AMAT 342 Lecture 20 11/7/19

Today: Product topology
Gluing (quotient spaces)

Last time: Examples of topological spaces
Subspace topology

We saw that if $(T, \mathcal{O}^T)$ is any topological space and $S \subset T$ is any subset, then we can define a topology $\mathcal{O}^S$ on S by

$$\mathcal{O}^S = \{ U \cap S \mid \underbrace{U \in \mathcal{O}^T}_{\text{i.e., } U \text{ is open}} \}$$

This is the subspace topology.

We saw that in the case that T is a metric space, this coincides with the metric topology on S.

Now, I want to tell a similar story about cartesian products.

Product topology

Let $X=(S^X, \mathcal{O}^X)$ and $Y=(S^Y, \mathcal{O}^Y)$ be topological spaces. The product of X and Y is the topological space $X \times Y = (S^X \times S^Y, \mathcal{O}^{X\times Y})$, where

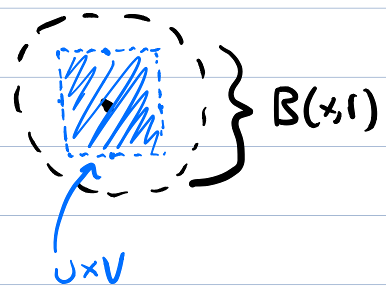
a set in $\mathcal{O}^{X\times Y}$ is a union of sets of the form $U \times V$ for $U \subset \mathcal{O}^X$ and $V \subset \mathcal{O}^Y$.

Proposition: The product topology on $\mathbb{R}\times\mathbb{R}=\mathbb{R}^2$ is the same as the metric topology on $\mathbb{R}^2$ w.r.t. the Euclidean metric

Proof: The idea here is that a union of open balls can be written as a union of open rectangles, and vice versa.

For any $x=(x_1,x_2) \in \mathbb{R}^n$ and $r>0$, note that $B(x,r)$ contains a set of the form $U\times V$, where $U, V \subset \mathbb{R}$ are open, and $x \in U \times V$. For example, we can take

$$U = (x_1 - \tfrac{1}{2}r,\ x_1 + \tfrac{1}{2}r)$$
$$V = (x_2 - \tfrac{1}{2}r,\ x_2 + \tfrac{1}{2}r).$$



Let's call $U \times V$ an open square.

For any open set $V \subset \mathbb{R}^2$ with respect to the metric topology, we can write $V = \bigcup_{x \in V} B(x, r_x)$ for some radii r_x. (We proved this earlier in any metric space.) That is, V is a union of balls centered at each of its points. Since each $B(x, r_x)$ contains an open square centered at x, V is also a union of open squares.

This shows that any open set in the metric topology on $\mathbb{R}^2$ is open in the product topology on $\mathbb{R} \times \mathbb{R}$.

Conversely, assume V is open in the product topology on $\mathbb{R} \times \mathbb{R}$. Then V is a union of sets of the form $I \times J$ where I, J are open in $\mathbb{R}$. If $x \in I \times J$, then since I and J are open $\underbrace{(x_1 - r, x_1 + r)}_{B(x_1, r)} \in I$ and $\underbrace{(x_2 - r, x_2 + r)}_{B(x_2, r)} \in J$ for some $r > 0$.

(This is using again the same fact that an open set in a metric space is a union of open balls centered at each of its points).

Thus, the square $S = (x_1 - r, x_1 + r) \times (x_2 - r, x_2 + r)$ is contained in $I \times J$. note that $B(x, r) \subset S \subset I \times J \subset V$. Thus, for every point $x \subset I \times J$, there is an open ball centered at x contained in V. this shows that V is the union of open balls in $\mathbb{R}^2$. Hence V is open in the metric topology on $\mathbb{R}^2$. ∎

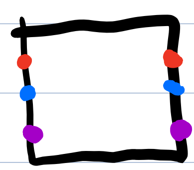
Note: We discussed different ways to put a metric on a cartesian product of metric spaces M_1 and M_2

For each of these metrics on the catesian product, the metric topology is the same as the product topology on the metric topologies of M_1 and M_2.

The proof is similar to the proof of the proposition above.

