AMAT 342 Lec. 19 11/5/2019

Today: Topological Spaces (continued)

- subspace topology
- product topology
- gluing (quotient spaces)

two constructions that allow us to reformulate everything we've seen about topology in a metric-free way.

Review

Proposition: A function $f: M \to N$ of metric spaces is continuous (in the ϵ-δ) sense if and only if $f^{-1}(U)$ is open for every open set U in N.

$\Rightarrow$ In topology, the metric matters only via the collection of open sets it generates.

Moreover, bothering with the extra structure of the metric when we just care about the open sets can be burdensome, making definitions and proofs more complicated than necessary.

This motivates us to develop topology in a metric-free way.

Definition: A <u>topological space</u> is a set T, together with a collection $\mathcal{O}^T$ of subsets of T such that:

1) arbitrary unions of elements of $\mathcal{O}^T$ are in $\mathcal{O}^T$
2) finite intersections of elements of $\mathcal{O}^T$ are in $\mathcal{O}^T$.
3) $T \in \mathcal{O}$
4) $\phi \in \mathcal{O}$.

$\mathcal{O}$ is called a <u>topology</u> on T or a <u>topological structure</u>.

Definition: A function $f: T \to T'$ of topological spaces is <u>continuous</u> if $f^{-1}(U)$ is open for each open set in T'.

<u>Example</u>: For any metric space (S, d), let $\mathcal{O}^d$ denote the collection of open sets. $(S, \mathcal{O}^d)$ is a topological space. We call $\mathcal{O}^d$ the <u>metric topology</u>.

<u>Note</u>: Most but not all topological spaces one encounters arise from a metric.

Example: Let T be a set and $\mathcal{O}^T$ = All subsets of T. Then $(T, \mathcal{O}^T)$ is a topological space. $\mathcal{O}^T$ is called the discrete topology, and we say that $(T, \mathcal{O}^T)$ is discrete.

Proposition: a topological space T is discrete if and only if the singleton set $\{x\}$ is open for all $x \in T$.

Pf: If T is discrete, then every subset is open, so $\{x\}$ is open. Conversely, for any $S \subset T$, $S = \bigcup_{x \in S} \{x\}$, so if $\{x\}$ is open, then S is open by property 1) of a topological space. ∎

Example: For M any finite metric space, the metric topology is discrete. for instance M could be a finite set in $\mathbb{R}^n$, with the Euclidean metric.

Explanation: Let r denote the minimum distance between two distinct points in M. Then $\forall x \in M$, $B(x, r/2) = \{x\}$, Since every singleton set $\{x\} \in M$ is open in the metric topology, This topology is discrete.

Remark: The discrete topology on any set <ins>always</ins> arises from a metric: For T any set, consider the metric d on T given by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Then for all $x \in T$, $B(x, \frac{1}{2}) = \{x\}$, so $\{x\}$ is open in the metric topology. Thus, by the prop., the metric topology on T is discrete.

Exercise: Suppose d and d' are topologically equivalent metrics on a set T and the metric topology of d is discrete. Which sets $S \subset T$ are open in the metric topology of d'?

Answer: All subsets $S \subset T$. Since the metric topology of d is discrete, $\mathcal{O}^d$ contains all subsets of T. But $\mathcal{O}^d = \mathcal{O}^{d'}$ because d and d' are topologically equivalent. So the metric topology of d' is discrete.

Example (the <ins>indiscrete topology</ins>, a.k.a. the <ins>trivial topology</ins>) For any set T, let $\mathcal{O}^T = \{\emptyset, T\}$. Then $(T, \mathcal{O}^T)$ is a topological space.

Prop:
If T has more than one point, the trivial topology is not the metric topology of any metric.

Proof: Let d be a metric on T and consider $x, y \in T$, $x \neq y$. Let $r = d(x,y)$. Then $B(x, \frac{r}{2})$ contains x but not y, so $B(x, \frac{r}{2})$ is neither $\emptyset$ nor T. ∎

The trivial topology is a rather artificial example. I introduce it mostly to give you a feel for the definition of topological space, and to show that this definition is general enough to allow for some wierd stuff.

Subspace topology We've seen that if M is a metric space and $S \subset M$ is any subset, then S inherits the structure of a metric space from M.

If the theory of topological spaces is to generalize the theory of metric spaces, something analogous should be true for topological spaces.

Definition:
Let $(T, \mathcal{O}^T)$ be a topological space and $S \subset T$ be any subset. Define the subspace topology $\mathcal{O}^S$ on S by

$$\mathcal{O}^S = \{ U \cap S \mid U \in \mathcal{O}^T \}.$$

Proposition: $(S, \mathcal{O}^S)$ is a topological space

this follows easily from elementary facts about unions and intersections of sets

The next definition provides some justification for the definition of subspace topology.

Proposition: Let M be a metric space, and let S be a subset of M, regared as a metric space. Then the metric topology on S is same as the subspace topology on S (with respect to the metric topology on M).

Proof: Exercise. In otherwerds, for metric spaces, the subspace topology is what you expect it to be.

From now on, subsets of topological spaces will be understood to be topological spaces, with the subspace topology.

Example: Consider the metric space $\mathbb{R}$, with the Euclidean metric. $[0,1)$ is not open in $\mathbb{R}$, because it contains the boundary point 0.

Let $S = [0,\infty)$. Regarding S as a metric space with the restriction of the Euclidean metric, we have $[0,1) = \underbrace{B(0,1)}_{\text{open ball in } S}$, so $[0,1)$ is open in the metric topology on S. On the other hand, $[0,1) = (-1,1) \cap S$. $(-1,1)$ is open in $\mathbb{R}$, so $[0,1)$ is open in the subspace topology on S, as guaranteed by the proposition.

Product topology

Let $X = (S^X, \mathcal{O}^X)$ and $Y = (S^Y, \mathcal{O}^Y)$ be topological spaces. The product space $X \times Y$ is the topological space w/ underlying set $S^X \times S^Y$ and $U \subset S^X \times S^Y$ open iff

U is a union of sets of the form

$U \times V$, where $U \subset \mathcal{O}^X$ and $V \subset \mathcal{O}^Y$.

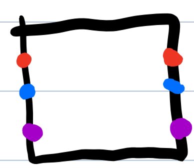
Note: We discussed different ways to put a metric on a cartesian product of metric spaces.

Each way we discussed yields the product topology. That's the motivation for this definition.

<u>Example</u>: The product topology on $\mathbb{R}\times\mathbb{R}=\mathbb{R}^2$ is the same as the metric topology on $\mathbb{R}^2$ w.r.t. the Euclidean metric

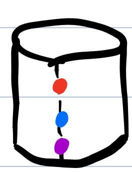
<u>Gluing (Quotient Spaces)</u>

Consider the square $I\times I$



Thinking of this as a piece of rubber, suppose we glue the left edge to the right edge, i.e., gluing $(0,y)$ to $(1,y)$ for all $y\in I$

We get a cylinder:



How do we model such gluing mathematically?
If the unglued object has a metric,
we can put a metric on the glued object, but this is awkward.
It's much cleaner to work abstractly, with topogical spaces.

The starting point for this is the following idea:
For any continuous surjection $f: T \to S$ we can always think of S as obtained from T by gluing.
Namely, $x, y \in T$ get glued together iff $f(x) = f(y)$.

So a gluing construction on S should involve constructing a space T and a continuous surjection
$\pi: S \to T$ so that $\pi(x) = \pi(y)$ if and only if we want to glue x to y.

To be continued...