Detecting Asset Price Bubbles in the Near-Explosive Random Coefficient Autoregressive Model.

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Abstract

This paper proposes a random-coefficient autoregressive model that can accomodate the pricing of assets under standard present-value relations, both according to fundamentals and in the presence of bubbles. The distribution of the random coefficient is parameterized in a local-asymptotics framework as a moderate deviation from a stochastic unit root. An application to inference regarding the dynamics of U.S. house prices shows the pertinence of the model.
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1 Introduction and motivations

The aim of this paper is to propose a random-coefficient autoregressive model that accommodates the pricing of assets under the standard present-value relation of Campbell and Shiller (1987) both according to fundamentals and in the presence of bubbles. To understand the rationale behind our modeling choice, consider a standard present value model where the price (P_t) of a unique asset at time t depends on the expected discounted (at rate $1 + R_{t+1}$) value of future associated cash flows. Campbell and Shiller (1987) show that the price of the asset at time t is function of the cash flow D_{t+1} it generates between t and $t+1$ as in:

$$P_t = \mathsf{E}_t \left[\frac{P_{t+1} + D_{t+1}}{1 + R_{t+1}} \right] \tag{1}$$

where $\mathsf{E}_t [\cdot]$ denotes the expectation conditional on information available at time t. $(1 + R_{t+1})^{-1}$ is often referred to as the Stochastic Discount Factor or pricing kernel. The present-value relation implies that the price can be decomposed into

$$P_t = F_t + B_t \tag{2}$$

where the so called *fundamental* price F_t is defined as the expectation, conditional on information available at t of discounted future cash flows:

$$F_t = \mathsf{E}_t \left[\sum_{j=1}^{\infty} \frac{D_{t+j}}{(1 + R_{t+1}) \dots (1 + R_{t+j})} \right]. \tag{3}$$

and satisfies the usual transversality (non-Ponzi) condition.

In expression (2), B_t denotes any process that satisfies the following "conditional exuberance" condition

$$B_t = \mathsf{E}_t \left[\frac{B_{t+1}}{1 + R_{t+1}} \right]. \tag{4}$$

There exist solutions to this condition for which B_t exhibits exponential growth and can be labeled as "bubbles", see inter alia, Blanchard (1979), Blanchard and Watson (1982), Hamilton (1986), West (1987), Evans (1991), Abreu and Brunnermeier (2003), see Lansing (2010) for a recent overview. Many of the proposed models can be represented as

$$B_t = a_t B_{t-1} + \eta_t \tag{5}$$

where η_t is a martingale difference sequence and e.g. $a_t > 1$ while the bubble lasts and $a_t = 0$ when it bursts (in Blanchard and Watson, 1982, $a_t = \frac{1+R}{\pi}$ with probability $\pi \in (0, 1)$ and zero otherwise). It is often assumed that it is the presence of deterministic breaks or regime switches that governs the inception and termination of bubbles (see e.g. Campbell *et al.*, 1996, and references therein). Diba and Grossman (1988b) conduct a thorough analysis of the causes of the emergence of a bubble of the form (5).

The literature has provided several techniques to test for the presence of a bubble, see Gürkaynak (2008) for an overview. They fall into two categories. The first relies on the link between the fundamental price P_t and the cash flow D_t. Systematic deviations between (functions of) the two are seen as indicative of the presence of a bubble. This is the spirit of the tests by West (1987) and Diba and Grossman (1988). Kenneth West uses a Haussman test for the difference of two estimators, thus testing both model misspecification and the presence of a bubble. Diba and Grossman resort to testing the null of cointegration between P_t and D_t, i.e. the null of the absence of a bubble. This approach has been criticized by Evans (1991) as it is not robust to periodically collapsing bubbles. Several authors have attempted to tackle the resulting size distortions, see inter alia Taylor and Peel (1998) and Van Norden and Vigfusson (1998) who allow for regime switching.

The other type of tests that has been proposed, and that is closer in spirit to our approach, only rely on testing whether prices (possibly log prices) are integrated of order 1 against an alternative that the first difference is not stationary. This approach relies on the idea that bubbles are 'more explosive' than stochastic trends and that the latter are in line with fundamental pricing. This also corresponds to the model of bubbles as 'charges' of Gilles (1989) and Gilles and LeRoy (1992). The null of the absence of a bubble can therefore be tested by a simple unit root test. Peter Phillips, Jun Yu and several coauthors have proposed in a stream of papers (see inter alia Phillips, Wu and Yu, 2011, and Phillips and Yu, 2009; respectively PWY and PY henceforth) to perform recursive Dickey-Fuller tests, where right-tailed rejection is indicative of a bubble. To estimate the inception and termination of the bubble, when the latter has been detected, these authors adapt the test size to the number of observations. They compute critical values using the distributions derived by Phillips and Magdalinos (2007, PM henceforth) under the assumption of a locally explosive root. This relies implicitly on the null that the process experiences deterministic breaks at the inception t_1 and burst t_2 of the bubble, as in its simplest version:

$$\begin{aligned} y_t &= y_{t-1} 1\{t < t_1 \text{ or } t > t_2\} + \delta_T y_{t-1} 1\{t_1 \le t \le t_2\} + \varepsilon_t \\ \delta_T &= 1 + \frac{c}{T^\alpha}, \quad c > 0, \quad \alpha \in (0,1) \end{aligned}$$

The Phillips-Yu approach has several advantages: (*i*) being univariate, it avoids the need to specify a structural model referring to the fundamentals; (*ii*) it is simple to use since it relies only on Dickey-Fuller tests and (*iii*) relying on functional central limit theorems, it is robust to short-run specification and powerful in the presence of a low magnitude or periodically collapsing bubble. Unfortunately, the methodology presents several drawbacks: (*i*) referring to deterministic breaks implies some trimming of observations at the beginning and end of the sample and does not allow for estimation of a unique model over the whole sample; also the deterministic breaks are not forecastable and so the timing of the bubble can only be made ex-post. This also implies that the estimators of (t_1, t_2) are biased. Also, (*ii*) the univariate setting relies on log linearization of the present-value model as in Campbell and Shiller (1988); this approximation introduces the discount rate as a function of the average dividend-price ratio. In the presence of bubbles, this average does not correspond to well defined population moments. Finally, (*iii*) the magnitude of the bubble is not related to its estimation whereas it seems intuitively important to tie the detection to the observed or estimated magnitude.

Specifying that the autoregressive coefficient is stochastic, this papers nests the Phillips-Yu models and alleviates some of the drawbacks mentioned above. We can draw inference on the whole sample and there is no need to resort to rolling/recursive windows to test the presence of a bubble and estimate its magnitude. In particular, there is no need to consider bubbles with a duration of small infinity, i.e. of order $O\left(\frac{\log T}{T}\right)$ in PWY. Also, we can do away with loglinearization. The emergence of the bubble relates to the value taken by the stochastic discount factor, hence improving the structural interpretation.

The remainder of the paper is as follows. In section 2, we provide a random-coefficient autoregressive process (RCAR) with loca-asymptotic parameterization. We then derive in section 3 the asymptotic properties of the process and parameter estimators. Section 4 presents the method of inference that we propose and simulations show their validity. We apply our methodology in section 5 to inference on the dynamic properties of U.S. house prices.

2 The model

The model we study in this paper belongs to the class of random-coefficient autoregressive (RCAR) models and is similar to those proposed and studies by Andel (1976), Nicholls and Quinn (1982), McCabe and Tremayne (1995) and Granger and Swanson (1997):

$$y_t = \rho_t y_{t-1} + \eta_t, \quad t = 1, \cdots, T; \tag{6}$$

where η_t is assumed to be i.i.d. $\mathsf{N}\left(0, \sigma^2\right)$ and ρ_t to be a nonnegative covariance stationary process. The RCAR model (6) is known (see Nicholls and Quinn, 1982, and Aue, Horváth and Steinebach, 2006) to admit a strictly non-anticipatory stationary solution if and only if

$$\mathsf{E}\left[\log|\rho_t|\right] < 0 \tag{7}$$

and a covariance stationary solution if

$$\mathsf{E}\left[\rho_t^2\right] < 1. \tag{8}$$

This model has a long pedigree in the econometric and statistical literatures. It has been studied for two main purposes.

First, it is a flexible model that nests the standard AR(1) and where the unit root hypothesis can take several forms: $\mathsf{E}[\rho_t] = 1$, or $\mathsf{E}\left[\rho_t^2\right] = 1$, see Granger and Swanson (1997) for a discussion. Several authors have proposed to perform tests of the unit root hypothesis using Lagrange-Multiplier tests within the RCAR model, see Leybourne, McCabe and Tremayne (1996), Hwang and Basawa (2005), Distasio (2008) and Aue and Horváth (2011). When $\mathsf{E}\left[\rho_t^2\right] > 1$, Hwang and Basawa (2005) denote this model an Explosive Random Coefficient Autoregressive model (ERCA) and study processes such that $\mathsf{E}\left[\rho_t^2\right] \geq 1$ and $\mathsf{E}[\log|\rho_t|] < 0$ (which are strictly stationary but do not possess finite second moments).

Second, expression (6) implies that y_t exhibits conditional heteroskedasticity: assume $\rho_t \sim iid\left(\rho, \sigma_\rho^2\right)$ then

$$\mathsf{E}\left[y_t|y_{t-1}\right] = \rho y_{t-1}, \quad \mathsf{Var}\left[y_t|y_{t-1}\right] = \sigma_\rho^2 y_{t-1}^2 + \sigma_\eta^2$$

see inter alia Tsay (1987), Yoon (2002), Hwang and Basawa (2005), Ling and Li (2006), Francq, Makarova and Zakoïan (2008) and Rahbek and Nielsen (2012). These authors, as well as others have also proposed functional forms that differ from (6) and that belong to the classes of double-autoregressive or bilinear processes.

Here we follow Aue (2008) and deviate from the existing literature on RCAR *à la* Granger-Swanson in the sense that we assume that both the expectation and variance of ρ_t are very close to unity: we model the moments using extensions to standard local-asymptotic frameworks so that as $T \to \infty$ $(\mathsf{E}[\rho_T], \mathsf{V}[\rho_T]) \to (1,0)$. This framework builds on Bobkoski (1983), Chan and Wei (1987), Phillips (1987) and more recent work of Giraitis and Phillips (2006), PM , PWY, and PY). We parameterize the distribution of ρ_t to ensure that its realizations take the form of local deviation from a unit root, possibly on the explosive side. The process we consider are formally defined as triangular arrays as the distribution of y_t, for $t \leq T$, is parameterized using the actual sample size T.

Throughout the paper, we make the following assumption.

Assumption 1

$$\rho_t = \exp\left\{\frac{\phi + \lambda T^{\alpha/2} u_t}{T^\alpha}\right\} \qquad \textit{with} \quad u_t \overset{i.i.d.}{\sim} \mathsf{N}(0,1) \tag{9}$$

where $(\phi, \lambda, \alpha) \in \mathbb{R} \times \mathbb{R}_+ \times (0,1)$, *and where* u_t *and* η_t *are mutually independent.*

Under assumption (9), the expectation of ρ_t is $\mathsf{E}[\rho_t] = \exp\left\{\left(\phi + \frac{1}{2}\lambda^2\right)/T^\alpha\right\}$ and its variance satisfies $\mathsf{V}[\rho_t] = \exp\left\{\frac{2\phi+\lambda^2}{T^\alpha}\right\}\left(\exp\left\{\frac{\lambda^2}{T^\alpha}\right\} - 1\right) = \frac{\lambda^2}{T^\alpha} + O\left(T^{-2\alpha}\right)$. So ρ_t admits the following stochastic expansion:

$$\rho_t = \mathsf{E}[\rho_t] + \frac{\lambda}{T^{\alpha/2}} u_t + \frac{\lambda^2}{2T^\alpha}\left(u_t^2 - 1\right) + O_p\left(T^{-2\alpha}\right) \tag{10}$$

It follows that the conditions (7) and (8) for strict and covariance stationarity write under assumption (9):

$$\mathsf{E}[\log|\rho_t|] = \frac{\phi}{T^\alpha} < 0 \Leftrightarrow \phi < 0$$
$$\mathsf{E}\left[\rho_t^2\right] = \exp\left\{2\frac{\phi + \lambda^2}{T^\alpha}\right\} < 1 \Leftrightarrow \phi + \lambda^2 < 0$$

We also notice the condition

$$\mathsf{E}[\rho_t] < 1 \Leftrightarrow \phi + \frac{1}{2}\lambda^2 < 0$$

which will be of interest to us.

The moderately explosive processes of PM and PY are obtained when considering $\lambda = 0$. The difference here is that $\rho_t \in [0, \infty)$: the autoregressive coefficient is allowed over time to enter the mean reversion region $(0,1)$, to be close to unity and to lie on the explosive side $(1, \infty)$. The model deviates non-trivially from that of Aue (2008) in that we allow for a greater role played by the stochastic variation in ρ_t : in his setting, $\mathsf{E}[\rho_t] - 1 = O(T^{-\alpha})$ with $\alpha \in (1/2, 1)$ and $\mathsf{V}[\rho_t] = o\left(T^{-1}\right)$ which implies that $\rho_t - \mathsf{E}[\rho_t]$ lies in a much tighter neighborhood of unity and so does not impact the explosiveness of y_t : $\mathsf{E}\left[\rho_t^2\right] = 2\mathsf{E}[\rho_t] - 1 + O\left(T^{-2\alpha}\right)$ so the conditions $\mathsf{E}\left[\rho_t^2\right] < 1$ and $\mathsf{E}[\rho_t] < 1$ are asymptotically equivalent. This differs from our setting which allows for richer dynamics. We rule out the assumption of fixed (non-local) parameterization, i.e. $\alpha = 0$. An empirical analysis of the RCAR with $\mathsf{E}[\rho_t] > 1$ and non-local parameters was made by Charemza and Deadman (1995) in the context of periodically collapsing bubbles (see also, Aue and Horváth, 2011, and Wang and Gosh, 2009). We show here that, following the recent work by P. C. B. Phillips and his coauthors, the introduction of a local-asymptotic framework yields benefits.

In order to show the sort of dynamics the model generates, figure 2 records simulations of the process over samples of $T = 1000$ observations using two sets of draws of (u_t, η_t). Exuberant periods become clearly more pronounced and explosive as ϕ increases or α decreases. For $\alpha = 1$, the processes exhibit near-unit roots as in Phillips (1987) and no type of what could be called a "bubble" seems to appear visually. As α decreases, some bubbles appear. Some local explosive pattern appears and disappears alternatively. Although, by visual inspection, some draws seem to exhibit volatility clustering (random draw 1, left column), this is generically not an observed pattern (see random draw 2).

