AMAT 342 Lecture 5, 9/10/18

Last time, we discussed continuous functions between subsets of Euclidean spaces.

Recall: informally, a continuous function $f: S \to T$ is a function that
- maps S to T without "tearing S" or to put it differently,
- sends nearby points to nearby points.

We also gave a formal definition of continuity using ϵ-δ language, which I'll not repeat here.

Recall the following key definition from last time:

Homeomorphism

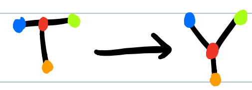
For S, T subsets of Euclidean spaces,
A function $f: S \to T$ is a homeomorphism if

1) f is a continuous bijection
2) The inverse of f is also continuous.

Last lecture, we looked one example illustrating this definition. We now consider several more.

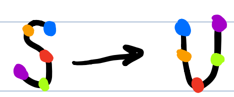
Example: Consider the capital letters as unions of curves in the plane with no thickness.

T is homeomorphic to Y:

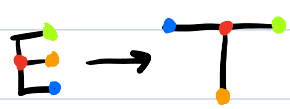


for example, one can define a homeomorphism T→Y which sends each of the colored points of T above to the point of Y of the same color.

S is homeomorphic to U:



E is homeomorphic to T:



O is not homeomorphic to S. Intuitively, any bijection O→S must "cut the O" somewhere, so cannot be continuous.

<u>Note</u>: In general, subsets of $\mathbb{R}^2$ with different "#'s of holes" are not homeomorphic. (Making this formal requires ideas from algebraic topology that we will not discuss right now.)

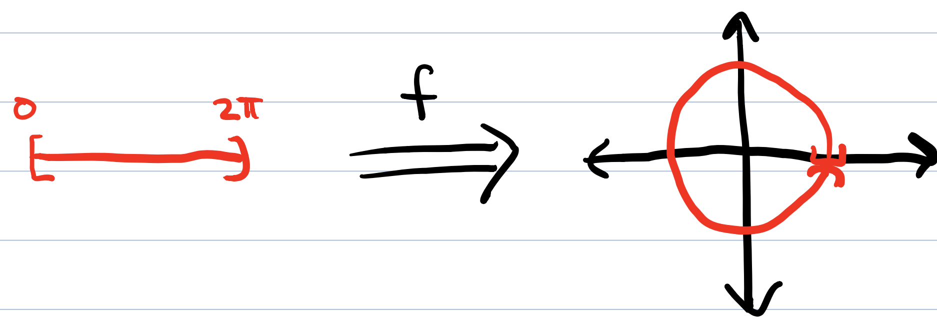
Example: B is not homeomorphic to any other letter, because B is the only capital letter with two holes.

Example X is not homeomorphic to Y.

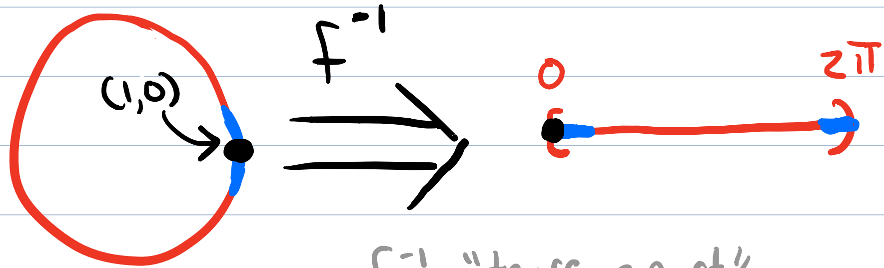
Explanation: X has a point where 4 line segments meet, Y does not. Using this, one can show that X and Y are not homeomorphic.

Example: The same type of argument shows that Y and S are not homeomorphic.

Example: Consider the function $f:[0,2\pi) \longrightarrow S^1$ from last lecture given by $f(x) = (\cos x, \sin x)$.



f is continuous, and we saw last lecture that it is a bijection. However, $f^{-1}: S^1 \longrightarrow [0,2\pi)$ is not continuous at $(1,0)$. (And therefore, f is not a homeomorphism.



f^{-1} "tears apart" any small neighborhood around $(1,0)$.

