AMAT 342 Lecture 24 11/25/19

Topics for rest of class: Only 3 lectures!!!

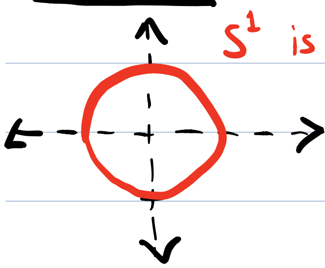
Manifolds
Cell complexes
Euler Characteristic
Topological Data Analysis

Business
- Exams back Tuesday
- Solutions out soon
- Homework out soon, due in 1 week.

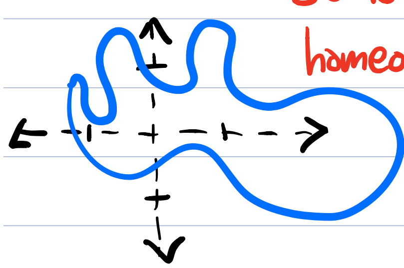
Manifolds

1-dimensional manifolds are <u>curves</u>

Examples

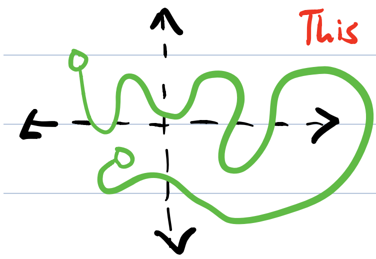


S^1 is a (1-D) manifold.



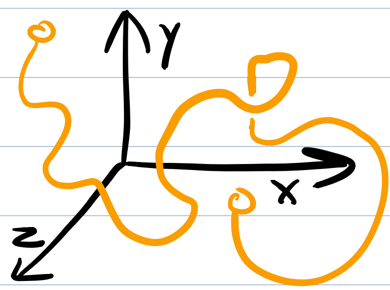
So is any topological space homeomorphic to S^1.

This is also a manifold

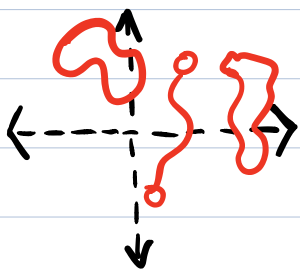


$(0,1) \subset \mathbb{R}$ is also a manifold. So is $\mathbb{R}$ itself!

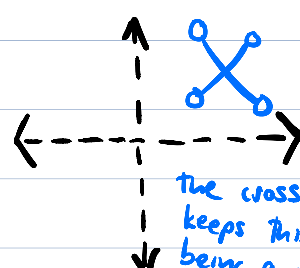
So is this curve in $\mathbb{R}^3$



A manifold can have multiple path components



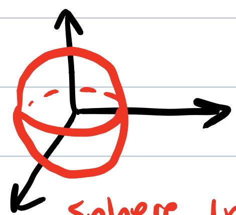
Here's something that's not a manifold:



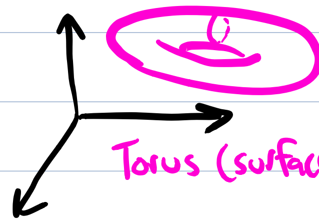
The crossing point keeps this from being a manifold!

2-D manifolds are surfaces

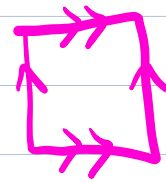
Examples

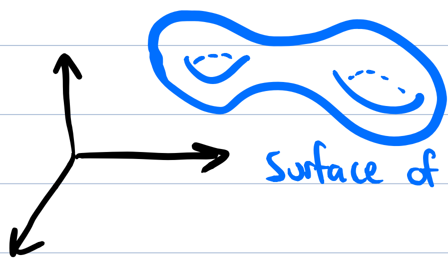


Sphere in $\mathbb{R}^3$

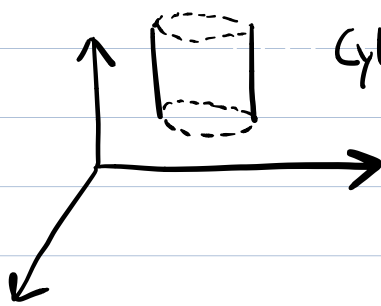


Torus (surface of a donut)

