Today:
- Functions (continued)
- Continuous functions
- Homeomorphisms

review from last time

Definition: Given sets S and T, a function f from S to T is a rule which assigns each $s \in S$ exactly one element in T.
- This element is denoted $f(t)$.

We call
S the domain of f.
T the codomain of f.

We write the function as $f: S \to T$.

Example: Let $S = \{1, 2\}$, $T = \{a, b\}$.
e can define a function $f: S \to T$ by $f(1) = a$, $f(2) = b$
$g: S \to T$ by $f(1) = a$, $f(2) = a$.

Example: We often specify a function by a formula, e.g.
$f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.
$f: \mathbb{R} \to \mathbb{R}$, $f(x) = 5e^{-x^2}$ OR

For $f: S \to T$ and $g: T \to U$, the composite $g \circ f: S \to U$ is the function given by $g \circ f(x) = g(f(x))$.

Image of a function (also called the range)

Definition: For a function $f: S \to T$ we define $\mathrm{im}(f)$ to be the subset of T given by

$$\mathrm{im}(f) = \{ t \in T \mid t = f(s) \text{ for some } s \in S \}.$$

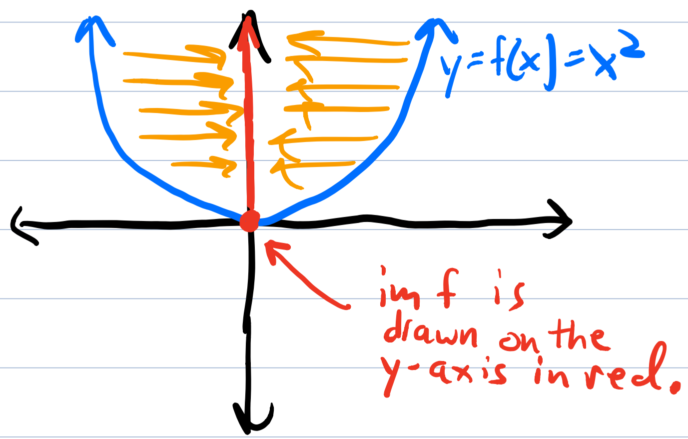
Intuitively, $\mathrm{im}(f)$ is the subset of T consisting of elements "hit by" f.

Example: For $S, T, f,$ and g as in the previous example,

$$\mathrm{im}(f) = \{a, b\} = T, \quad \mathrm{Im}(g) = \{a\}.$$

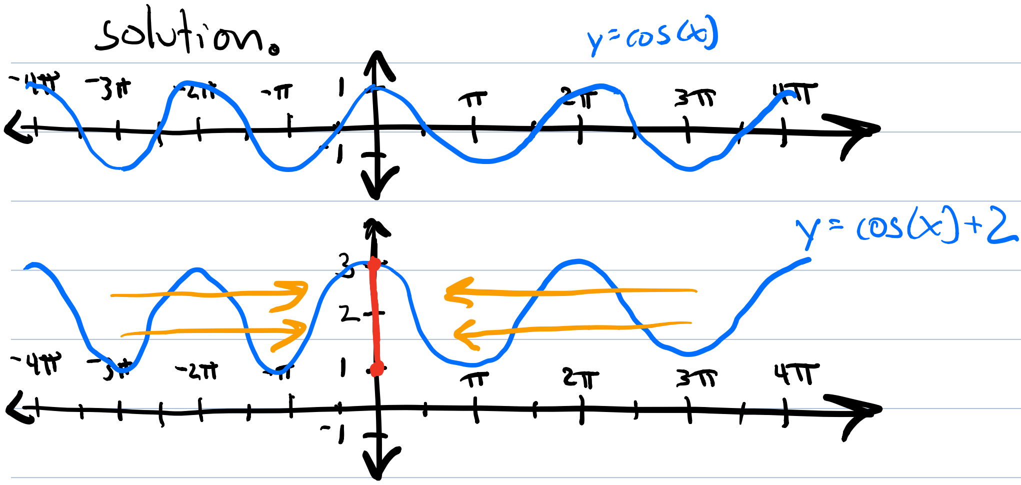
Example:

for $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$,
$\operatorname{im} f = [0, \infty)$



Exercise: For $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = \cos(x) + 2$, what is $\operatorname{im} f$?

solution.



