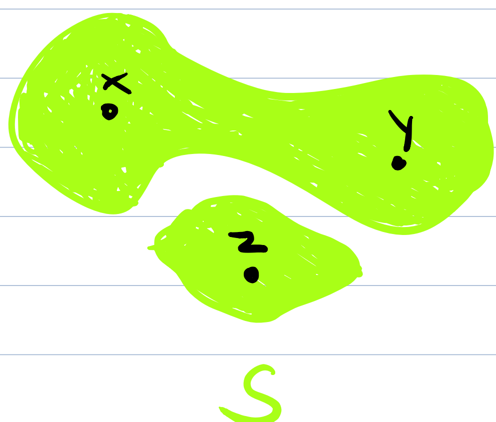
AMAT 342 Lecture 10 9/26/19

Today: Path components continued
Metric Spaces

Review: For $S \subset \mathbb{R}^n$, we defined an equivalence relation $\sim$ on S by taking $x \sim y$ if and only if $\exists$ a path from x to y.



$x \sim y$, but $x \not\sim z$ and $y \not\sim z$.

An equivalence class of $\sim$ is called a <u>path component</u> of S.

We denote the set of path components of S by $\pi(S)$.

Proposition: If $f: S \to T$ is a homeomorphism, then there is a bijection from $\pi(S)$ to $\pi(T)$.

That is, $\pi(S)$ and $\pi(T)$ have the same # of elements, possibly infinite.)

<u>Proof</u>: We started the proof last time but did not finish. I'll review the parts from last time, then complete the proof.

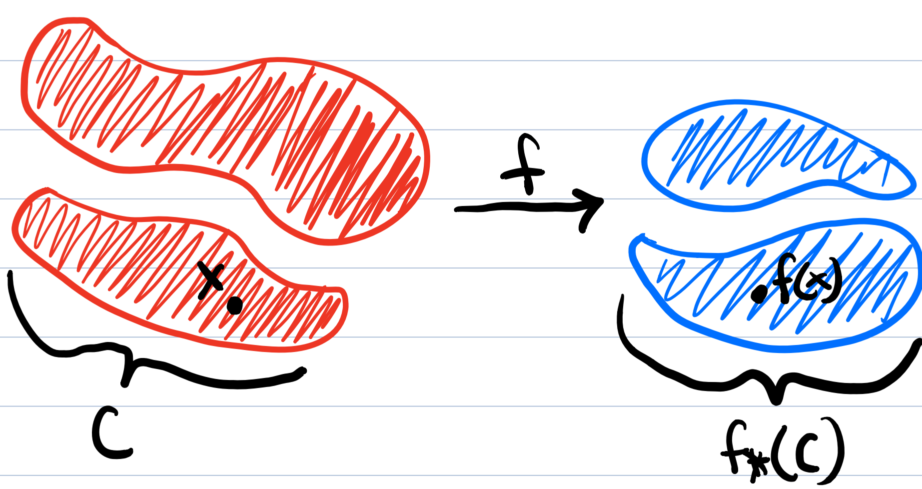
For $f: S \to T$ any continuous map, define a function

$f_*: \pi(S) \to \pi(T)$ by the formula

$$f_*([x]) = [f(x)].$$

call this the map on path components induced by f

That is, if $C \in \pi(S)$, choose $x \in C$. Define $f_*(C) =$ the path component containing $f(x)$.



We showed that $f_*(C)$ is independent of the choice of $x \in C$, so f_* is well defined.

<u>2 facts</u>: 1) For any $S \subset \mathbb{R}^n$, $Id^S_* = Id^{\pi(S)}$.

In words, the map on path components induced by the identity is the Identity.

2) For any continuous maps $f: S \to T$, $g: T \to U$

$$(g \circ f)_* = g_* \circ f_*.$$

All of that was review. Now let's pick up where we left off and finish the proof.

Assume $f: S \to T$ is a homeomorphism.
Then f, f^{-1} are both continuous, and we have

$f^{-1} \circ f = Id^S$

$f \circ f^{-1} \; Id^T$

$$\text{Thus, } (f^{-1} \circ f)_* = \underbrace{Id^S_*}_{= Id^{\pi(S)}} \Rightarrow f^{-1}_* \circ f_* = Id^{\pi(S)}$$

$$(f \circ f^{-1})_* = \underbrace{Id^T_*}_{= Id^{\pi(T)}} \Rightarrow f_* \circ f^{-1}_* = Id^{\pi(T)}.$$

Thus, $f_*: \pi(S) \to \pi(T)$ is invertible, with inverse f^{-1}_*. Therefore f_* is a bijection. ∎