3 Asymptotic properties

In this section, we derive asymptotic properties for the RCAR model defined in (6) and assumption (9) that will be useful when building hypothesis tests in the next session. Proofs are given in the Appendix.

First, the following proposition provides a Functional Central Limit Theorem for the model.

Proposition 1 *Let the process y_t be defined for $t \geq 0$ as in (6)-(9), with $y_0 = 0$. For $r \in [0, T^{1-\alpha}]$ and as $T \to \infty$, it holds*

$$T^{-\alpha/2} y_{[rT^\alpha]} \Rightarrow K_{\phi,\lambda}(r) \equiv \int_0^r \exp\left\{(r-s)\phi + \lambda(W_r - W_s)\right\} dB_s$$

where W, B are two independent standard Brownian motion such that, for $(s,v) \in [0,1]^2$,

$$T^{-1/2}\left(\sum_{t=1}^{[sT]} u_t, \sigma^{-1}\sum_{t=1}^{[vT]} \eta_t\right) \Rightarrow (W_s, B_v),$$

$[\cdot]$ denoting the integer part.

Corollary 2 *The limiting process $K_{\phi,\lambda}(r)$ rewrites as*

$$K_{\phi,\lambda}(r) = J_\phi(r) - e^{r\phi + \lambda W_r}\left(\frac{\lambda^2}{2}\int_0^r X_s Y_s ds - \lambda \int_0^r X_s Y_s dW_s\right) \tag{11}$$

where the Ornstein-Uhlenbeck process $J_\phi(r) = \int_0^r e^{(r-s)\phi} dB_s$ is the limit of $T^{-\alpha/2} y_{[rT^\alpha]}$ when ρ_t is nonstochastic ($\lambda = 0$) as in PM, and where $(X_s, Y_v) = \left(e^{-\lambda W_s}, e^{-\phi v} J_\phi(v)\right)$.

The process $K_{\phi,\lambda}(r)$ defined in Proposition 1 is distributed as

$$K_{\phi,\lambda}(r) \sim \mathsf{N}\left(0, \sigma_\eta^2 \int_0^r e^{2\left(\phi+\lambda^2\right)s} ds\right), \quad \text{for } r \in [0, T^{1-\alpha}]$$

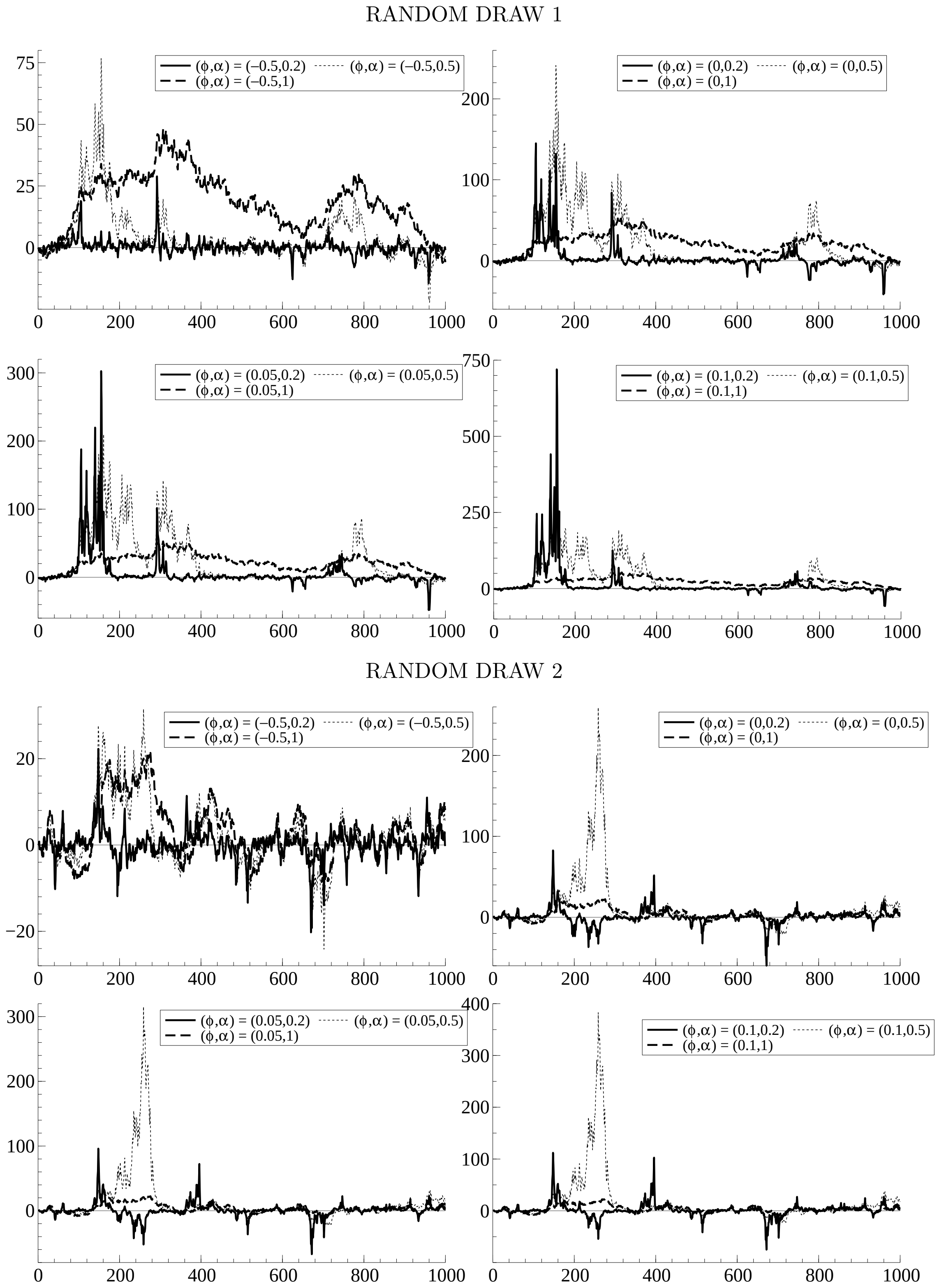

Figure 1: Simulated realizations from the model of autoregressive conditional exuberance for different parameter values.

where the integral $\int_0^r e^{2\left(\phi+\lambda^2\right)s} ds = \frac{e^{2\left(\phi+\lambda^2\right)r}-1}{2(\phi+\lambda^2)}$ if $\phi + \lambda^2 \neq 0$ and is equal to r otherwise. It shows that the parameter

$$c = \phi + \lambda^2 \tag{12}$$

plays a major role for the asymptotics of the process: if $c < 0$, then $K_{\phi,\lambda}(r)$ admits a stationary version given by

$$K^*_{\phi,\lambda}(r) = e^{cr} K^*_{\phi,\lambda}(0) + K_{\phi,\lambda}(r) \quad \textit{with} \quad K^*_{\phi,\lambda}(0) \sim \mathsf{N}\left(0, \frac{\sigma_\eta^2}{-2c}\right)$$

whereas it is not the case for $c \geq 0$. Hence several cases arise since

$$y_T = \begin{cases} O_p\left(T^{\frac{\alpha}{2}}\right) & \text{if} \quad c < 0 \\ O_p\left(T^{\frac{1}{2}}\right) & \text{if} \quad c = 0 \\ O_p\left(e^{cT^{1-\alpha}} T^{\frac{\alpha}{2}}\right) & \text{if} \quad c > 0 \end{cases}$$

Equation (11) shows how the stochastic nature of the autoregressive coefficient modifies the asymptotic distribution of y_t. In addition this formulation may be useful for efficient computing of the Monte Carlo simulations.

We are interested in studying the properties of the OLS estimator $\widehat{\rho}$ in the regression of y_t on y_{t-1}. For this purpose, we introduce the following random variables.
If $\lambda^2 < \phi$, let $X = \sqrt{2(\phi - \lambda^2)}\sigma_\eta^{-1} \int_0^\infty e^{-\phi s - \lambda W_s} dB_s$ which is distributed $\mathsf{N}(0,1)$. If $\lambda^2 \geq \phi$, X is not defined but we introduce instead $X^* \sim \mathsf{N}(0,1)$ such that , when $T \to \infty$, $X^*_{T^{1-\alpha}} \Rightarrow X^*$, with

$$X^*_{T^{1-\alpha}} = \begin{cases} \sigma_\eta^{-1}\sqrt{2(\lambda^2 - \phi)} e^{-\left(\lambda^2-\phi\right)T^{1-\alpha}} \int_0^{T^{1-\alpha}} e^{-(\phi s + \lambda W_s)} dB_s & \text{if} \quad \lambda^2 - \phi > 0 \\ \sigma_\eta^{-1} T^{-\frac{1-\alpha}{2}} \int_0^{T^{1-\alpha}} e^{-(\phi s + \lambda W_s)} dB_s & \text{if} \quad \lambda^2 = \phi \end{cases}$$

Similarly, we define the following random variables:

$$\left(Z_{T^{1-\alpha}}, Y_{T^{1-\alpha}}, V_{T^{1-\alpha}}\right) = \left(\int_0^{T^{1-\alpha}} e^{2(\phi r + \lambda W_r)} dr \,,\, \sigma_\eta^{-1} \int_0^{T^{1-\alpha}} e^{\phi r + \lambda W_r} dB_r \,,\, \int_0^{T^{1-\alpha}} e^{\phi r + \lambda W_r} dW_r\right)$$

satisfying, as $T \to \infty$,

$$\left(\frac{Z_{T^{1-\alpha}}}{E\left(Z_{T^{1-\alpha}}\right)}, \frac{Y_{T^{1-\alpha}}}{\sqrt{Var\left(Y_{T^{1-\alpha}}\right)}}, \frac{V_{T^{1-\alpha}}}{\sqrt{Var\left(V_{T^{1-\alpha}}\right)}}\right) \Rightarrow (Z, Y, V)$$

Y, V being independent standard normal random variables and Z a random variable with unit expectation (see Matsumoto and Yor, 2005, theorems 7.2 and 7.4(ii)). When $\lambda = 0$, $Z = 1$ with probability one. We now state our main result.

Theorem 3 *Let the process* (y_t) *be defined as in (6) under assumption (9) for* $t \geq 0$, *with* $y_0 = 0$. *The OLS estimator* $\widehat{\rho}$ *in the regression of* y_t *on* y_{t-1} *satisfies the following properties as* $T \to \infty$:

- *if* $c < 0$: $\begin{cases} T^{\frac{1+\alpha}{2}} \left(\widehat{\rho} - E\left(\rho_t\right)\right) \Rightarrow N\left(0, 3\lambda^2 - 2c\right) \\ T^{\frac{1+\alpha}{2}} \left(\widehat{\rho} - \rho_t\right) \Rightarrow N\left(0, -2c\right) \end{cases}$
- *if* $c = 0$ *and* $\lambda \neq 0$: $\begin{cases} T^{\frac{1+\alpha}{2}} \left(\widehat{\rho} - \rho_t\right) \Rightarrow \sqrt{2\phi} \frac{Y}{XZ} \\ T^{(1-\alpha)} e^{-\lambda^2 T^{1-\alpha}} \left(\widehat{\rho} - \mathsf{E}\left[\rho_t\right]\right) \Rightarrow \frac{1}{2}\frac{V}{Z} \end{cases}$
- *if* $c > 0$ *and if*

$$\lambda^2 < \phi : \begin{cases} T^{\alpha} e^{\left(\phi + \lambda^2\right) T^{1-\alpha}} \left(\widehat{\rho} - \rho_t\right) \Rightarrow 2\sqrt{\phi^2 - \lambda^4} \frac{Y}{XZ} \\ e^{\phi T^{1-\alpha}} \left(\widehat{\rho} - \mathsf{E}\left[\rho_t\right]\right) \Rightarrow \lambda \frac{\phi + \lambda^2}{\sqrt{\phi + 2\lambda^2}} \frac{V}{Z} \end{cases}$$

$$\lambda^2 = \phi : \begin{cases} T^{\frac{1+\alpha}{2}} e^{2\lambda^2 T^{1-\alpha}} \left(\widehat{\rho} - \rho_t\right) \Rightarrow \frac{Y}{X^* Z} \\ T^{\frac{1-\alpha}{2}} e^{\lambda^2 T^{1-\alpha}} \left(\widehat{\rho} - \mathsf{E}\left[\rho_t\right]\right) \Rightarrow \frac{2\lambda^2}{\sqrt{3}} \frac{V}{Z} \end{cases}$$

$$\lambda^2 > \phi : \begin{cases} T^{\alpha} e^{2\lambda^2 T^{1-\alpha}} \left(\widehat{\rho} - \rho_t\right) \Rightarrow 2\sqrt{\lambda^4 - \phi^2} \frac{Y}{X^* Z} \\ T^{\alpha/2} e^{\phi T^{1-\alpha}} \left(\widehat{\rho} - \mathsf{E}\left[\rho_t\right]\right) \Rightarrow \frac{\phi + \lambda^2}{\sqrt{\phi + 2\lambda^2}} \frac{V}{Z} \end{cases}$$

In all parameter combinations, we provide two asymptotic results for $\widehat{\rho} - \rho_t$ and $\widehat{\rho} - \mathsf{E}\left[\rho_t\right]$. This shows how the stochastic nature of the autoregressive parameter affects the properties of the estimator. The asymptotic distribution of $\widehat{\rho} - \rho_t$ is here comparable to the results of PM where ϕ (the only localizing parameter since they assume $\lambda = 0$) is here replaced with c. When $c < 0$, the presence of the stochastic root does not affect the asymptotic normality of $\widehat{\rho}$. The only difference in this case is that $\lambda \neq 0$ leads to higher variance.

By contrast, when $c \geq 0$ the results above differ from those of PM. First when $c = 0$, the presence of nonzero λ implies that $\widehat{\rho} - \rho_t$ does not converge at a rate $O_p\left(T\right)$ as in the standard unit-root setting (since $c = 0$ and $\lambda = 0$ imply that $\rho_t \equiv 1$). The theorem shows that that the estimator $\widehat{\rho}$ is not consistent for its expectation in only one case, when $c = 0$ and $\lambda \neq 0$ so $\mathsf{E}\left[\rho_t\right] = 1$ but $\rho_t \not\equiv 1$.[1] The presence of a stochastic coefficient ρ_t implies for all cases where $c \geq 0$ that the asymptotic distribution contains a factor $1/Z$ that disappears when $\lambda = 0$. This is the main impact of $\lambda \neq 0$ on the asymptotic distribution of $\widehat{\rho} - \rho_t$ which is otherwise similar to that in PM.

