Notes for final review, friday 12/13/19

Review outline

- Sets + Functions
- Homeomorphism + Isotopy
- Abstract Topological Spaces
- Metric Spaces
- Gluing / Quotient spaces

Cartesian Products

Definition For sets $S_1, \dots, S_n$

$$\underbrace{S_1 \times S_2 \times \cdots \times S_n}_{\text{Cartesian product of } S_1,\dots,S_n} = \{(x_1, x_2, \dots, x_n) \mid x_i \in S_i \ \forall\, i \in \{1, \dots, n\}\}.$$

Exercise from HW: Sketch $[1,2] \times [3,4] \subseteq \mathbb{R}^2$

Exercise Sketch $I \times I \times [1,2]$ as a subset of $\mathbb{R}^3$

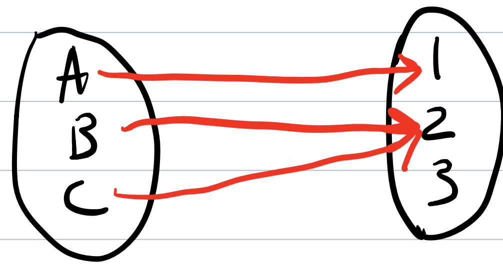
(recall: $I \subset \mathbb{R}$ is the interval $[0,1]$.)

Functions: The image of a function $f: S \to T$ is the subset of T given by

$$\mathrm{im}(f) = \{ y \in T \mid y \in f(x) \text{ for some } x \in S \}.$$

<u>Example</u>: $f: \{A, B, C\} \to \{1, 2, 3\}$

$$f(A) = 1$$
$$f(B) = 2$$
$$f(C) = 2$$

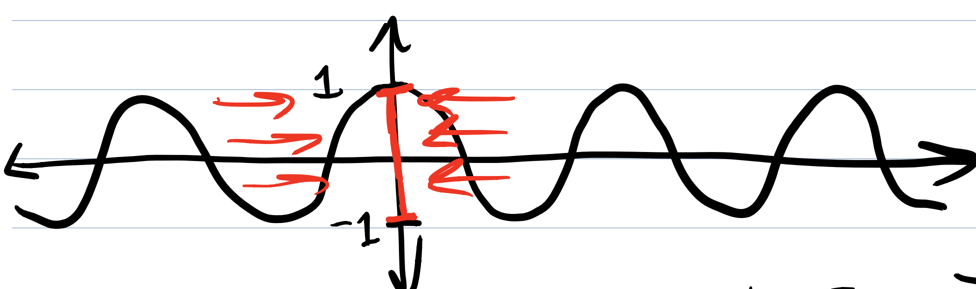


$$\mathrm{im}(f) = \{1, 2\}.$$

<u>Example</u>: $g(x): \mathbb{R} \to \mathbb{R}$,
$$g(x) = \cos(x) + 2.$$

What is $\mathrm{im}(g)$?

First, what the image $\cos: \mathbb{R} \to \mathbb{R}$



$$\mathrm{im}(\cos) = [-1, 1] \Rightarrow \mathrm{im}(g) = [1, 3].$$

Inverse Images (a.k.a. preimages)

For $f: S \to T$ any function and $U \subset T$,

$f^{-1}(U)$ is a subset of S, given by

$f^{-1}(U) = \{x \in S \mid f(x) \in U\}$.

Example

For $f: \{A, B, C\} \to \{1, 2, 3\}$ as above,
and

Then $f^{-1}(\{2,3\}) = \{x \in \{A, B, C\} \mid f(x) \in \{2, 3\}\}$.

$\Rightarrow \quad = \{B, C\}$.

$f^{-1}(\{3\}) = \emptyset$.

Example $g^{-1}(\{0\}) = \{x \in \mathbb{R} \mid \cos x + 2 = 0\} =$

$\emptyset$,

$g^{-1}((0,5)) = \{x \in \mathbb{R} \mid \cos x + 2 \in (0,5)\}$
since $\operatorname{im}(g) = [1,3] \subset (0,5)$, $g^{-1}(0,5) = \mathbb{R}$.

A function $f: S \to T$ is

- **injective** if $f(x) = f(y)$ only when $x = y$.
- **surjective** if $\mathrm{im}(f) = T$.
- **bijective** if both injective and surjective

Ex: f is neither injective nor surjective.