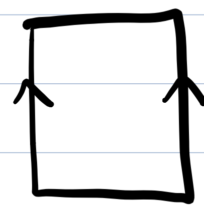


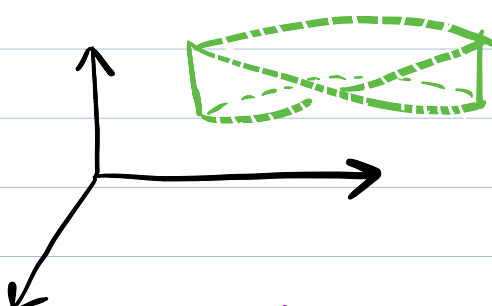


Surface of a two-holed donut

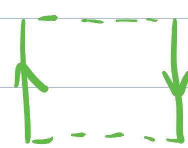


Cylinder

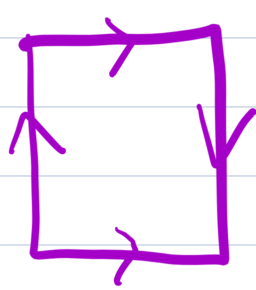




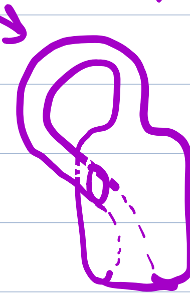
Mobius band



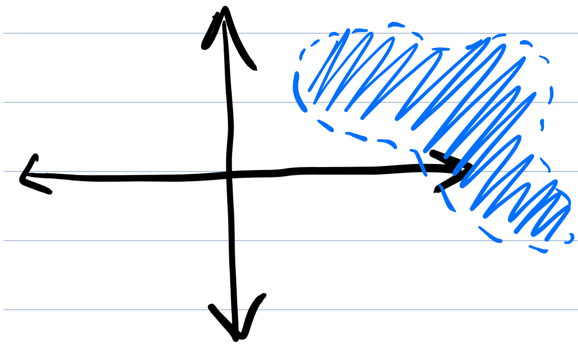
(Embeds in $\mathbb{R}^4$, not $\mathbb{R}^3$)



Klein bottle

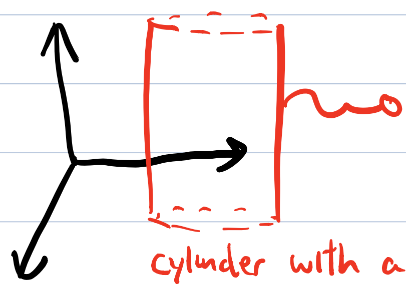


Open sets in $\mathbb{R}^2$ are also 2-D manifolds



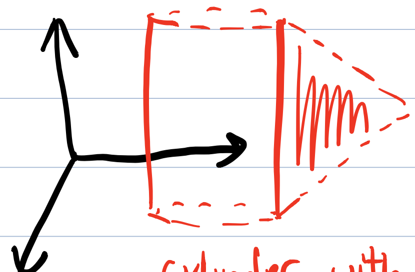
In particular, $\mathbb{R}^2$ is a manifold!

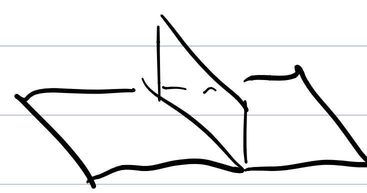
Not a manifold



cylinder with a tail

Not a manifold

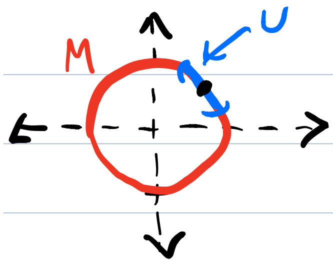




cylinder with a fin

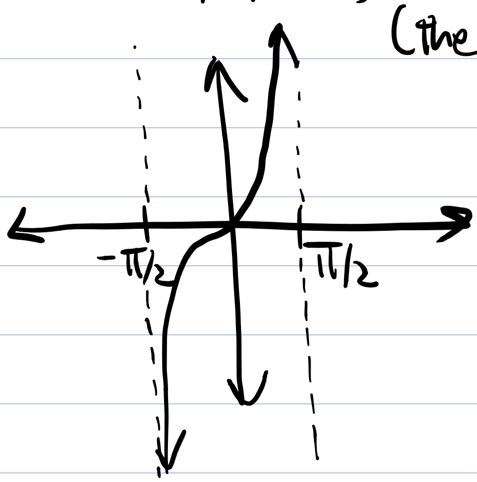
Definition: For $n \geq 1$. Here and throughout we assume $\mathbb{R}^m$ has the usual Euclidean metric / topology.
An n-dimensional (topological) manifold is a subspace $M \subset \mathbb{R}^m$ (for some $m \geq n$) such that for each $x \in M$, there exists $U \subset M$ with $x \in U$ and U homeomorphic to $\mathbb{R}^n$.