The asymptotic distribution of $\widehat{\rho} - \mathsf{E}\left[\rho_t\right]$ is notably different in that $\rho_t - \mathsf{E}\left[\rho_t\right]$ is the determining element that drives the results. The distribution does not depend on B and defined as the ratio of two uncorrelated variables driven by W.

In the next section, we show how theorem 3 can be used to conduct inference on model parameters.

4 Inference

4.1 Confidence Sets

The DGP we consider uses a local-asymptotic parameterization and it is well known that localizing parameters may not be consistently estimable using standard techniques (see Phillips, 1987).[2]

[1] This result which holds even in the fixed-parameter case where $\alpha = 0$ does not seem to have been established in the literature.

[2] We hence rule out nonlinear extensions to the Kalman filter and the particle filter.

Yet, these are the parameters of interest, and the key assumption we want to test is whether $(\phi, \lambda) \neq (0, 0)$, i.e. whether the market price differs from its fundamental.

To conduct inference, we resort to the standard technique that consists in inverting a test statistic. There exists now a significant literature where such an approach is used for inference in the near-unit root framework (originated from Stock, 1991). This technique is also common in the context of weak instruments where there exists no fully robust estimation method, but robust tests can be constructed (see Anderson and Rubin, 1949, Dufour, 1997, and Staiger and Stock, 1997). Papers that discuss the mechanics of the inversion of robust tests to form confidence sets include Zivot, Startz, and Nelson (1998), Dufour and Jasiak (2001), and Dufour and Taamouti (2005), for a general overview see Andrews and Stock (2005) and references therein.

The technique relies on introducing a scalar function $\tau_{\theta,T}$ (a test statistic) of $Y_T = (y_1, ..., y_T)'$ that satisfies

$$\tau_{\theta,T}(Y_T) \Rightarrow \tau_\theta(Y) \tag{13}$$

with $\theta = (\phi, \lambda, \sigma_\eta^2, y_0)' \in \Theta$. Under the null $H_0 : \theta = \theta_0$, Stock (1991) constructs asymptotic $(1-\varphi)\%$ confidence sets as $\Theta^* \subset \Theta$ consisting of the values θ^* which are not rejected by $\tau_{\theta^*,T}(Y_T)$ at size φ. The finite sample corrections of Andrews (1993), Hansen (1999) and the two-sided Romano and Wolf (2001) have been shown by Mikusheva (2007, see also 2012) to be valid also. Elliott and Stock (2001) discuss it and refine it, using the Elliott, Rothenberg and Stock (1996) unit-root test. In this setting, the least rejected parameter θ^* may constitute a biased estimator of θ but median-unbiased estimation is feasible under the weak convergence assumption, provided that the quantile function is monotonic (Stock, 1991, Andrews, 1993, 1994). When τ is a continuously-updated GMM statistic, θ^* can be seen as the continuously updated estimator (see Stock, Wright and Yogo, 2002) and it inherits its properties.

Here we conduct inference under the null

$$H_0 : (\phi, \lambda) = (\phi_0, \lambda_0).$$

Since $y_t - \mathsf{E}_{H_0}[\rho_T]\, y_{t-1} = (\rho_t - \mathsf{E}_{H_0}[\rho_T])\, y_{t-1} + \eta_t \equiv v_t$, we use the moment condition:

$$\mathsf{Cov}\left(y_t - \mathsf{E}_{H_0}[\rho_T]\, y_{t-1}, y_{t-1}\right) \underset{H_0}{=} 0$$

The test we choose for simplicity follows the pseudo Dickey-Fuller autoregression

$$y_t - \mathsf{E}_{H_0}[\rho_T]\, y_{t-1} = \beta y_{t-1} + \eta_t \tag{14}$$

and we set $\tau_{\theta,T}$ to be the OLS estimator $\widehat{\beta}$ scaled by the asymptotic rate given in theorem 3. Confidence sets are obtained by grid search over all possible values of (ϕ, λ). The parameter σ_η^2 is a scaling that does not affect the asymptotic distribution of $\widehat{\beta}$ so we may fix it to unity. Also, α is not identified using the method above so we fix it also, to $1/2$ in the empirical application.

Alternative test statistics have been proposed in the literature: the locally best invariant Lagrange-Multiplier test of Leybourne *et al.* (1996) which was modified by Distaso (2008) was shown in its original version not to be consistent under the unit root hypothesis against explosive alternatives (see Nagakura, 2009) so we do not use it although we have not analyzed its modified version. Also, Aue and Horvath (2011) propose a Quasi-Maximum Likelihood Estimator that is consistent for $(\mathsf{E}[\rho_t], \mathsf{V}[\rho_t])$ conditional on knowing σ_η^2. Since $\widehat{\beta}$ above is scale invariant, the latter seems preferable to us. Also Hwang and Basawara suggest a weighted least-squares estimator of $\mathsf{E}[\rho_t]$ that is consistent (and asymptotically equivalent to the QMLE of Nicholls and Quinn, 1982). Yet, a test based on this estimator requires estimating $\mathsf{V}[\rho_t]$ and σ_η^2 in a first-step. Estimators thereof where suggested by Schick (1996), under the assumption of covariance stationarity but the properties of these when y_t is explosive are not established. Also, in the context of near-integrated AR(1) models, Jansson and Moreira (2006) recommend a Likelihood Ratio test, see also Müller (2011).

4.2 Evaluation

4.2.1 Power

The $\tau_{\theta,T}$ statistic from the pseudo Dickey-Fuller regression was chosen above for its simplicity, but it may not be the most efficient. A significant issue here concerns the powers of the test. In the spirit of PWY, it may seem natural to perform a one-sided right-tail test for H_0 since the alternative we consider is that of bubbles. Yet, as the sign of parameter λ is not identified – so we have assumed it positive – the right-tail test holds very little power to reject $\lambda_0 = 0$ under the alternative that $\lambda \neq 0$. By contrast a two-sided test is much more powerful.

To assess the power of the inference technique that we propose, figures 3,2 and 4 report, for a given value of $\mathsf{E}[\rho_t] = \rho$, the asymptotic rejection probabilities of the null $H_0 : (\phi, \lambda) = (T^\alpha \log \rho, 0)$ at the nominal size of 10% under the alternative (ϕ, λ) which preserves $\mathsf{E}[\rho_t] = \rho$ (this is indexed by λ). Each figure considers a different value of $\alpha \in \{1/4, 1/2, 3/4\}$.

Starting with figure 2 where $\alpha = 1/2$, we see that our method of inference only rejects the null of a non-stochastic root with a high probability when λ is large or $\phi + \lambda^2/2 > 0$, i.e. $\mathsf{E}[\rho_t] > 1$. Highest power is achieved when λ is large, close to unity with ϕ close to zero, i.e. when the source of explosiveness is stochastic, not deterministic. Notice that as $\phi + \lambda^2/2$ increases, the rejection probabilities converge to values in the range 0.40-0.50. This confirms the analysis by Evans (1991) that stochastic bubbles, being non-permanent by nature, can be difficult to detect even when their magnitude is large.

Now figure 3 considers the case where $\alpha = 1/4$, which we have shown in the simulations of section 2 to generate more explosive patterns. Correspondingly, the proposed technique is even more capable of rejecting the incorrect null of a nonstochastic autoregressive coefficient when the autoregressive root is close to unity, i.e. $\phi + \lambda^2/2 \approx 0$. Yet, rejection probabilities remain low for $\phi + \lambda^2/2 \leq 0$ and stabilize at around 0.50 when $\phi + \lambda^2/2 > 0$. Finally, figure 4 presents the case where $\alpha = 3/4$ and we then see that the power is very low at all values, unless λ is large. This shows that for large values of α, the resulting dynamics may not differ significantly enough from an AR(1).

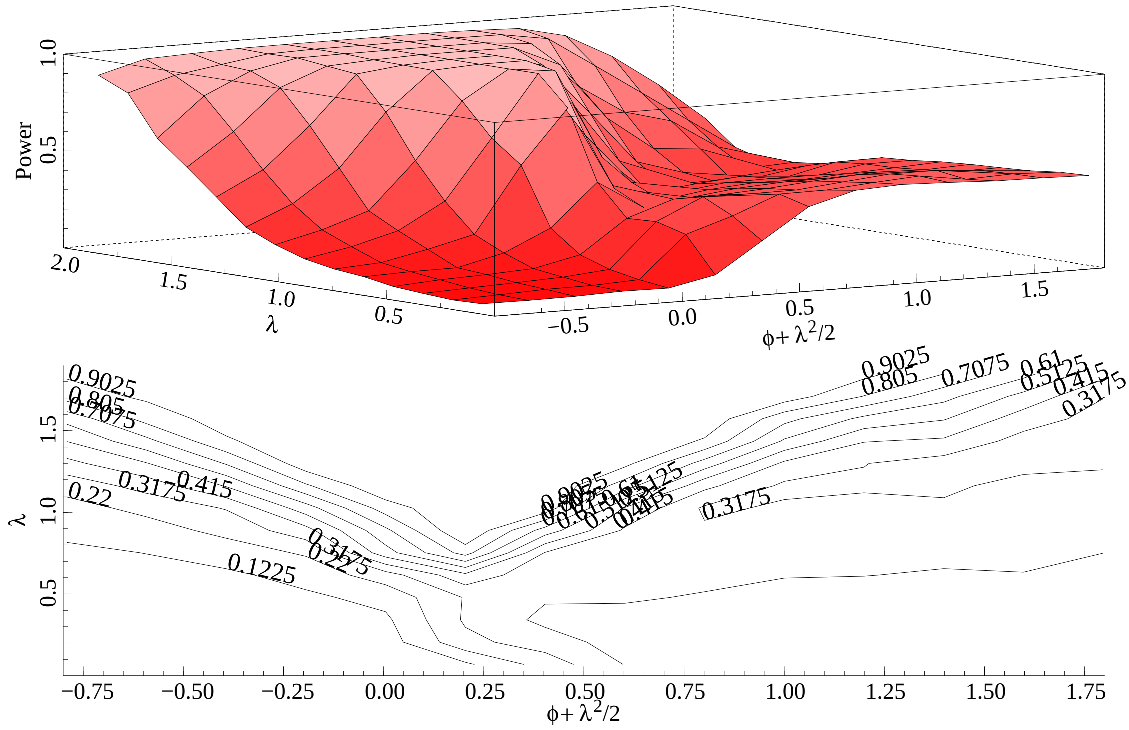

Figure 2: Rejection probabilities at the nominal size of 5% corresponding to the null $H_0 : (\phi, \lambda) = (\phi_0, 0)$ under the alternative that the process follows a random coefficient autoregressive model with $\mathsf{E}[\rho_t] = \exp(\phi_0 T^{-\alpha})$. The value of $\alpha = 0.50$.

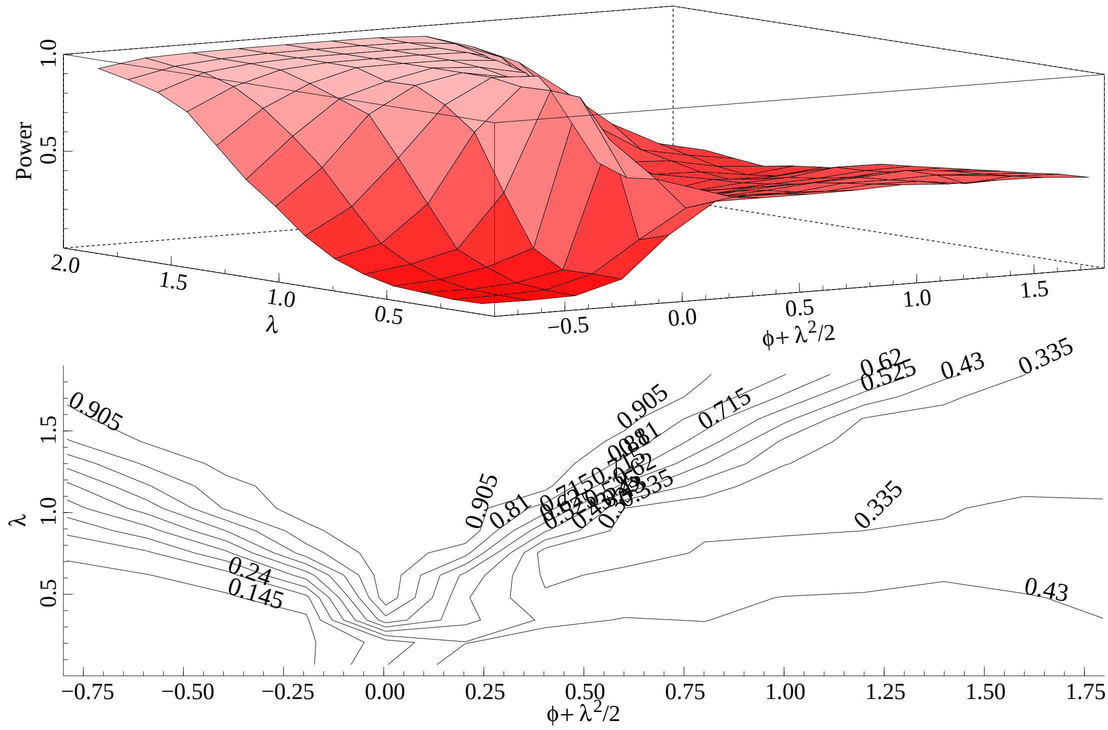

Figure 3: Rejection probabilities at the nominal size of 5% corresponding to the null $H_0 : (\phi, \lambda) = (\phi_0, 0)$ under the alternative that the process follows a random coefficient autoregressive model with $\mathsf{E}[\rho_t] = \exp(\phi_0 T^{-\alpha})$. The value of $\alpha = 0.25$.