Ex: g is neither injective nor surjective.

$f: S \to T$ is called **invertible** if $\exists$ a function $g: T \to S$

s.t. $g \circ f = Id_S$, $f \circ g = Id_T$ — identity functions on S and T

g is called the inverse of f.

Fact: f is bijective iff f is invertible.

Exercise Suppose $f: S \to T$ is injective. Is there a bijection from S to $\mathrm{im}(f)$?

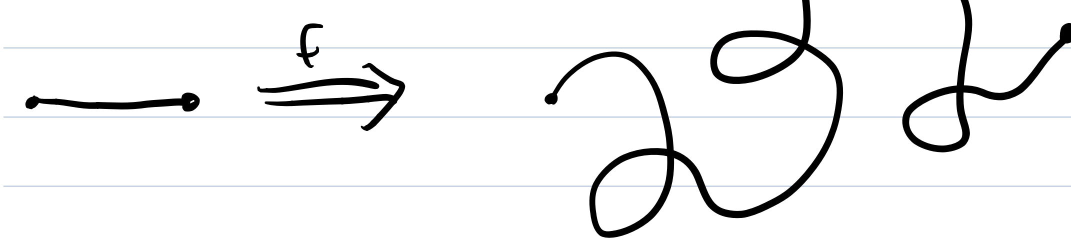
Ans: Yes. Let $\tilde{f}(x): S \to \mathrm{im}(f)$ be given by $\tilde{f}(x) = f(x)$. Clearly this a surjection since $y \in \mathrm{im}(f) \Rightarrow y = f(x) = \tilde{f}(x)$ for some $x \in S$.
But $\tilde{f}(x) = \tilde{f}(y) \Rightarrow f(x) = f(y) \Rightarrow x = y$, so $\tilde{f}$ is also injective.

Continuous Functions

Intuitive interpretation 1: For S, T subsets of Euclidean spaces, $f: S \to T$ is continuous if f "puts S into T without tearing S."

Intuitive interpretation 2: f is continuous if f "maps nearby points to nearby points."

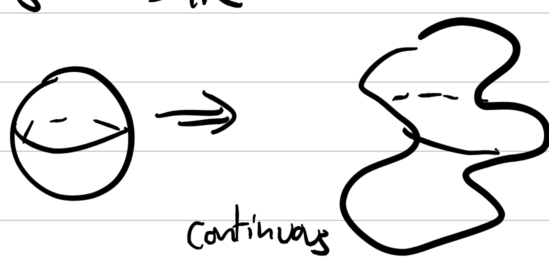
Illustration $f: I \to \mathbb{R}^2$



continuous

not continuous

$f: S^2 \to \mathbb{R}^3$



continuous

In this class we defined continuity 3 times, in increasing generality:

1st, for subsets of Euclidean space $\Big\}$ ϵ-δ def
2nd, for metric spaces
3rd, for topological spaces (open set def).

In this review, I'll emphasize the 2nd and 3rd settings.

Metric Spaces

A metric on a set S is a function

$d: S \times S \to [0, \infty)$ such that

1) $d(x,y) = 0$ iff $x = y$.
2) $d(x,y) = d(y,x)$ $\forall x, y \in S$.
3) $d(x,y) + d(y,z) \geq d(x,z)$ $\forall x, y, z \in S$.

Examples We define 3 metrics on $\mathbb{R}^n$: d_2, d_1, d_{max}

$$d_2(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$ "Euclidean distance"

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$ "Manhattan distance"

$$d_{max}(x,y) = \max_{1 \le i \le n} |x_i - y_i|$$ "max norm distance"

- Edit distance on genetic sequences
- RMSD on 3-D structures of a protein
- Any subset of a metric space inherits a metric.

Notation/Terminology: Let $M = (S, d)$ be a metric space

"$x \in M$" means $x \in S$

"$U \subset M$" means $U \subset S$.

Let $M' = (S', d')$ be another metric space

A "function $f: M \to M'$" is understood to mean a function $f: S \to S'$.

Continuity

For metric spaces M and N w/ metrics d_M, d_N, $f: M \to N$ is continuous at $x \in M$ if $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ s.t. $d_M(x,y) < \delta \Rightarrow d_N(f(x), f(y)) < \epsilon$.

