AMAT 342 Lecture 15

- Finish with RMSD
- Return | review exams
- Topology of metric spaces.

Recall: $RMSD: O^n \times O^n \to [0,\infty)$ is given by

$$RMSD(P,P') = \min_{\varphi \in E} \frac{1}{\sqrt{n}} d_2(V(P), V(\varphi(P'))).$$

ordinary Euclidean distance

where: O^n = Set of all length n ordered lists of points in $\mathbb{R}^3$.
E = set of all rigid motions of $\mathbb{R}^3$
$V: O^n \to \mathbb{R}^{3n}$ is given by

$$V\Big(\big((x_1,y_1,z_1),(x_2,y_2,z_2),\ldots,(x_n,y_n,z_n)\big)\Big)$$
$$=(x_1,y_1,z_1,x_2,y_2,z_2,\ldots,x_n,y_n,z_n).$$

RMSD satisfies properties 2 and 3 of a metric. [proof omitted]
Also $RMSD(P,P)=0$ $\forall$ $P \in O^n$, because $Id_{\mathbb{R}^3}: \mathbb{R}^n \to \mathbb{R}^n$ is a rigid motion.

But we can have $\mathrm{RMSD}(P,Q)=0$ but $P\neq Q$.

we saw an example of this last lecture.

$\Rightarrow$ property 1 of a metric does not hold.

Namely, let $Q=\varphi(P)$ for any rigid motion φ.

Then φ^{-1} is a rigid motion, and

$$d_2\left(V(P), V(\varphi^{-1}(Q))\right) = d_2\left(V(P), V(\varphi^{-1}(\varphi(P)))\right)$$
$$= d_2(V(P), V(P)) = 0$$

so $\mathrm{RMSD}(P,Q)=0$.

We'll modify RMSD to get a metric.

Define an equivalence relation $\sim$ on $\mathcal{O}^n$ by

$P\sim Q$ iff $\exists$ a rigid motion $\varphi:\mathbb{R}^3\to\mathbb{R}^3$ with $\varphi(P)=Q$.

<u>Fact</u>: $\mathrm{RMSD}(P,Q)=\mathrm{RMSD}(P',Q')$ if $P\sim P'$ and $Q\sim Q'$

(<u>Exercise</u>: Prove this).

As a consequence, $\mathrm{RMSD}: O^n \times O^n \to [0,\infty)$ descends to a genuine metric on $O^n/\sim$.

Specifically, we define

$$\overline{\mathrm{RMSD}} : O^n/\sim \times O^n/\sim \to [0,\infty) \text{ by}$$

$$\overline{\mathrm{RMSD}}([P],[Q]) = \mathrm{RMSD}(P,Q).$$

By the fact*, this function is well defined.

<u>Exercise</u>: Prove that $\overline{\mathrm{RMSD}}$ is a metric.

<u>Exam</u>: Total points = 33 (Excludes 1 exam taken a week later)
Mean = 23.1
Median = 22.25

<u>Curve</u>: Take the number of points you lost and multiply it by .43 That's the number of points you lost in the curved score.

curved mean ≈ 87.

Exam review

4. Prove that if $f: S \to T$ and $g: T \to U$ is a homeomorphism, then $g \circ f: S \to U$ is a homeomorphism.

Pf: Since f and g are homeomorphisms

f and g are continuous,

f^{-1} and g^{-1} are continuous,

$\Rightarrow g \circ f$ is continuous and $f^{-1} \circ g^{-1}$ is continuous.

(Composition of continuous functions is continuous).

Note that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

$$\left[\begin{array}{l} (g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ g^{-1} = Id_U. \\ (f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ f = Id_S. \end{array}\right]$$

Thus $g \circ f$ is invertible, hence a bijection.

6. Let

$$S = \{(1, y) \mid y \in I\} \subset \mathbb{R}^2$$

$$T = \{(2, y) \mid y \in I\} \subset \mathbb{R}^2$$

a. Sketch S and T

b. Give an explicit homeomorphism $f: S \to T$

$f(1,y) = (2,y)$.

c. What is f^{-1}? $f^{-1}(2,y) = (1,y)$.

d. Give an explicit expression for an isotopy

$h: S \times I \to \mathbb{R}^2$ from S to T

$h((1,y),t)) = (1+t, y)$

e. Is h invertible?

No. $\text{im}(h) = [1,2] \times I \Rightarrow h$ is not surjective

$\Rightarrow h$ is not a bijection

$\Rightarrow h$ is not invertible.

However, h can be "reversed" to give an isotopy $\overline{h}$ from T to S:

$\overline{h}: S \times I \to \mathbb{R}^2$, $\overline{h}((1,y),t) = h((1,y),(1-t)) = (1+(1-t),y)$

$= ((2-t), y)$

$\bar{h}$ is not an inverse of h.

7. Let T_k denote a subset of the unit circle obtained by removing k distinct points from S^1.

How many path components does T_k have?

<u>Ans</u>: k.

8. Prove that if $f: S \to T$ is an embedding then S and $\operatorname{im} f$ have the same # of path components.

<u>Pf</u>: f is an embedding means that $\tilde{f}: S \to \operatorname{im}(f)$ is a homeomorphism, where $\tilde{f}(x) = f(x)$ $\forall x$.

