AMAT 342 Lecture 4, Sept 5, 2019

Continuous Functions

As noted last time, we will consider the continuity of functions between subsets of Euclidean spaces.

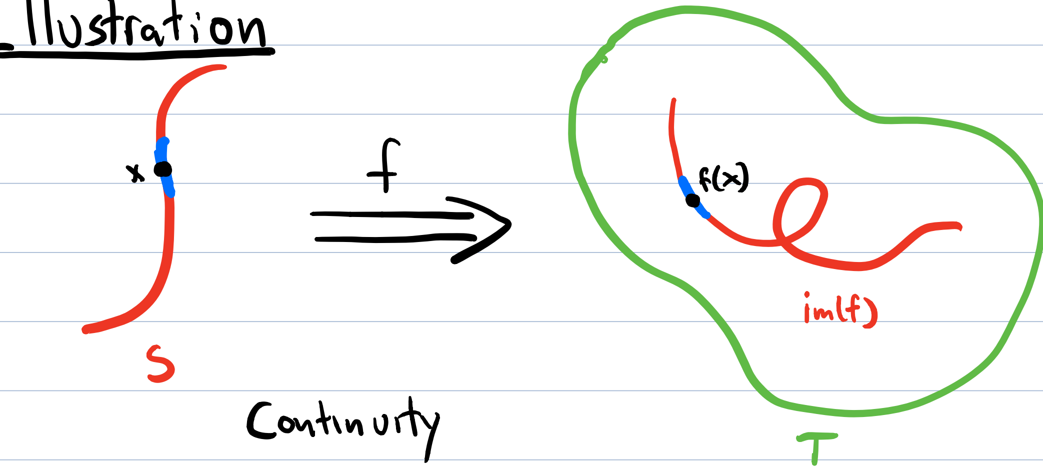
For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$
$y = (y_1, \ldots, y_n) \in \mathbb{R}^n$,

let $d(x,y)$ denote the Euclidean distance between x and y, i.e.,

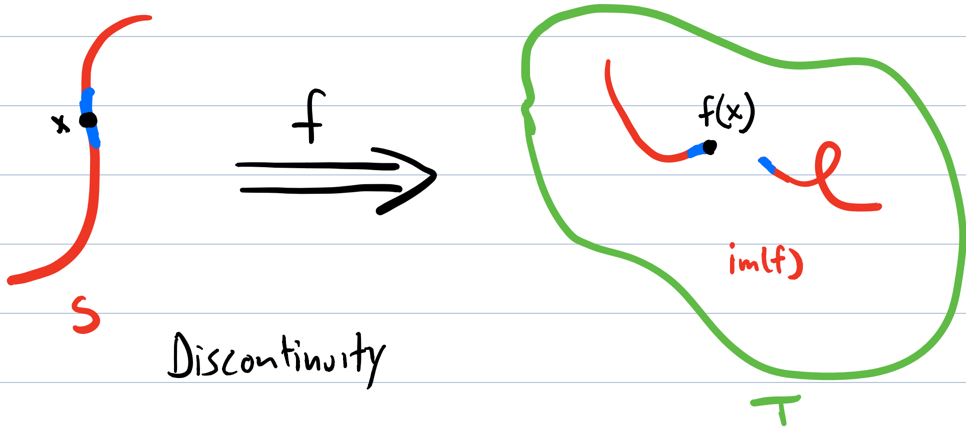
$$d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2}$$
$$= \|x-y\|.$$

Let $S \subset \mathbb{R}^m$ and $T \subset \mathbb{R}^n$ for some $n, m \geq 1$.
Intuitively, a function $f: S \to T$ is continuous if f maps nearby points to nearby points.