$\operatorname{im}(\cos) = [-1, 1]$, so $\operatorname{im}(f) = [1, 3]$

Exercise:

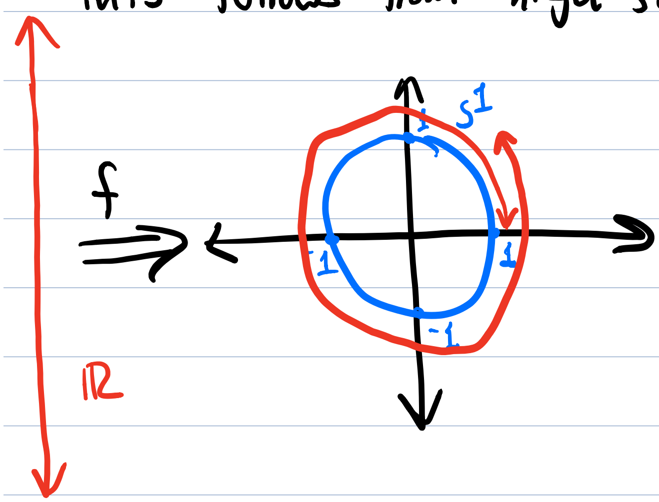
Let $f: \mathbb{R} \to \mathbb{R}^2$ be given by
$f(x) = (\cos x, \sin x)$.

What is $\operatorname{im} f$?

Solution: $\operatorname{im} f = S^1$, where S^1 denotes the unit circle, i.e.,

$$S^1 = \{(a,b) \in \mathbb{R}^2 \mid a^2 + b^2 = 1\}.$$

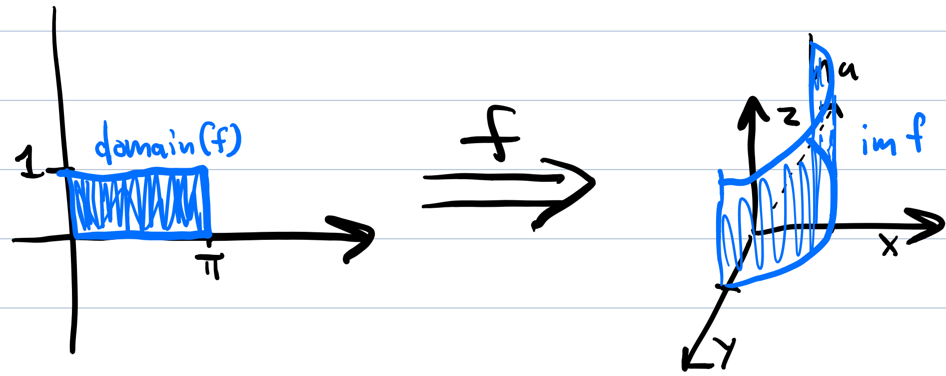
This follows from high school trig.



f "wraps" $\mathbb{R}$ around S^1 an infinite number of times

Example: Let $f: [0, \pi] \times I \to \mathbb{R}^3$ be given by
by $f(x,y) = (\cos x, \sin x, y)$

$\operatorname{im} f$ is a half-cylinder.



Injective, Surjective, and Bijective Functions

We say a function $f: S \to T$ is

injective (or 1-1) if $f(s) = f(t)$ only when $s = t$.

surjective (onto) if $\mathrm{im}(f) = T$.

bijective (a bijection) if f is both injective and surjective.

Example: $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is neither injective nor surjective.

