AMAT 342 8/27/19 (first class)

Like virtually all of modern mathematics, topology relies heavily on the language of sets and functions. We begin with a review of this language.

<u>Review of Sets and functions</u>

We begin with the following informal definition of a set.

<u>Definition</u>: A set is a collection of distinct elements.

- This is perhaps not a very satisfying definition, because we haven't defined a collection, nor have we defined elements.
- It is possible to introduce sets in a more formal, axiomatic way.
- However, this informal definition will be fine for our purposes.

The definition is best made clear via some examples.

<u>Example</u>: $S_1 = \{a, b\}$ is a set.

S_1 is the set whose elements are the letters a and b.

- As shown here, we can specify a set by listing its elements in curly braces.
- The order in which we write the elements doesn't matter.

<u>Example</u>: $S_2 = \{0, 1, 2\}$ is a set.

<u>Example</u>: $S_3 = \{\}$ is a set, called the empty set.

also written as ϕ.

$\{\}$ is the unique set with no elements.

Example: We can have a set whose elements are themselves sets.

E.g., for S_1, S_2 as just defined,

$S_4 = \{S_1, S_2\} = \{\{a,b\}, \{0,1,2\}\}$ is a set.

Example: $S_5 = \{a, b, 0, 1, 2\}$ is a set.
This example shows that a set is allowed to have elements of different types.

NOTE: A set is not allowed to contain itself.

Example: $S_1 = \{a, b, S_1\}$ is not a valid set.

Example: A set can have a single element, e.g. $\{1\}$. Such a set is called a "singleton set."

Example: Infinite sets are allowed.

- The natural numbers $\{0,1,2,3,\ldots\}$ form a set, denoted $\mathbb{N}$.
- The integers $\{\ldots,{}^-3,{}^-2,{}^-1,0,1,2,3,\ldots\}$ form a set, denoted $\mathbb{Z}$.

Example: The real numbers form a set, denoted $\mathbb{R}$.
This example is very important in topology.

There are multiple ways one can define a real number. Here is the way that seems most friendly:

A real number is a decimal number with (possibly) an infinite number of digits to the right of the decimal.

$$\left.\begin{array}{l} .25 \\ 14.25 \\ 3.3333\ldots \\ \pi = 3.14159\ldots \end{array}\right\} \text{each of these is a real number}$$

also written as $3.\overline{33}$

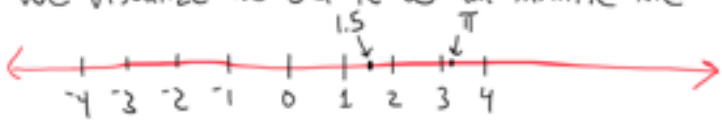
NOTE: Any integer or rational number (i.e., fraction) can be written as an infinite decimal, so these are real numbers.

Important Detail: A decimal number ending in an infinite string of 9's is regarded as the same real number as the one obtained by removing the infinite string of 9's and then increasing the last digit of what remains by 1.

Examples: $1.79999\ldots = 1.8$

$0.0999\ldots = .1$

We visualize the set $\mathbb{R}$ as an infinite line:



Points on the line are real numbers.

Set Notation

Containment

- If a set S contains an element a, we write $a \in S$.
- Otherwise, we write $a \notin S$.

Example: $^-1 \notin \mathbb{N}$, but $^-1 \in \mathbb{Z}$.

Equality of Sets

We say sets S and T are equal, and write $S = T$, if the elements of S and T are exactly the same.

Otherwise, we write $S \neq T$.

Example: If $S = \{a, b, c\}$ and $T = \{b, c, a\}$, then $S = T$

(as noted earlier, order in which elements are listed doesn't matter)

Example: If $S = \{\{a,b\}, \{0,1,2\}\}$ and $T = \{a, b, 0, 1, 2\}$, then $S \neq T$.

S has 2 elements, T has 5 elements.

Subsets We say a set S is a subset of a set T, and write $S \subset T$, if every element of S is an element of T.

(otherwise, we write $S \not\subset T$.)

Example: $\{0,1\} \subset \{0,1,2\}$, but $\{0,1,2\} \not\subset \{0,1\}$

doesn't contain 2.

Example: $\underbrace{\mathbb{N}}_{\substack{\text{natural numbers} \\ \{0,1,2,\ldots\}}} \subset \underbrace{\mathbb{Z}}_{\text{integers}} \subset \underbrace{\mathbb{R}}_{\text{real numbers}}$

Question: Is a set S a subset of itself?

Ans: Yes, by definition, because every element of S is an element of S!

Example: The empty set $\emptyset$ is a subset of any set S.

$\emptyset$ has no elements, so this is vacuously true.

Exercise: What are all the subsets of $\{1,2\}$?

Specifying a Subset via a Property

We sometimes specify a set via a property satisfied by its elements.

For example, let E denote the even integers. We can write

$$E = \{\ldots, {}^-6, {}^-4, {}^-2, 0, 2, 4, 6, \ldots\}$$

but we can also say:

" E is the set of integers x such that x is divisible by 2.

We can write this in symbols as:

$$E = \{x \in \mathbb{Z} \mid x \text{ is divisible by } 2\}.$$

The symbol "$|$" translates to "such that."

Note: Sometimes ":" is used in place of "$|$"; e.g. $\{x \in \mathbb{Z} : x \text{ is divisible by } 2\}$.

Exercise: Use the notation introduced above to give an expression for the set of prime numbers as a subset of $\mathbb{N}$.

Intervals

Intervals are subsets of $\mathbb{R}$ that play an especially important role in topology.

Several types:

For $a, b \in \mathbb{R}$, $a < b$,

$(a,b) = \underbrace{\{x \in \mathbb{R} \mid a < x < b\}}$.

this is the set of real numbers greater than a and less than b.

$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$.

$(a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}$.

and for $a \leq b$,

$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$