AMAT 342 Lecture 22

Today: Finish Quotient Spaces
Homework problem review

Recall: If $T=(S,\mathcal{O}^S)$ is any topological space and $\sim$ is an equivalence relation on S, the *quotient space* $T/\sim$ is given by
$T/\sim\, = (S/\sim, \mathcal{O}^\sim)$, where

$$\mathcal{O}^\sim = \{U \subset S/\sim \mid \pi^{-1}(U) \text{ is open in } T\}$$

(where $\pi: S \to S/\sim$ is defined by $\pi(x)=[x]$.)

Thus, U is open in $T/\sim$ iff $\pi^{-1}(U)$ is open in T.

Note: π is surjective.
Note: We can regard π as a function from T to $T/\sim$.

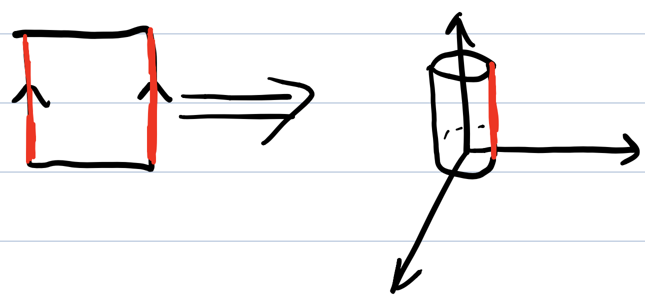
Prop: $\pi: T \to T/\sim$ is continuous. **Pf.** U open in $T/\sim \Rightarrow \pi^{-1}(U)$ open in T. ∎

So π is a continuous surjection

Motivation / Intuition for Quotient spaces (continued)

In general, we should think of a gluing operation as a continuous surjection.

For example, consider $f: I \times I \to S^1 \times I \subset \mathbb{R}^3$
$f(x,y) = (\cos(2\pi x), \sin(2\pi x), y)$



Intuitively, f glues x and y together iff $f(x) = f(y)$.

Let's reframe this in terms of an equivalence relation:
Last lecture, we defined the equivalence relation $\sim$ on $I \times I$ by iff $(y_1 = y_2)$ AND $(x_1 = x_2$ OR $x_1, x_2 \in \{0,1\})$.

Note that $f(x) = f(y)$ iff $x \sim y$.

In this example, f seems to "correctly" do the gluing specified by $\sim$.

So perhaps can take this as inspiration for a more intuitive definition of the quotient space.

Bad def: Given a topological space T and an equivalence relation $\sim$ on T, define the quotient space to be the codomain X of any continuous surjection $f: T \to X$ such that $f(x) = f(y)$ iff $x \sim y$.

Issue: It's not clear that such f exists. (actually it does...)

A Deeper Problem: We can have
$$f: T \to X$$
$$f' \; T \to X'$$
with
$$f(x) = f(y) \text{ iff } x \sim y$$
$$f'(x) = f'(y) \text{ iff } x \sim y$$

but X and X' not homeomorphic.

So this definition doesn't make sense.

Example: $T = X = [0, 2\pi)$ $\quad x \sim y$ iff $x = y$.

$X' = S^1$.

$f: T \to T = Id_T$

$f: T \to S^1$, $f'(x) = (\cos x, \sin x)$

Then both f and f' are continuous bijections (in particular, they are surjections), such that $\underbrace{f(x) = f(y) \text{ iff } x \sim y}_{\text{i.e. } f \text{ is injective}}$ and $\underbrace{f'(x) = f'(y) \text{ iff } x \sim y}_{\text{i.e., } f' \text{ is injective.}}$.

However $X = [0, 2\pi)$ is not homeomorphic to $X' = S^1$.

(Intuitively S^1 is glued together more than $[0, 2\pi)$.)

To fix the "bad definition" we would need to also require that the continuous surjection $f: T \to X$ "glue stuff together as little as possible," in some sense.

The definition of quotient space we have given satisfies such a property, as made clear by the next definition.

<u>Proposition</u>: For any topological space T, equivalence relation $\sim$ on T, and continuous surjection $f: T \to X$ such that $f(x) = f(y)$ whenever $x \sim y$, there is a unique continuous surjection $\tilde{f}: T/\sim \longrightarrow X$ such that $f = \tilde{f} \circ \pi$.