Example $f: \mathbb{R} \to S^1$ given by

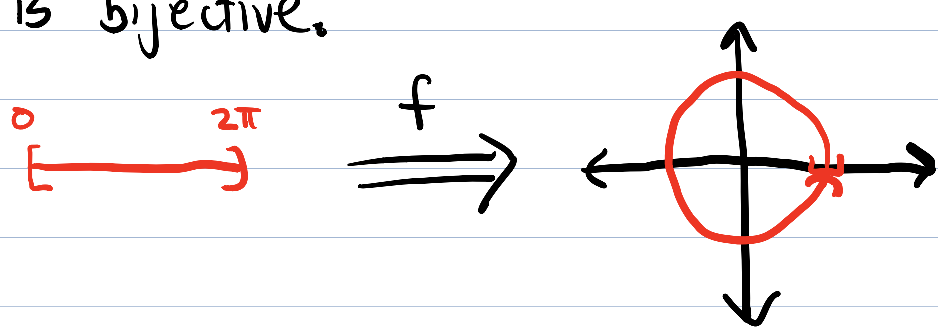
$f(x) = (\cos x, \sin x)$ is surjective but not injective.

e.g. $f(0) = f(2\pi) = (1,0)$.

Example $f: [0, 2\pi) \to S^1$ given by

$f(x) = (\cos x, \sin x)$

is bijective.



Bijections and Inverses

For S any set, the identity function on S, is the function

$Id_S : S \to S$ given by $Id_S(x) = x \ \forall x \in S$.

Fuctions $f: S \to T$ and $g: T \to S$ are said to be <u>inverses</u> if

$$g \circ f = Id_S \quad \text{and} \quad f \circ g = Id_T.$$

function composition

<u>Fact:</u> A function $f: S \to T$ has an inverse $g: T \to S$ if and only if f is a bijection.

<u>Example</u> Let $f: [0, 2\pi) \to S^1$ be bijection of the previous example.

We define the inverse $g: S^1 \to [0, 2\pi)$ to be the function which maps $y \in S^1$ to the angle θ $\overrightarrow{Oy}$ makes with the positive x-axis (in radians).

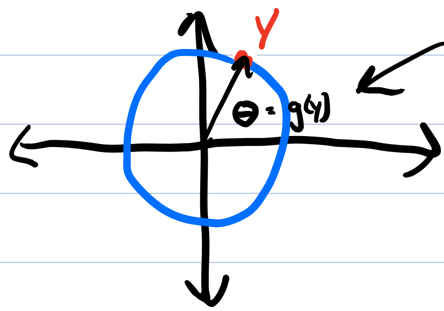


Illustration of g.

Continuous functions

Topology begins with the notion of continuous functions. You have already encountered these in your calculus classes.

It's possible to give a very abstract, general definition of continuous functions. We may do this later, but for now, we will take a more concrete approach.

We consider the continuity of a functions between subsets of Euclidean spaces.

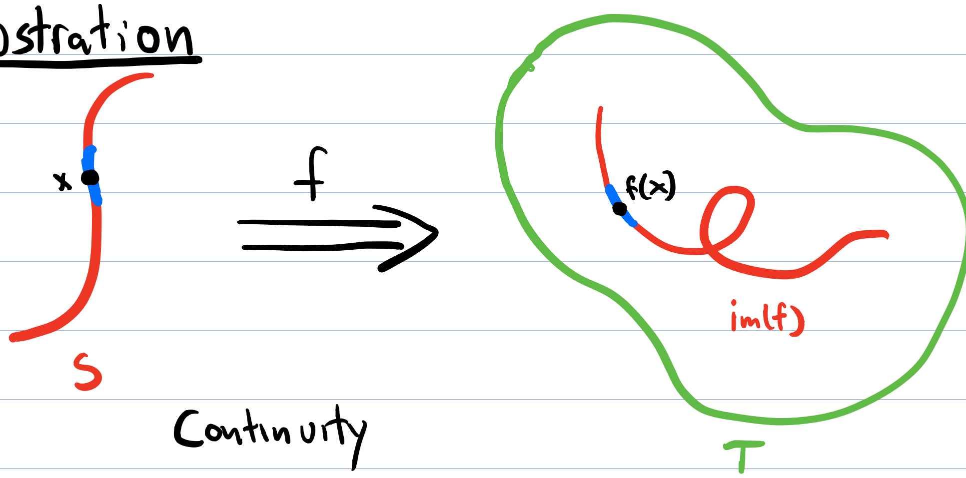
For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$
$y = (y_1, \ldots, y_n) \in \mathbb{R}^n$,