Note: The fact that f is not a homeomorphism doesn't imply that $[0, 2\pi)$ and S^1 are not homeomorphic. But in fact they are not. (They have different #'s of holes.)

Basic Facts About Homeomorphisms.

- Clearly, if $f: S \to T$ a homeomorphism, then f^{-1} is a homeomorphism.

- If $f: S \to T$ and $g: T \to U$ are homeomorphisms, then $g \circ f: S \to T$ is a homeomorphism (w/ inverse $f^{-1} \circ g^{-1}$)

as an immediate consequence, if X and Y are homeomorphic, and Y and Z are homeomorphic, then X and Z are homeomorphic.

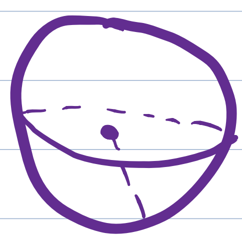
Isotopy

All of the pair of homeomorphic spaces we've seen so far are topologically equivalent in a sense that's stronger than homeomorphism, called isotopy.

The definition of isotopy is closer to the "rubber-sheet geometry" idea of continuous deformation that we introduced on the first day.

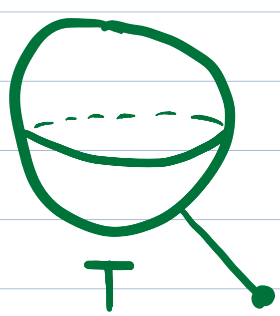
Motivating example

Let $S, T \subset \mathbb{R}^3$ be as illustrated:



S

S is a unit circle with a line segment attached to one point. The line segment points inward.



T

T is also a unit circle with a line segment attached to the same point, but now line segment points outward.

S and T are homeomorphic.

However, if S and T were made of rubber, we couldn't deform S into T without tearing. The line segment would have to pass through the sphere.

Formally, we express this idea using <u>isotopy</u>.

[lecture ended around here.]

To define isotopy, we need to first define <u>homotopies</u> and <u>embeddings</u>

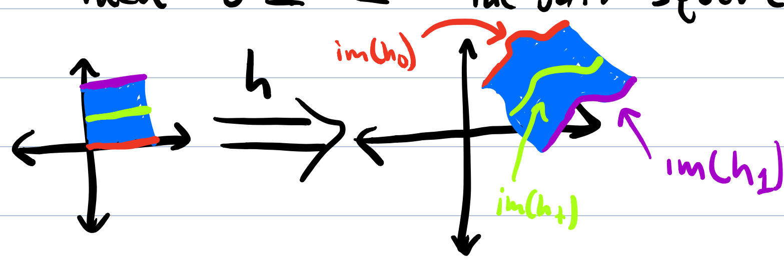
Homotopy is a notion of continuous deformation for <u>functions</u> (rather than spaces).

For S a set, $h: S \times I \to T$ a continuous function and $t \in I$, let $h_t : S \to T$ be given by $h_t(x) = h(x,t)$.

<u>Interpretation</u>: we can think of h as a family of continuous functions $\{h_t \mid t \in I\}$ from S to T evolving in time. (We interpret t as time.) The continuity of h means that h_t "evolves continuously" as t changes.

Example: $S = I$, $T = \mathbb{R}^2$.

Then $S \times I = I^2 =$ The unit square.