Intuition: We think of U as a small region in M around x.



<u>First important observation</u>: $\mathbb{R}^n$ is homeomorphic to an open ball in $\mathbb{R}^n$.

For example, in the case $n=1$, the function $\tan : (-\pi/2, \pi/2) \longrightarrow \mathbb{R}$ is a homeomorphism (the inverse is arctan).



Note that $(-\pi/2, \pi/2) = B(0, \frac{\pi}{2}) \subset \mathbb{R}$, so $B(0, \frac{\pi}{2})$ and $\mathbb{R}$ are homeomorphic.

More generally, for $n \geq 1$, the function $f : \underset{\subset (\mathbb{R}^n, d_2)}{B(0, \pi/2)} \longrightarrow \mathbb{R}^n$ given by

$$f(\vec{x}) = \tan(\|\vec{x}\|)\vec{x}$$

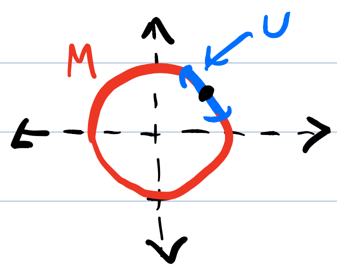
is a homeomorphism, where $\|\vec{x}\|$ = Euclidean dist. from $\vec{x}$ to $\vec{0}$.

Fact: For B, B' any open balls in $\mathbb{R}^n$, B and B' are homeomorphic.

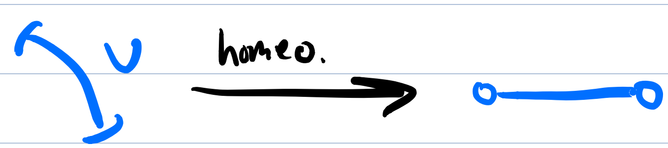
Pf: Easy exercise.

So $\mathbb{R}^n$ is homeomorphic to any open ball in $\mathbb{R}^n$!

Upshot: In the definition of a manifold, we can take U to be homeomorphic to an open ball in $\mathbb{R}^n$.



It is intuitively clear that for each $x \in S^1$, $\exists$ $U \subset S^1$ containing x such that U is homeomorphic to an open ball = open interval in $\mathbb{R}$.



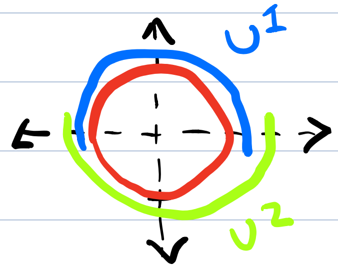
Hence, S^1 is a manifold.

Let's <u>prove</u> that S^1 is 1-D manifold. Fix small $\delta > 0$.

Define $U^1, U^2 \subset S^1$ by

$$U^1 = \{(\cos x, \sin x) \mid x \in (-\delta, \pi+\delta)\}$$

$$U^2 = \{(\cos x, \sin x) \mid x \in (\pi-\delta, 2\pi+\delta)\}$$

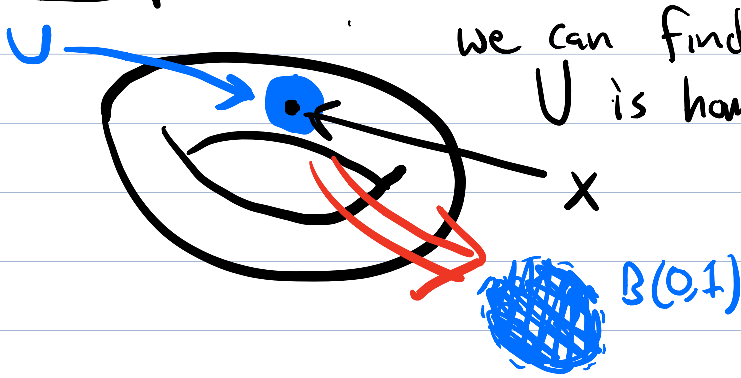


Clearly, $S^1 = U^1 \cup U^2$, so for any $x \in S^1$, $x \in U^1$ or $x \in U^2$!.