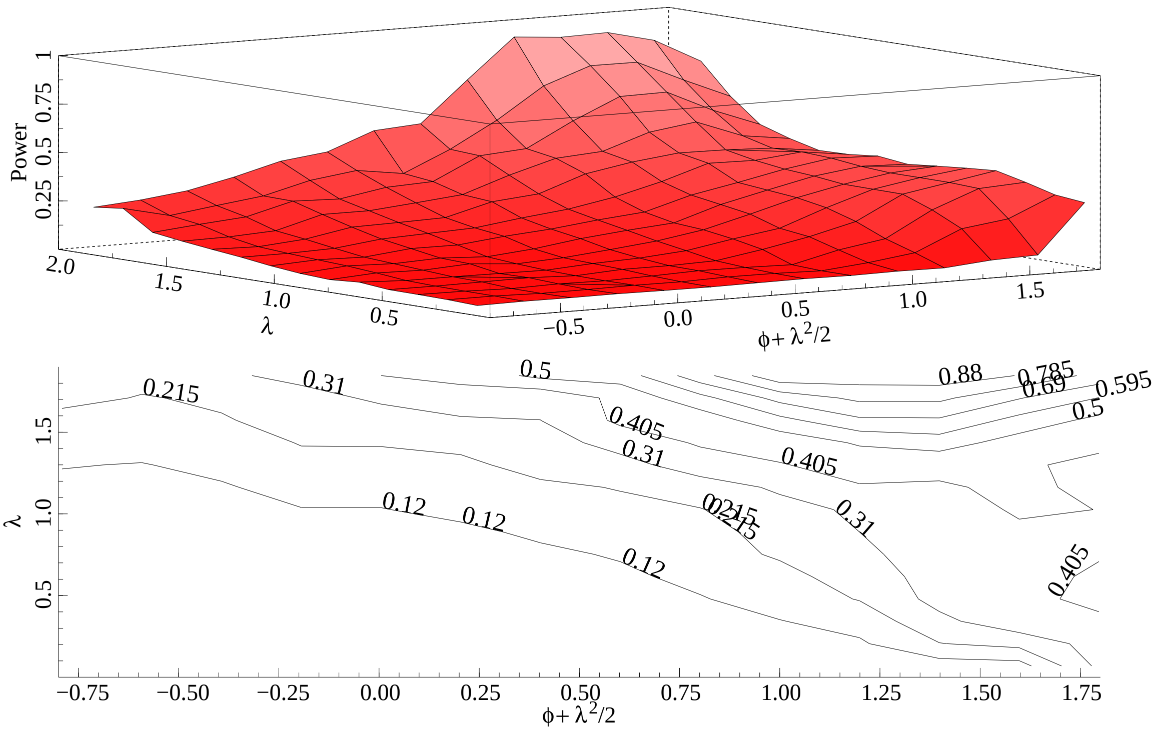

Figure 4: Rejection probabilities at the nominal size of 5% corresponding to the null $H_0 : (\phi, \lambda) = (\phi_0, 0)$ under the alternative that the process follows a random coefficient autoregressive model with $\mathsf{E}[\rho_t] = \exp(\phi_0 T^{-\alpha})$. The value of $\alpha = 0.75$.

4.2.2 Monte Carlo: finite samples

We now provide a short evaluation of the finite sample probability coverage of confidence intervals. Contrary to the issue of power in the previous section, Monte Carlo simulations show that the two-sided intervals have very poor probability coverage but that one-sided intervals against an explosive alternative perform reasonably. Table 1 reports the simulated finite sample ($T = 300$) coverage probability of 95% confidence intervals constructed using the asymptotic distribution under $\alpha = 1/2$. The table shows that the asymptotic distribution is only correct for negative ϕ and λ is large (above unity). When $\phi \leq 0$, and λ is low, the confidence sets are too narrow. When $\phi > 0$, the test is liberal and the confidence intervals too wide when λ is low; the test is conservative and the intervals too narrow (even empty) when λ is large. Unreported simulations show that confidence intervals at a lower nominal probability behave accordingly.

The method of asymptotic inference that was introduced by Stock (1991) was modified by Hansen (1999) who recommend the use of a so called *grid bootstrap.* Such bootstrap aims at replacing the use of the asymptotic distribution (13) by the finite-sample bootstrap distribution whose critical values can be obtained by repetitive sampling from the empirical distribution of the errors $v_t \equiv y_t - \mathsf{E}_{H_0}[\rho_T]\, y_{t-1}$ (which are observed under H_0). Given the possible strong dependence in y_t, it is important to correct the standard bootstrap. We used for this purpose the Maximum Entropy bootstrap (see Vinod, 2006) which is known to perform well in the presence of strong dependence.

Noticing that

$$v_t = \left[\frac{\lambda}{T^{\alpha/2}} u_t + \frac{\lambda^2}{2T^{\alpha}}\left(u_t^2 - 1\right) + O_p\left(T^{-2\alpha}\right)\right] y_{t-1} + \eta_t$$

is asymptotically serially uncorrelated, it appears possibly sufficient to use a boostrapping technique that is immune to heteroskedasticity, such as the wild bootstrap. The lower rows of table 1 show that when $\phi \geq 1$, the bootstraped confidence intervals are better when λ is large. So we use it also in the empirical applications.

5 Application to Housing Prices

We follow Campbell and Shiller (1987) in assuming that the cash flow D_t is integrated of order 1, and, for simplicity, that it follows a random walk $D_t = D_{t-1} + \zeta_t$, with ζ_t white noise. We also assume that R_t is *i.i.d.* stationary and independent of D_t. This implies that, $F_t = \frac{1}{R} D_t$ where $R < \infty$ satisfies $(1 + R)^{-1} = \mathsf{E}\left[(1 + R_t)^{-1}\right]$. Under the simplifying assumption that the ex-post return $r_{t+1} = \frac{P_{t+1}+D_{t+1}}{P_t} - 1$ is constant and equal to R, the present-value relation (1) then admits the solution (with minimal number of state variables, see McCallum, 1983)

$$\Delta P_t = (1 + (1 - \delta) R + \delta R_t) \Delta P_{t-1} - \zeta_t \qquad (15)$$

for some $\delta \in [0, 1]$ (see derivation in the appendix). Bubbles occur when $(1 - \delta) R + \delta R_t > 0$.

We apply our methodology to the seasonally adjusted monthly Case-Shiller housing market price index maintained by Standard and Poor's (288 observations). The series is presented figure 5: the price exhibits sustained growth over the 1987-2005 period followed by a sharp collapse. The

Finite Sample probability coverage for a nominal 95% Confidence Interval								
$\lambda =$	0	0.1	0.2	0.3	0.5	1	1.5	2
Asymptotic Distibution								
$\phi =$ -0.2	0.97	0.97	0.98	0.96	0.93	0.97	0.98	0.98
-0.1	0.94	0.94	0.92	0.90	0.93	0.97	0.97	0.98
0	0.94	0.79	0.84	0.89	0.92	0.95	0.95	0.95
0.1	1.00	1.00	1.00	1.00	1.00	0.00	0.00	0.00
0.2	0.99	1.00	1.00	1.00	1.00	0.00	0.00	0.00
0.3	0.98	1.00	1.00	1.00	1.00	1.00	0.00	0.00
0.4	0.96	1.00	1.00	1.00	1.00	1.00	0.00	0.00
Standard Bootstrap								
$\phi =$ -0.2	1.00	1.00	1.00	1.00	0.93	0.98	0.98	0.98
-0.1	1.00	1.00	1.00	1.00	0.93	0.97	0.98	0.98
0	1.00	0.99	0.98	0.96	0.95	0.96	0.97	0.98
0.1	1.00	1.00	1.00	1.00	1.00	0.95	0.96	0.97
0.2	1.00	1.00	1.00	1.00	1.00	0.95	0.96	0.98
0.3	1.00	1.00	1.00	1.00	1.00	0.94	0.96	0.97
0.4	1.00	1.00	1.00	1.00	1.00	0.93	0.96	0.96
Normal Wild Bootstrap								
$\phi =$ -0.2	1.00	1.00	1.00	1.00	0.93	0.97	0.98	0.98
-0.1	1.00	1.00	1.00	1.00	0.94	0.97	0.98	0.98
0	1.00	0.99	0.97	0.97	0.96	0.97	0.97	0.98
0.1	1.00	1.00	1.00	1.00	1.00	0.98	0.97	0.98
0.2	1.00	1.00	1.00	1.00	1.00	1.00	0.96	0.97
0.3	1.00	1.00	1.00	1.00	1.00	1.00	0.96	0.97
0.4	1.00	1.00	1.00	1.00	1.00	1.00	0.99	0.97
Maximum Entropy Bootstrap								
$\phi =$ -0.2	1.00	1.00	1.00	1.00	0.93	0.99	0.98	0.98
-0.1	1.00	1.00	1.00	1.00	0.94	0.97	0.98	0.98
0	1.00	1.00	1.00	0.99	0.97	0.96	0.97	0.98
0.1	1.00	1.00	1.00	1.00	1.00	0.96	0.97	0.98
0.2	1.00	1.00	1.00	1.00	1.00	0.94	0.96	0.97
0.3	1.00	1.00	1.00	1.00	1.00	0.94	0.96	0.96
0.4	1.00	1.00	1.00	1.00	1.00	0.94	0.95	0.96

Table 1: Simulated Finite Sample Probability Coverage of one-sided confidence intervals at an asymptotic nominal probability of 0.95, together without distributions obtained using the standard, Gaussian wild, and Maximum Entropy bootstraps. The simulated sample size is $T = 300$ with $\alpha = 1/2$.

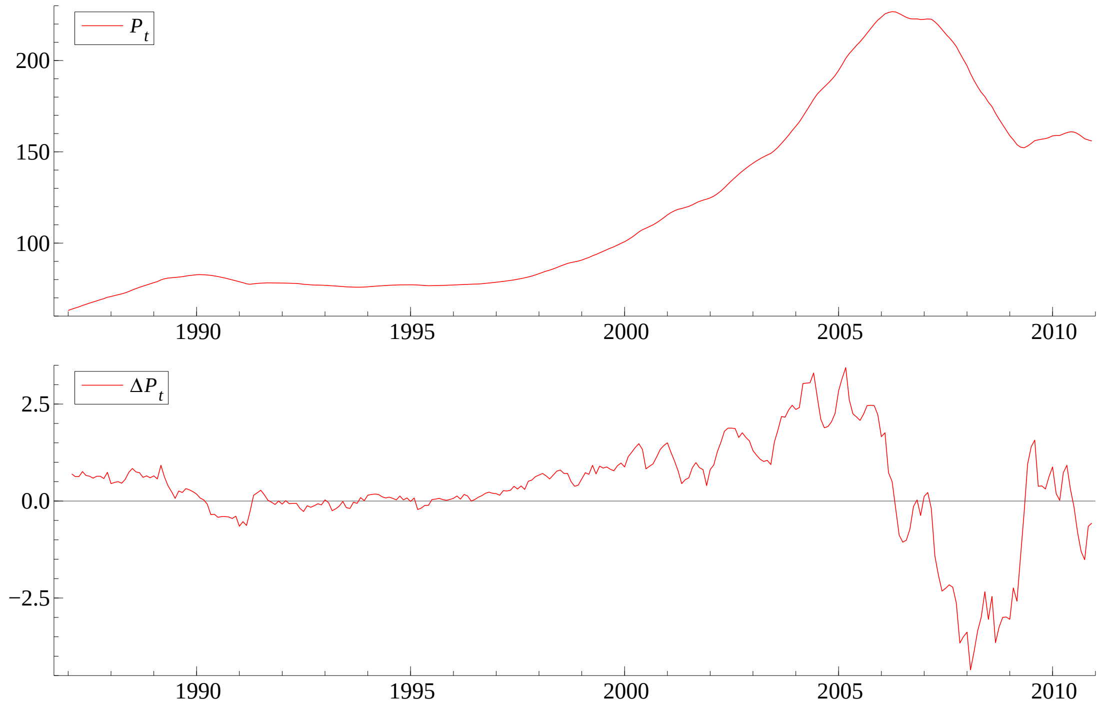

Figure 5: The seasonally adjsuted monthly Case-Shiller Housing price index for the United-States (P_t) and its first-order difference ΔP_t.

figure shows that the difference in prices seem to exhibit patterns similar to those that arise under model (6)-(9). RCAR models such as (6) have also been used in the literature for the price level or log price of an asset, see e.g. Leybourne, McCabe and Mills (1996), Gonzalo and Lee (1998). Yet, we believe that because the model does not preclude negative values of the process, it is better suited for differences, since differencing an explosive process does not remove the explosive root.

The model is applied to the solution to the present value model, expression (15), using the expansion presented in (10). To construct confidence sets, we perform grid searches using uniform draws of the parameters $\phi \in (-1, 1)$ and $\lambda \in (0, 1)$, setting $\alpha = 1/2$.[3]

Figure 6 record parameter draws which were not rejected using the asymptotic distribution under the null. These parameter values were concentrated around the value $\phi = 0$, which we showed to exhibit maximum asymptotic power. Corresponding statistics are presented in table 5. The least rejected parameter combination yields for an explosive root $\mathsf{E}[\rho_t] = 1.07$ with $(\phi, \lambda) = (0.016, 1.58)$ so $c = \phi + \lambda^2 >> 0$. Yet the null that the process follows a pure random walk is only rejected at the 0.12 significance level.

[3]The reported results in this preliminary version of the paper still use too few parameter draws, namely 1000).

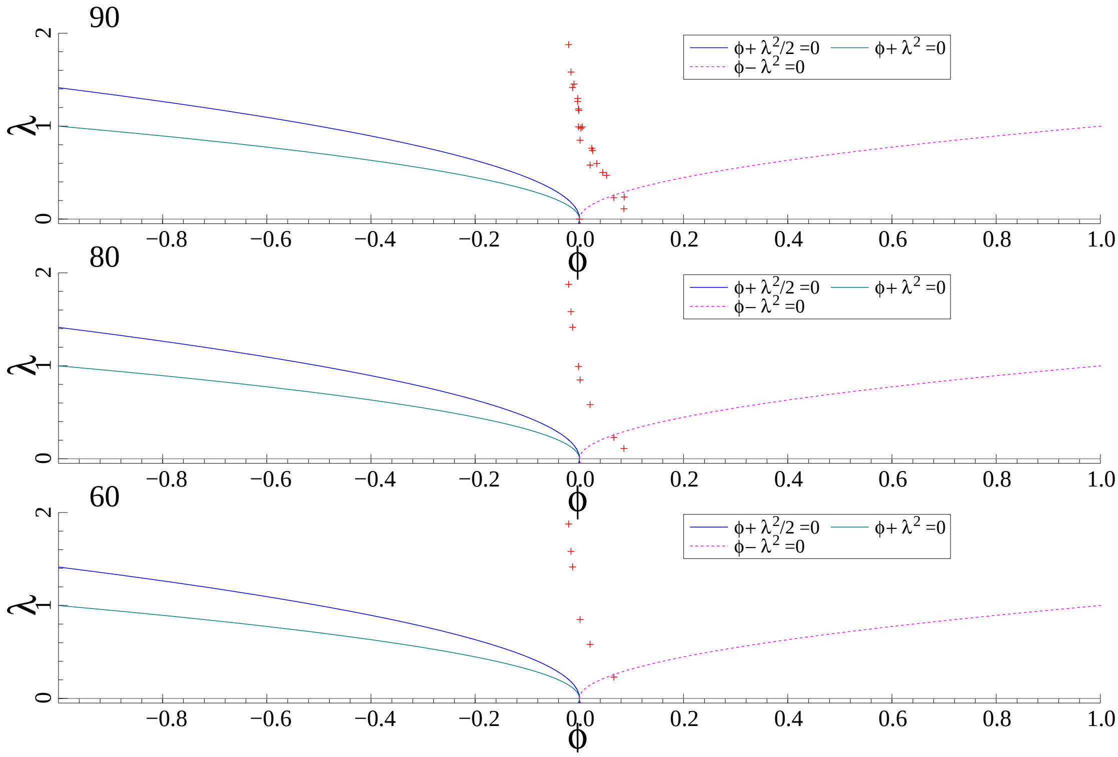

Figure 6: The figure reports parameter combinations which are not rejected at the .9, .8 and .6 probabilities according to the asymptotic distribution ot the OLS estimator $\widehat{\beta}$ in expression (14).

