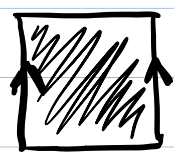
AMAT 342 Lec 21 11/12/19

Today: Quotient spaces

Last lecture, we gave the definition of quotient spaces, but didn't offer any intuition for the definition. Today, we'll provide that intuition.

Review

Quotient spaces formalize the idea of gluing.

Example  We can glue the left edge of the square $T = I \times I$ to the right edge to create a cylinder

Formally, we do this by defining an equivalence relation $\sim$ on T as follows:

$$(x_1, y_1) \sim (x_2, y_2) \text{ iff } y_1 = y_2 \text{ AND } (x_1 = x_2 \text{ OR } x_1, x_2 \in \{0,1\}).$$

Idea: We want $T/\sim$ to be homeomorphic to a cylinder. For that to make sense, we have to put a topological structure on $T/\sim$.

We give $T/\sim$ the quotient topology.

Let's review the quotient topology in full generality:

Formal Definition of quotient topology

For S any set and $\sim$ an equivalence relation on S, define $\pi: S \to S/\sim$ by $\pi(x) = [x]$. Thus π sends x to its equivalence class.

For $T = (S, O^S)$ a topological space and $\sim$ an equivalence relation on S, define the quotient topology $O^\sim$ by

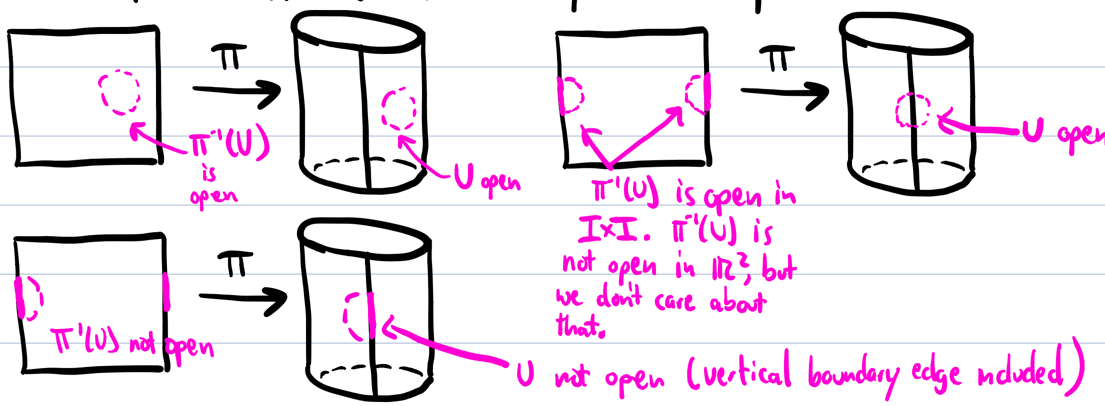
$$O^\sim = \{U \subset S/\sim \mid \underbrace{\pi^{-1}(U) \text{ is open}}_{\text{this means } \pi^{-1}(U) \in O^S}\}.$$

Thus, $U \subset S/\sim$ is open in the quotient topology iff $\pi^{-1}(U)$ is open in T.

The topological space $(S/\sim, O^\sim)$ is denoted $T/\sim$ and is called the quotient space.

In the example of the cylinder, $S = I \times I$ and $\mathcal{O}^S$ is the metric topology coming from the Euclidean metric.

Let's look at a few examples of open sets.

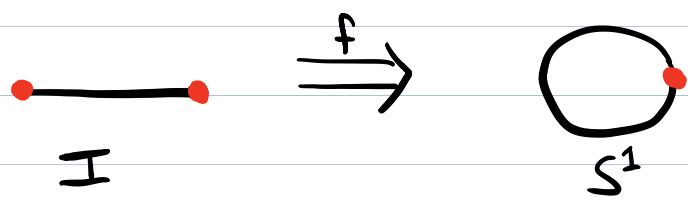


Remark: The surjection $\pi : S \to S/\sim$ can be considered as a continuous function $\pi : T \to T/\sim$.

Intuition/Motivation for Definition of Quotient Space

In general, we can think of a continuous surjection as a gluing operation.

For example, Consider $f: I \to S^1$, $f(x) = (\cos x, \sin x)$

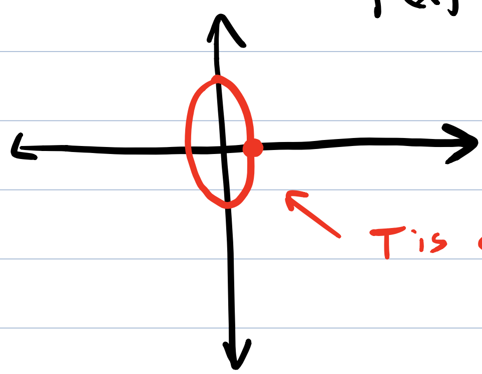


Then $f(0)=f(1)$, so we can think of f as a gluing operation on I, which glues 0 to 1. $f(x)\neq f(y)$ for any other $x\neq y\in I \Rightarrow f$ doesn't do any other gluing.