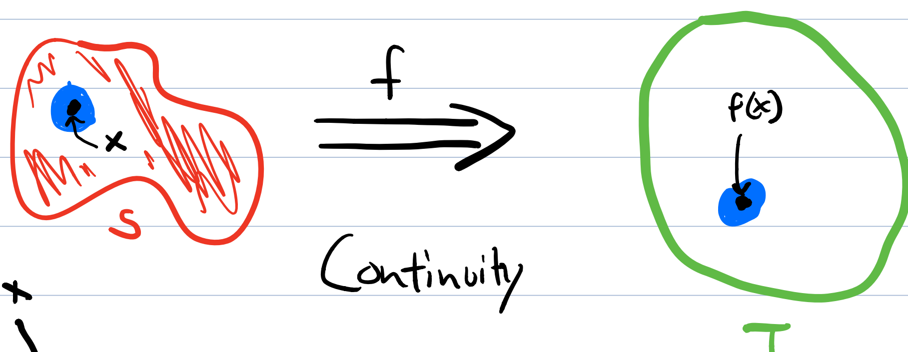
Illustration



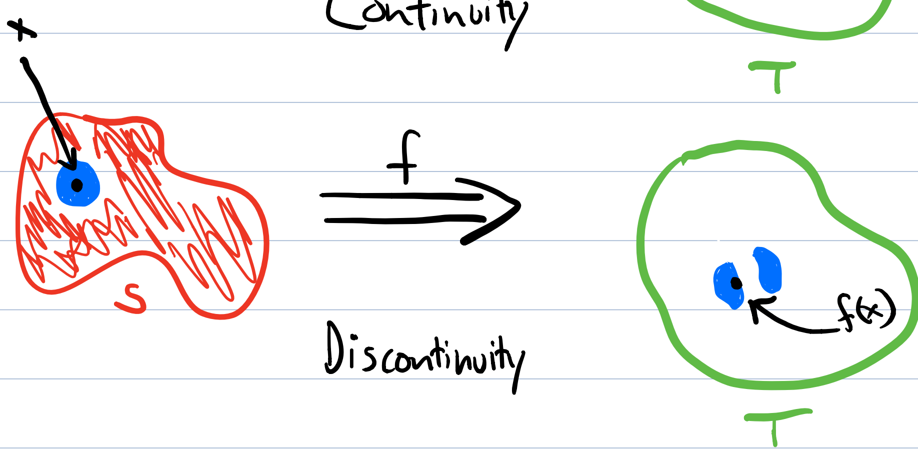
Continuity



Discontinuity



Continuity



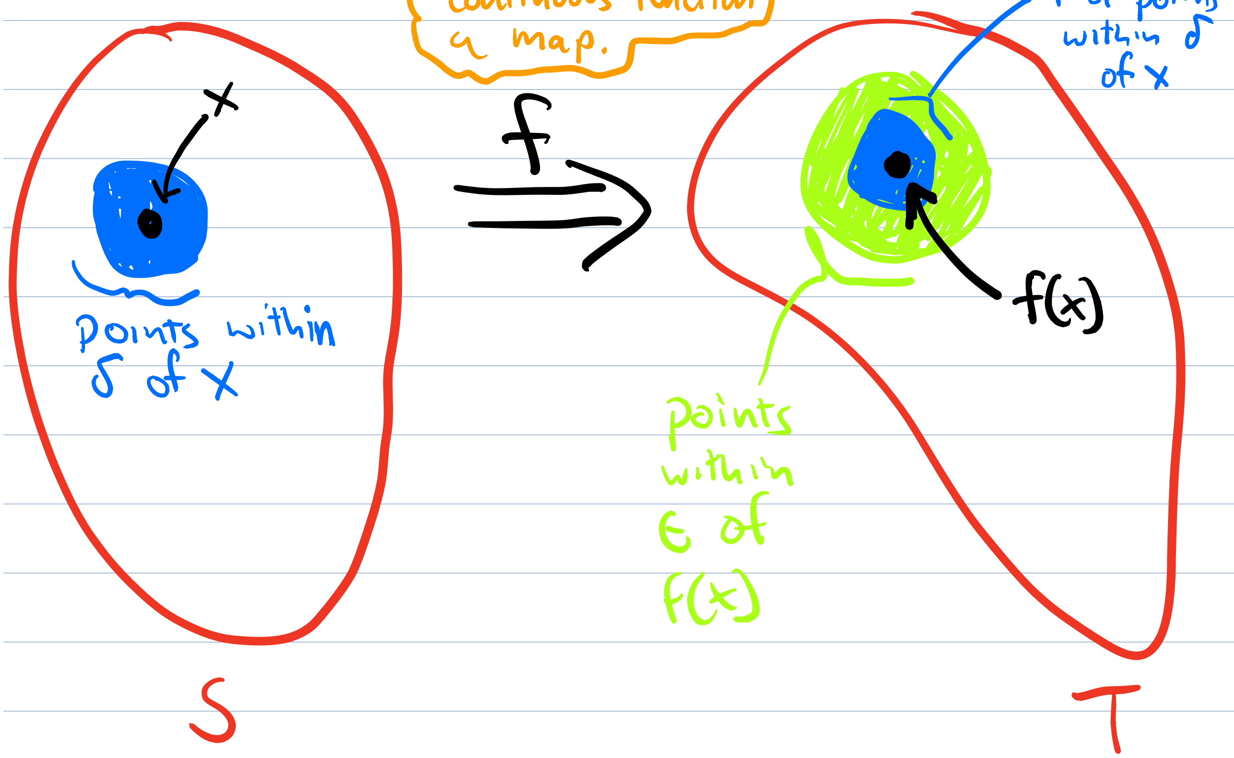
Discontinuity

Formal Definition

We say $f: S \to T$ is continuous at $x \in S$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in S$ and $d(x,y) < \delta$, then $d(f(x), f(y)) < \epsilon$.