<u>Proposition</u>: homeomorphic spaces have the same # of path components.

$\Rightarrow$ S and $\operatorname{im}(f)$ have same # of path components

Metrics and topology

Metric space definition of continuity:

Let M and N be metric spaces with metrics d_M, d_N.

A function $f: M \to N$ is continuous at $x \in M$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$d_M(x,y) < \delta \Rightarrow d_N(f(x), f(y)) < \epsilon.$$

f is said to be continuous if it is continuous at each $x \in M$.

(This definition generalizes the definition for Euclidean subspaces considered earlier).

Example: Let M be any metric space and take N to be $\mathbb{R}$ with the Euclidean metric.

For any $x \in M$, the function $d^x: M \to \mathbb{R}$ given by $d^x(y) = d_M(x,y)$ is a continuous function.

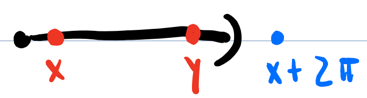
Pf: Exercise.

With this definition of continuity, the definition of homeomorphism extends immediately to metric spaces:

For metric spaces M and N, $f: M \to N$ is a <u>homeomorphism</u> if

1) f is a continuous bijection
2) f^{-1} is also continuous.

<u>Example</u>: Consider the metric d on $[0, 2\pi)$ given by $d(x,y) = \min\left(|x-y|,\ |(x+2\pi)-y)|,\ |(x-2\pi)-y|\right)$



take S^1 to have usual Euclidean metric

Then the function $f: ([0, 2\pi), d) \to S^1$ given by $f(t) = (\cos t, \sin t)$ is a homeomorphism.

The definition of isotopy also extends, but we'll not get into the details of this.

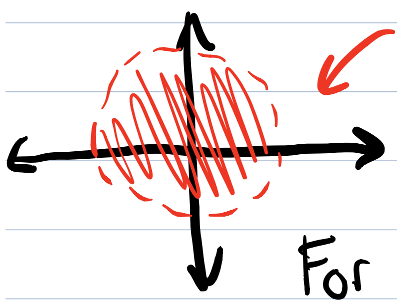
An alternate discription of continuity

Open Sets

Let M be a metric space. For $x \in M$ and $r > 0$, the open ball in M of radius r, centered at x, is the set

$$B(x,r) = \{y \in M \mid d_M(x,r) < r\}.$$

Example: For $M = \mathbb{R}^2$ with the Euclidean distance. $B(\vec{0},1)$ looks like this



disc of radius 1 centered at the origin, with the boundary not included.

For $M = \mathbb{R}$, the open ball of radius r centered at x is just the interval $(x-r, x+r)$.

A subset of M is called <u>open</u> if it is a union of (possibly infinitely many) open balls.

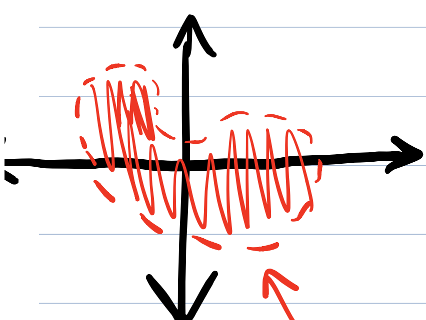
The <u>empty set</u> is always considered open.

M itself is open: $M = \bigcup_{x \in M} B(x,1)$

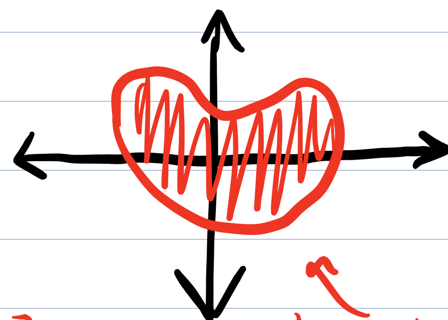
Fact: A region in $\mathbb{R}^n$ is open if it contains none of its boundary points.

this is an informal statement because I haven't defined "boundary points." It can be made formal, but I will not go into the details.

Illustration: Dashed line = boundary not included
Solid line = boundary included



open subset of $\mathbb{R}^2$

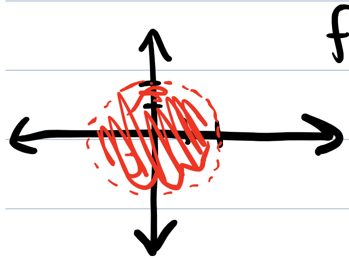


Subset which is not open.

Fundamental Fact: Whether a function of of metric spaces $f: M \to N$ is continuous depends only on the open sets of M, N and not otherwise on the metric! (this is made precise by the proposition below)

Notation: For $f: S \to T$ any function and $U \subset T$, $f^{-1}(U) = \{x \in S \mid f(x) \in U\}$.

Example: Let $f: \mathbb{R}^2 \to [0, \infty)$ be given by
$f(x) = d_2(x, 0)$.



$f^{-1}([0,2))$ = the open ball of radius 2 centered at 0.

Proposition: A function $f: M \to N$ of metric spaces is continuous if and only if $f^{-1}(U)$ is open for every open subset of N.

Proof: Exercise.