Each $h_t : I \to \mathbb{R}^2$ species a curve in $\mathbb{R}^2$.
As t increases, these cuves evolve continuously

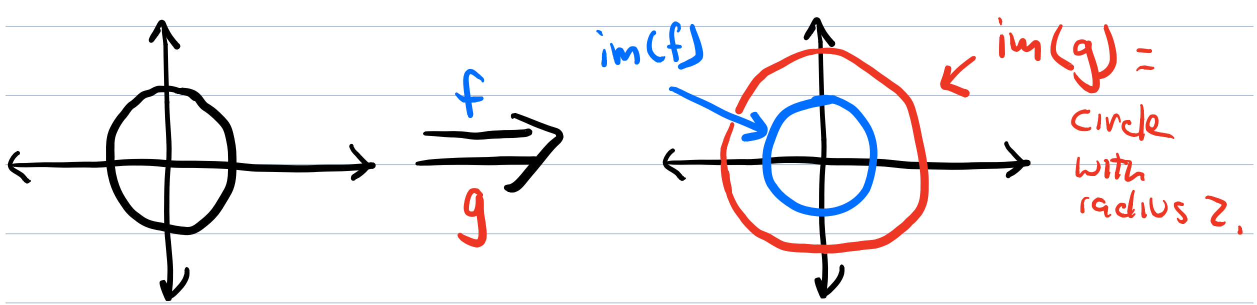
Definition: For continuous maps $f, g : S \to T$ a *homotopy* from f to g is a continuous map

$$h : S \times I \to T$$

such that $h_0 = f$ and $h_1 = g$.

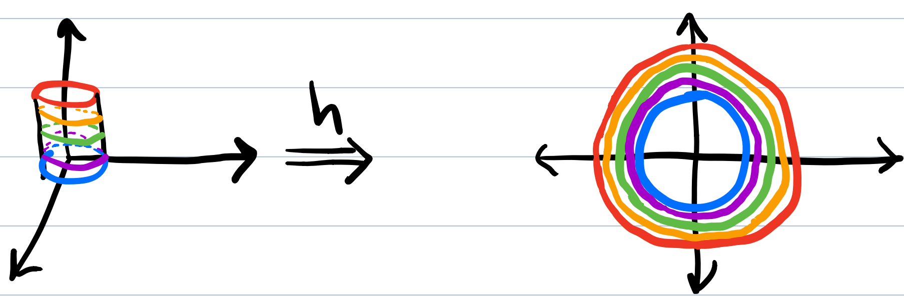
Example $f, g : S^1 \to \mathbb{R}^2$ $\quad f(\vec{x}) = \vec{x}$ (f is the inclusion map.)

unit circle in $\mathbb{R}^2$. $\quad g(\vec{x}) = 2\vec{x}$



Let $h: S^1 \times I \to \mathbb{R}^2$ be given by
$h(x,t) = (1+t)\vec{x}$.
Then $h_t: S^1 \to \mathbb{R}^2$ is given by $h_t(x) = (1+t)\vec{x}$,
and clearly $h_0 = f$, $h_1 = g$.

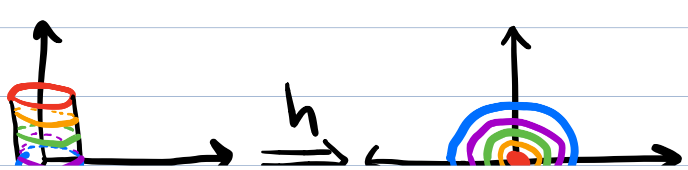
$S^1 \subset \mathbb{R}^2$ and $I \subset \mathbb{R}$, so $S^1 \times I \subset \mathbb{R}^3$.
In fact, $S^1 \times I$ is a cylinder, and the following illustrates h:

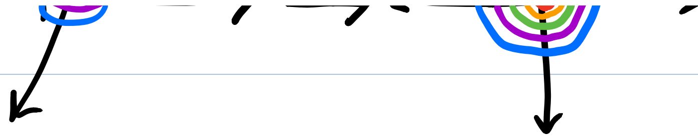


$\mathrm{im}(h_t)$ is shown above for $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

Example: Let $f: S^1 \to \mathbb{R}^2$ be the inclusion map, and
This example is similar to the last one and will be skipped in class.
let $g: S^1 \to \mathbb{R}^2$ be given by
$g(x) = (0,0)$ for all $x \in S^1$.

