AMAT 342 8/29/19 (Lecture 2)

Last time, we started a review of sets and functions, and today we'll continue this.

Plan for today / next week

- Finish reviewing sets/functions
- Continuity
- Isotopy, homeomorphism
- Material from Carlson Ch. 1.

Sets + functions, Continued

Subsets We say a set S is a subset of a set T, and write $S \subset T$, if every element of S is an element of T.

(otherwise, we write $S \not\subset T$.)

Example: $\{0,1\} \subset \{0,1,2\}$, but $\{0,1,2\} \not\subset \{0,1\}$ doesn't contain 2.

Example: $\underbrace{\mathbb{N}}_{\substack{\text{natural numbers} \\ \{0,1,2,\ldots\}}} \subset \underbrace{\mathbb{Z}}_{\text{integers}} \subset \underbrace{\mathbb{R}}_{\text{real numbers}}$

Question: Is a set S a subset of itself?

Ans: Yes, by definition, because every element of S is an element of S!

Example: The empty set $\emptyset$ is a subset of any set S.

$\emptyset$ has no elements, so this is vacuously true.

Exercise: What are all the subsets of $\{1,2\}$?

Specifying a Subset via a Property

We sometimes specify a set via a property satisfied by its elements.

For example, let E denote the even integers. We can write

$$E = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$$

but we can also say:

"E is the set of integers x such that x is divisible by 2.

We can write this in symbols as:

$$E = \{x \in \mathbb{Z} \mid x \text{ is divisible by } 2\}.$$

The symbol "$|$" translates to "such that."

Note: Sometimes "$:$" is used in place of "$|$", e.g.

$$\{x \in \mathbb{Z} : x \text{ is divisible by } 2\}.$$

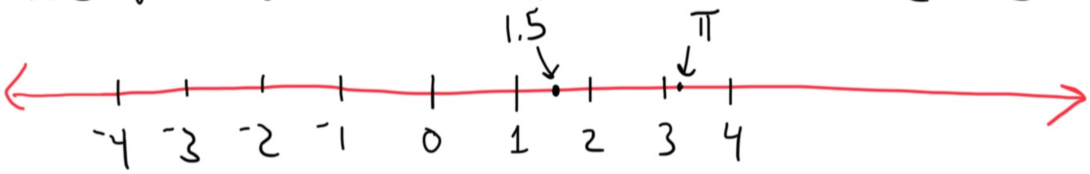
Exercise: Use the notation introduced above to give an expression for the set of prime numbers as a subset of $\mathbb{N}$.

Solution: The primes are the set

$$\{x \in \mathbb{N} \mid x \geq 2 \text{ and } x \text{ is divisible only by } 1 \text{ and } x\}.$$

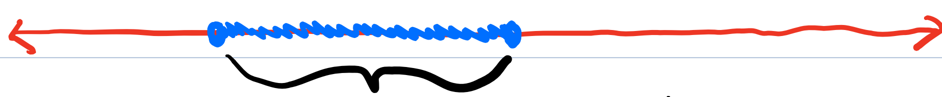
Intervals are subsets of $\mathbb{R}$ that are particularly important in topology.

We visualize the set $\mathbb{R}$ as an infinite line:

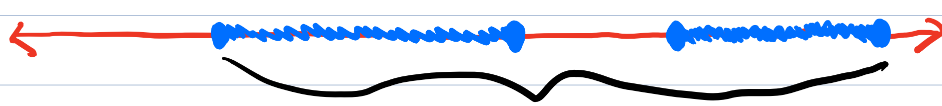


Points on the line are real numbers.

Definition: An **interval** is a non-empty subset of $\mathbb{R}$ consisting of a single connected component.



an interval (in blue)



not an interval

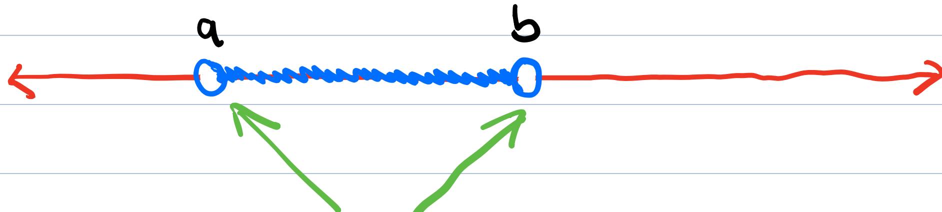
Note: We haven't formally defined connected component, but the intuitive meaning should be clear. Later, we will give a formal definition.

