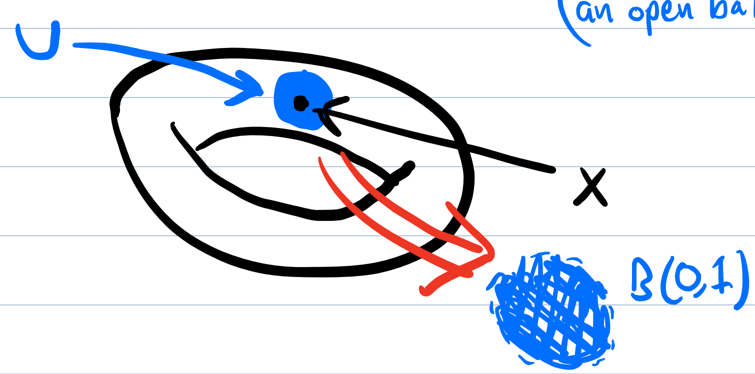
AMAT 342 Lecture 25

Today: More about manifolds

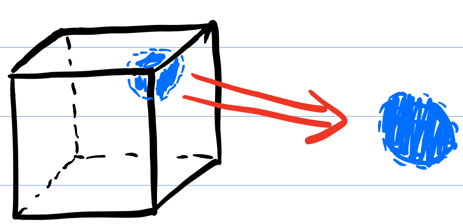
Recall the definition:

Definition: For $n \geq 1$.
An n-dimensional (topological) manifold is a subspace $M \subset \mathbb{R}^m$ (for some $m \geq n$) such that for each $x \in M$, there exists $U \subset M$ with $x \in U$ and U homeomorphic to $\mathbb{R}^n$.

(equivalently, homeomorphic to an open ball in $\mathbb{R}^n$.)



Remark: A topological manifold can have edges and corners. For example, the surface of a cube is a manifold.



This is homeomorphic to the sphere.

In advanced mathematics, one sometimes wants to consider manifolds with no edges or corners. To do so, one can use ideas from multivariable calculus to define smooth manifolds. We'll not worry about the details here.

Why do we care about manifolds? (continued from last time)

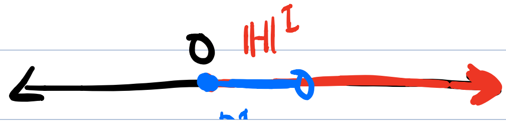
- Solutions to equations are often manifolds (or almost manifolds, in some sense)

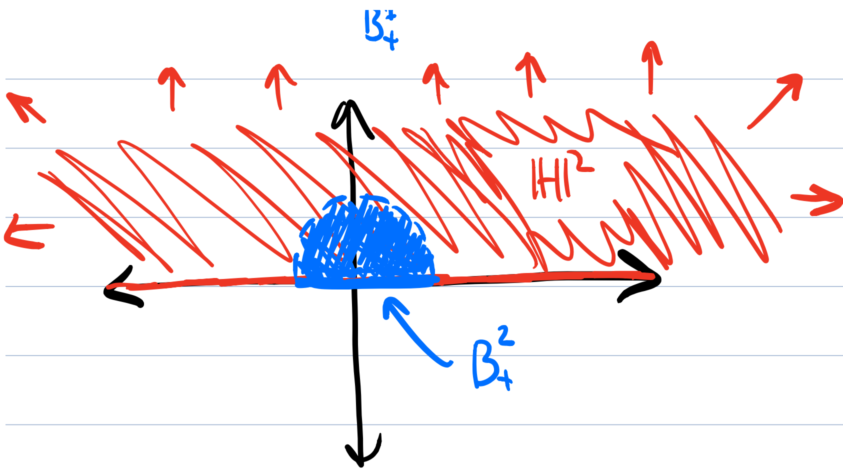
- Surfaces of objects are manifolds
  $\Rightarrow$ Manifolds are important in computer vision, graphics, physical modeling.

- Manifolds are fundamental in physics
  (The theory of general relativity describes the gravitation in terms of the curvature of a 4-D manifold.)

Manifolds-with-Boundary

For $n \geq 1$, let $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.





Let $B_+^n = \{\vec{x} \in \mathbb{H}^n \mid d_2(0,x) < 1\}$ = "upper half ball centered at 0 in $\mathbb{R}^n$".