Moreover, $\alpha_1 : (-\delta, \pi+\delta) \to U^1$, $\alpha_1(x) = (\cos x, \sin x)$
$\alpha_2 : (\pi-\delta, 2\pi+\delta) \to U^2$, $\alpha_2(x) = (\cos x, \sin x)$

are homeomorphisms (I'll skip the proof of this part).
Thus U^1 and U^2 are homeomorphic to $\mathbb{R}$.

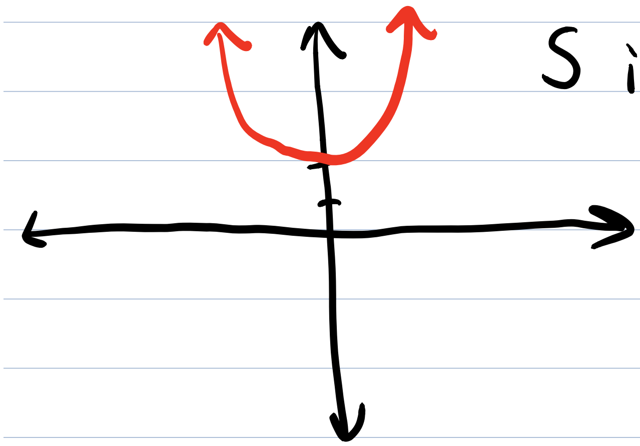
<u>Example</u>: Consider the torus T. For each $x \in T$, we can find $x \in U \subset T$ s.t. U is homeomorphic to $B(0,1)$



Why manifolds are important.

- Very often, the solutions to equations are manifolds.

Example Let $S \subset \mathbb{R}^2$ be the set of solutions to the equation $y = x^2 + 2$.



S is a parabola. This is a manifold.

Similarly, S^1 is the set of solutions to $x^2+y^2=1$.

S^2 is the set of solutions to $x^2+y^2+z^2=1$.

Is the torus the solution to any equation? Yes.

Fact: T is homeomorphic to $S^1 \times S^1 \subset \mathbb{R}^4$.

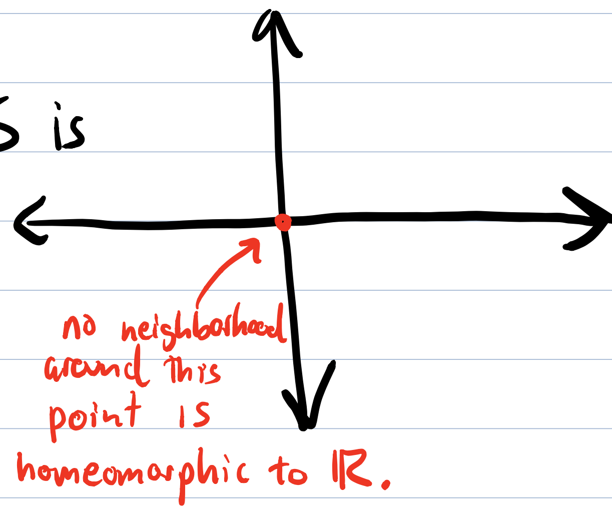
$S^1 \times S^1$ is the set of simultaneous solutions in $\mathbb{R}^4$ to the equations

$$w^2+x^2=1$$
$$y^2+z^2=1$$

Here is a simple equation whose solution set is not a manifold:

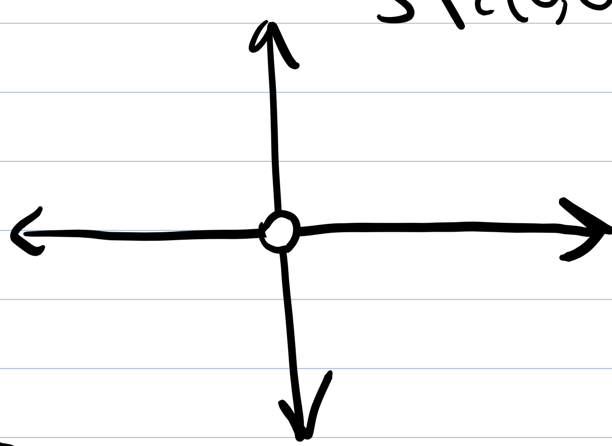
$$xy = 0.$$

Solution set S is the union of the x and y axes.



But S is <u>almost</u> a manifold: $S \setminus \{(0,0)\}$ is a manifold

$S \setminus \{(0,0)\}$.



$\{(0,0)\}$ is called a <u>singularity</u>. This is typical behavior for the set of solutions of polynomial equations.