There are 9 different types of intervals,
Let $a, b \in \mathbb{R}$ with $a < b$.

4 with finite endpoints:

Open interval:

$$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}.$$

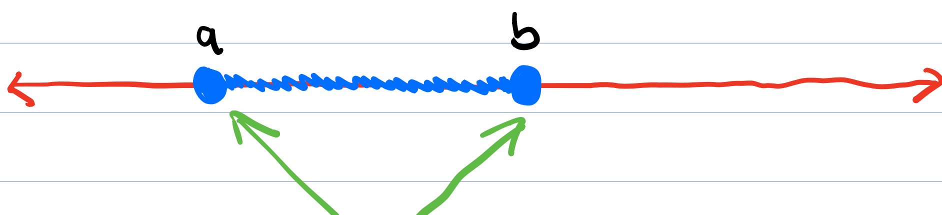


the empty circles indicate that a and b are not in the interval.

(In words, (a,b) is the set of all real numbers greater than a and less than b.)

Closed interval:

$$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

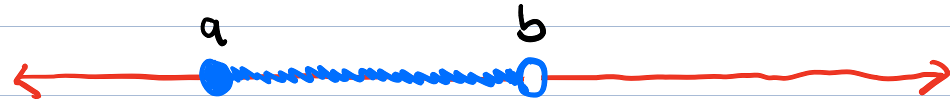


the filled-in circles indicate that a and b are in the interval.

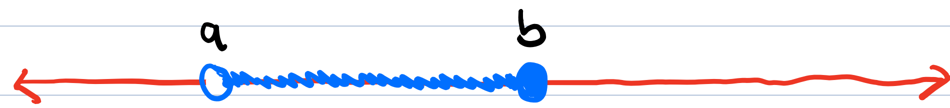
Aside:
By convention, $[a,a] = \{a\}$.

Half-open intervals:

$$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\},$$

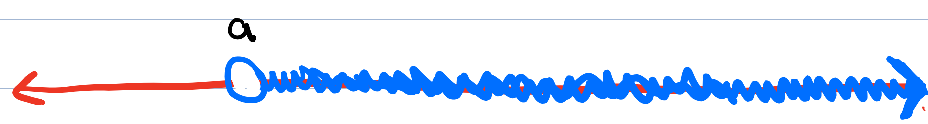


$$(a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}.$$



<u>5 interval types with at least one "infinite endpoint."</u>

$$(a,\infty) = \{x \in \mathbb{R} \mid a < x\}.$$



$$[a,\infty) = \{x \in \mathbb{R} \mid a \leq x\}.$$

$$(-\infty,a) = \{x \in \mathbb{R} \mid x < a\}.$$

$$(-\infty,a] = \{x \in \mathbb{R} \mid x \leq a\}.$$

$$(-\infty,\infty) = \mathbb{R}.$$

Note: The closed interval $[0,1]$ is denoted by I.

Cartesian Products:

Definition: For sets S and T, the Cartesian product of S and T, denoted $S \times T$, is the set of all ordered pairs (x,y) with $x \in S$ and $y \in T$.

In symbols, we write this as:

$$S \times T = \{(x,y) \mid x \in S,\ y \in T\}.$$

Note: Here we are using parentheses to denote an ordered pair (s,t). But just before, we used parentheses to denote an open interval.

These are the notational conventions that are typically used. It is a bit unfortunate that the same notation is used for two different things. In practice, though, this rarely causes confusion, as it usually clear from context what is meant.

<u>Example</u>: For $S=\{1,2\}$ and $T=\{a,b\}$,

$$S\times T=\{(1,a),(1,b),(2,a),(2,b)\}.$$

<u>Example</u>: By definition, $\mathbb{R}\times\mathbb{R}$ is the set of ordered pairs of real numbers.

