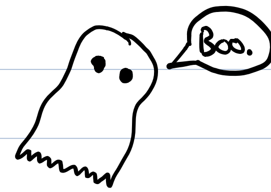
AMAT 342 Lecture 18 10/31/19



Today: Topologically equivalent metric spaces
Topological spaces
Gluing

Review

Proposition: A function $f: M \to N$ of metric spaces is continuous if and only if $f^{-1}(U)$ is open for every open subset of N.

For the proof, see the notes from Lec. 17.

Upshot: The structure of a metric matters topologically only by way of the collection of open sets it determines.

This motivates the following definition:

Def:
Two metrics d_1 and d_2 on a set S are called topologically equivalent if

(S, d_1) and (S, d_2) have the same open sets.

Interpretation: Topologically equivalent metrics look the same through the lens of topology.

Note: Examples of topologically equivalent metrics are common.

Fact: If there are positive constants $0 < \alpha, \beta$ such that $\forall x,y \in S$,
$\alpha\, d_1(x,y) \leq d_2(x,y) \leq \beta\, d_1(x,y)$, then d_1 and d_2 are topologically equivalent.

Example: Recall that we defined several metrics on $\mathbb{R}^n$: d_2, d_1, and d_{max}.

By the fact, these are topologically equivalent.
In fact, $d_2 \leq d_1 \leq \sqrt{n}\, d_2$,
$d_{max} \leq d_1 \leq n\, d_{max}$ } Straightforward exercise.

Lecture ended here.

Note: When I write $d_2 \leq d_1$, e.g., I mean that $d_2(x,y) \leq d_1(x,y)$ $\forall x,y \in \mathbb{R}^n$. And similarly for the other inequalities here.

Let's prove that $d_2 \le d_1 \le \sqrt{n}\, d_2$:

$d_2(x,y) \le d_1(x,y)$ iff

$d_2(x,y)^2 \le d_1(x,y)^2 \le n\, d_2^2$.

$$d_2(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

$\Rightarrow$

$$d_2(x,y)^2 = \sum_{i=1}^{n} (x_i - y_i)^2$$

$$d_1(x,y)^2 = \left(\sum_{i=1} |x_i - y_i|^2\right)^2 =$$

$$\sum_{i=1} (x_i - y_i)^2 + \text{other non-negative terms}$$

so $d_2(x,y)^2 \le d_1(x,y)^2$.

The proof that $d_1(x,y) \le n\, d_2$ is an application of a standard result from linear algebra called the Cauchy-Schwartz inequality, which says that $\forall\, v, w \in \mathbb{R}^n$

$\left| \sum_i v_i w_i \right|^2 \le d_2(v,0)^2 \cdot d_2(w,0)^2$. Applying this with $v = (|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|)$

$w = (1, 1, \ldots, 1)$ gives the result.

The proof that $d_{max} \le d_1 \le n\, d_{max}$ is just as easy. I will skip it.

Example: The intristic metric d_{int} and extrinsic metric d_2 on S^1 are topologically equivalent.

In fact, $d_2 \le d_{int} \le \frac{\pi}{2} d_2$.

Metrics on Cartesian Products

Let M and N be metric spaces with metrics d_M, d_N.

How do I define a metric d on $M \times N$?

Motivation:

To extend the definition of homotopy to metric spaces, we need to talk about a continuous function $h: M \times I \to N$ where M, N are metric spaces. But for that we "need" a metric structure on $M \times I$.

as we'll explain shortly, we need slightly less.

There are multiple options, e.g.

- $d((m_1,n_1),(m_2,n_2)) = d_M(m_1,m_2) + d_N(n_1,n_2)$
  $d((m_1,n_1),(m_2,n_2)) = \max(d_M(m_1,m_2) + d_N(n_1,n_2))$
  $d((m_1,n_1),(m_2,n_2)) = \sqrt{d_M(m_1,m_2)^2 + d_N(n_1,n_2)^2}$

But it turns out that these are topologically equivalent!

Topology without Metrics (Abstract Formulation of Topology)

Idea: If all that matters in topology is the open sets of a metric space, then we should formulate the basic definitions of topology without mentioning metrics at all!

