AMAT 342 Lecture 17, 10/29/19

Today: Open sets and continuity, continued
Gluing

Review:

Let $M=(S^M, d^M)$ be a metric space.

convention: we abuse notation slightly and write S^M simply as M, where convenient. So for example, we may write $x \in M$.

For $x \in M$ and $r > 0$, the open ball in M of radius r, centered at x, is the set

$$B(x,r) = \{y \in M \mid d_M(x,y) < r\}.$$

A subset of M is called open if it is a union of (possibly infinitely many) open balls.

Notation for unions: Let A be set and suppose that for each $x \in A$, I have a set S_x. Then we write the union of all the sets S_x as $\bigcup_{x \in A} S_x$.

Example: $A = \{1, 2\}$. $S_1 = [1,2]$, $S_2 = [2,3]$

$\bigcup_{x \in \{1,2\}} S_x = S_1 \cup S_2 = [1,3]$.

2 important examples of empty sets:

The <u>empty</u> <u>set</u> is always considered open.

M itself is open: $M = \underbrace{\bigcup_{x \in M} B(x,1)}$

this denotes the union
of all $B(x,1)$ where $x \in M$.

Explanation: $B(x,1) \subset M$ for all $x \in M$, so
$\bigcup_{x \in M} B(x,1) \subset M$.

On the other hand, for any $y \in M$,
$y \in B(y,1) \subset \bigcup_{x \in M} B(x,1)$,

so $M \subset \bigcup_{x \in M} B(x,1)$.

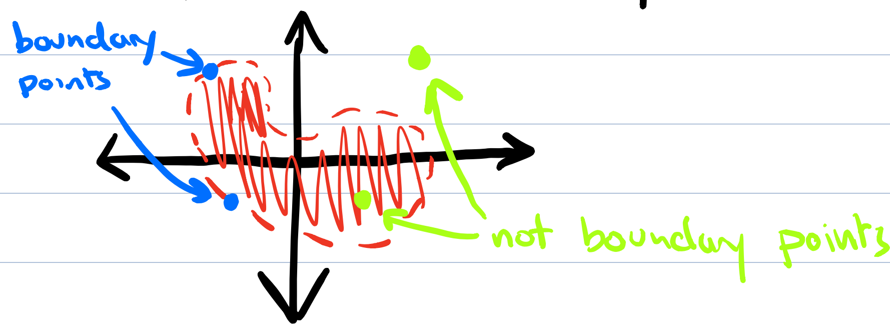
Since $\bigcup_{x \in M} B(x,1) \subset M$ and

$M \subset \bigcup_{x \in M} B(x,1)$,

the two sets are equal.

Definition:

For S a subset of M, a boundary point of S is a point $x \in M$ such that every open ball around x contains a point in S and a point not in S.



Proposition: A subset of M is open iff it contains none of its $\underbrace{\text{boundary points}}$.

Proof is a homework exercise.

Hint: First show that for any open ball $B(x,r) \subset M$ $y \in B(x,r)$, $B(y,r') \subset B(x,r)$ for some $r' > 0$.

This hint is of interest beyond the proof of the proposition.

Proof: Let $d = d_M(x,y)$. Then $\boxed{0 \leq d < r}$, where the second inequality holds because $y \in B(x,r)$.

Let $r' = r - d$, which implies $d = r - r'$

Thus the inequalities in the pink box yield

$$0 \le r - r' < r \;\underset{\substack{\text{multiply}\\\text{everything}\\\text{by } -1}}{\Rightarrow}\; -r < r' - r \le 0 \;\underset{\substack{\text{add}\\ r}}{\Rightarrow}\; 0 < r' \le r$$