We denote $\mathbb{R}\times\mathbb{R}$ as $\mathbb{R}^2$.

More generally, given sets $S_1, S_2, \dots, S_n$,

The Cartesian product

$S_1\times S_2\times\cdots\times S_n$ is the set of

ordered lists $(x_1, x_2, \dots, x_n)$ where $x_i\in S_i$ for each i.

In symbols,

$$S_1\times S_2\times\cdots\times S_n=\{(x_1,x_2,\dots,x_n)\mid x_i\in S_i\ \forall\, i\}.$$

this symbol means

"for all."

<u>Example</u>: For T any set, we denote

$$\underbrace{T \times T \times \cdots \times T}_{n \text{ copies of } T} \text{ by } \mathbb{R}^n.$$

In particular, this gives a definition of $\mathbb{R}^n$ as a set:

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ copies of } \mathbb{R}} = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R} \ \forall i\}.$$

The following fact is obvious, but will be important to us in this course.

<u>Fact</u>: If $S_1, S_2, \ldots, S_n$ and $T_1, T_2, \ldots, T_n$

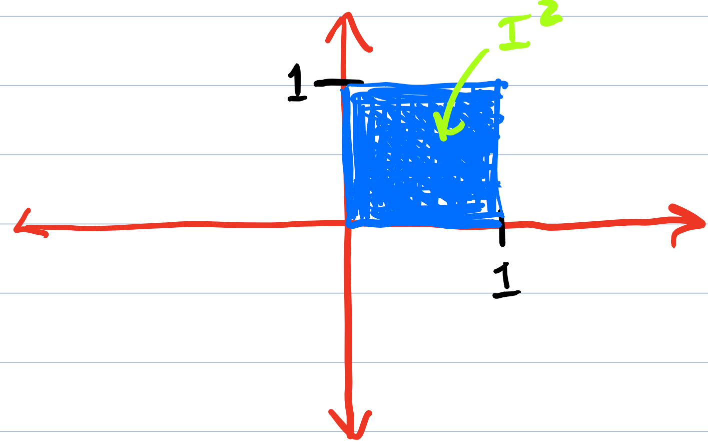
are sets with $T_i \subset S_i$ for each $i \in \{1, \ldots, n\}$, then

$$T_1 \times T_2 \times \cdots \times T_n \subset S_1 \times S_2 \times \cdots \times S_n$$

<u>Example</u> $I \subset \mathbb{R}$, so $I^n \subset \mathbb{R}^n$.

I^n is called the n-dimensional unit cube.

Illustration for $n=2$



<u>Exercise</u>: Sketch the subset $\{0,1\}^3 \subset \mathbb{R}^3$.

What is the geometric relationship between $\{0,1\}^3 \subset \mathbb{R}^3$ and I^3?

<u>Note</u>: For now, I am not going to review unions, intersections, and complements, three fundamental definitions from set theory. These are essential to developing the foundations of topology, but in this course, we will not need them, at least for a while.

Functions In a sense, topology is all about studying continuous functions. Before we can talk about continuous functions, we need to review some basic ideas about functions.

Definition: Given sets S and T, a function f from S to T is a rule which assigns each $s \in S$ exactly one element in T.

- This element is denoted $f(s)$.

We call

S the domain of f.

T the codomain of f.

We write the function as $f: S \to T$.

Example: Let $S = \{1, 2\}$, $T = \{a, b\}$.

We can define a function $f: S \to T$ by $f(1) = a$, $f(2) = b$

(Lecture ended here.) $g: S \to T$ by $f(1) = a$, $f(2) = a$.

Example: We often specify a function by a formula, e.g.

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. OR

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$.

Ima e of a function (also called the range)

Definition: For a function $f: S \to T$ we define $\mathrm{im}(f)$ to be the subset of T given by

$$\mathrm{im}(f) = \{ t \in T \mid t = f(s) \text{ for some } s \in S \}.$$

Intuitively, $\mathrm{im}(f)$ is the subset of T consisting of elements "hit by" f.

Example: For S, T, f, and g as in the previous example,

$$\mathrm{im}(f) = \{a, b\} = T, \quad \mathrm{im}(g) = \{a\}.$$