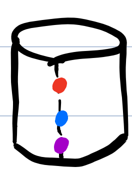
Gluing (Quotient Spaces)

Consider the square $I \times I$

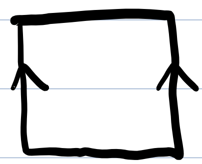


Thinking of this as a piece of rubber, suppose we glue the left edge to the right edge, i.e., gluing $(0,y)$ to $(1,y)$ for all $y \in I$

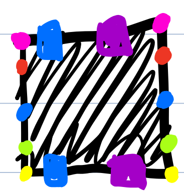
We get a cylinder:



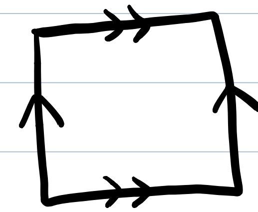
We represent this gluing schematically by marking the left and right edges of the square $I \times I$ with an arrowhead as shown:



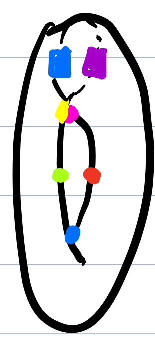
2. What if we also glue the top edge to the bottom edge, i.e. glue



$(x, 0)$ to $(x, 1)$

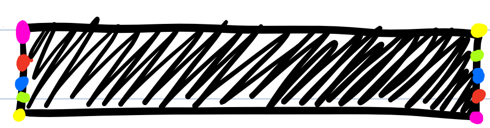


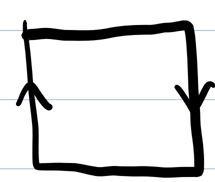
schematic representation



We get a torus, i.e. surface of a donut.

3. What if in the first example, we instead glue $(0, y)$ to $(1, 1-y)$





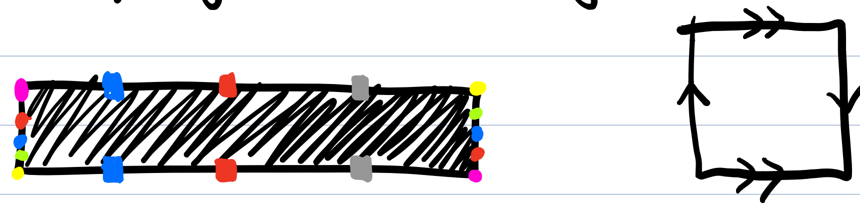
shematic representation
(opposite arrow directions indicate the twist)

that is, we glue opposite sides with a twist!

We get the Mobius band

This is a surface with one side!

4. Suppose now that in the above example, we also glue the top edge to the bottom edge:

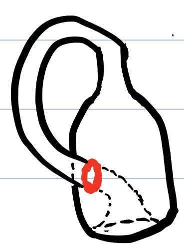


That is, we glue $(0,y)$ to $(1,1-y)$ $\forall y \in I$, and

$$(x,0) \text{ to } (x,1) \ \forall x \in I.$$

We get a surface K we call the Klein bottle.

Fact: K admits no embedding into $\mathbb{R}^3$. The figure below illustrates the image of a non-injective map $f: K \to \mathbb{R}^3$.



f is "almost injective"; all points in $\mathrm{im}(f)$ are mapped to by a unique point of K, except for

The points in the red circle, which are hit by two points.

If we consider $j: \mathbb{R}^3 \to \mathbb{R}^4$, $j(x,y,z) = (x,y,z,0)$, then $j \circ f$ can be perturbed to an embedding: We can use the extra coordinate to perturb away the "self-intersection."

Thus, K embeds in $\mathbb{R}^4$.

Topogists love the Klein bottle because it is a surface w/ no boundary that is "non-orientable."

Informally, non-orientable means the surface has no separate inside and outside. This can be made formal but we won't get into that now.

Quotient Spaces (formalizing gluing)

A nice application of the abstract definition of a topological space.

Plan :

Given a topological space $T = (S, \mathcal{O}^S)$ and an equivalence relation $\sim$ on S, define a topology $\mathcal{O}^\sim$ on $S/\sim$. We denote $(S/\sim, \mathcal{O}^\sim)$ by $T/\sim$

$T/\sim$ is called a quotient space, and $\mathcal{O}^\sim$ is called the quotient topology (Definition of $\mathcal{O}^\sim$ will be given below)

Interpretation: points $x,y \in T$ get glued together if and only if $x \sim y$.