Intersections: For any sets S and T, the intersection of S and T, denoted $S \cap T$, is the set consisting of all elements contained in **both** S and T.

Example: $S = \{1, 2, 3\}$ $T = \{2, 3, 4\}$

$S \cap T = \{2, 3\}$.

More generally, for sets $S_1, S_2, \ldots S_n$, we can define the intersection $S_1 \cap S_2 \cap \cdots \cap S_n$.

Key properties of open subsets of a metric space M

1) Union of open sets is open
2) Intersection of finitely many open sets is open
3) M is open
4) $\emptyset$ is open.

The only one of these properties that not already clear is the second one. This is an easy exercise.

Proof of property 2): Consider open sets $U_1, \dots, U_n \subset M$.
Let $U = U_1 \cap U_2 \cap \cdots \cap U_n$.
By the "consequence of the hint" from last lecture, if $x \in U$, then $B(x, r_i) \subset U_i$ for some $r_i > 0$. Thus, letting $r_x = \min(r_1, \dots, r_n)$, we have $B(x, r_x) \subset U$. Since every point of U is contained in an open ball in U, U is open. ∎

Let's use these properties as inspiration for a definition

Definition: A topological space is a pair (T, O^T), where:

- T is a set
- O^T is a collection of subsets of T, called the open sets, satisfying properties 1)-4) above

Note that there is no mention of a metric here.

We call O^T a topology on T, or a topological structure on T.

Example: For any metric space (S, d), call the collection of open sets O. Then (S, O) is a topological space. We call O the metric topology.

Example: Let T be a set and O^T = All subsets of T. Then (T, O^T) is a topological space. O^T is called the discrete topology. Note: Every subset is open in the discrete topology!

For example, Let $S \subset \mathbb{R}^n$ be finite. Regard S as a metric space. Let d_{min} be the minimum distance between two distinct points in S. For each $x \in S$, $B(x, d_{min}) \subset S$ is equal to $\{x\}$. Thus, any subset of S is a union of open balls, hence is open in S. So the metric topology is discrete.

Note: Most examples of topological spaces one sees via metrics, as in the above example. But not all do.

Definition: A function $f: X \to Y$ between topological spaces is called <u>continuous</u> if $f^{-1}(U)$ is open whenever U is open.

Example: If X has the discrete topology, then since every subset of X is open, $f^{-1}(U)$ is always open. So then for any topological space f and function $f: X \to Y$, f is continuous!

From now on, we use the language of topological spaces, but for concreteness, you can think of metric spaces, or subsets of Euclidean spaces.

Subspace topology We've seen that if M is a metric space and $S \subset M$ is any subset, then S inherits the structure of a metric space from M.

If the theory of topological spaces is to generalize the theory of metric spaces, something analogous should be true for topological spaces.

Definition:
Let $(T, \mathcal{O}^T)$ be a topological space and $S \subset T$ be any subset. Define the *subspace topology* $\mathcal{O}^S$ on S by

$$\mathcal{O}^S = \{ U \cap S \mid U \in \mathcal{O}^T \}.$$

Proposition: $(S, \mathcal{O}^S)$ is a topological space

this follows easily from elementary facts about unions and intersections of sets

The next definition provides some justification for the definition of subspace topology.

Proposition: Let M be a metric space, let S be a subset of M, regarded as a metric space, and let $\mathcal{O}^M$ and $\mathcal{O}^S$ denote the respective subspace topologies. Then $\mathcal{O}^S$ is the subspace topology.

Proof: exercise.

From now on, subsets of topological spaces will be understood to be topological spaces, with the subspace topology.

Product topology

Let $X = (S^X, O^X)$ and $Y = (S^Y, O^Y)$ be topological spaces.

The product space $X \times Y$ is the topological space w/ underlying set $S^X \times S^Y$ and $U \subset S^X \times S^Y$ open iff

U is a union of sets of the form

$U \times V$, where $U \subset O^X$ and $V \subset O^Y$.

Note: We discussed different ways to put a metric on a cartesian product of metric spaces.

Each way we discussed yields the product topology. That's the motivation for this definition.