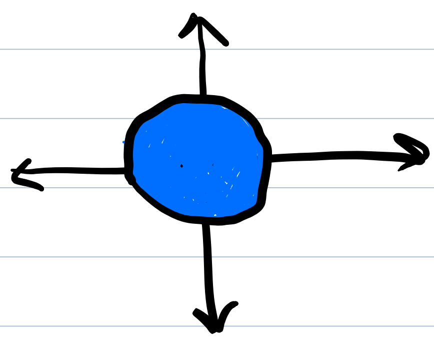
AMAT 342 Lec 26 12/5/19 Last lecture!

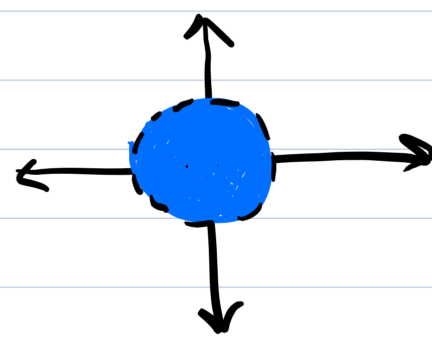
<u>Today</u>: Classification of Surfaces (2-D manifolds)
Topological Data Analysis (ams.org/opportunities)

<u>Review</u> $S \subset \mathbb{R}^n$ is <u>closed</u> if S contains all of its boundary points.

<u>Examples</u> $D^2 = \{(x,y) \mid x^2+y^2 \leq 1\}$ is closed

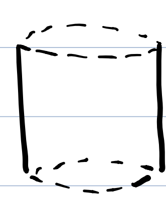


$B(0,1) = \{(x,y) \mid x^2+y^2 < 1\}$ is not closed

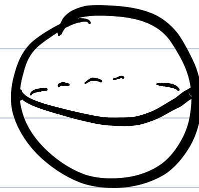


$S^1 \times I$ is closed

$S^1 \times (0,1)$ is not closed (not open either)



S^2 is closed



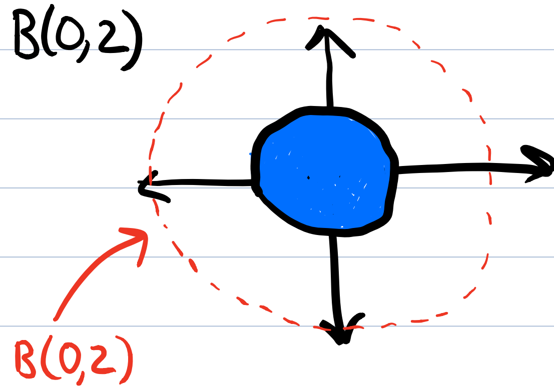
Def: $S \subset \mathbb{R}^n$ is **bounded** if $S \subset B(x,r)$ for some $x \in \mathbb{R}^n$, $r > 0$.

r is allowed to be quite large.

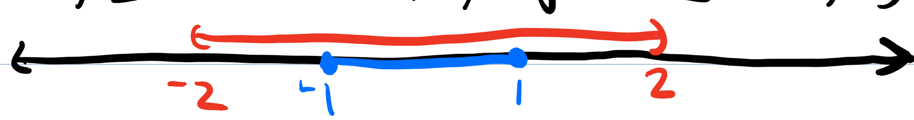
Examples:

- Every one of the examples I drew above is bounded.

e.g. $D^2 \subset B(0,2)$



$[-1,1] \subset \mathbb{R}$ is bounded, e.g. $[-1,1] \subset (-2,2)$



$[0,\infty)$ is not bounded.

- $\mathbb{R}^n$ is not bounded for any n.
- $\{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$ is not bounded

Def: $S \subset \mathbb{R}^n$ is called compact if it is both closed and bounded.

Remark: Compactness can be defined in more generality, using open-sets language. But we won't bother with that here.