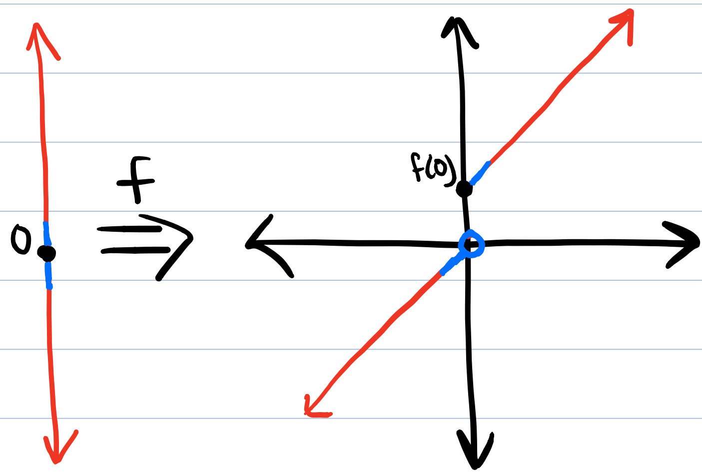
We say f is continuous if it is continuous at all $x \in S$.

Sometimes we call a continuous function a map.



Interpretation: You give me any positive ϵ no matter how small. Continuity at x means that I can choose a positive δ such that points within distance δ of x map under f to points within distance ϵ of $f(x)$. (I'm allowed to choose δ as small as I want, as long as it's positive.)

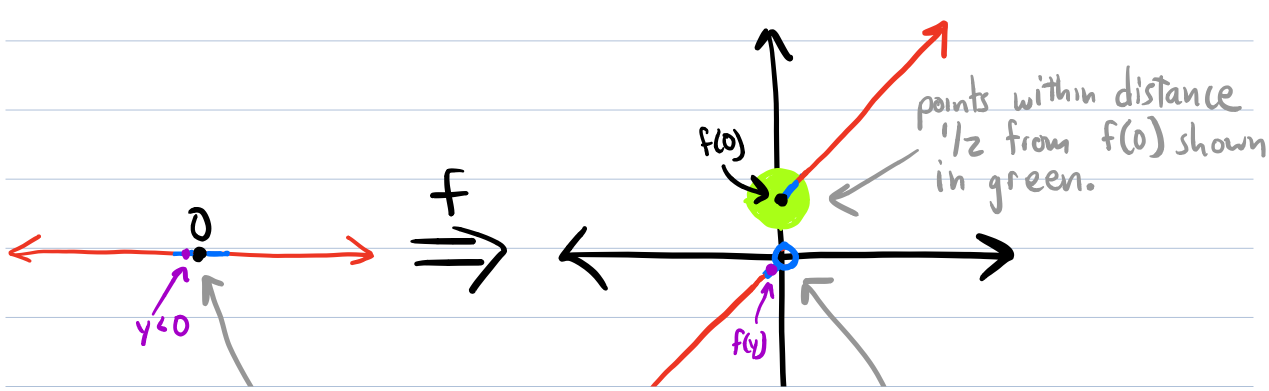
Example: Consider $f: \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = \begin{cases} (x, x+1) & \text{if } x < 0 \\ (x, x) & \text{if } x \geq 0 \end{cases}$

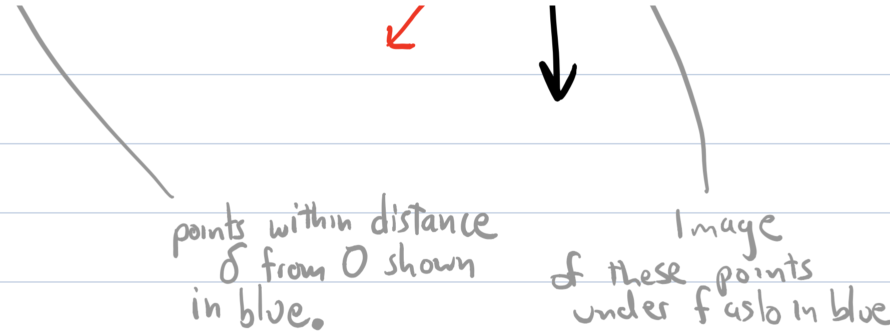


Since f "splits the line" at 0, we expect that f is not continuous. Lets check this using the formal definition of continuity.