let $d(x,y)$ denote the Euclidean distance between x and y, i.e.,

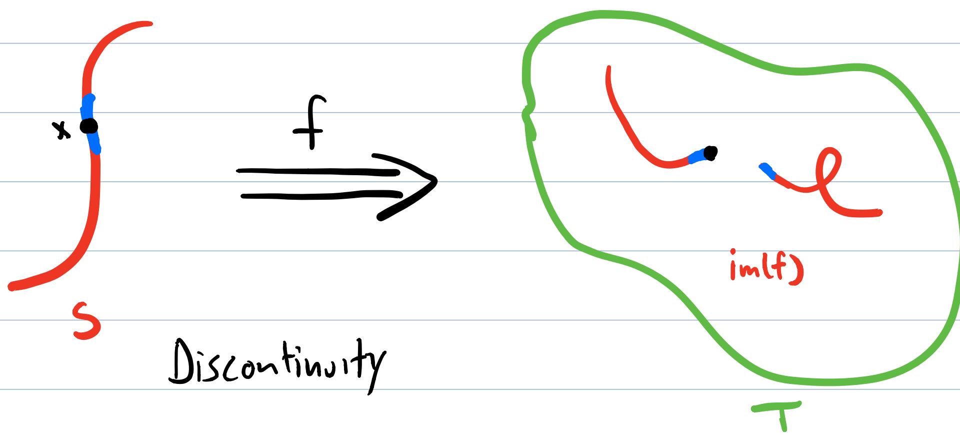
$$d(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2}$$
$$= \|x-y\|.$$

> Let $S \subset \mathbb{R}^m$ and $T \subset \mathbb{R}^n$ for some $n, m \geq 1$.
> Intuitively, a function $f: S \to T$ is continuous if f maps nearby points to nearby points.