Exercise: Which of the following spaces are compact?

a) $\mathbb{R}^2$ × not bounded

b) $\{(x,x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ × not bounded

c) S^2 (sphere) ✓

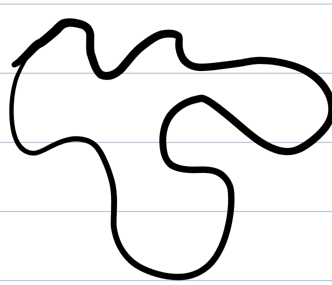
d) Torus  ✓

e) $S^1 \times (0,1)$ × not closed

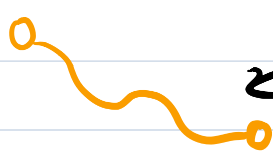
Classification of Curves + Surfaces

Theorem: Every path connected compact 1-D manifold is homeomorphic to S^1

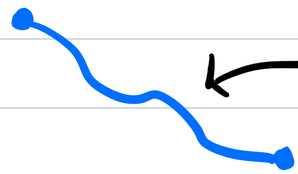
Examples



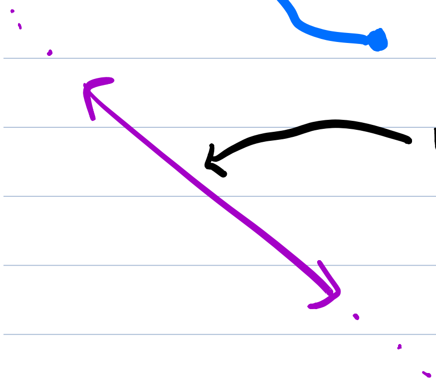
← compact



← not compact (not closed)



not a manifold (but is a manifold with boundary)



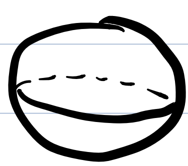
not compact (not bounded)

Case of Surfaces

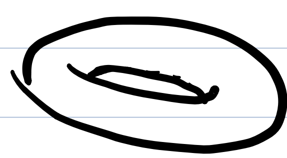
Def: A *surface* is a 2-D manifold

Informal Def: A path connected surface is orientable if it has a separate inside and outside.

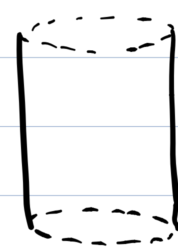
Orientable



S^2

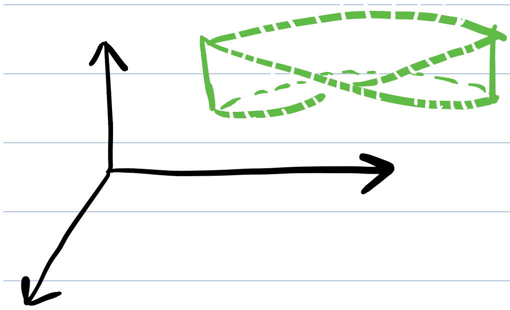


Torus

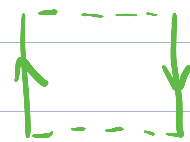


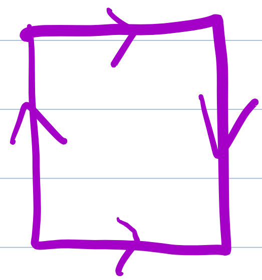
$S^1 \times (0,1)$

Non-orientable

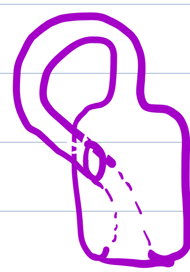


Mobius band with the edge circle removed is a manifold (without boundary.)

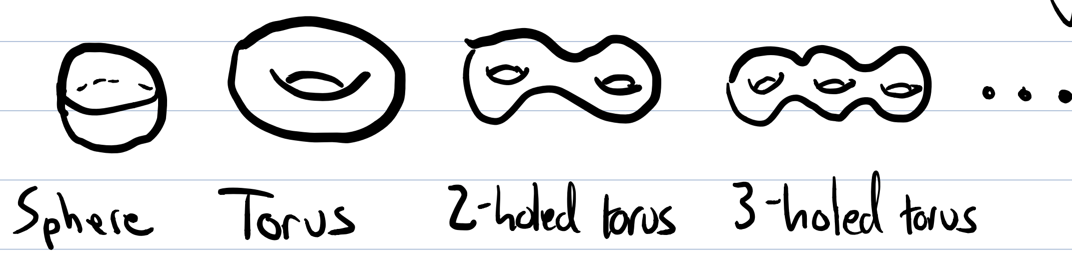




Klein bottle



Theorem: Any path connected, compact, orientable surface is homeomorphic to one of the following:



(If we regard the sphere as a 0-holed torus, then these are the n-holed tori, $n \geq 0$)

For details on the proof, see Carlsson.