	$\widehat{\rho}$ [*pvalue*]	Least Rejected $(\phi, \lambda)^*$	Confidence Interval 0.90	0.80	0.60	Test $(\phi, \lambda) = (0, 0)$ *pvalue*
Asymptotic	0.972 [0.738]	$(-0.016, 1.58)$ $\mathsf{E}[\rho_t^*] = 1.076$	$\lambda : [0, 1.58]$ $\phi : [-0.016, 0.085]$	$\lambda : [0.11, 1.58]$ $\phi : [-0.016, 0.085]$	$\lambda : [0.58, 1.58]$ $\phi : [-0.016, 0.020]$	0.112
Bootstrap						
Wild bootstrap	0.972 [0.99]	$(-0.93, 1.06)$ $\mathsf{E}[\rho_t^*] = 0.978$	$\lambda : [-0.040, 1.41]$ $\phi : [-1.00, 0.085]$	$\lambda : [-0.035, 1.37]$ $\phi : [-1.00, 0.00]$	$\lambda : [-0.028, 1.22]$ $\phi : [-0.98, -0.077]$	0.246
Max. Entropy	0.972 [0.42]	$(-0.99, 0.026)$ $\mathsf{E}[\rho_t^*] = 0.943$	$\lambda : [0, 0.51]$ $\phi : [-1.00, -0.80]$	$\lambda : [0, 0.51]$ $\phi : [-1.00, -0.88]$	$\lambda : [0.026, 0.026]$ $\phi : [-0.99, -0.99]$	0.000

Table 2: The table reports statistics regarding inference on the dynamics of house prices. *pvalues* were computed under the null of the least rejected parameter values.

Figures 7 and 8 together with the lower part of table 5 present the inference results under the wild (standard normal) bootstrap and the maximum entropy bootstrap of Vinod (2006). The confidence sets under the bootstrap and Maximum entropy methods are much different from the asymptotic, in that they are much wider and centered on negative values of ϕ. According to the Wild bootstrap, most values such that $\mathsf{E}[\rho_t] \leq 0$ cannot be rejected. The maximum entropy bootstrap by contrast yields confidence sets which cover only very negative values of ϕ.

Conclusion

The paper aims also to provide an empirical application to validate the model of local-asymptotic RCAR and show its applicability. On a theoretical side, it seems important to relax the assumption that u_t is i.i.d. since the latter is unlikely to hold in practice. Some persistence in the stochastic discount factor is indeed expected.

The empirical application we present here needs to be assessed further. In particular, it seems important to assess the properties of the bootstrap techniques. In turn, inference about the parameters (ϕ, λ) will allow to draw conclusions about the probabilities that bubbles appear or terminate.

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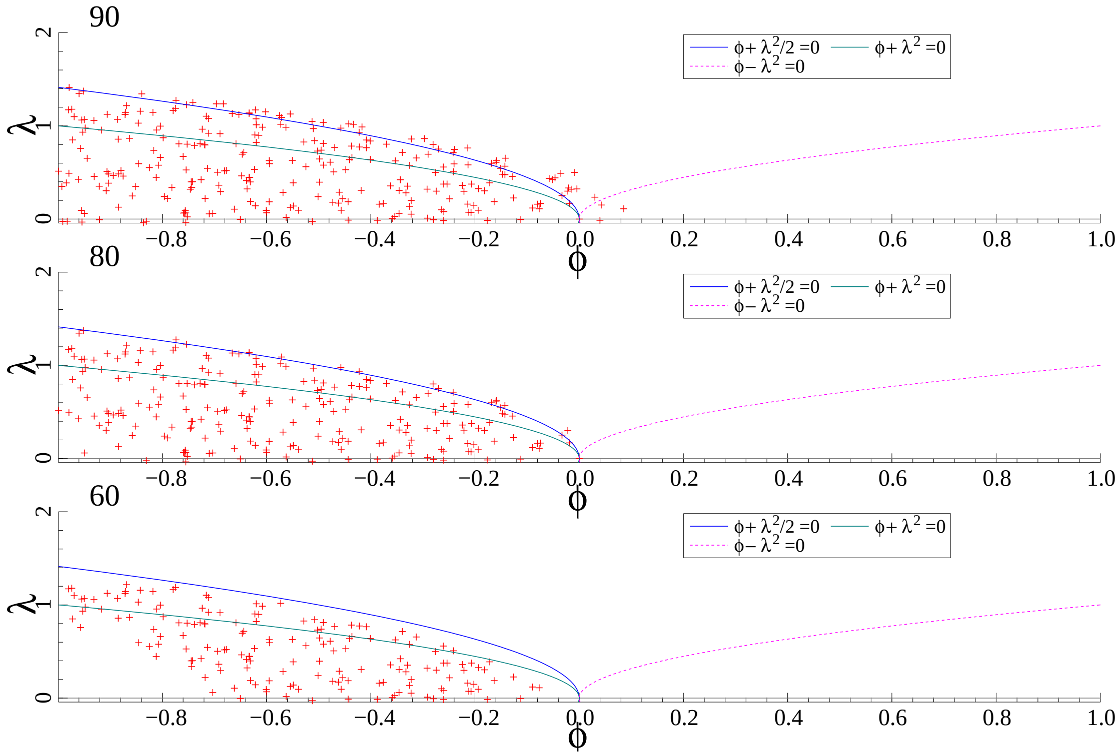

Figure 7: The figure reports parameter combinations which are not rejected at the .9, .8 and .6 probabilities according to the distribution ot the OLS estimator $\widehat{\beta}$ in expression (14). The distribution of $\widehat{\beta}$ was computed using the wild Bootstrap with standard normal weights.

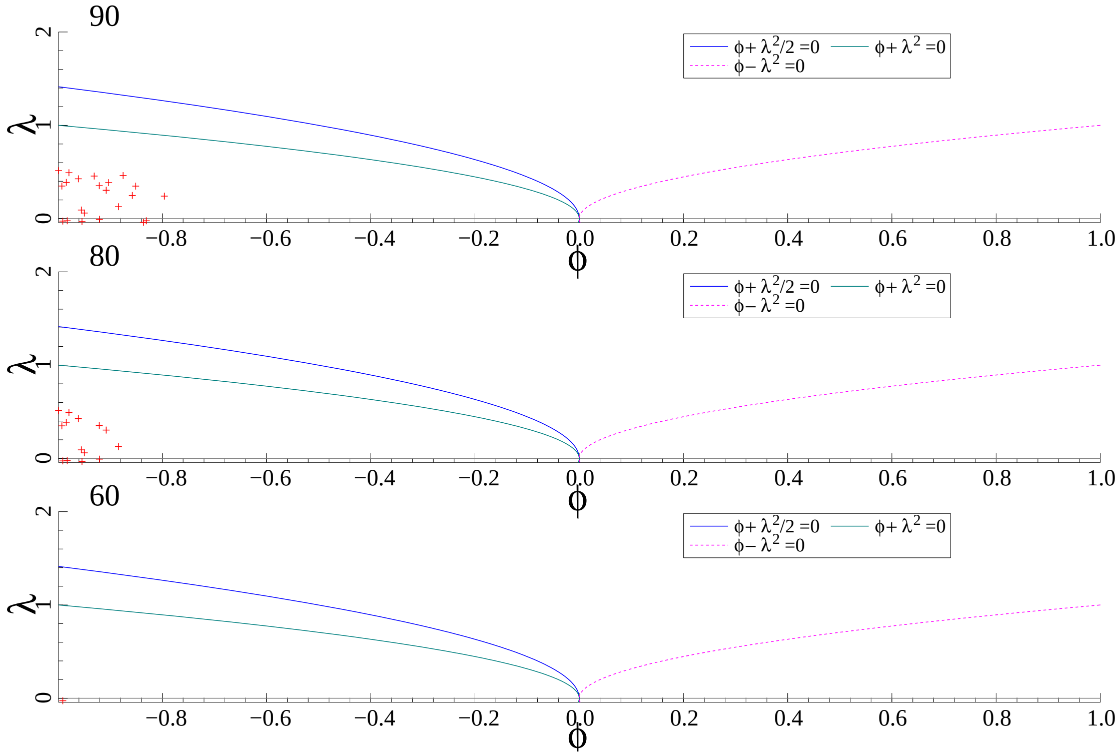

Figure 8: The figure reports parameter combinations which are not rejected at the .9, .8 and .6 probabilities according to the asymptotic distribution ot the OLS estimator $\widehat{\beta}$ in expression (14). The distribution of $\widehat{\beta}$ was tabulated using the Maximum Entropy bootstrap of Vinod (2006).

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Appendix

Proof of Proposition 1

We write, given y_0, and setting $\prod_{j=0}^{-1} \rho_j := 1$

$$
\begin{aligned}
y_t &= \left(\prod_{j=0}^{t-1} \rho_{t-j} \right) y_0 + \sum_{i=0}^{t-1} \left(\prod_{j=0}^{i-1} \rho_{t-j} \right) \eta_{t-i} \\
&= \left(\prod_{i=1}^{t} \rho_i \right) y_0 + \sum_{i=1}^{t} \left(\prod_{j=i+1}^{t} \rho_j \right) \eta_i \\
&= \exp\left\{ \frac{tT^{-\alpha/2}\phi + \lambda S_t}{T^{\alpha/2}} \right\} y_0 + \sum_{i=1}^{t} \exp\left\{ \frac{(t-i)T^{-\alpha/2}\phi + \lambda(S_t - S_i)}{T^{\alpha/2}} \right\} \eta_i.
\end{aligned}
$$

We evaluate the increment $y_t - y_0$ using the blocking method as in Phillips and Magdalinos (2005). Setting, for $t = 1$ to T, $t = [jT^\alpha] + k$ ($[x]$ denoting the integer part of x) for $j = 0, \cdots, [T^{1-\alpha}] - 1$, and $k = 1, \cdots, [T^\alpha]$, and letting $k = [pT^\alpha]$ for some $p \in [0,1]$, we can write

$$\begin{aligned}
\frac{1}{T^{\alpha/2}}(y_{[jT^\alpha]+[pT^\alpha]} - y_0) &= \frac{1}{T^{\alpha/2}}\left(\exp\left\{\frac{[jT^\alpha]+[pT^\alpha]}{T^\alpha}\phi + \lambda\frac{S_{[jT^\alpha]+[pT^\alpha]}}{T^{\alpha/2}}\right\} - 1\right) y_0 \quad + \\
&\quad \sigma \sum_{i=1}^{[jT^\alpha]+[pT^\alpha]} \exp\left\{\frac{[jT^\alpha]+[pT^\alpha]-i}{T^\alpha}\phi + \lambda\frac{S_{[jT^\alpha]+[pT^\alpha]} - S_i}{T^{\alpha/2}}\right\} \frac{\eta_i}{\sqrt{\sigma^2 T^\alpha}} \\
&= \frac{1}{T^{\alpha/2}}\left(\exp\left\{\frac{[jT^\alpha]+[pT^\alpha]}{T^\alpha}\phi + \lambda\frac{S_{[jT^\alpha]+[pT^\alpha]}}{T^{\alpha/2}}\right\} - 1\right) y_0 \quad + \\
&\quad \sigma \int_0^{j+p} \exp\left\{\frac{[jT^\alpha]+[pT^\alpha]-[sT^\alpha]}{T^\alpha}\phi + \lambda\frac{S_{[jT^\alpha]+[pT^\alpha]} - S_{[sT^\alpha]}}{T^{\alpha/2}}\right\} dB_{T^\alpha}(s)
\end{aligned}$$

using Proposition A1 in Phillips and Magdalinos (2004) in the last equality, where $B_{T^\alpha}(s) := \frac{1}{\sigma T^{\alpha/2}} \sum_{i=1}^{[sT^\alpha]} \eta_i$.

When applying the FCLT to $\tilde{S}_T(s) := \frac{S_{[sT^\alpha]}}{\sqrt{T^\alpha}}$ $(0 \le s \le 1)$, we obtain that the process $\left(\tilde{S}_T\right)$ converges in distribution, as $T \to \infty$, to a BM on $[0,1]$ that we denote by W.

By the same theorem, we can also say that the process (B_{T^α}) defined above converges in distribution , as $T \to \infty$, to a BM on $[0,1]$ that we denote by B and which is independent of W by assumption on the sequences (u_i) and (η_j).

Then we can deduce (using e.g. Th.8.3.1 in Liptser and Shiryaev, 1989) that

$$\int_0^{j+p} \exp\left\{\frac{[jT^\alpha]+[pT^\alpha]-[sT^\alpha]}{T^\alpha}\phi + \lambda\frac{S_{[jT^\alpha]+[pT^\alpha]} - S_{[sT^\alpha]}}{T^{\alpha/2}}\right\} dB_{T^\alpha}(s)$$

converges, as $T \to \infty$, to

$$\int_0^r \exp\left\{(r-s)\phi + \lambda\left(W_r - W_s\right)\right\} dB_s, \quad \text{with} \quad r = j+p.$$

This last integral can be written as

$$e^{r\phi+\lambda W_r}\int_0^r X_s dY_s \quad \text{where} \quad X_s = e^{-\lambda W_s} \quad \text{and} \quad dY_s = e^{-s\phi} dB_s.$$

The covariation process of two independent BM being identically 0 (see e.g. Klebaner, 2005,, th 4.19), the stochastic integration by parts reduces to the usual integration by parts formula and provides

$$e^{r\phi+\lambda W_r}\int_0^r X_s dY_s = \int_0^r e^{(r-s)\phi} dB_s - e^{r\phi+\lambda W_r}\int_0^r Y_s dX_s = J_\phi(r) - e^{r\phi+\lambda W_r}\int_0^r Y_s dX_s \quad (16)$$

where $J_\phi(r) = \int_0^r e^{(r-s)\phi} dB_s$ which corresponds to the limit obtained in Phillips and Magdalinos (2005).