We specify a homotopy $h: S^1 \times I \to \mathbb{R}^2$ from f to g by
$h(\vec{x}, t) = (1-t)\vec{x}$





Note that $\mathrm{im}(h_t)$ is a circle for $t<1$ and a point for $t=1$. As above, $\mathrm{im}(h_t)$ is shown for $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

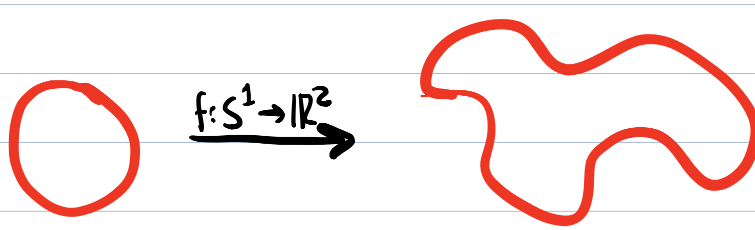
Embeddings

<u>Recall:</u> for any function $f: S \to T$, there is an associated function onto the image of f, namely

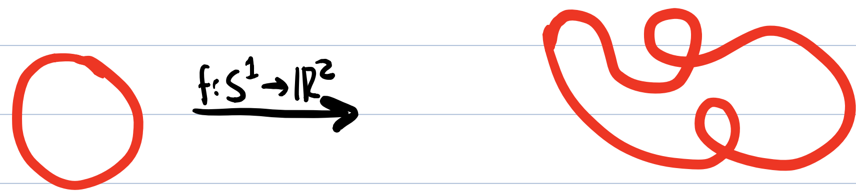
$$\tilde{f}: S \to \mathrm{im}(f)$$

given by $\tilde{f}(x) = f(x)$. That is f and $\tilde{f}$ are given by the same rule, but the codomain of $\tilde{f}$ is as small as possible.

<u>Def:</u> A continuous map $f: S \to T$ is an <u>embedding</u> if f is a $\underbrace{\text{homeomorphism onto its image}}_{\text{i.e., } \tilde{f} \text{ is a homeomorphism}}$. (for concreteness, think of T as $\mathbb{R}^n$)



embedding



not an embedding

Fact: Any embedding is an injection but not every continuous injection is an embedding.

Proof of injectivity: If $\tilde{f}$ is a homeomorphism then it is bijective, hence injective. $f = j \circ \tilde{f}$, where $j: \mathrm{im}(f) \to T$ is the inclusion map. j is injective. The composition of two injective functions is injective, so f is injective. $\blacksquare$

Example: The following illustrates that a continuous injection is not necessarily an embedding

Consider $f: [0, 2\pi) \to \mathbb{R}^2$, $f(x) = (\cos x, \sin x)$. We seen above that $\tilde{f}$ is a bijection but not a homeomorphism.

Isotopy

Definition: For $S, T \subset \mathbb{R}^n$ an isotopy from S to T is a homotopy $h: S \times I \to \mathbb{R}^n$ such that

$$h_0 = Id_S, \quad im(h_1) = T,$$

$$h_t : S \to \mathbb{R}^n \text{ is an embedding for all } t \in I.$$

If there exists an isotopy from S to T, we say S and T are isotopic.

Interpretation: - $im(h_t)$ is the snapshot at time t of a continuous deformation from S to T.
- Continuity of h ensures that these "snapshots" evolve continuously in time.

Example: Let $T \subset \mathbb{R}^2$ be the circle of radius 2 centered at the origin.

The homotopy $h: S^1 \times I \to \mathbb{R}^2$, $h(\vec{x}, t) = (1+t)\vec{x}$ in the example above is an isotopy from S to T.

Note: If S and T are isotopic, then they are homeomorphic; for h any isotopy from S to T, h_1 is a homeomorphism from S to T.

Explanation: $h_1 : S \to \mathbb{R}^n$ is an embedding, hence a homeomorphism onto its image. But $\mathrm{im}(h_1) = T$.

Example: Let L be the left semi-circle in $\mathbb{R}^2$, i.e.,