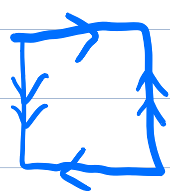
Non-oriented version

Theorem: There is a sequence of surfaces
$K_1, K_2, K_3, \ldots$

such that any path connected, compact, non-orientable surface is homeomorphic to one of the K_i.

K_2 = Klein bottle

K_1 = "the projective plane" (Embeds in $\mathbb{R}^4$ but not $\mathbb{R}^3$)



Equivalent definition: K_1 is the quotient space of the sphere obtained by gluing each $\vec{x}$ to $-\vec{x}$.

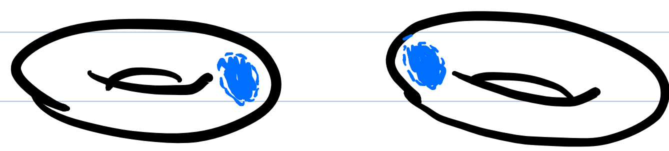
Fact: Any non-orientable surface embeds in $\mathbb{R}^4$ but not in $\mathbb{R}^3$.

For $n \geq 2$, K_n is defined as the connected sum of n copies of the projective plane.

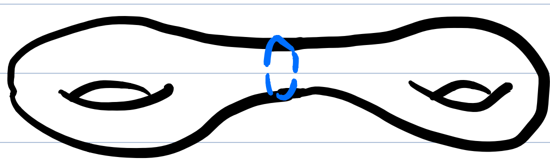
In general, for surfaces M, N, the connected sum $M \oplus N$ of M and N is given by

1) Removing a small open disc from M and N
2) Gluing the resulting surfaces along the boundaries of the two removed disks.

<u>Example</u> If T is a torus, $T \oplus T$ is a 2-holed torus



T $\qquad\qquad$ T



$T \oplus T$

Again, for details see Carlsson.

Topology + Data (What I've been working on for the last 10^+ years)

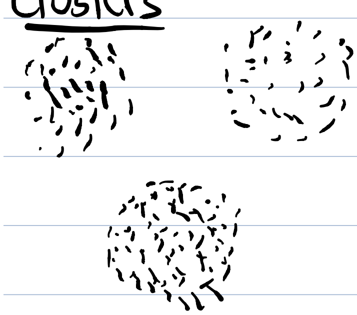
A relatively new branch of applied math
- Very active area of research

Goal: Use topology to study the **shape** of (possibly) high-dimensional data.

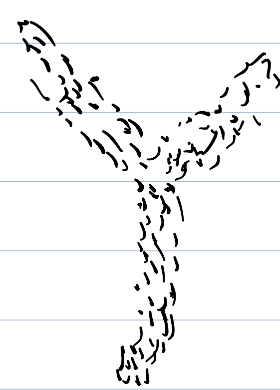
Data = Finite set of points in $\mathbb{R}^n$

Shape of data = Things like

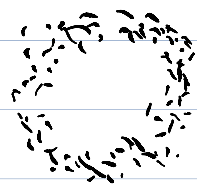
Clusters



Tendrils

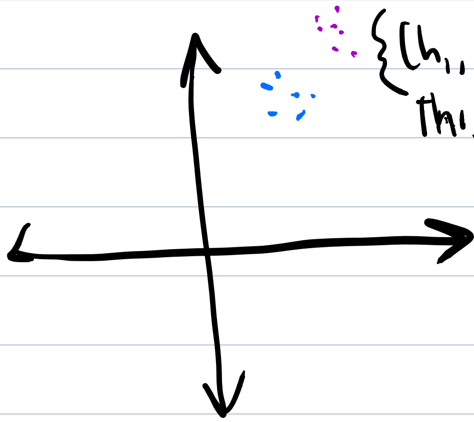


Loops



Example setting: Let (h_i, w_i) denote the height and weight of the i^{th} student in class.

$\{(h_1, w_1), \dots, (h_{16}, w_{16})\} \subset \mathbb{R}^2$ will like this



(2 clusters, because men on average are heavier and taller than women.)

If we record height, weight, age, blood pressure, and time to run a mile, then we get a set of points in $\mathbb{R}^5$, not $\mathbb{R}^2$.

To use topology to study the data, we first thicken the data:

Let $X \subset \mathbb{R}^n$ be the data

Choose $r > 0$, and consider the set

$$T(x, \delta) = \bigcup_{y \in X} B(y, r)$$