Since X_s satisfies the SDE $dX_s = \frac{\lambda^2}{2} X_s ds - \lambda X_s dW_s$, the second term on the RHS of (16) can be written as

$$e^{r\phi+\lambda W_r}\left(\frac{\lambda^2}{2}\int_0^r X_s Y_s ds - \lambda\int_0^r X_s Y_s dW_s\right) \quad (17)$$

with $Y_s = \int_0^s e^{-u\phi} dB_u = e^{-s\phi} B_s + \phi\int_0^s e^{-u\phi} B_u du$.

Proof of Thoerem 2

5.0.3 Proposition

We first prove the following:We use the notation $S_{yy} = \sum_{t=1}^{T} y_t^2$, $S_{y\eta} = \sum_{t=1}^{T} y_{t-1}\eta_t$ and $S_{yyu} = \sum_{t=1}^{T} y_{t-1}^2 u_t$.

Proposition 4 *Let the process (y_t) be defined as in (6)-(9) for $t \geq 0$, with $y_0 = 0$. As $T \to \infty$ and for $x \in \{yy, y\eta, yyu\}$*

$$\sigma_\eta^{-2}\phi_x^T S_x \Rightarrow U_x$$

where (ϕ_T, U_x) are defined as follows

	ϕ_T^{yy}	$\phi_T^{y\eta}$	ϕ_T^{yyu}
$c<0$	$\sqrt{-2c}T^{-1-\alpha}$	$\sqrt{-2c}T^{-\frac{1+\alpha}{2}}$	$\frac{2}{\sqrt{3}}cT^{-\frac{1+2\alpha}{2}}$
$c=0$	$2\left(\phi-\lambda^2\right)T^{-(1+\alpha)}$	$\sqrt{2\left(\phi-\lambda^2\right)}T^{-\frac{1+\alpha}{2}}$	$2\sqrt{\phi+2\lambda^2}e^{-2\lambda^2T^{1-\alpha}}T^{-\frac{3\alpha}{2}}$
$c>0$			
$\lambda^2<\phi$	$4\left(\phi^2-\lambda^4\right)e^{-2cT^{1-\alpha}}T^{-2\alpha}$	$2\sqrt{\phi^2-\lambda^4}e^{-cT^{1-\alpha}}T^{-\alpha}$	$4\left(\phi-\lambda^2\right)\sqrt{\phi+2\lambda^2}e^{-\left(c+\lambda^2\right)T^{1-\alpha}}T^{-\frac{1+2\alpha}{2}}$
$\lambda^2=\phi$	$2\left(\phi+\lambda^2\right)e^{-2cT^{1-\alpha}}T^{-(1+\alpha)}$	$\sqrt{2\left(\phi+\lambda^2\right)}e^{-cT^{1-\alpha}}T^{-\frac{1+\alpha}{2}}$	$2\sqrt{\phi+2\lambda^2}e^{-\left(c+\lambda^2\right)T^{1-\alpha}}T^{-\frac{1+2\alpha}{2}}$
$\lambda^2>\phi$	$4\left(\lambda^4-\phi^2\right)e^{-4\lambda^2T^{1-\alpha}}T^{-2\alpha}$	$2\sqrt{\lambda^4-\phi^2}e^{-2\lambda^2T^{1-\alpha}}T^{-\alpha}$	$4\left(\lambda^2-\phi\right)\sqrt{\phi+2\lambda^2}e^{-\left(4\lambda^2-\phi\right)T^{1-\alpha}}T^{-3\alpha/2}$

and

	U_{yy}	$U_{y\eta}$	U_{yyu}
$c<0$	1	$\mathsf{N}(0,1)$	$\mathsf{N}(0,1)$
$c=0$	X^2Z	XY	X^2V
$c>0$			
$\lambda^2<\phi$	X^2Z	XY	X^2V
$\lambda^2=\phi$	$X^{*2}Z$	X^*Y	$X^{*2}V$
$\lambda^2>\phi$	$X^{*2}Z$	X^*Y	$X^{*2}V$

Proof.

Recall that $c = \phi + \lambda^2$.

- *Case $c < 0$*

  From Proposition 1, we have

$$T^{-\alpha/2}y_{[n^\alpha r]} \Rightarrow K_{\phi,\lambda}(r) \sim \mathsf{N}\left(0, \frac{e^{2cr}-1}{2c}\sigma_\eta^2\right)$$

  Let us introduce $K_{\phi,\lambda}^*(r) = e^{cr}K_{\phi,\lambda}^*(0) + K_{\phi,\lambda}(r)$ with $K_{\phi,\lambda}^*(0) \sim \mathsf{N}\left(0, \frac{\sigma_\eta^2}{-2c}\right)$, so $K_{\phi,\lambda}^*(r) \sim \mathsf{N}\left(0, -\frac{\sigma_\eta^2}{2c}\right)$ and is stationary.

  Then

$$T^{-(1+\alpha)}\sum_{t=1}^{T} y_t^2 \Rightarrow -\frac{\sigma_\eta^2}{2c}$$

$$T^{-\frac{1+\alpha}{2}}\sum_{t=1}^{T} y_{t-1}\eta_t \Rightarrow \mathsf{N}\left(0, -\frac{\sigma_\eta^4}{2c}\right)$$

The result concerning $\sum_{t=1}^{T} y_{t-1}^2 u_t$ similary follows from the martingale $\xi_t = T^{-\frac{1+2\alpha}{2}} y_{t-1}^2 u_t$ which admits conditional variance $\sum_{t=1}^{T} \mathsf{E}_{t-1}\left(\xi_t^2\right) = T^{-(1+2\alpha)} \sum_{t=1}^{T} y_{t-1}^4 \Rightarrow \frac{3\sigma_\eta^4}{4c^2}$. A Lindberg condition ensures then that

$$T^{-\frac{1+2\alpha}{2}} \sum_{t=1}^{T} y_{t-1}^2 u_t \Rightarrow \mathsf{N}\left(0, \frac{3\sigma_\eta^4}{4c^2}\right)$$

- *Case* $c \geq 0$

  ▷ Assume $\lambda^2 < \phi$.

  Note that under this assumption, $X = \int_0^\infty e^{-\phi s - \lambda W_s} dB_s$ is well defined.

  To cover both cases $c > 0$ and $c = 0$ given in Proposition 2, we introduce

$$\phi_{n^{1-\alpha}} = \begin{cases} \frac{e^{cn^{1-\alpha}}}{\sqrt{2c}}, & c > 0 \\ n^{\frac{1-\alpha}{2}} & c = 0 \end{cases}$$

  The proof follows the main steps given in Phillips & Magdalinos (2004); hence we keep their notation, namely set $T = n$ and let $\kappa_n = n^\alpha \left[n^{1-\alpha}\right]$ and $q = n^{1-\alpha} - \left[n^{1-\alpha}\right]$; we also assume w.l.o.g. $\sigma_\eta = 1$.

  – First let us consider the sample variance. of y_t. We can write

$$\begin{aligned} \frac{1}{n^{2\alpha}} \sum_{t=1}^{n} y_t^2 &= \frac{1}{n^{2\alpha}} \sum_{j=0}^{\left[n^{1-\alpha}\right]-1} \sum_{k=1}^{[n^\alpha]} y_{[n^\alpha j]+k}^2 + \frac{1}{n^{2\alpha}} \sum_{t=[\kappa_n]}^{n} y_t^2 + O_p\left(n^{-\alpha}\right) \\ &= U_{1n} + U_{2n} + O_p\left(n^{-\alpha}\right) \end{aligned}$$

  On one hand, we have

$$U_{2n} = \int_0^q \left(\frac{1}{n^{\alpha/2}} y_{[\kappa_n]+[n^\alpha p]}\right)^2 dp + O_p\left(n^{-2\alpha}\right)$$

  where

$$n^{-\alpha/2} y_{[\kappa_n]+[n^\alpha p]} \Rightarrow \int_0^{\left[n^{1-\alpha}\right]+p} e^{\phi\left(\left[n^{1-\alpha}\right]+p-s\right)+\lambda\left(W_{\left[n^{1-\alpha}\right]+p}-W_s\right)} dB_s$$

  then

$$\begin{aligned} U_{2n} &= \int_0^q e^{2\left(\phi\left(\left[n^{1-\alpha}\right]+p\right)+\lambda W_{\left[n^{1-\alpha}\right]+p}\right)} \left(\int_0^{\left[n^{1-\alpha}\right]+p} e^{-\phi s-\lambda W_s} dB_s\right)^2 dp + o_p(1) \\ &= \left(\int_0^{\left[n^{1-\alpha}\right]+q} e^{-\phi s-\lambda W_s} dB_s\right)^2 \int_0^q e^{2\left(\phi\left(\left[n^{1-\alpha}\right]+p\right)+\lambda W_{\left[n^{1-\alpha}\right]+p}\right)} dp + o_p(1) \\ &= \left(\int_0^{\left[n^{1-\alpha}\right]+q} e^{-\phi s-\lambda W_s} dB_s\right)^2 \left(\int_0^{\left[n^{1-\alpha}\right]+q} e^{2(\phi s+\lambda W_s)} ds - \int_0^{\left[n^{1-\alpha}\right]} e^{2(\phi s+\lambda W_s)} ds\right) dp. \end{aligned}$$

  Let ψ_n^2 such that $\psi_{n^{1-\alpha}}^2 := \mathsf{E}\left[\int_0^q e^{2\phi\left(\left[n^{1-\alpha}\right]+p\right)+2\lambda W_{\left[n^{1-\alpha}\right]+p}} dp\right] = e^{2c\left[n^{1-\alpha}\right]} \frac{e^{2cq}-1}{2c}$, hence

$$e^{-2c\left[n^{1-\alpha}\right]} \psi_{n^{1-\alpha}}^2 \to \begin{cases} \frac{e^{2cq}-1}{2c} & c > 0 \\ q & c = 0 \end{cases}$$

  and $\psi_{n^{1-\alpha}}^2 \sim \phi_{n^{1-\alpha}}^2 - \phi_{\left[n^{1-\alpha}\right]}^2$

It follows that

$$\phi_{[n^{1-\alpha}]}^{-2} U_{2n} \Rightarrow \begin{cases} X^2 (Z - Z^*) \frac{e^{2cq}-1}{2c} & c > 0 \\ X^2 (Z - Z^*) q & c = 0 \end{cases}$$

where Z is defined by

$$\phi_{n^{1-\alpha}}^{-2} \int_0^q e^{2\left(\phi([n^{1-\alpha}]+p)+\lambda W_{[n^{1-\alpha}]+p}\right)} \Rightarrow Z \quad \text{s.t.} \quad E[Z] = 1.$$

On the other hand, we have

$$\begin{aligned} \phi_{[n^{1-\alpha}]}^{-2} U_{1n} &= \phi_{[n^{1-\alpha}]}^{-2} \int_0^{[n^{1-\alpha}]} \left(n^{-\alpha/2} y_{[n^\alpha r]}\right)^2 dr + o_p(1) \\ &= \phi_{[n^{1-\alpha}]}^{-2} \int_0^{[n^{1-\alpha}]} e^{2(\phi r + \lambda W_r)} \left(\int_0^r e^{-\phi s - \lambda W_s} dB_s\right)^2 dr + o_p(1) \\ &= \left(\int_0^{[n^{1-\alpha}]} e^{-\phi s - \lambda W_s} dB_s\right)^2 \phi_{[n^{1-\alpha}]}^{-2} \int_0^{[n^{1-\alpha}]} e^{2(\phi r + \lambda W_r)} dr + o_p(1) \end{aligned} \tag{18}$$

Let us prove this last equation.

$$\begin{aligned} y_t &= \sum_{i=0}^{t-1} \exp\left(\frac{\phi}{n^\alpha} i + \frac{\lambda}{n^{\alpha/2}} \sum_{j=t-i+1}^{t} u_j\right) \eta_{t-i} \\ &= \sum_{i=1}^{t} \exp\left(\frac{\phi}{n^\alpha}(t-i) + \frac{\lambda}{n^{\alpha/2}}(U_t - U_i)\right) \eta_i \\ y_t^2 &= \sum_{i=1}^{t} \exp\left(2\frac{\phi}{n^\alpha}(t-i) + 2\frac{\lambda}{n^{\alpha/2}}(U_t - U_i)\right) \eta_i^2 \\ &+ 2\sum_{i=1}^{t} \sum_{j=i+1}^{t} \exp\left(\frac{\phi}{n^\alpha}(t-i+t-j) + \frac{\lambda}{n^{\alpha/2}}(2U_t - U_i - U_j)\right) \eta_i \eta_j \\ &= \exp\left(\frac{2\phi}{n^\alpha} t + \frac{2\lambda}{n^{\alpha/2}} U_t\right) \sum_{i=1}^{t} \exp\left\{-2\left(\frac{\phi}{n^\alpha} i + \frac{\lambda}{n^{\alpha/2}} U_i\right)\right\} \eta_i^2 \\ &= \exp\left(\frac{2\phi}{n^\alpha} t + \frac{2\lambda}{n^{\alpha/2}} U_t\right) \left[\sum_{i=1}^{t} \exp\left(-\frac{\phi}{n^\alpha} i - \frac{\lambda}{n^{\alpha/2}} U_i\right) \eta_i\right]^2 \end{aligned}$$

and

$$\begin{aligned} \sum_{k=1}^{t} y_k^2 &= \sum_{k=1}^{t} \exp\left(\frac{2\phi}{n^\alpha} k + \frac{2\lambda}{n^{\alpha/2}} U_k\right) \left[\sum_{i=1}^{k} \exp\left(-\frac{\phi}{n^\alpha} i - \frac{\lambda}{n^{\alpha/2}} U_i\right) \eta_i\right]^2 \\ &= \left(\sum_{k=1}^{t} \exp\left(\frac{2\phi}{n^\alpha} k + \frac{2\lambda}{n^{\alpha/2}} U_k\right)\right) \left[\sum_{i=1}^{t} \exp\left(-\frac{\phi}{n^\alpha} i - \frac{\lambda}{n^{\alpha/2}} U_i\right) \eta_i\right]^2 + R_t \end{aligned}$$

where $R_t = -\sum_{k=1}^{t-1} \exp\left(\frac{2\phi}{n^\alpha} k + \frac{2\lambda}{n^{\alpha/2}} U_k\right) \left[\sum_{i=k+1}^{t} \exp\left(-\frac{\phi}{n^\alpha} i - \frac{\lambda}{n^{\alpha/2}} U_i\right) \eta_i\right]^2$ can be shown to be negligible.