Suppose $z \in B(y, r')$. Then by the triangle inequality,

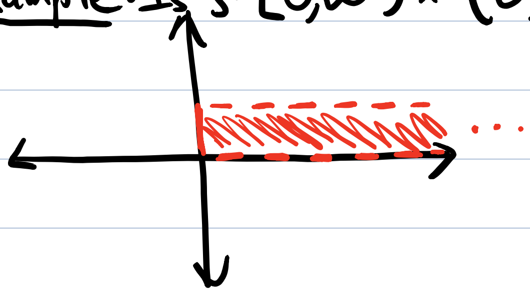
$$\begin{aligned} d_M(x,z) &\le d_M(x,y) + d_M(y,z) \\ &= d + d_M(y,z) \\ &\le d + r' \\ &= r - r' + r' \\ &= r. \end{aligned}$$

Thus $z \in B(x, r)$. This shows that $B(y,r') \subset B(x,r)$, as claimed.

<u>Consequence of hint</u>: If $U \subset M$ is open, then for any $x \in U$, $B(x,r) \subset U$ for some $r > 0$.

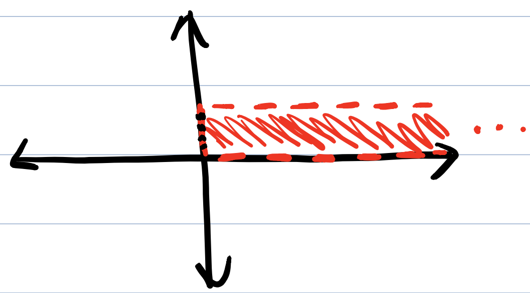
<u>Proof</u>: If $x \in U$, then since U is a union of open balls, $x \in B(y, r')$ for some open ball $B(y,r') \subset U$. By the hint, $B(x,r) \subset B(y,r') \subset U$. ∎

Example: Is $S = [0,\infty) \times (0,1)$ open?



Ans: No. For example, $(0, \frac{1}{2})$ is a boundary point of S.

Example: $(0,\infty) \times (0,1)$ is open.



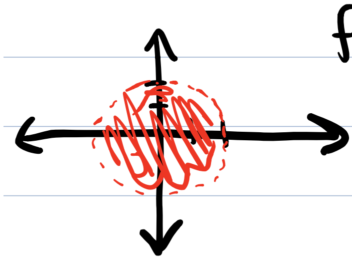
Fundamental Fact: Whether a function of of metric spaces $f: M \to N$ is continuous depends only on the open sets of M, N and not otherwise on the metric! (this is made precise by the proposition below)

To explain this, we need one new bit of set theory notation.

Notation: For $f: S \to T$ any function and $U \subset T$, $f^{-1}(U) = \{x \in S \mid f(x) \in U\}$.

In words, $f^{-1}(U)$ is the set of elements in S that map to U.

Example: Let $f: \mathbb{R}^2 \to [0,\infty)$ be given by
$f(x) = d_2(x,0) = \sqrt{x_1^2 + x_2^2}$



$f^{-1}([0,2))$ = the open ball of radius 2 centered at 0.

$f^{-1}(\{1\}) = S^1$

Proposition: A function $f: M \to N$ of metric spaces is continuous if and only if $f^{-1}(U)$ is open for every open subset of N.

Proof: Assume f is continuous, let U be an open set in N, and let x be an element of $f^{-1}(U)$. To show that $f^{-1}(U)$ is open, it suffices to show that $B(x,r) \subset f^{-1}(U)$ for some $r > 0$.
Since $x \in f^{-1}(U)$, $f(x) \in U$. Since U is open, it follows from the consequence of the hint above that U contains some open ball $B(f(x), \epsilon)$. By continuity, $\exists\, \delta > 0$ such that if $d_M(y,x) < \delta$, then $d_N(f(x), f(y)) < \epsilon$.
Thus f maps each point of $B(x,\delta)$ to a point in $B(f(x),\epsilon) \subset U$.
Therefore $B(x,\delta) \subset f^{-1}(U)$. Thus $f^{-1}(U)$ is open.