$f: M \to N$ is a homeomorphism if

1) f is a continuous bijection
2) f^{-1} is also continuous.

Recall notation: If $f: S \to T$ is any function, $\tilde{f}: S \to \mathrm{im}(f)$ is the map given by $\tilde{f}(x) = f(x)$.

Embeddings: Let M, N be metric spaces. $f: M \to N$ is called an embedding if

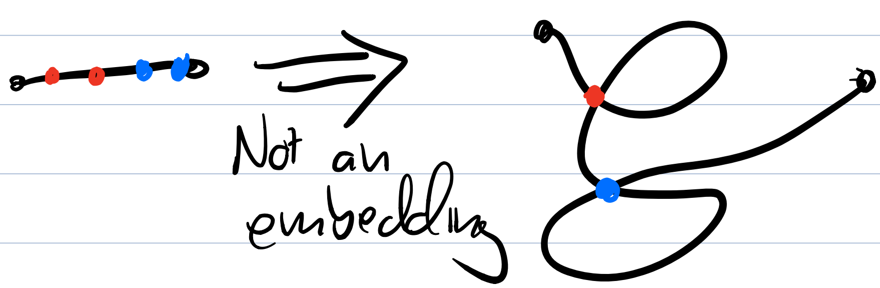
$\tilde{f}: M \to im(f)$ is a homeomorphism.

Remark $im(f) \subset N$ so we regard $im(f)$ as a metric space by restricting the metric on N.

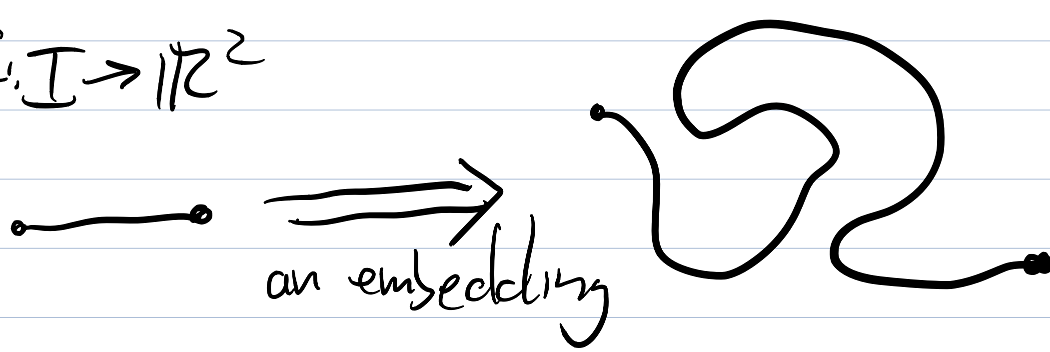
Intuition: An embedding is a function that doesn't do any "gluing"

Illustration:

$f: I \to \mathbb{R}^2$



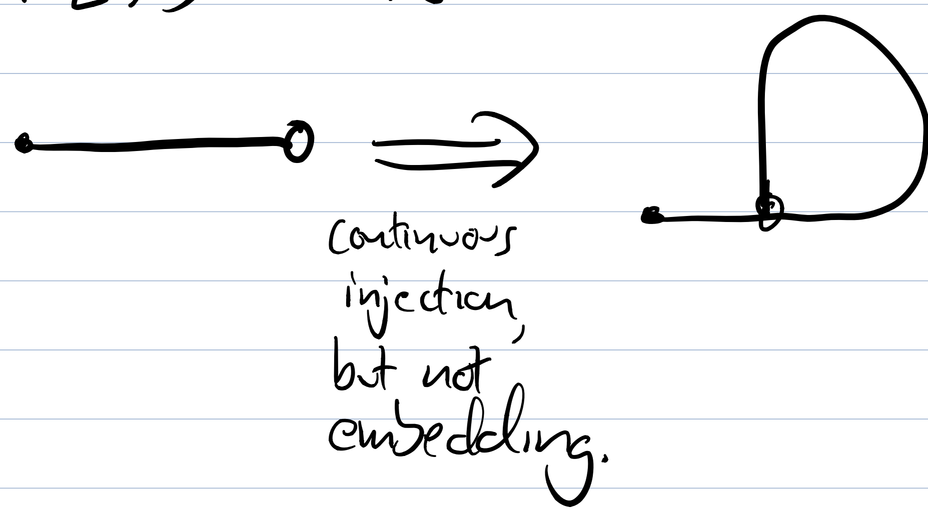
$f: I \to \mathbb{R}^2$



Remark: An embedding is a continuous injection, but the converse is not true

Example

$f : [0,1) \longrightarrow \mathbb{R}^2$



Homotopy + Isotopy

For simplicity, I define these for subsets of $\mathbb{R}^n$, w/ Euclidean distance.