Example: $T = I \times I$. Define $\sim$ by

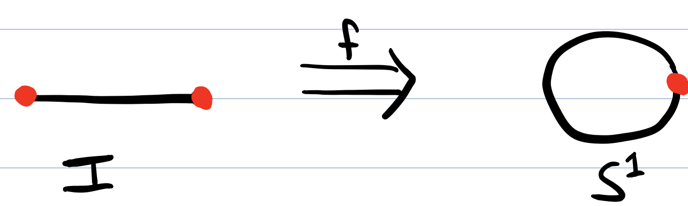
$$(x_1,y_1) \sim (x_2,y_2) \text{ iff } y_1 = y_2 \text{ AND } (x_1 = x_2 \text{ OR } x_1, x_2 \in \{0,1\})$$

Then $T/\sim$ is the cylinder and this formalizes the gluing of the square to form the cylinder shown above.

Before going into the definition of the quotient topology, we'll provide some intuition/motivation.

In general, we can think of a continuos surjection as a gluing operation.

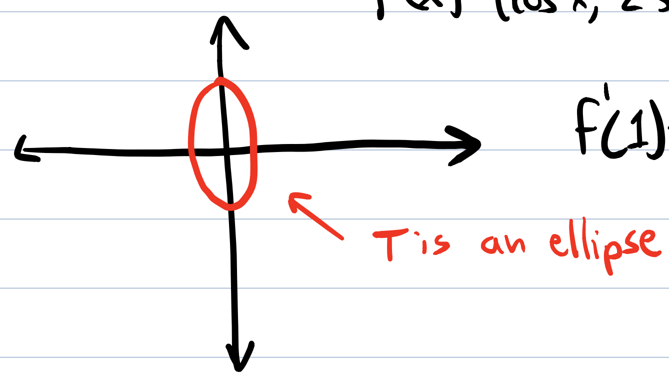
For example, Consider $f: I \to S^1$; $f(x) = (\cos x, \sin x)$



Then $f(0) = f(1)$, so we can think of f as a gluing operation on I, which glues 0 to 1. $f(x) \neq f(y)$ for any other $x \neq y \in I$.

Now, there are many other continuous surjections $f': I \to T$ with with similar gluing behavior, e.g.

Let $T = \{ (\cos x, 2 \sin x) \mid x \in [0, 2\pi) \} \subset \mathbb{R}^2$
$f'(x) = (\cos x, 2 \sin x)$.



$f'(1) = f'(0) = (1,0)$, and $f'(x) \neq f'(y)$ for all other $x \neq y \in I$.

We would like to define this kind of gluing in some kind of canonical way, that doesn't depend on an arbitrary choice of T, g. That's one motivation for the definition of a quotient space.

Important note: Suppose we have continuous surjections $f: S \to T$ and $f': S \to T'$ such that $f(x) = f(y)$ iff $f'(x) = f'(y)$.

Are T and T' always homeomorphic? No!

<u>Example</u>: $S = [0, 2\pi)$
$T = S \qquad f : [0,2\pi) \to [0, 2\pi) = Id_{[0,2\pi)}$
$T' = S^1. \qquad f' : [0,2\pi) \to [0, 2\pi) = S^1.$

Then both f and f' are continuous bijections (in particular, they are surjections), but the codomains $[0, 2\pi)$ and S^1 are not homeomorphic.

Intuitively S^1 is glued together more than $[0, 2\pi)$.

The definition of the quotient topology is chosen so that (in a precise sense), for any topological space T and equivalence relation $\sim$ on T, $T/\sim$ glues stuff together as little as possible, subject to the constraint that equivalent points get glued together.

<u>Formal Definition of quotient topology</u>
For S any set and $\sim$ an equivalence relation on S, define $\pi : S \to S/\sim$ by $\pi(x) = [x]$. Thus π sends x to its equivalence class.

For $T = (S, O^S)$ a topological space and $\sim$ an equivalence relation on S, define the <u>quotient topology</u> $O^\sim$ by

$$O^\sim = \{ A \subset S \mid \pi^{-1}(A) \text{ is open} \}.$$

Proposition: The map $\pi : T \to T/\sim$ is continuous.

Pf: By definition, $\pi^{-1}(U)$ is open for all open sets $U \in T/\sim$.

The following proposition tells us that the quotient space contruction glues stuff together as little as possible, subjet to the constraint that equivalent points be glued together.

Proposition: For any topological space T, equivalence relation $\sim$ on T, and continuous surjection

$f : T \to S$, there is a unique continuous surjection $\tilde{f} : \tilde{T} \to S$ such That $f = \tilde{f} \circ \pi$.

Thus, S is obtained from $\tilde{T}$ by gluing more stuff.