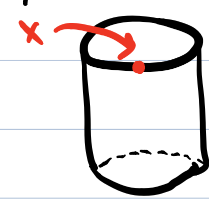
$$= \mathbb{H}^n \cap \underbrace{B(0,1)}_{\text{Ball in } \mathbb{R}^n \text{ wrt Euclidean metric}}.$$

<u>Fact</u>: B_+^n and $\mathbb{H}^n$ are homeomorphic

The homeomorphism $B(0,1) \to \mathbb{R}^n$ discussed last lecture restricts to a homeomorphism $B_+^n \to \mathbb{H}^n$.

<u>Fact</u>: $\mathbb{H}^n$ and $\mathbb{R}^n$ are not homeomorphic.

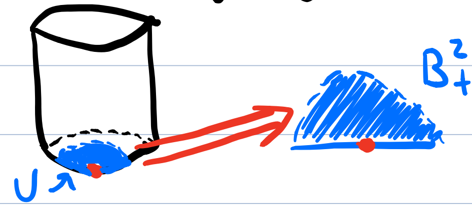
<u>An example</u>



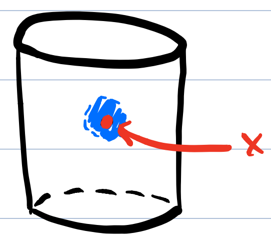
$S^1 \times I$ is not a manifold:

For a point $x \in S^1 \times \{0,1\}$, i.e., x lies on the top or bottom circle) there is no $U \subset S^1 \times I$ containing x and homeomorphic to $\mathbb{R}^n$.

However, a small region U around such x is homeomorphic to B_+^2 (to $\mathbb{H}^2$)



For $x \in S^1 \times I$, not lying on the top or bottom circular edge, there is $U \subset S^1 \times I$ containing x and homeomorphic to $\mathbb{R}^2$.



Thus, for every $x \in S^1 \times I$ there is $U \subset S^1 \times I$ such that U is homeomorphic to either $\mathbb{R}^2$ or $\mathbb{H}^2$.

We call an object with this property a manifold with boundary.

Def: For $n \geq 1$, an n-D manifold with boundary is a subspace $M \subset \mathbb{R}^m$ for some m such that for each $x \in M$, there exists $U \subset M$ containing x with U homeomorphic to $\mathbb{R}^n$ or to $\mathbb{H}^n$.

Def: If $\exists$ $U \subset M$ homeo to $\mathbb{R}^n$ and $x \in U$, we call x an interior pt. Otherwise, we call x a manifold boundary point.

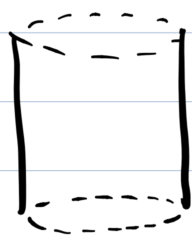
Note: A manifold boundary point is always a boundary point, as defined earlier, but the converse is false.

In fact, every point in $S^1 \times I \subset \mathbb{R}^3$ is a boundary point, but only the top and bottom circles of $S^1 \times I$ contain manifold boundary points.

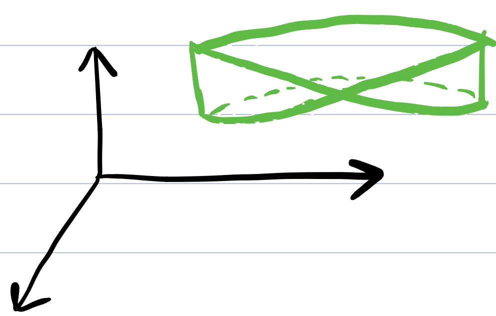
<u>Remark</u>: It is immediate from the definitions that a manifold is a manifold with boundary. But the reverse is not true.

Sometimes we call a manifold a "manifold without boundary" to avoid confusion.

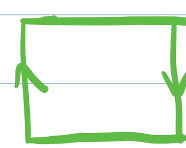
<u>Example</u>

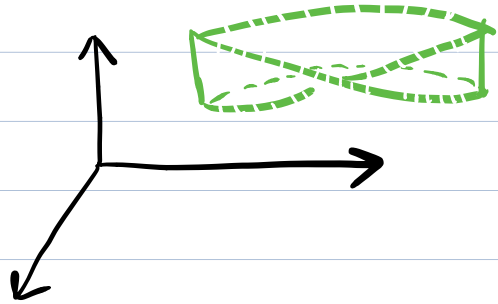


$S^1 \times (0,1)$ is the cylinder with top and bottom circles removed. This is a manifold without boundary.