Thus, X is obtained from $T/\sim$ by gluing more stuff. That is, $\pi: T \to T/\sim$ clues stuff together as little as possible, among maps that glue x and y together if $x \sim y$.

Proof: Define $\tilde{f}$ by $\tilde{f}([x]) = f(x)$. If $[x] = [y]$, then $x \sim y$ so $\tilde{f}([x]) = f(x) = f(y) = \tilde{f}([y])$, so this is well defined, and it is clear that $f = \tilde{f} \circ \pi$. If $\tilde{f}' : T/\sim \to X$ also satisfies $f = \tilde{f}' \circ \pi$. Then $\tilde{f}'([x]) = \tilde{f}' \circ \pi(x) = f(x) = \tilde{f} \circ \pi(x) = \tilde{f}([x])$, so $\tilde{f}' = \tilde{f}$. This gives the claimed uniqueness property. If $y \in X$, then since f is surjective, $y = f(x)$ for some x, and then $y = \tilde{f}([x])$, so $\tilde{f}$ surjective. If $U \subset X$ is open, then $f^{-1}(U)$ is open because f is continuous. $f^{-1}(U) = \pi^{-1}(\tilde{f}^{-1}(U))$, so by the definition of the quotient topology $\tilde{f}^{-1}(U)$ is open. Hence, $\tilde{f}$ is continous ∎

Remark: The proposition can be adapted into an (equivalent) definition of the quotient space, but we won't do that here.

Summary: The quotient space $T/\sim$ is obtained from T by doing as little gluing as possible, subject to the constraint that x is glued to y in $T/\sim$ whenever $x \sim y$.

Exam Similar format to last time

- one page of handwritten notes allowed, front and back
- covers homeworks 4-6.
- may be a question on the subspace topology.
- Edit distance will be an exam.
- not on this exam: Gluing, quotient topology product topology, RMSD.
- at least one proof
- at least one definition
- at least one problem directly from the HW.

Homework Problems

Problem set 5, # 5.

Prove that a subset S of a metric space M is open iff it contains none of its boundary points.

Def: For S any subset of M, $x \in M$ is a **boundary point** if each open ball centered at x contains a point in S and a point not in S.

Pf: Suppose S contains none of its boundary points. For any $x \in S$, x is not a boundary point. Therefore, for some open ball $B(x, r_x)$, $B(x, r_x) \subset S$ or $B(x, r_x) \cap S = \phi$. But $x \in B(x, r_x)$, So $B(x, r_x) \cap S \neq \phi \Rightarrow B(x, r_x) \subset S$. (Choosing such a ball $B(x, r_x)$

$\forall x \in S$, we have $S = \bigcup_{x \in S} B(x, r_x)$, so S is open.

<u>Key fact</u>: If S is an open subset of M, then for each $x \in S$, $B(x,r) \subset S$ for some $r > 0$.

If S is open, then by the key fact, no point $x \in S$ is a boundary point. ∎

<u>HW 6.</u> #1.e. Let S be a finite subset of $\mathbb{R}^2$. What is the boundary of S? Is S open?

For any $x \in S$ and ball B centered at x, B contains x. B clearly also contains points in X. So x is a boundary point. If $x \notin S$, then a very small ball around x contains no points in S. Thus x is not a boundary point, So $\text{Boundary}(S) = S$.

$\Rightarrow$ If S is non-empty, S is not open.

Problem 2. For $M = [-1, 1]$ w/ the Euclidean metric, which of the following are open subsets of M?

a. $\{1\}$. **Not open.** If it is open, then it contains an open ball centered at 1, by the key fact. But any open ball centered at 1 is of the form
$$B(1, r) = \begin{cases} (1-r, 1] & \text{if } r \le 2 \\ [-1, 1] & \text{if } r > 2. \end{cases}$$

$\{1\}$ contains no such set.

b. $(0,1) = \underbrace{B(\tfrac{1}{2}, \tfrac{1}{2})}_{\text{open ball in } M.}$ so $(0,1)$ is open in M

c. $[0,1)$ is not open because 0 is a boundary point.

d. $(0,1]$ is open because $(0,1] = B(1,1)$.