# Main Topics

Sets
- Subsets, Cartesian products

Functions
- Images of functions
- Injections, surjections, bijections, inverses.

Continuous functions
- Intuitive geometric interpretation
- [Rigorous ($\epsilon$-$\delta$) definition of continuity will not be covered, except perhaps as a bonus question.]
- Properties of continuity, which guarantee that functions that we hope are continuous typically are in fact continuous. (Will not be emphasized heavily).

Homeomorphism

Homotopy

Embeddings

Isotopy
- Intuitive geometric idea
- Formal definition
- Not on exam: Surprising isotopies, like unlinking the 2-holed donut

Equivalence relations

Path components / Path connectedness

Metric Spaces

## Subsets

**Exercise:** Let $S \subset \mathbb{R}^2$ be the set of points of distance at most 1 from some point on the $x$-axis.

Express $S$ in "bracket notation" by filling in the blank in the following expression:

$$S = \{(x,y) \in \mathbb{R}^2 \mid \_\_\_\_\_\_\_\_ \}$$

## Cartesian Products

**Definition** For sets $S_1, \ldots, S_n$

$$\underbrace{S_1 \times S_2 \times \cdots \times S_n}_{\text{Cartesian product of } S_1,\ldots,S_n} = \{(x_1, x_2, \ldots, x_n) \mid x_i \in S_i \ \forall\, i \in \{1,\ldots,n\}\}.$$

**Exercise from HW:** Sketch $I \times \{1,2,3,4\}$.

**Exercise** Sketch $I \times I \times \{0,1,2\}$ as a subset of $\mathbb{R}^3$

(recall: $I \subset \mathbb{R}$ is the interval $[0,1]$.)

**Functions:** The **image** of a function $f: S \to T$ is the subset of $T$ given by

$$\mathrm{im}(f) = \{ y \in T \mid y \in f(x) \text{ for some } x \in S \}.$$

A function $f: S \to T$ is

- **injective** if $f(x) = f(y)$ only when $x = y$.
- **surjective** if $\mathrm{im}(f) = T$.
- **bijective** if both injective and surjective

$f: S \to T$ is called **invertible** if $\exists$ a function $g: T \to S$

s.t. $g \circ f = Id_S$
$\quad\;\; f \circ g = Id_T$
identity functions on $S$ and $T$

$g$ is called the inverse of $f$.

**Fact:** $f$ is bijective iff $f$ is invertible.

**Exercise** Suppose $f: S \to T$ is invertible. What is $\mathrm{im}(f)$? Explain your answer.

**Ans:** $f$ is invertible, so $f$ is bijective, hence surjective. Thus $\mathrm{im}(f) = T$.

**Exercise** Show that if $f: S \to T$ and $g: T \to U$ are bijections, then $g \circ f: S \to U$ is a bijection.

## Continuous functions

**Intuitive interpretation 1:** For $S, T$ subsets of Euclidean spaces, $f: S \to T$ is continuous if $f$ "puts $S$ into $T$ without tearing $S$."

**Intutive interpretation 2:** $f$ is continuous if $f$ "maps nearby points to nearby points."

**Def:** $f: S \to T$ is a homeomorphism if

1) $f$ is a continuous bijection
2) $f^{-1}$ is also continuous.

**Intuition:** $f$ is a bijection which puts $S$ into $T$ without either tearing or gluing $S$.

**Easy facts:** 1) Inverse of a homeomorphism is a homeomorphism
2) Composition of homeomorphisms is a homeomorphism

**Exercise:** Show that if $g$ and $g \circ f$ are homeomorphisms, then so is $f$.

**Exercise:** Give an example where $g \circ f$ is a homeomorphism, but $f$ is not a homeomorphism.

**Isotopy:**

A way of formalizing the idea of "continuous deformation"

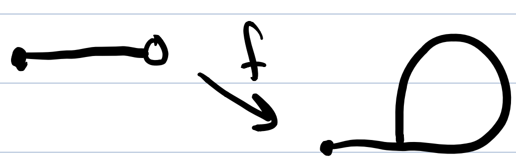
- Concerns two subsets $S, T \subset \mathbb{R}^n$ (for the same $n$)
- Models evolution of a geometric object in time.

**Embeddings:** A continuous map $f: S \to T$ is an **embedding** if the induced map

$$\tilde{f}: S \to \mathrm{im}(f) \text{ is a homeomorphism.}$$

An embedding is a continuous injection, but embeddings also disallow certain kinds of self-gluing.

$$f: [0,1) \to \mathbb{R}^2$$



not an embedding

embedding

**Def:**
For $S, T \subset \mathbb{R}^n$, an isotopy from $S$ to $T$ is a homotopy (i.e. continuous function)

$$h : X \times I \to \mathbb{R}^n$$

such that

1) $\mathrm{im}(h_0) = S$
2) $\mathrm{im}(h_1) = T$
3) $h_t$ is an embedding $\forall t \in I$.