Therefore

$$U_{1n} = \frac{1}{n^{2\alpha}} \sum_{j=0}^{\left[n^{1-\alpha}\right]-1} \sum_{k=1}^{[n^\alpha]} y^2_{[n^\alpha j]+k}$$

$$= \left(n^{-\alpha} \sum_{k=1}^{\left[n^\alpha\left[n^{1-\alpha}\right]\right]} \exp\left(\frac{2\phi}{n^\alpha}k + \frac{2\lambda}{n^{\alpha/2}}U_k \right) \right) \left[\frac{1}{n^{\alpha/2}} \sum_{i=1}^{\left[n^\alpha\left[n^{1-\alpha}\right]\right]} \exp\left(-\frac{\phi}{n^\alpha}i - \frac{\lambda}{n^{\alpha/2}}U_i \right) \eta_i \right]^2$$

$$+ R_{[n^\alpha[n^{1-\alpha}]]}$$

where

$$\frac{1}{n^{\alpha/2}} \sum_{i=1}^{\left[n^\alpha\left[n^{1-\alpha}\right]\right]} \exp\left(-\frac{\phi}{n^\alpha}i - \frac{\lambda}{n^{\alpha/2}}U_i \right) \eta_i = \int_0^{\left[n^{1-\alpha}\right]} e^{-\phi n^\alpha s - \lambda W(n^\alpha s)} dB_{n^\alpha}(s) + o_p(1)$$

$$\Rightarrow \int_0^\infty e^{-\phi s - \lambda W(s)} dB(s)$$

$$\left(n^{-\alpha} \sum_{k=1}^{\left[n^\alpha\left[n^{1-\alpha}\right]\right]} \exp\left(\frac{2\phi}{n^\alpha}k + \frac{2\lambda}{n^{\alpha/2}}U_k \right) \right) = \int_0^{\left[n^{1-\alpha}\right]} e^{2\phi s + 2\lambda W(s)} ds + o_p(1)$$

$$\phi^{-1}_{[n^{1-\alpha}]} \left(n^{-\alpha} \sum_{k=1}^{\left[n^\alpha\left[T^{1-\alpha}\right]\right]} \exp\left(\frac{2\phi}{n^\alpha}k + \frac{2\lambda}{n^{\alpha/2}}U_k \right) \right) = \phi^{-1}_{n^\alpha[n^{1-\alpha}]} \int_0^{\left[n^{1-\alpha}\right]} \exp\left(\frac{2\phi}{n^\alpha}\left[kn^a\right] + \frac{2\lambda}{n^{\alpha/2}} U_{[kn^a]} \right) dk$$

with
$E\left(\int_0^{\left[n^{1-\alpha}\right]} \exp\left(\frac{2\phi}{n^\alpha}\left[kn^a\right] + \frac{2\lambda}{n^{\alpha/2}} U_{[kn^a]} \right) dk \right) = \phi^{-1}_{[n^{1-\alpha}]} \int_0^{\left[n^{1-\alpha}\right]} \exp\left(2c\frac{[kn^a]}{n^\alpha} \right) dk \to 1.$
Hence the result (18).
Now, combining the results for U_{2n} and U_{1n} and noticing that

$$\phi^{-1}_{[n^{1-\alpha}]} \int_0^{\left[n^{1-\alpha}\right]} e^{2(\phi r + \lambda W_r)} dr \Rightarrow Z^*$$

provides

$$\frac{\phi^{-2}_{n^{1-\alpha}}}{n^{2\alpha}} \sum_{t=1}^n y_t^2 \Rightarrow X^2 Z$$

– Let us look now at the covariance terms.
Let

$$\left(\frac{e^{cn^{1-\alpha}}}{c} \right)^{-1} Y_{n^{1-\alpha}} = ce^{-cn^{1-\alpha}} \int_0^{n^{1-\alpha}} e^{\phi r + \lambda W_r} dB_r \Rightarrow Y \sim \mathsf{N}(0,1)$$

then

$$\frac{\phi^{-1}_{n^{1-\alpha}}}{n^\alpha} \sum_{t=1}^n y_{t-1}\eta_t = \left(\int_0^{n^{1-\alpha}} e^{-(\phi s + \lambda W_s)} dB_s \right) \phi^{-1}_{n^{1-\alpha}} \left(\int_0^{n^{1-\alpha}} e^{\phi r + \lambda W_r} dB_r \right) + o_p(1)$$

$$\Rightarrow XY$$

Let us check the convergence of $\frac{\phi^{-2}_{n^{1-\alpha}}}{n^{\alpha}} \sum_{t=1}^{n} y_{t-1}^2 u_t$ where

$$\begin{aligned}
&\sum_{t=0}^{T-1} y_t^2 u_{t+1} \\
&= \sum_{t=0}^{T-1} \exp\left(\frac{2\phi}{T^\alpha}t + \frac{2\lambda}{T^{\alpha/2}}U_t\right)\left[\sum_{i=1}^{t} \exp\left(-\frac{\phi}{T^\alpha}i - \frac{\lambda}{T^{\alpha/2}}U_i\right)\eta_i\right]^2 (U_{t+1} - U_t) \\
&= \left(\sum_{k=1}^{T-1} \exp\left(\frac{2\phi}{T^\alpha}k + \frac{2\lambda}{T^{\alpha/2}}U_k\right)(U_{k+1} - U_k)\right)\left[\sum_{i=1}^{T} \exp\left(-\frac{\phi}{T^\alpha}i - \frac{\lambda}{T^{\alpha/2}}U_i\right)\eta_i\right]^2 \\
&+ R_t^* \qquad (19)
\end{aligned}$$

The latter summation in the previous expression was defined previously. We focus on the stochastic integral. Again, we must use a Lindberg Condition, this time regarding

$$\zeta_{k+1} = \left(T^{\alpha/2}\frac{e^{2\left(\phi+2\lambda^2\right)n^{1-\alpha}}}{2\sqrt{\phi+2\lambda^2}}\right)^{-1} \exp\left(\frac{2\phi}{T^\alpha}k + \frac{2\lambda}{T^{\alpha/2}}U_k\right)(U_{k+1} - U_k)$$

with

$$T^{-\alpha}\left(\frac{e^{4\left(\phi+2\lambda^2\right)n^{1-\alpha}}}{4\left(\phi+2\lambda^2\right)}\right)^{-1}\sum_k \mathsf{E}\left[\zeta_{k+1}^2|I_k\right] = T^{-\alpha}\left(\frac{e^{4\left(\phi+2\lambda^2\right)n^{1-\alpha}}}{4\left(\phi+2\lambda^2\right)}\right)^{-1}\sum_k \exp\left(\frac{4\phi}{T^\alpha}k + \frac{4\lambda}{T^{\alpha/2}}U_k\right)$$

We have

$$\sum_{k=1}^{T-1}\frac{\exp\left(\frac{2\phi}{T^\alpha}k + \frac{2\lambda}{T^{\alpha/2}}U_k\right)}{\frac{e^{2\left(\phi+2\lambda^2\right)T^{1-\alpha}}}{2\sqrt{\phi+2\lambda^2}}}\frac{U_{k+1}-U_k}{T^{\alpha/2}} = 2\sqrt{\phi+2\lambda^2}e^{-2\left(\phi+2\lambda^2\right)T^{1-\alpha}}\int_0^{T^{1-\alpha}} e^{2(\phi r+\lambda W_r)}dW_r + o_p(1)$$

so

$$\frac{2\sqrt{\phi+2\lambda^2}e^{-2\left(\phi+2\lambda^2\right)T^{1-\alpha}}}{T^{3\alpha/2}}\sum_{t=1}^{T} y_{t-1}^2 u_t = \left(\int_0^{T^{1-\alpha}} e^{-(\phi s+\lambda W_s)}dB_s\right)^2 \frac{\int_0^{T^{1-\alpha}} e^{2(\phi r+\lambda W_r)}dW_r}{\sqrt{\frac{e^{4\left(\phi+2\lambda^2\right)T^{1-\alpha}}-1}{4\left(\phi+2\lambda^2\right)}}} + o_p(1)$$

and

$$\frac{2\sqrt{\phi+2\lambda^2}e^{-2\left(\phi+2\lambda^2\right)T^{1-\alpha}}}{T^{3\alpha/2}}\sum_{t=1}^{T} y_{t-1}^2 u_t \Rightarrow X^2 V$$

and if $\phi + 2\lambda^2 = 0$, then $T^{-\frac{1+2\alpha}{2}}\sum_{t=1}^{T} y_{t-1}^2 u_t \Rightarrow X^2 V$.

▷ Assume $\lambda^2 \geq \phi$.

The main difference is that, now, $\int_0^{n^{1-\alpha}} e^{-(\phi s+\lambda W_s)}dB_s$ diverges as $n \to \infty$. Since it is normally distributed, we only need to scale it by it standard deviation. It comes

$$\frac{\int_0^{n^{1-\alpha}} e^{-(\phi s+\lambda W_s)}dB_s}{\sqrt{\int_0^{n^{1-\alpha}} \mathsf{E}\left[e^{-2(\phi s+\lambda W_s)}\right]ds}} = \frac{\int_0^{n^{1-\alpha}} e^{-(\phi s+\lambda W_s)}dB_s}{\sqrt{\int_0^{n^{1-\alpha}} e^{2(-\phi+\lambda^2)s}ds}} = \frac{\int_0^{n^{1-\alpha}} e^{-(\phi s+\lambda W_s)}dB_s}{\sqrt{\frac{e^{2\left(-\phi+\lambda^2\right)n^{1-\alpha}}-1}{2\left(-\phi+\lambda^2\right)}\sigma_\eta^2}}$$

so

$$if \phi + \lambda^2 > 0 : X^*_{n^{1-\alpha}} = e^{-\left(-\phi+\lambda^2\right)n^{1-\alpha}}\int_0^{n^{1-\alpha}} e^{-(\phi s+\lambda W_s)}dB_s \Rightarrow X^* \sim \mathsf{N}\left(0, \frac{\sigma_\eta^2}{2\left(-\phi+\lambda^2\right)}\right)$$

$$if \phi + \lambda^2 = 0 : X^*_{n^{1-\alpha}} = n^{-\frac{1-\alpha}{2}}\int_0^{n^{1-\alpha}} e^{-(\phi s+\lambda W_s)}dB_s \Rightarrow X^* \sim \mathsf{N}\left(0, \sigma_\eta^2\right)$$

Hence the result of the proposition.

Proof of Theorem 3

Theorem 3 will be directly deduced from the results obtained in Proposition 4.

- *Case* $c < 0$

  Since we have, as $T \to \infty$,

$$y_t = \Big(E(\rho_t) + \lambda T^{-\alpha/2} u_t + O_p\left(T^{-\alpha}\right)\Big) y_{t-1} + \eta_t$$

  and noticing that $\sum_{t=1}^T y_{t-1}^2 u_t$ is asymptotically uncorrelated with $\sum_{t=1}^T y_{t-1}\eta_t$, then the OLS estimator given by $\widehat{\rho} = \frac{\sum_{t=1}^T y_{t-1} y_t}{\sum_{t=1}^T y_{t-1}^2}$ satisfies

$$\begin{aligned}\widehat{\rho} - E(\rho_t) &= \lambda T^{-\alpha/2} \frac{\sum_t y_{t-1}^2 u_t}{\sum_t y_{t-1}^2} + \frac{\sum_t y_{t-1}\eta_t}{\sum_t y_{t-1}^2} \\ &= \lambda \frac{T^{-1-\alpha} T^{-\frac{1+2\alpha}{2}} \sum_t y_{t-1}^2 u_t}{T^{-\frac{1+\alpha}{2}} T^{-1-\alpha} \sum_t y_{t-1}^2} + \frac{T^{-1-\alpha} T^{-\frac{1+\alpha}{2}} \sum_t y_{t-1}\eta_t}{T^{-\frac{1+\alpha}{2}} T^{-1-\alpha} \sum_t y_{t-1}^2}\end{aligned}$$

  from which we deduce, using Proposition 2,

$$T^{\frac{1+\alpha}{2}}\left(\widehat{\rho} - E(\rho_t)\right) = \lambda \frac{T^{-\frac{1+2\alpha}{2}} \sum_t y_{t-1}^2 u_t}{T^{-1-\alpha} \sum_t y_{t-1}^2} + \frac{T^{-\frac{1+\alpha}{2}} \sum_t y_{t-1}\eta_t}{T^{-1-\alpha} \sum_t y_{t-1}^2} \Rightarrow N\left(0, 3\lambda^2 - 2c\right)$$

  When u_t is observed, we can use the result $T^{\frac{1+\alpha}{2}}\left(\widehat{\rho} - \rho_t\right) \Rightarrow N(0, -2c)$.

