Wiley & Sons, 1983.

[17] M. Haiman and W. Schmitt. Incidence algebra antipodes and Lagrange inversion in one and several variables. *J. Combin. Theory Ser. A*, 50:172–185, 1989.

[18] M. Hazewinkel. *Formal Groups and Applications.* Academic Press, New York, 1978.

[19] S. A. Joni and G.-C. Rota. Coalgebras and bialgebras in combinatorics. *Stud. Appl. Math.*, 61:93–139, 1979.

[20] I. Kozma. Witt vectors and complex cobordism. *Topology*, 13:389–394, 1974.

[21] S. Lang. *Algebra.* Addison-Wesley, Reading, MA, 1965.

[22] A. Lascoux and M.-P. Schützenberger. Formulaire raisonné des fonctions symétriques. Publ. univ. paris 7, 1985.

[23] C. Lenart. Formal group laws and symmetric functions. Preprint, University of Manchester, 1995.

[24] C. Lenart. Necklace algebras and Witt vectors associated with formal group laws. submitted to *Discrete Mathematics*, 1996.

[25] C. Lenart and N. Ray. Chromatic polynomials of partition systems. to appear in *Discrete Math.*

[26] C. Lenart and N. Ray. Hopf algebras of set systems. submitted to *Discrete Mathematics*, 1995.

[27] C. Lenart and N. Ray. Some applications of incidence Hopf algebras to formal group theory and algebraic topology. Preprint, University of Manchester, 1995.

[28] I. G. Macdonald. *Symmetric Functions and Hall Polynomials.* Oxford Mathematical Monographs. Oxford University Press, Oxford, second edition, 1995.

[29] N. Metropolis and G.-C. Rota. Witt vectors and the algebra of necklaces. *Adv. Math*, 50:95–125, 1983.

[30] J. W. Milnor and J. D. Stasheff. *Characteristic Classes*, volume 76 of *Ann. of Math. Stud.* Princeton University Press, Princeton, NJ, 1974.

[31] R. Mullin and G.-C. Rota. On the foundations of combinatorial theory III: theory of binomial enumeration. Graph Theory and Its Applications, pages 168–213. Academic Press, New York, 1970.

[32] W. Nichols and M. Sweedler. Hopf algebras and combinatorics. In R. Morris, editor, *Umbral calculus and Hopf algebras*, volume 6 of *Contemp. Math.*, pages 49–84. Amer. Math. Soc., Providence, RI, 1980.

[33] D. C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres*. Academic Press, New York, 1986.

[34] N. Ray. Extensions of umbral calculus; penumbral coalgebras and generalised Bernoulli numbers. *Adv. Math.*, 61:49–100, 1986.

[35] N. Ray. Symbolic calculus: a 19th century approach to MU and BP. In J. D. S. Jones and E. Rees, editors, *Proceedings of 1985 Durham Symposium on Homotopy Theory*, volume 117 of *London Math. Soc. Lecture Note Ser.*, pages 195–238, Cambridge, 1987. Cambridge University Press.

[36] N. Ray. Loops on the 3-sphere & umbral calculus. In M. Mahowald and S. Priddy, editors, *Algebraic Topology*, volume 96 of *Contemp. Math.*, pages 297–302. Amer. Math. Soc., Providence, RI, 1989.

[37] N. Ray. Tutte algebras of graphs and formal group theory. *Proc. London Math. Soc.*, 65:23–45, 1992.

[38] N. Ray and W. Schmitt. Coclosure operators and chromatic polynomials. *Proc. Natl. Acad. Sci. USA*, 87:4685–4687, 1990.

[39] N. Ray and W. Schmitt. Combinatorics of the Novikov and Steenrod algebras. Preprint, University of Manchester, 1995.

[40] N. Ray, W. Schmitt, and C. Wright. Colouring simplicial complexes. *Ars Combin.*, 29A:161–169, 1990.

[41] N. Ray and C. Wright. Colourings and partition types: a generalised chromatic polynomial. *Ars Combin.*, 25B:277–286, 1988.

[42] C. Reutenauer. On symmetric functions related to Witt vectors and the free Lie algebra. *Adv. Math.*, 110:234–246, 1995.

[43] J. Riordan. *Combinatorial Identities.* Robert E. Krieger Publishing Co., Huntington, New York, reprint edition, 1979.

[44] G.-C. Rota. On the foundations of combinatorial theory I: theory of Möbius functions. *Z. Wahrsch. Verw. Gebiete*, 2:340–368, 1964.

[45] B. E. Sagan. Congruences via Abelian groups. *J. Number Theory*, 20:210–237, 1985.

[46] W. Schmitt. Antipodes and incidence coalgebras. *J. Combin. Theory Ser. A*, 46:264–290, 1987.

[47] W. R. Schmitt. Incidence Hopf algebras. *J. Pure Appl. Algebra*, 96:299, 1994.

[48] D. Segal. The cooperation on $MU_*(\mathbb{C}P^\infty)$ and $MU_*(\mathbb{H}P^\infty)$ and the primitive generators. *J. Pure Appl. Algebra*, 14:315–322, 1979.

[49] R. Stanley and J. Stembridge. On immanants of Jacobi-Trudi matrices and permutations with restricted position. *J. Combin. Theory Ser. A*, 62:261–279, 1993.

[50] R. P. Stanley. *Enumerative Combinatorics.* Wadsworth & Brooks/Cole, Monterey, CA, 1986.

[51] R. P. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.*, 111:166, 1995.

[52] V. Strehl. Cycle counting for isomorphism types of endofunctions. *Bayreuth. Math. Schr.*, 40:153–167, 1992.

[53] M. E. Sweedler. *Hopf Algebras*. W. A. Benjamin, New York, 1969.

[54] K. Varadarajan and K. Wehrhahn. Aperiodic rings, necklace rings, and Witt vectors. *Adv. Math.*, 81:1–29, 1990.

[55] D. Wagner. The partition polynomial of a finite set system. *J. Combin. Theory Ser. A*, 56:138–159, 1991.

[56] T. Zaslavsky. Generalized matchings and generalized Hermite polynomials. In A. Hajnal, L. Lovász, and V. T. Sos, editors, *Finite and infinite sets*, volume 37 of *Coll. Math. Soc. Janos Bolyai*, pages 851–865. North-Holland, 1984.