Conversely, assume $f^{-1}(U)$ is open for every open subset of N, and let $x \in M$. We show that f is continuous at x. Let $\epsilon > 0$. $B(f(x), \epsilon)$ is open, so $f^{-1}(B(f(x)), \epsilon)$ is also open. Let us call this set V. Clearly $x \in V$. Thus by the consequence of the hint, $B(x, \delta) \subset V$ for some $\delta > 0$. We then have that for any $y \in B(x, \delta)$, $f(y) \in B(f(x), \epsilon)$, i.e. if $y \in M$ is such that $d_M(x,y) < \delta$, then $d_N(f(x), f(y)) < \epsilon$. This shows that f is continuous at x. ∎

Philosophical implications:

- In topology, we study geometric objects via the continuous functions between them.

(The continuous functions are what matter in topology)

- Thus, in view of the proposition, the specific choice of metric on a metric space matters topologically only insofar as this determines the open subsets of the metric space.

This motivates the following definition:

<u>Def</u>:
Two metrics d_1 and d_2 on a set S are called <u>topologically</u> <u>equivalent</u> if

(S, d_1) and (S, d_2) have the same open sets.

<u>Interpretion</u>: Topologically equivalent metrics look the same through the lens of topology.

<u>Note</u>: Examples of topologically equivalent metrics are <u>common</u>.

<u>Fact</u>: If there are positive constants $0 < \alpha, \beta$ such that $\forall x, y \in S$,
$\alpha d_1(x,y) \leq d_2(x,y) \leq \beta d_2(x,y)$, then d_1 and d_2 are topologically equivalent.

<u>Example</u>: Recall that we defined several metrics on $\mathbb{R}^n$: d_2, d_1, and d_{max}.

By the fact, these are topologically equivalent.

In fact, $d_2 \le d_1 < \sqrt{n}\, d_2$,
$d_{max} \le d_1 \le n\, d_2$.

Straightforward exercise.

Lecture ended here.

Example: The intristic an extrinsic metric on S^1 are topologically equivalent.

In fact, $d_{ext} \le d_{int} \le \frac{\pi}{2} d_{ext}$.

Metrics on Cartesian Products

Let M and N be metric spaces with metrics d_M d_N.

How do I define a metric d on $M \times N$?

Motivation:

To extend the definition of homotopy to metric spaces, we need to talk about a continuous function $h: M \times I \to N$ where M, N are metric spaces. But then we need a metric structure on $M \times I$.

There are multiple options, e.g.

- $d((m_1, n_1), (m_2, n_2)) = d_M(m_1, m_2) + d_N(n_1, n_2)$

  $d((m_1, n_1), (m_2, n_2)) = \max(d_M(m_1, m_2) + d_N(n_1, n_2))$

But it turns out that these are <u>topologically</u> <u>equivalent</u>!

Topology without Metrics (Abstract Formulation of Topology)

Idea: If all that matters in topology is the open sets of a metric space, then perhaps it would be simpler to give the basic definitions of topology without mentioning metrics at all.

Key properties of open subsets of a metric space M

1) Union of open sets is open
2) Intersection of finitely many open sets is open
3) M is open
4) $\emptyset$ is open.

The only one of these properties that not already clear is the second one. This is an easy exercise.

Lets use these facts as inspiration to make a definition

Definition: A topological space is a pair (T, O), where:

- T is a set
- O is a collection of subsets of T.

Note That

called the <u>open</u> <u>sets</u>, satisfying properties 1)-4) above

There is no mention of a metric here.

<u>Definition</u>: A function $f: X \to Y$ between topological spaces is called <u>continuous</u> if $f^{-1}(U)$ is open whenever U is open.

<u>Example</u>: For any metric space (S, d), call the collection of open sets $\mathcal{O}$. Then $(S, \mathcal{O})$ is a topological space.

<u>Note</u>: Most examples of topological spaces arise via metrics, as in the above example. But not all do.

From now on, we use the language of topological spaces, but for concreteness, you can think of metric spaces, or subsets of Euclidean spaces.