- *Case* $c = 0$

  The convergence of the OLS estimator as $T \to \infty$ comes from the following convergences.

$$\frac{\sqrt{2(\phi - \lambda^2)} T^{-\frac{1+\alpha}{2}} \sum_{t=1}^T y_{t-1}\eta_t}{2(\phi - \lambda^2) T^{-(1+\alpha)} \sum_{t=1}^T y_t^2} = \frac{T^{\frac{1+\alpha}{2}}}{\sqrt{2(\phi - \lambda^2)}} \frac{\sum_{t=1}^T y_{t-1}\eta_t}{\sum_{t=1}^T y_t^2} \Rightarrow \frac{Y}{XZ}$$

$$\frac{\sqrt{2(\phi - \lambda^2)} T^{-\frac{1+\alpha}{2}} \sum_{t=1}^T y_{t-1}\eta_t}{4(\phi - \lambda^2)\sqrt{(\phi + 3\lambda^2)(\phi + 2\lambda^2)} e^{-2\lambda^2 T^{1-\alpha}} T^{-(1+\alpha)} \sum_{t=1}^T y_t^2} = \frac{e^{2\lambda^2 T^{1-\alpha}} T^{\frac{1+\alpha}{2}}}{\sqrt{2(\phi - \lambda^2)}\sqrt{(\phi + 3\lambda^2)(\phi + 2\lambda^2)}} \frac{\sum_{t=1}^T y_{t-1}\eta_t}{\sum_{t=1}^T y_t^2} \Rightarrow \frac{Y}{XZ^*}$$

$$\frac{\frac{4(\phi - \lambda^2)\sqrt{\phi + 2\lambda^2} e^{-(\phi + 2\lambda^2) T^{1-\alpha}}}{T^{3\alpha/2}} \sum_{t=1}^T y_{t-1}^2 u_t}{2(\phi - \lambda^2) T^{-(1+\alpha)} \sum_{t=1}^T y_t^2} = 2\sqrt{\phi + 2\lambda^2} e^{-\lambda^2 T^{1-\alpha}} T^{(1-\alpha/2)} \frac{\sum_{t=1}^T y_{t-1}^2 u_t}{\sum_{t=1}^T y_t^2} \Rightarrow \frac{V}{Z}$$

$$\frac{\frac{4(\phi - \lambda^2)\sqrt{\phi + 2\lambda^2} e^{-(\phi + 2\lambda^2) T^{1-\alpha}}}{T^{3\alpha/2}} \sum_{t=1}^T y_{t-1}^2 u_t}{4(\phi - \lambda^2)\sqrt{(\phi + 3\lambda^2)(\phi + 2\lambda^2)} e^{-2\lambda^2 T^{1-\alpha}} T^{-(1+\alpha)} \sum_{t=1}^T y_t^2} = \frac{T^{1-\alpha/2} e^{\lambda^2 T^{1-\alpha}}}{\sqrt{(\phi + 3\lambda^2)}} \frac{\sum_{t=1}^T y_{t-1}^2 u_t}{\sum_{t=1}^T y_t^2} \Rightarrow \frac{V}{Z^*}$$

- *Case* $c > 0$

  - If $\lambda^2 < \phi$

$$\frac{\frac{2\sqrt{\phi^2 - \lambda^4} e^{-(\phi + \lambda^2) T^{1-\alpha}}}{T^\alpha} \sum_t y_{t-1}\eta_t}{\frac{4(\phi^2 - \lambda^4) e^{-2(\phi + \lambda^2) T^{1-\alpha}}}{T^{2\alpha}} \sum_t y_{t-1}^2} = \frac{T^\alpha e^{(\phi + \lambda^2) T^{1-\alpha}}}{2\sqrt{\phi^2 - \lambda^4}} \frac{\sum_t y_{t-1}\eta_t}{\sum_t y_{t-1}^2} \Rightarrow \frac{Y}{XZ}$$

$$\begin{aligned}\frac{\frac{2\sqrt{\phi^2 - \lambda^4} e^{-(\phi + \lambda^2) T^{1-\alpha}}}{T^\alpha} \sum_t y_{t-1}\eta_t}{\frac{4(\phi - \lambda^2)\sqrt{(\phi + 3\lambda^2)(\phi + 2\lambda^2)} e^{-2(\phi + 2\lambda^2) T^{1-\alpha}}}{T^{2\alpha}} \sum_t y_{t-1}^2} &= \frac{T^\alpha e^{(\phi + 3\lambda^2) T^{1-\alpha}}}{2\sqrt{\phi^2 - \lambda^4}\sqrt{(\phi + 3\lambda^2)(\phi + 2\lambda^2)}} \frac{\sum_t y_{t-1}\eta_t}{\sum_t y_{t-1}^2} \\ &\Rightarrow \frac{Y}{X} \frac{1}{Z^*}\end{aligned}$$

and

$$\frac{\frac{4\left(\phi-\lambda^2\right)\sqrt{\phi+2\lambda^2}e^{-\left(\phi+2\lambda^2\right)T^{1-\alpha}}}{T^{3\alpha/2}}\sum_{t=1}^{T}y_{t-1}^2u_t}{\frac{4(\phi^2-\lambda^4)e^{-2(\phi+\lambda^2)T^{1-\alpha}}}{T^{2\alpha}}\sum_t y_{t-1}^2} = \frac{\sqrt{\phi+2\lambda^2}T^{\alpha/2}e^{\phi T^{1-\alpha}}}{(\phi+\lambda^2)}\frac{\sum_{t=1}^{T}y_{t-1}^2u_t}{\sum_t y_{t-1}^2} \Rightarrow \frac{V}{Z}$$

$$\frac{\frac{4\left(\phi-\lambda^2\right)\sqrt{\phi+2\lambda^2}e^{-\left(\phi+2\lambda^2\right)T^{1-\alpha}}}{T^{3\alpha/2}}\sum_{t=1}^{T}y_{t-1}^2u_t}{\frac{4(\phi-\lambda^2)\sqrt{(\phi+3\lambda^2)(\phi+2\lambda^2)}e^{-2\left(\phi+2\lambda^2\right)T^{1-\alpha}}}{T^{2\alpha}}\sum_t y_{t-1}^2} = \frac{T^{\alpha/2}e^{\left(\phi+2\lambda^2\right)T^{1-\alpha}}}{\sqrt{(\phi+3\lambda^2)}}\frac{\sum_{t=1}^{T}y_{t-1}^2u_t}{\sum_t y_{t-1}^2} \Rightarrow \frac{V}{Z^*}$$

– If $\lambda = \phi$

$$\frac{\frac{\sqrt{2(\phi+\lambda^2)}e^{-\left(\phi+\lambda^2\right)T^{1-\alpha}}}{T^{\frac{1+\alpha}{2}}}\sum_{t=1}^{T}y_{t-1}\eta_t}{\frac{2(\phi+\lambda^2)e^{-2(\phi+\lambda^2)T^{1-\alpha}}}{T^{1+\alpha}}\sum_{t=1}^{T}y_t^2} = \frac{T^{\frac{1+\alpha}{2}}e^{\left(\phi+\lambda^2\right)T^{1-\alpha}}}{\sqrt{2\left(\phi+\lambda^2\right)}}\frac{\sum_{t=1}^{T}y_{t-1}\eta_t}{\sum_{t=1}^{T}y_t^2} \Rightarrow \frac{Y}{X^*Z}$$

$$\frac{\frac{\sqrt{2(\phi+\lambda^2)}e^{-\left(\phi+\lambda^2\right)T^{1-\alpha}}}{T^{\frac{1+\alpha}{2}}}\sum_{t=1}^{T}y_{t-1}\eta_t}{\frac{2\sqrt{(\phi+3\lambda^2)(\phi+2\lambda^2)}e^{-2(\phi+2\lambda^2)T^{1-\alpha}}}{T^{1+\alpha}}\sum_{t=1}^{T}y_t^2} = \frac{\sqrt{(\phi+\lambda^2)}T^{\frac{1+\alpha}{2}}e^{\left(\phi+3\lambda^2\right)T^{1-\alpha}}}{\sqrt{2\left(\phi+3\lambda^2\right)\left(\phi+2\lambda^2\right)}}\frac{\sum_{t=1}^{T}y_{t-1}\eta_t}{\sum_{t=1}^{T}y_t^2} \Rightarrow \frac{Y}{X^*Z^*}$$

$$\frac{\frac{2\sqrt{\phi+2\lambda^2}e^{-\left(\phi+2\lambda^2\right)T^{1-\alpha}}}{T^{\frac{1+2\alpha}{2}}}\sum_{t=1}^{T}y_{t-1}^2u_t}{\frac{2(\phi+\lambda^2)e^{-2(\phi+\lambda^2)T^{1-\alpha}}}{T^{1+\alpha}}\sum_{t=1}^{T}y_t^2} = \frac{\sqrt{\phi+2\lambda^2}T^{1/2}e^{\phi T^{1-\alpha}}}{(\phi+\lambda^2)}\frac{\sum_{t=1}^{T}y_{t-1}^2u_t}{\sum_{t=1}^{T}y_t^2} \Rightarrow \frac{V}{Z}$$

$$\frac{\frac{2\sqrt{\phi+2\lambda^2}e^{-\left(\phi+2\lambda^2\right)T^{1-\alpha}}}{T^{\frac{1+2\alpha}{2}}}\sum_{t=1}^{T}y_{t-1}^2u_t}{\frac{2\sqrt{(\phi+3\lambda^2)(\phi+2\lambda^2)}e^{-2(\phi+2\lambda^2)T^{1-\alpha}}}{T^{1+\alpha}}\sum_{t=1}^{T}y_t^2} = \frac{T^{1/2}e^{\left(\phi+2\lambda^2\right)T^{1-\alpha}}}{\sqrt{(\phi+3\lambda^2)}}\frac{\sum_{t=1}^{T}y_{t-1}^2u_t}{\sum_{t=1}^{T}y_t^2} \Rightarrow \frac{V}{Z^*}$$

– If $\lambda^2 > \phi$

$$\frac{\frac{2\sqrt{\lambda^4-\phi^2}e^{-2\lambda^2T^{1-\alpha}}}{T^{\alpha}}\sum_{t=1}^{T}y_{t-1}\eta_t}{\frac{4(\lambda^4-\phi^2)e^{-4\lambda^2T^{1-\alpha}}}{T^{2\alpha}}\sum_{t=1}^{T}y_t^2} = \frac{T^{\alpha}e^{2\lambda^2T^{1-\alpha}}}{2\sqrt{\lambda^4-\phi^2}}\frac{\sum_{t=1}^{T}y_{t-1}\eta_t}{\sum_{t=1}^{T}y_t^2} \Rightarrow \frac{Y}{X^*Z}$$

$$\frac{\frac{2\sqrt{\lambda^4-\phi^2}e^{-2\lambda^2T^{1-\alpha}}}{T^{\alpha}}\sum_{t=1}^{T}y_{t-1}\eta_t}{\frac{2(\lambda^2-\phi)\sqrt{(\phi+3\lambda^2)(\phi+2\lambda^2)}e^{-6\lambda^2T^{1-\alpha}}}{T^{2\alpha}}\sum_{t=1}^{T}y_t^2}$$

$$= \frac{\sqrt{\lambda^2+\phi}T^{\alpha}e^{4\lambda^2T^{1-\alpha}}}{\sqrt{(\phi+3\lambda^2)\left(\phi+2\lambda^2\right)\left(\lambda^2-\phi\right)}}\frac{\sum_{t=1}^{T}y_{t-1}\eta_t}{\sum_{t=1}^{T}y_t^2} \Rightarrow \frac{Y}{X^*Z^*}$$

$$\frac{\frac{4\left(\lambda^2-\phi\right)\sqrt{\phi+2\lambda^2}e^{-\left(4\lambda^2-\phi\right)T^{1-\alpha}}}{T^{3\alpha/2}}\sum_{t=1}^{T}y_{t-1}^2u_t}{\frac{4(\lambda^4-\phi^2)e^{-4\lambda^2T^{1-\alpha}}}{T^{2\alpha}}\sum_{t=1}^{T}y_t^2} = \frac{\sqrt{\phi+2\lambda^2}T^{\alpha/2}e^{\phi T^{1-\alpha}}}{(\phi+\lambda^2)}\frac{\sum_{t=1}^{T}y_{t-1}^2u_t}{\sum_{t=1}^{T}y_t^2} \Rightarrow \frac{V}{Z}$$

$$\frac{\frac{4\left(\lambda^2-\phi\right)\sqrt{\phi+2\lambda^2}e^{-\left(4\lambda^2-\phi\right)T^{1-\alpha}}}{T^{3\alpha/2}}\sum_{t=1}^{T}y_{t-1}^2u_t}{\frac{2(\lambda^2-\phi)\sqrt{(\phi+3\lambda^2)(\phi+2\lambda^2)}e^{-6\lambda^2T^{1-\alpha}}}{T^{2\alpha}}\sum_{t=1}^{T}y_t^2} = \frac{T^{\alpha/2}e^{\left(\phi+2\lambda^2\right)T^{1-\alpha}}}{2\sqrt{(\phi+3\lambda^2)}}\frac{\sum_{t=1}^{T}y_{t-1}^2u_t}{\sum_{t=1}^{T}y_t^2} \Rightarrow \frac{V}{Z^*}$$

5.1 Present Value Model

Consider the standard definition of an ex-post asset return

$$r_{t+1} = \frac{P_{t+1}+D_{t+1}}{P_t} - 1$$

see e.g. Campbell, Lo and McKinlay (1996, expression (7.1.1)) and assume r_{t+1} constant and equal R^c then

$$P_t = \frac{P_{t+1} + D_{t+1}}{1 + R^c}$$

which is compatible with

$$\Delta P_t = (1 + (1-\delta) R + \delta R_t) \Delta P_{t-1} - \zeta_t$$

where R_t is *iid* and $\mathsf{E}\left[(1+R_t)^{-1}\right] = (1+R)^{-1}$.
Proof:

$$\Delta P_t = (1 + (1-\delta) R + \delta R_t) \Delta P_{t-1} - \zeta_t$$

implies that

$$\begin{aligned} P_{t+1} + D_{t+1} &= P_t + (1 + (1-\delta) R + \delta R_{t+1}) \Delta P_t - \zeta_{t+1} + D_t + \zeta_{t+1} \\ \frac{P_{t+1} + D_{t+1}}{1 + R_{t+1}} &= \frac{P_t + (1 + (1-\delta) R + \delta R_{t+1}) \Delta P_t}{1 + R_{t+1}} + \frac{D_t}{1 + R_{t+1}} \\ &= \frac{P_t + (1 + (1-\delta) R) \Delta P_t}{1 + R_{t+1}} + \delta \frac{R_{t+1}}{1 + R_{t+1}} \Delta P_t + \frac{D_t}{1 + R_{t+1}} \end{aligned}$$

Now, if

$$P_t = \mathsf{E}_t \frac{P_{t+1} + D_{t+1}}{1 + R_{t+1}}$$

then

$$\begin{aligned} P_t &= \frac{P_t + (1 + (1-\delta) R) \Delta P_t}{1 + R} + \frac{D_t}{1 + R} + \delta \mathsf{E}_t \left[\frac{1 + R_{t+1}}{1 + R_{t+1}} - \frac{1}{1 + R_{t+1}}\right] \Delta P_t \\ &= \frac{P_t + (1 + R) \Delta P_t}{1 + R} + \frac{D_t}{1 + R} \\ &= \frac{P_t + D_t}{1 + R} + \Delta P_t \end{aligned}$$

i.e.

$$P_{t-1} = \frac{P_t + D_t}{1 + R}$$

qed.