**Note:** It follows from the definition that $X$ is homeomorphic to both $S$ and $T$.

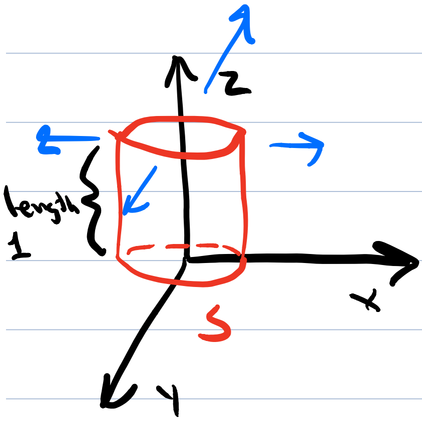
Thus, $S, T$ isotopic $\Rightarrow$ $S, T$ homeomorphic.

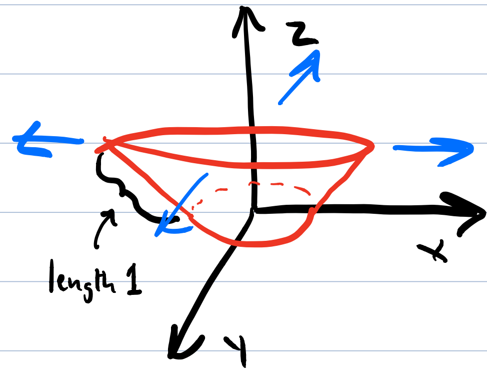
**Note:** If $S, T$ are isotopic we can always take $X = S$ and $h_0 : S \to \mathbb{R}^n$ to be the inclusion.

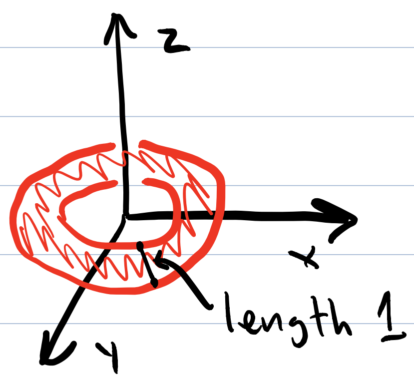
**Example from HW**

Let $\underset{\mathbb{R}^3}{\underset{\cap}{S}}$ = the cylinder $S^1 \times I$, and $\underset{\mathbb{R}^3}{\underset{\cap}{T}}$ = the annulus in the xy-plane with inner radius 1.

outer radius 2.




How to find an elplicit isotopy from $\overset{\mathbb{R}^n}{\overset{\cup}{S}}$ to $\overset{\mathbb{R}^n}{\overset{\cup}{T}}$ in practice

- Find an explicit homeomorphism $f: S$ to $T$
- Modify the expression for $f$ to get an isotopy $h: S \times I \to \mathbb{R}^n$ with

  $h_0 =$ the inclusion $S \hookrightarrow \mathbb{R}^n$

  $h_1 = f.$

<u>Exercise</u>: Let $S = \{(1,y) \mid y \in I\}$

$T = \{(x,0) \mid x \in [1,2]\}$

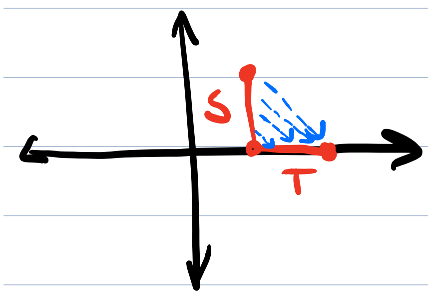
Find an explicit isotopy from $S$ to $T$.

$f: S \to T$ given by $f((1,y)) = (1+y, 0)$
is a homeomorphism.

$\left(f^{-1}: T \to S \text{ is given by } f^{-1}(x,0) = (1, x-1)\right)$

An isotopy $h: S \times I \to \mathbb{R}^2$ is given by

$$h((1,y),t)) = (1 + ty, (1-t)y).$$




<u>Equivalence</u> <u>Relations</u> [you should review relations yourself]

A relation $\sim$ on a set $S$ is an <u>equivalence relation</u> if

1) $x \sim y$ for all $x \in S$
2) $x \sim y \Rightarrow y \sim x \quad \forall x, y \in S$
3) $x \sim y$ and $y \sim z \Rightarrow x \sim z \quad \forall x, y, z \in S$.

For $x \in S$ define the equivalence class of $x$ by

$[x] = \{y \in S \mid y \sim x\}$.

**Path Components**

For $S \subset \mathbb{R}^n$, define an equivalence relation $\sim$ on $S$ by $x \sim y$ iff $\exists$ a path $\gamma$ from $x$ to $y$.

**Def:** A path component of $S$ is an equivalence class of of $\sim$.

**Prop:** If $S$ and $T$ are homeomorphic, they have the same # of path components (possibly infinite).

**Metric Spaces** All the topological concepts we've introduced so far generalize to metric spaces.

**Def:** A metric space is a pair $(S, d)$, where $S$ is a set and $d: S \times S \to [0, \infty)$ is a function such that

1) $d(x,y) = 0$ iff $x = y$
2) $d(x,y) = d(y,x)$ $\forall x, y \in S$
3) $d(x,z) \leq d(x,y) + d(x,z)$ $\forall x, y, z \in S$.