AMAT 342 Homework 6. Due Tuesday Nov. 12, 2019

1. For each of the following sets $S \subset \mathbb{R}^2$, give an explicit description of the boundary of the set (with respect to the Euclidean metric on $\mathbb{R}^2$), and say whether or not S is open.

a. $S = S^1$,
b. $S = \{(x, y) \mid x > 0\}$,
c. $S = \{(x, y) \mid x \geq 0\}$,
d. $S = \{(x, 0) \mid x \in \mathbb{R}\}$,
e. $S =$ a finite set.

2. For $M = [-1, 1]$ with the Euclidean metric, which of the following are open subsets of M?

a. $\{1\}$,
b. $(0, 1)$,
c. $[0, 1)$,
d. $(0, 1]$.

3. Let d and d' be metrics on a set S. Prove the following fact from class: If there exist positive constants α, β such that

$$\alpha d(x, y) \leq d'(x, y) \leq \beta d(x, y)$$

for all $x, y \in S$, then d and d' are topologically equivalent. [HINT: According to a result proven in class and the notes, if U is an open subset of (S, d), then for each $x \in U$, U contains an open ball B of (S, d) centered at x. Show that B contains an open ball of (S, d') centered at x. Also show that the same is true with the roles of d and d' reversed.]

4. Show that for $d_{\max}$ and d_1 the metrics on $\mathbb{R}^n$ described in class, we have $d_{\max} \leq d_1 \leq n\, d_{\max}$.

5. Give an example of a pair of topologically inequivalent metrics on $\mathbb{R}$.

6. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Give an explicit expression for $f^{-1}(U)$, for each of the following sets $U \subset \mathbb{R}$:

a. $U = [1, 4]$,
b. $U = [-4, -1]$,
c. $U = \{2\}$,
d. $U = (0, 1)$.

7. Let $O = \{\mathbb{R}, \emptyset\} \cup \{(-\infty, a] \mid a \in \mathbb{R}\}$. Is O a topology on $\mathbb{R}$? Explain your answer.

8. Let $f : S \to T$ be a continuous function of topological spaces where S is non-empty and has the trivial topology, and T has the discrete topology. Prove that f is a constant function, i.e., $\mathrm{im}(f)$ contains a single element.