Proof that f is not continuous

Let $\epsilon = 1/2$.





No matter how small we take δ, if $y<0$ and $d(0,y)<\delta$, then $d(f(0), f(y)) > \frac{1}{2}$.
Hence f is not continuous at 0.

Examples of continuous functions.

Elementary functions $U \to \mathbb{R}$ (with $U \subseteq \mathbb{R}$) from calculus are continuous at each point where they are defined, e.g.:

- $\sin x$, $\cos x$, $\log x$, e^x, polynomials
- sums, products, and quotients of these.

4 facts (moral: functions that you think would be continuous usually are).

1) If $f: S \to T$ and $g: T \to U$ are both continuous, then $g \circ f: S \to U$ is continuous.

2) If $S \subset T \subset \mathbb{R}^n$, then the inclusion map $j: S \to T$ given by $j(x)=x$ is continuous.

3) If $U \subset \mathbb{R}^m$ and $f_1, f_2, \dots, f_n : U \to \mathbb{R}$ are continuous, then $(f_1, f_2, \dots, f_n) : U \to \mathbb{R}^n$, given by $(f_1, f_2, \dots, f_n)(x) = (f_1(x), f_2(x), \dots, f_n(x))$ is continuous.

4) If $f : S \to T$ is continuous then the the map $\tilde{f} : S \to \mathrm{im}(f)$ defined by $\tilde{f}(x) = f(x)$ is continuous.

In this class, we won't spend too much time worrying about the rigorous definition of continuity, but I do want you to be familiar with it.

Homeomorphism

For S, T subsets of Euclidean spaces, A function $f : S \to T$ is a homeomorphism if

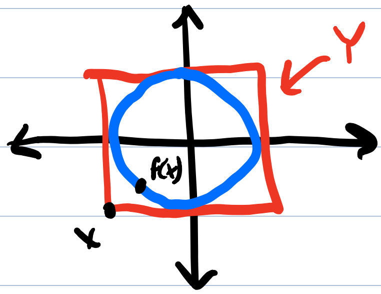
1) f is a continuous bijection ← bijection = has inverse
2) The inverse of f is also continuous.

Homeomorphism is the main notion of continuous deformation we'll consider in this course.

If $\exists$ a homeomorphism $f : S \to T$, we say S and T are homeomorphic.

In this class, "topologically equivalent" = homeomorphic.

Example Let $Y \subset \mathbb{R}^2$ be the square of side length 2, embedded in the plane as shown



The function $f: Y \to S^1$ given by $f(x) = \frac{x}{\|x\|}$ is a homeomorphism.

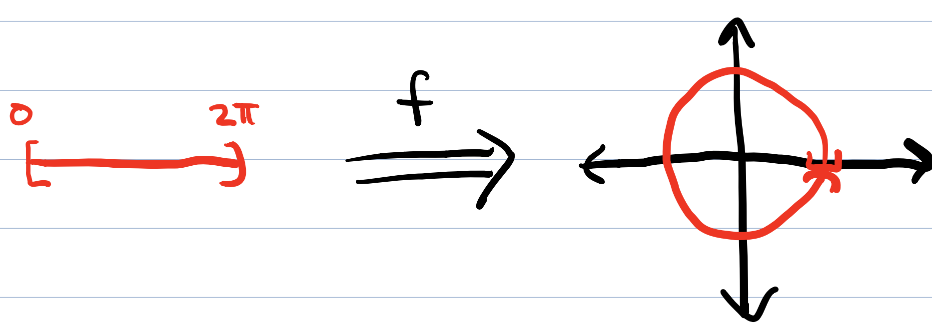
where $\|x\|$ = distance of x to origin $= \sqrt{x_1^2 + x_2^2}$

By facts above, this is continuous.