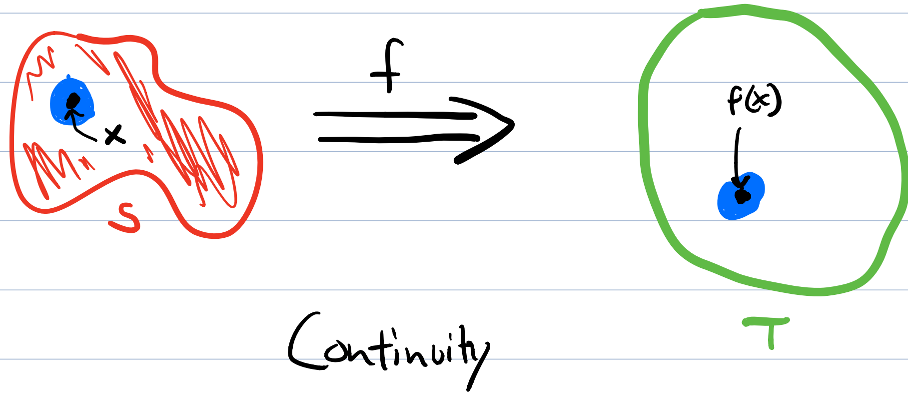
Illustration

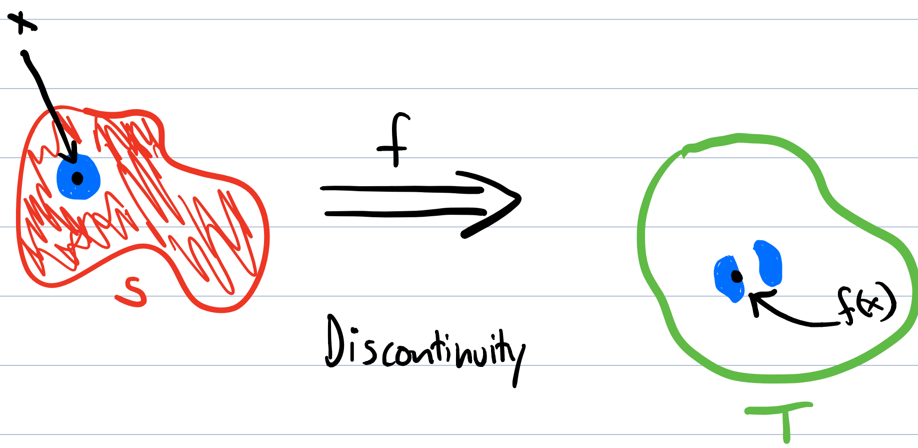


Continuity



Discontinuity



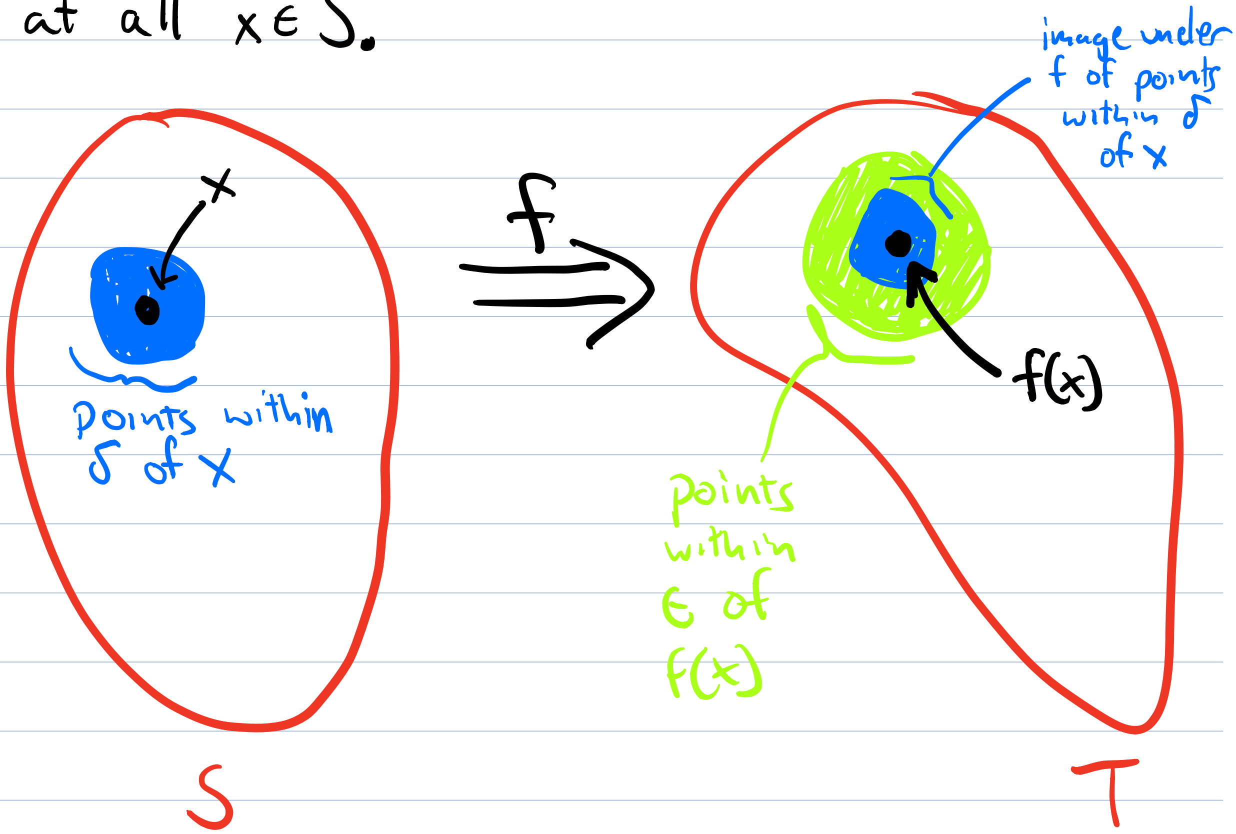


[Lecture ended somewhere around here.]

Formal Definition

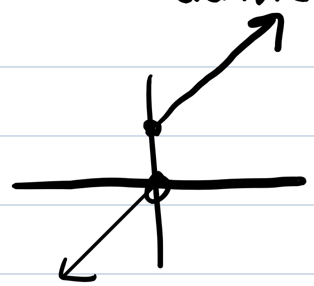
We say $f: S \to T$ is <u>continuous</u> at $x \in S$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in S$ and $d(x,y) < \delta$, then $d(f(x), f(y)) < \epsilon$.

We say f is <u>continuous</u> if it is continuous at all $x \in S$.



<u>Interpretation</u>: You give me any positive ϵ no matter how small. Continuity at x means that I can find a positive δ such that points within distance δ of x map to points within distance ϵ of x. (I'm allowed to choose δ as small as I want, as long as it's positive.)

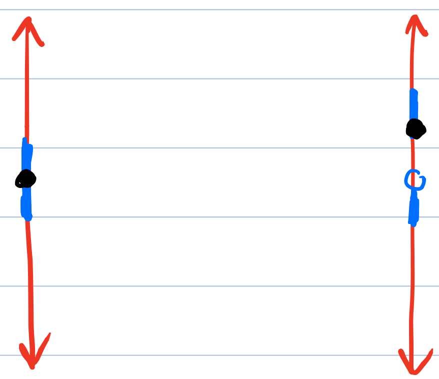
Example: Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} x & \text{if } x < 0 \\ x+1 & \text{if } x \geq 0 \end{cases}$