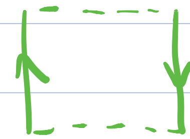


Mobius band is a manifold-with-boundary but not a (manifold without boundary)

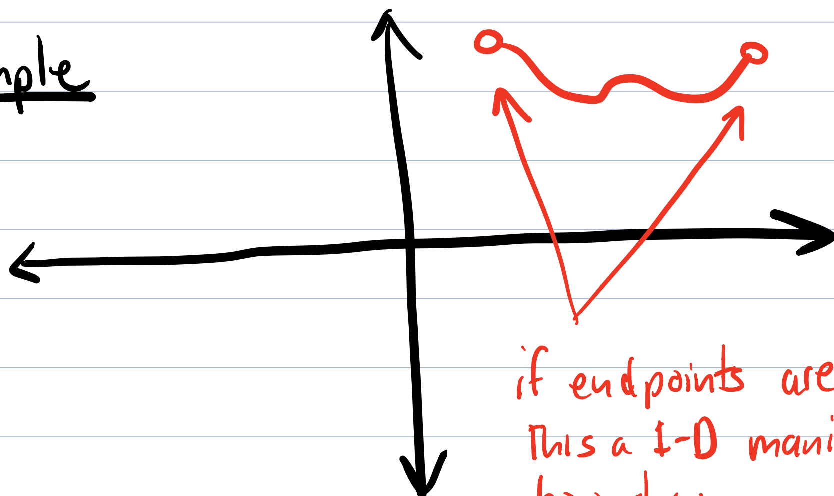




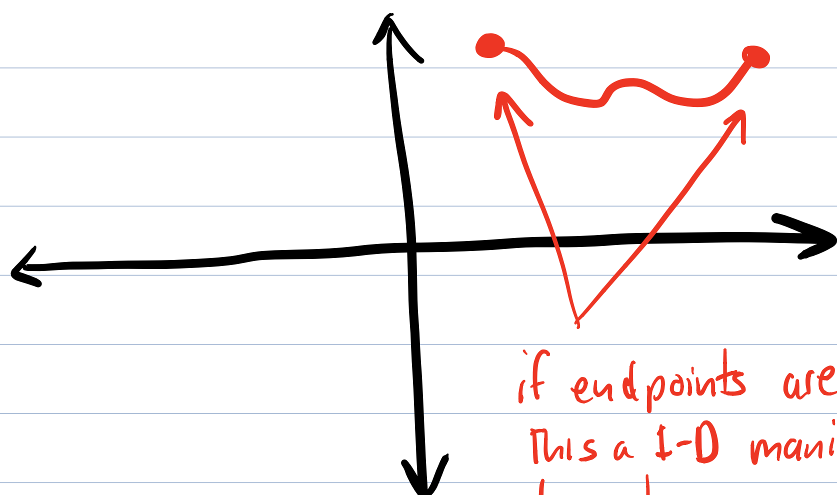
Mobius band with the edge circle removed is a manifold without boundary.



Example



if endpoints are not included, This a 1-D manifold without boundary



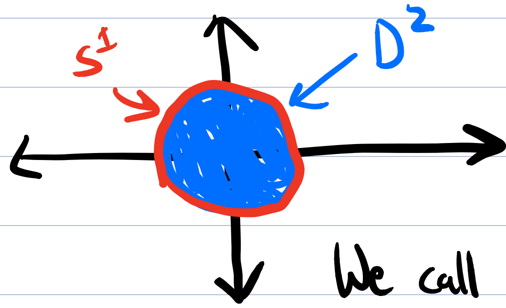
if endpoints are included, This a 1-D manifold with boundary

Cell complexes

Define $S^{n-1} \subseteq D^n \subseteq \mathbb{R}^n$ by

$$D^n = \{\vec{x} \in \mathbb{R}^n \mid d_2(\vec{x}, 0) \leq 1\}$$
$$S^{n-1} = \{\vec{x} \in \mathbb{R}^n \mid d_2(\vec{x}, 0) = 1\}.$$

D^n is a manifold-with-boundary.
The set of (manifold) boundary points is S^{n-1}.



In this case, manifold boundary points and boundary points in the earlier sense are the same.

We call D^n a closed disk.

D^0 is considered to be a point.
S^0 is considered to be the empty set

by

A <u>cell complex</u> is a topological space built by iteratively gluing together closed disks along their boundary.