AMAT 342 Lec 16 10/24/19

Today
- Metric spaces and continuity
- The open-set perspective on continuity (point-set topology)

Plan for next few lectures
- Quotient Spaces (how to glue stuff together in topology)
- Manifolds (curves, surfaces)

Metrics and topology

Metric space definition of continuity.
Let $M=(S^M, d^M)$ and $N=(S^N, d^N)$ be metric spaces.

Def: A function $f: M \to N$ is simply a function with domain S^M and codomain S^N.

Def:
A function $f: M \to N$ is continuous at $x \in M$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$d_M(x,y) < \delta \Rightarrow d_N(f(x), f(y)) < \epsilon.$$

f is said to be continuous if it is continuous at each $x \in M$.

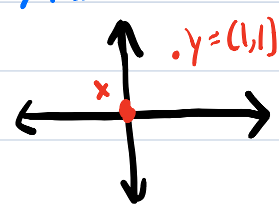
(This definition generalizes the definition for Euclidean subspaces considered earlier).

<u>Example</u>: Let M be a metric space w/ metric d and take N to be $\mathbb{R}$ with the Euclidean metric.

For any $x \in M$, the function $d^x : M \to \mathbb{R}$ given by $d^x(y) = d_M(x,y)$ is a continuous function.

<u>Pf</u>: Exercise.

For example, take $M = \mathbb{R}^2$, $d = d_2$, and $x = 0$.



$d^x((1,1)) = d((0,0),(1,1)) = \sqrt{2}$

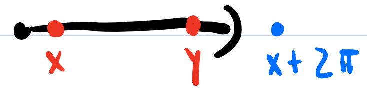
With this definition of continuity, the definition of homeomorphism extends immediately to metric spaces:

For metric spaces M and N, $f: M \to N$ is a <u>homeomorphism</u> if

1) f is a continuous bijection
2) f^{-1} is also continuous.

<u>Example</u>: Consider the metric d on $[0, 2\pi)$ given

by $d(x,y) = \min\left(|x-y|, |(x+2\pi)-y|, |(x-2\pi)-y|\right)$



take S^1 to have usual Euclidean metric

Then the function $f: ([0, 2\pi), d) \to S^1$ given by $f(t) = (\cos t, \sin t)$ is a homeomorphism.

(we won't bother proving this)

Example* In the notation of last lec., $M = \left(O^2/\sim, \overline{RMSD}\right)$ is homeomorphic to $N = ([0,\infty), d_2)$.

A very abstract example!

For example, $f: M \to N$ given by
$$f([(a,b)]) = \tfrac{1}{2} d_2(a,b)$$
a homeomorphism.

Partial explanation:
A function $f: M \to N$ of metric spaces is called an isometry if $d^M(a,b) = d^N(f(a), f(b))$. That is, an isotopy "preserves the metric."

Fact: Any isometry is a homeomorphism
Proof: exercise

—

In the example ★ above, f is an isometry. To prove this one derives an explicit expression for RMSD on O^2:

$$\mathrm{RMSD}\big((a,b),(c,d)\big) = \frac{|d_2(a,b)-d_2(b,c)|}{2}$$ ← proof is a bit of a pain

So far, we've talked about extending the def. of homeomorphism to metric spaces. The other basic topological notions we've considered, like

- homotopy
- embeddings
- isotopy
- path components

also extend readily.

One subtlety: If M is a metric space, is $M \times I$ a metric space?

To be discussed later, perhaps

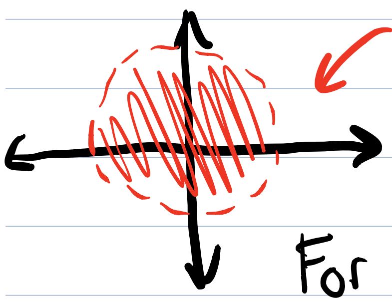
An alternate discription of continuity

Open Sets

Let M be a metric space. For $x \in M$ and $r > 0$, the open ball in M of radius r, centered at x, is the set

$$B(x,r) = \{y \in M \mid d_M(x,y) < r\}.$$

Example: For $M = \mathbb{R}^2$ with the Euclidean distance.
$B(\vec{0}, 1)$ looks like this



disc of radius 1 centered at the origin, with the boundary not included.

For $M = \mathbb{R}$, the open ball of radius r centered at x is just the interval $(x-r, x+r)$.

A subset of M is called <u>open</u> if it is a union of (possibly infinitely many) open balls.

The <u>empty set</u> is always considered open.

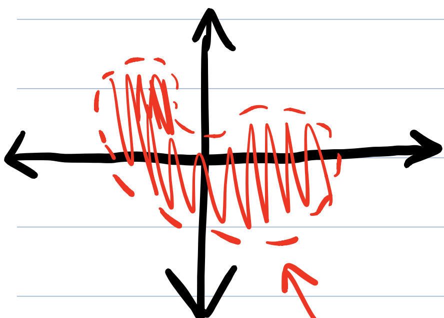
M itself is open: $M = \bigcup_{x \in M} B(x,1)$

<u>Fact</u>: A subset of M is open if it contains none of its $\underbrace{\text{boundary points}}$.