f is not continuous at 0. To see this, consider $\epsilon = 1$. For all $\delta > 0$, there is some $y \in \mathbb{R}$ with $d(x,y) < \delta$ and $f(y) < 0$. For example, we can take $y = -\frac{\delta}{2}$.

Then $d(f(x), f(y)) > 1 = \epsilon$.

This shows f is not continuous at 0.



In this class, we won't spend much time worrying about the rigorous definition of continuity, but I do want you to be familiar with it.

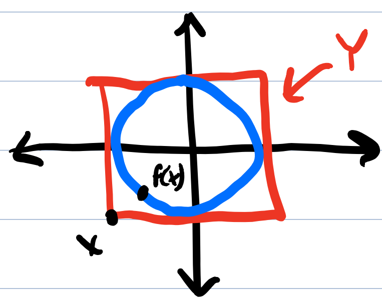
Homeomorphism

For S, T subsets of Euclidean spaces,
A function $f: S \to T$ is a <u>homeomorphism</u>
if

1) f is a continuous bijection
2) The inverse of f is also continuous.

Homeomorphism is the main notion of continuous deformation we'll consider in this course.

Example Let $Y \subset \mathbb{R}^2$ be the square of side length 2, embedded in the plane as shown



The function $f: Y \to S^1$ given by
$f(x) = \frac{x}{\|x\|}$ is a homeomorphism.
where $\|x\|$ = distance of x to origin
$= \sqrt{x_1^2 + x_2^2}$

by standard calculus, this is continuous.
It is intuitive clear that this is a bijection with a continuous inverse. The inverse can be written down, but we won't bother.