(This allows me to avoid worrying about what the Cartesian product of metric spaces should be.)

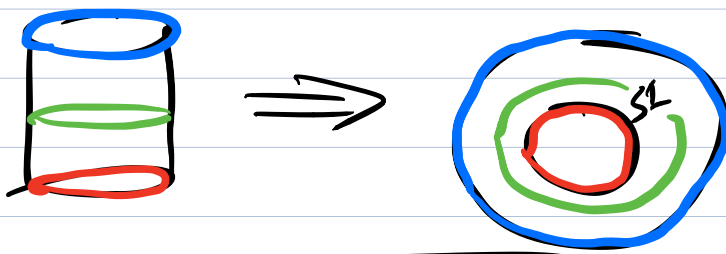
For $S \subset \mathbb{R}^m$ and $T \subset \mathbb{R}^n$ we call a function

$h: S \times I \to T$ a <u>homotopy</u>.

Example: $S = S^1$ (unit circle
$T = \mathbb{R}^2$

$S \times I =$ Cylinder

$h: S \times I \to \mathbb{R}^2$, $h(\vec{x}) = (1+t)\vec{x}$



for $t \in I$, we define $h_t : S \to T$ by
$h_t(x) = h(x,t)$.

Think of the homotopy as a continuous deformation of the map h_0 to the map h_1 via the intermediate maps h_t.

We think of t as time.

To talk about continuous deformation of spaces (sets) one considers the sets $\mathrm{im}(h_t) \subset \mathbb{R}^4$.

Isotopy For $S, T \subset \mathbb{R}^n$, an isotopy from S to T is a homotopy $h: X \times I \to \mathbb{R}^n$ such that

1) $im(h_0) = S$
2) $im(h_1) = T$
3) h_t is an embedding $\forall t$.

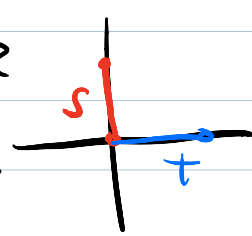
Fact: If $\exists$ such an isotopy, then S, T and X are all homeomorphic.

Fact: If S and T are isotopic, then we can find an isotopy from S to T with $X = S$ and $im(h_0) = Id_S$.

How to construct an isotopy from S to T

1) Take $X = S$, $h_0 = Id_S$, $h_1 =$ a homeomorphism $f: S \to T$.
2) Find the other h_t.

Example $S = I \times \{0\} \subset \mathbb{R}^2$
$T = \{0\} \times I \subseteq \mathbb{R}$

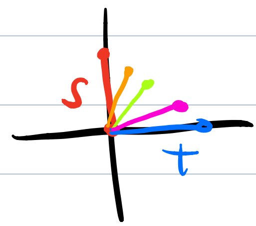


Consider **a** **homeo**: $f: S \to T$, $f(x, 0) = (0, x)$

Now define $h: S \times I \to \mathbb{R}^2$ by

$$h((x,0),t)) = (x(1-t), tx).$$

This is an isotopy from S to T.



<u>Open Sets</u> / <u>Open balls</u> [An alternate discription of continuity]

Let M be a metric space w/ metric d.

For $x \in M$ and $r > 0$, the open ball in M of radius r centered at O is given by

$$B(x,r) = \{y \in M \mid d(x,y) < r\}.$$

<u>Exam 2 questions</u>:

Can $B(x,r)$ be empty? No, because $d(x,x) = 0 < r$, so $x \in B(x,r)$.

Can we have $B(x,r) = B(x,r')$ when $r \neq r'$? Yes.

Take $M = \{0,1\}$ w/ Euclidean metric.

$B(0,\frac{1}{2}) = B(0,\frac{1}{3}) = \{0\}$.

__Def__: A subset of M is __open__ if it is a (possibly infinite) union of open balls.

__Remark__: M is always open and so is the empty set.

The rest will have to be improvised...