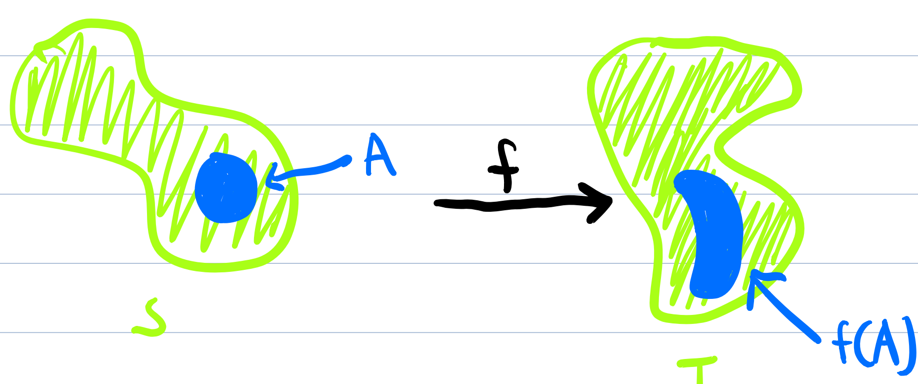
Homework Hint: Problem 4 asks you to prove that if $f: X \to Y$ is continuous surjection and X has k path components, then Y has at most k path components.

To prove this, show that $f_*: \pi(X) \to \pi(Y)$ is a surjection.

Application: Consider the symbols $+$, $=$, and $\div$ as subsets of $\mathbb{R}^2$.
$|\pi(+)| = 1$, $|\pi(=)| = 2$, $|\pi(\div)| = 3$. Thus none is homeomorphic to any other.

Application: We prove that as unions of curves w/ no thickness, X and Y are not homeomorphic.
[The argument will be skipped in class]

Fact: If $f: S \to T$ is a homeomorphism and $A \subset S$, then A and $f(A)$ are homeomorphic, where $f(A) = \{y \in T \mid y = f(x) \text{ for some } x \in A\}$.



proof of fact: Let $\iota: A \to S$ be the inclusion. $\mathrm{im}(f \circ j) = \check{f}(A)$. Since f is a bijection, so $\widetilde{f \circ j} : A \to f(A)$. It follows from the facts about continuity stated in an earlier lecture that $\widetilde{f \circ j}$ is continuous. Moreover, if $j' : f(A) \to T$ is the inclusion, $(\widetilde{f \circ j})^{-1} = \widetilde{f^{-1} \circ j'}$, and this is continuous by the same reasoning.

<u>Proof that X and Y are not homeomorphic</u>:

Let $X' \subset X$ be obtained by removing the center point p. $|\pi(X')| = 4$. Note that there no way to remove a single point from Y to get $Y' \subset Y$ with $|\pi(Y)| = 4$

If we have a homeomorphism $f : X \to Y$, then $f(X')$ is obtained from Y by removing $f(p)$, and $|\pi(f(X'))| = |\pi(X')| = 4$ by the prop., which is impossible. Thus, no homeomorphism $f : X \to Y$ can exist.

Topology Beyond Subsets of Euclidean Space

So far in this course, we've only considered continuity of functions $f: S \to T$ where S and T subsets of Euclidean spaces.

we sometimes use the word "subspace"

Hence, all the topological concepts we've introduced so far, e.g.,

- homeomorphism
- isotopy
- path components

have been defined in class only for Euclidean subspaces.

However, these ideas make sense in much more generality, and that extra generality can be extremely useful.

In fact, there are two levels to this extra generality. We discuss first level now.

Recall our definition of a continuous function between Euclidean subspaces:

<u>Formal</u> <u>Definition</u> <u>of</u> <u>Continuity</u>

We say $f: S \to T$ is <u>continuous</u> at $x \in S$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in S$ and $d(x,y) < \delta$, then $d(f(x), f(y)) < \epsilon$.

We say f is <u>continuous</u> if it is continuous at all $x \in S$.

from Lec 3

<u>Important observation</u>: The only way we are using the fact that S and T are Euclidean subspaces is through their distance functions.

$\Rightarrow$ Continuity should make sense for any functions between sets endowed with some reasonable definition of a distance.

There are many extremely important examples, beyond the Euclidean subspaces we've already seen.

To explain this formally, we introduce <u>metric spaces</u>.

A metric space is a set S, together with a function $d: S \times S \to [0,\infty)$ satisfying:

1) $d(x,y)=0$ if and only if $x=y$.
2) $d(x,y)=d(y,x)$ [symmetry]
3) $d(x,z) \leq d(x,y)+d(y,z)$ $\forall$ $x,y,z \in S$ [triangle inequality].

We denote the metric space as (S,d). We call d a metric.

