High School Math Problems
2016
Week 17
Problem and Solution

Let x_1 and x_2 be the roots of the equation $(2k+1)\,x^2 - (k+2)\,x + k - 1 = 0$. For what values of k do we have that

$$x_1^3 x_2 + x_1 x_2^3 = \frac{3k - 3k^3 - 6}{(2k+1)^3}?$$

Solution:

From Vieta's formulas we have that

$$\begin{aligned}
x_1^3 x_2 + x_1 x_2^3 &= x_1 x_2 \left(x_1^2 + x_2^2\right) \\
&= x_1 x_2 \left[(x_1 + x_2)^2 - 2x_1 x_2\right] \\
&= \frac{k-1}{2k+1}\left[\left(\frac{k+2}{2k+1}\right)^2 - 2 \cdot \frac{k-1}{2k+1}\right] \\
&= \frac{k-1}{2k+1} \cdot \frac{(k+2)^2 - 2\,(k-1)\,(2k+1)}{(2k+1)^2} \\
&= (k-1) \cdot \frac{(k+2)^2 - 2\,(k-1)\,(2k+1)}{(2k+1)^3} \\
&= (k-1) \cdot \frac{k^2 + 4k + 4 - 2\,(2k^2 - k - 1)}{(2k+1)^3} \\
&= (k-1) \cdot \frac{k^2 + 4k + 4 - 4k^2 + 2k + 2}{(2k+1)^3} \\
&= (k-1) \cdot \frac{-3k^2 + 6k + 6}{(2k+1)^3} \\
&= \frac{-3k^3 + 9k^2 - 6}{(2k+1)^3}.
\end{aligned}$$

By hypothesis we then have that

$$\begin{aligned}
&\frac{-3k^3+9k^2-6}{(2k+1)^3} = \frac{3k-3k^3-6}{(2k+1)^3} \\
&-3k^3+9k^2-6 = 3k-3k^3-6 \\
&9k^2-6 = 3k-6 \\
&9k^2-3k = 0 \\
&k(3k-1) = 0 \\
&k=0 \text{ or } k=\frac{1}{3}.
\end{aligned}$$