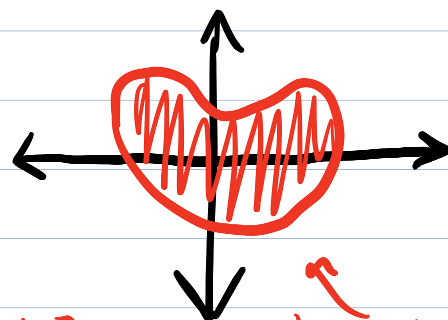
this is an informal statement because

I haven't defined boundary points.
It can be made formal, but I will not go into the details.

Illustration: Dashed line = boundary not included
Solid line = boundary included



open subset of $\mathbb{R}^2$

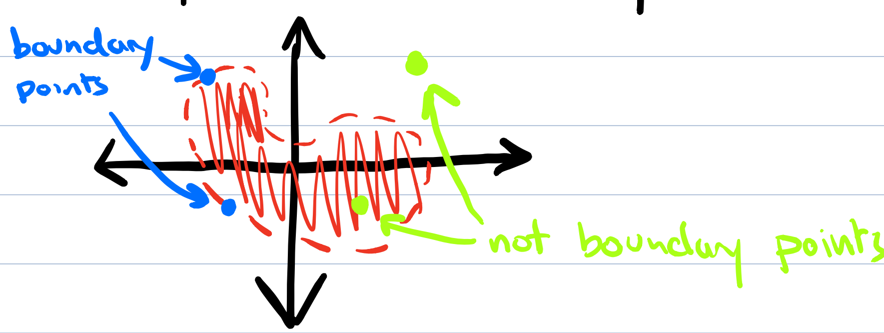


subset which is not open.

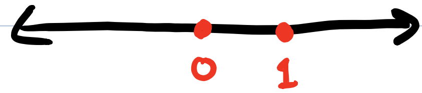
Let's make this precise:

I'll start with the Euclidean case, for concreteness.

For S a subset of M, a **boundary point** of S is a point $x \in M$ such that every open ball around x contains a point in S and a point not in S.



Exercise: What are the boundary points of $I = [0,1] \subset \mathbb{R}$?



Exercise: What are the boundary points of

$$D = \{(x,y) \mid x^2+y^2 \le 1\} \subseteq \mathbb{R}^2 ?$$

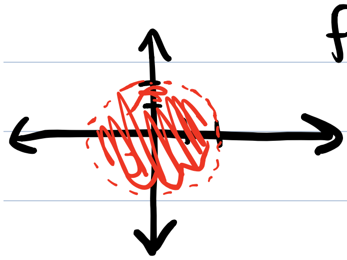
Exercise: What are the boundary points of S^1?

Note: The definition of boundary point in fact makes sense for a subset of any metric space.

Fundamental Fact: Whether a function of of metric spaces $f: M \to N$ is continuous depends only on the open sets of M, N and not otherwise on the metric! (this is made precise by the proposition below)

Notation: For $f: S \to T$ any function and $U \subset T$, $f^{-1}(U) = \{x \in S \mid f(x) \in U\}$.

Example: Let $f: \mathbb{R}^2 \to [0,\infty)$ be given by $f(x) = d_2(x,0)$.



$f^{-1}([0,2))$ = the open ball of radius 2 centered at O.

<u>Proposition</u>: A function $f: M \to N$ of metric spaces is continuous if and only if $f^{-1}(U)$ is open for every open subset of N.

<u>Proof</u>: Exercise.

<u>Philosophical</u> <u>implications</u>:

- In topology, we study geometric objects via the continuous functions between them.

(The continuous functions are what matter in topology)

- Thus, in view of the proposition, the specific choice of metric on a metric space matters topologically only insofar as this determines the open subsets of the metric space.

This motivates the following definition:

<u>Def</u>:
Two metrics d_1 and d_2 on a set S are called <u>topologically equivalent</u> if

(S, d_1) and (S, d_2) have the same open sets.

<u>Interpretation</u>: Topologically equivalent metrics look the same through the lens of topology.

<u>Note</u>: Examples of topologically equivalent metrics are <u>common</u>.

<u>Fact</u>: If there are positive constants $0 < \alpha, \beta$ such that $\forall x, y \in S$,
$\alpha d_1(x,y) \leq d_2(x,y) \leq \beta d_2(x,y)$, then d_1 and d_2 are topologically equivalent.

<u>Example</u>: Recall that for $p \in [1, \infty)$, we defined the metric

d_p on $\mathbb{R}^n$ by $d_p(x,y) = \sqrt[p]{\sum_{i=1}^{n} |x_i - y_i|^p}$.

<u>Well known fact</u>: For all $p, q \in [1, \infty)$, d_p and d_q are topologically equivalent.

<u>Example</u>: The intristic and extrinsic metric on

S^1 are topologically equivalent.

Metrics on Cartesian Products

Let M and N be metric spaces with metrics d_M d_N.

How do I define a metric d on $M \times N$?

Motivation:

To extend the definition of homotopy to metric spaces, we need to talk about a continuous function $h: M \times I \to N$ where M, N are metric spaces. But then we need a metric structure on $M \times I$.

There are multiple options, e.g.

- $d((m_1, n_1), (m_2, n_2)) = d_M(m_1, m_2) + d_N(n_1, n_2)$

  $d((m_1, n_1), (m_2, n_2)) = \max(d_M(m_1, m_2) + d_N(n_1, n_2))$

But it turns out that these are topologically equivalent!