Example: The familar example: $S=\mathbb{R}^n$, $d_2: \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$,

$$d_2(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\dots+(x_n-y_n)^2}$$

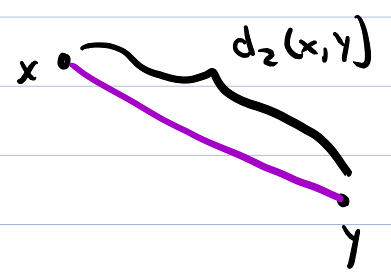
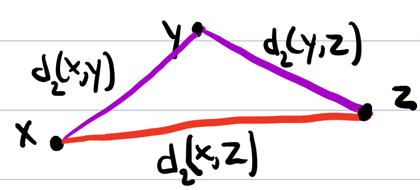


Illustration of the triangle inequality (case that x, y, z don't all lie on the same line)

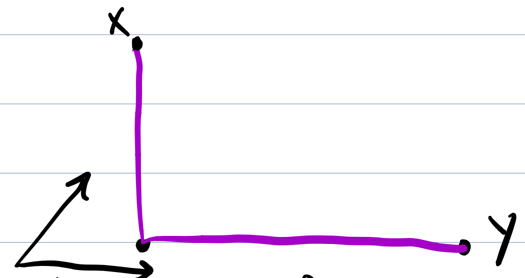


$d_2(x,z) \leq d_2(x,y)+d_2(y,z)$ because the length of any side of a triangle is less than the sum of the lengths of the other two sides. Hence the name "triangle inequality".

Example:

$S = \mathbb{R}^n$, $d_1 : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$

$$d_1(x,y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|$$



$d_1(x,y)$ is the sum of these two edge lengths.

d_1 is sometimes called the Manhattan distance or taxicab distance.

e.g. $d_1((1,3),(3,4)) = |1-3| + |3-4| = 2+1 = 3$

Let's check that this is a metric.

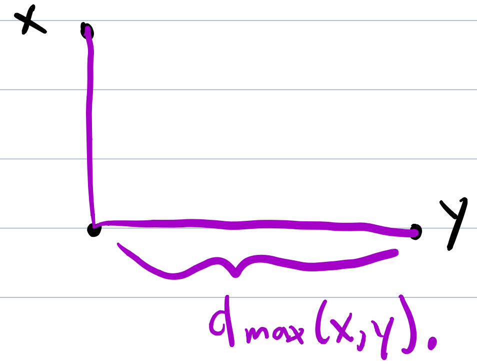
1) Clearly $d_1(x,x) = 0$ for all $x \in \mathbb{R}^n$.
   If $x \neq y$, then $x_k \neq y_k$ for some $k \in \{1,\ldots,n\}$
   so $0 < |x_k - y_k| \leq d_1(x,y)$, so $0 < d_1(x,y)$.

2) $d_1(x,y) = d_1(y,x)$ because $|x_k - y_k| = |y_k - x_k|$ for all $k \in \{1,\ldots,n\}$.

3) $d_1(x,z) \leq d_1(x,y) + d_1(y,z)$ because
   $|x_k - z_k| \leq |x_k - y_k| + |y_k - z_k|$
   (explanation: $|a+b| \leq |a| + |b|$. Take $a = x_k - y_k$, $b = y_k - z_k$.

Example: $S = \mathbb{R}^n$, $d_{max}: \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$,

$$d_{max}(x,y) = \max\left(|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\right)$$



This is also a metric.

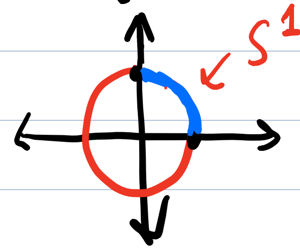
[Lecture ended here]

Fact: If (M, d^M) is a metric space, $S \subset M$, and $d^S: S \times S \to [0,\infty)$ is the restriction of d^M to $S \times S$ (i.e., $d^S(x,y) = d^M(x,y)\ \forall x,y \in S$), then (S, d^S) is a metric space.

That is, subspaces of metric spaces inherit the structure of a metric space in the obvious way.

But in many cases, there is another construction of a metric on a subspace, the intrinsic metric.

Example: Define a metric d on S^1 by $d(x,y)$ = minimum length of an arc in S^1 connecting x and y.



This is called the intrinsic metric on S^1.

e.g. $d((1,0),(0,1)) = \frac{\pi}{2}$ because minimum length of an arc from $(1,0)$ to $(0,1)$ is $\frac{1}{4}(\text{circumference of } S^1) = \frac{2\pi}{4} = \frac{\pi}{2}$.

By comparison $d_2((1,0),(0,1)) = \sqrt{1^2+1^2} = \sqrt{2}$.

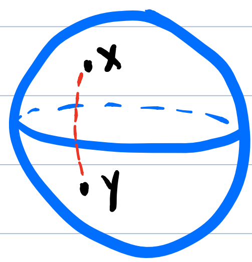
length of the straight line connecting $(1,0)$ and $(0,1)$

More generally, the intrinsic metric d can be defined on a very large class of subsets $S \subset \mathbb{R}^n$ as follows:

$d(x,y)$ = minimum length of a differentiable path $\gamma: I \to S$ from x to y. (Since codomain of γ is S, $\mathrm{im}(\gamma)$ is required to lie in S.)

As in calculus, length$(\gamma) := \int_0^1 |\gamma'(t)|\, dt$.

For example, we can take S to be a sphere in $\mathbb{R}^3$



$\leftarrow$ $d(x,y)$ is the length of the shortest curve connecting x and y.

or any other surface in $\mathbb{R}^3$.