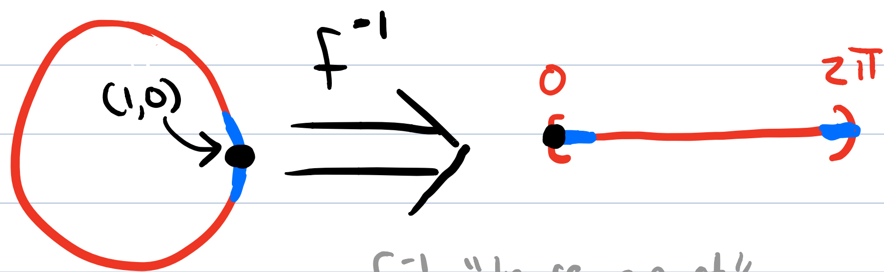
It is intuitively clear that this is a bijection with a continuous inverse. The inverse can be written down, but we won't bother.

[Lecture ended here.]

Example: Consider the function $f: [0, 2\pi) \longrightarrow S^1$ from last lecture given by $f(x) = (\cos x, \sin x)$.



f is continuous, and we saw last lecture that it is a bijection. However, $f^{-1}: S^1 \to [0, 2\pi)$ is not continuous at $(1,0)$. (And therefore, f is not continuous.)



f^{-1} "tears apart" any small neighborhood around $(1,0)$.

<u>Note</u>: The fact that f is not a homeomorphism doesn't imply that $[0, 2\pi)$ and S^1 are not homeomorphic. In fact they are not, and we will explain why soon.

<u>Basic Facts About Homeomorphisms.</u>

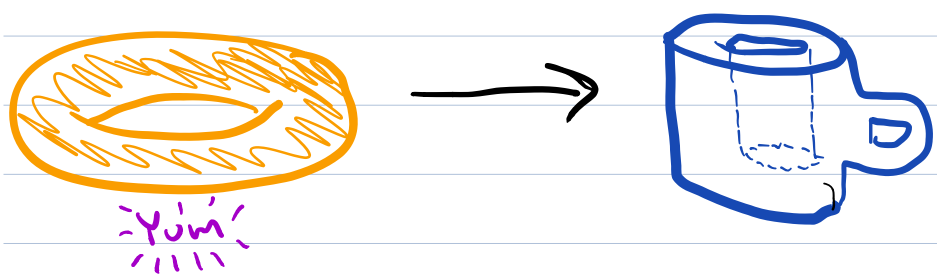
- Clearly, if $f: S \to T$ a homeomorphism, then f^{-1} is a homeomorphism.

- If $f: S \to T$ and $g: T \to U$ are homeomorphisms, then $g \circ f: S \to T$ is a homeomorphism (w/ inverse $f^{-1} \circ g^{-1}$)

Example: Returning to examples from the 1st day of class, consider the capital letters as unions of curves (no thickness)

D and O are homeomorphic
T, Y, and J, E, and F are G homeomorphic
C, S, and Z homemorphic.
X and K are homeomorphic (at least, the way I write K.)

Example: The donut and coffee mug are homeomorphic



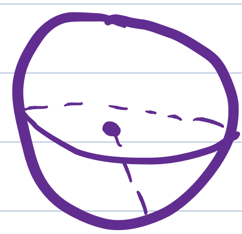
Isotopy

All of the pair of homeomorphic spaces we've seen so far are topologically equivalent in a sense that's stronger than homeomorphism, called isotopy.

The definition of isotopy is closer to the "rubber-sheet geometry" idea of continuous deformation that we introduced on the first day.

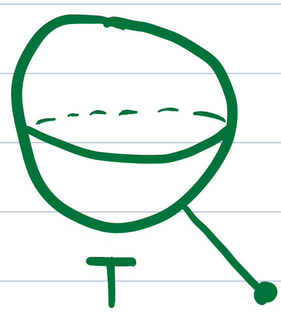
Motivating example

Let $S, T \subset \mathbb{R}^2$ be as illustrated:



S

S is a unit circle with a line segment attached to one point. The line segment points inward.



T

T is also a unit circle with a line segment attached to the same point, but now line segment points outward.

It is clear that there is a homeomorphism $f: S \to T$.
However, it's also clear that if S and T were made out of stretchy rubber, there is no way we could deform S into T without tearing. The line segment would have to pass through the sphere.

Formally, we express this idea using isotopy.

To define isotopy, we need to first define homotopy. Homotopy is a notion of continuous deformation for functions (rather than spaces).

For S a set and $h : S \times I \to T$ a function, and $t \in I$, let $h_t : S \to T$ be given by $h_t(x) = h(x,t)$.

Definition: For continuous maps $f, g : S \to T$ a homotopy between f and g is a continuous map

$$h : S \times I \to T$$

such that $h_0 = f$, $h_1 = g$.