Now, there are many other continuous surjections $f':I\to T$ with with similar gluing behavior, e.g.

Let $T=\{(\cos x, 2\sin x) \mid x\in[0,2\pi)\}\subset\mathbb{R}^2$

$f'(x)=(\cos x, 2\sin x)$.



T is an ellipse

$f'(1)=f'(0)=(1,0)$, and $f'(x)\neq f'(y)$ for all other $x\neq y\in I$.

We would like to define this kind of gluing in some kind of <u>canonical</u> <u>way</u>, that doesn't depend on an arbitrary choice of T, g. That's one motivation for the definition of a quotient space.

<u>More</u> <u>motivation</u>:

With the above in mind, here's a naive idea for defining the quotient space which fails (but whose failure will help us understand the actual definition):

Bad def: Given a topological space T and an equivalence relation $\sim$ on T, define the quotient space as the codomain of any continuous surjection $f: T \to X$ such that $f(x) = f(y)$ iff $x \sim y$.

The examples above show that there can be two different continuous surjections $f: T \to X$, $f': T \to X'$ such that $X \neq X'$.
(define $\sim$ on I by $x \sim y$ if $x = y$ or $x, y \in \{0, 1\}$.

But if X and X' were always homeomorphic, this wouldn't be so bad, since homeomorphic spaces are considered to be "topologically equivalent."

However, X and X' needn't even be homeomorphic!
So the bad definition is really problematic.

Example: $T = X = [0, 2\pi)$ $\quad x \sim y$ iff $x = y$.
$X' = S^1$.
$f: T \to T = Id_T$
$f': T \to S^1$, $f'(x) = (\cos x, \sin x)$

Then both f and f' are continuous bijections (in particular, they are surjections), such that $\underbrace{f(x) = f(y) \text{ iff } x \sim y}_{\text{i.e. } f \text{ is injective}}$ and $\underbrace{f'(x) = f'(y) \text{ iff } x \sim y}_{\text{i.e., } f' \text{ is injective.}}$

However $X = [0, 2\pi)$ is not homeomorphic to $X' = [0, 2\pi)$.

(Intuitively S^1 is glued together more than $[0, 2\pi)$.)

To fix the "bad definition" we would need to also require that the continuous surjection $f: T \to X$ "glue stuff together as little as possible", in some sense.

The definition of quotient space we have given satisfies such a property, as made clear by the next definition.

<u>Proposition</u>: For any topological space T, equivalence relation $\sim$ on T, and continuous surjection $f: T \to X$ such that $f(x) = f(y)$ whenever $x \sim y$, there is a unique continuous surjection $\tilde{f}: T/\sim \longrightarrow X$ such that $f = \tilde{f} \circ \pi$.

Thus, X is obtained from $T/\sim$ by gluing more stuff.

<u>Proof</u>: Define $\tilde{f}$ by $\tilde{f}([x]) = f(x)$. If $[x] = [y]$, then $x \sim y$ so $\tilde{f}([x]) = f(x) = f(y) = \tilde{f}([y])$, so this is well defined, and it is clear that $f = \tilde{f} \circ \pi$. If $\tilde{f}': T/\sim \to X$ also satisfies $f = \tilde{f}' \circ \pi$. Then $\tilde{f}'([x]) = \tilde{f}' \circ \pi(x)$ $= f(x) = \tilde{f} \circ \pi(x) = \tilde{f}([x])$, so $\tilde{f}' = \tilde{f}$. This gives the claimed uniqueness property. If $y \in X$, then since f is surjective,

$y = f(x)$ for some x, and then $y = \tilde{f}([x])$, so $\tilde{f}$ surjective.
If $U \subset X$ is open, then $f^{-1}(U)$ is open because f is continuous.
$f^{-1}(U) = \pi^{-1}(\tilde{f}^{-1}(U))$, so by the definition of the quotient topology $\tilde{f}^{-1}(U)$ is open. Hence, $\tilde{f}$ is continuous ■

<u>Remark</u>: The proposition can be adapted into an (equivalent) definition of the quotient space, but we won't do that here.

<u>Summary</u>: The quotient space $T/\sim$ is obtained from T by doing as little gluing as possible, subject to the constraint that x is glued to y in $T/\sim$ whenever $x \sim y$.