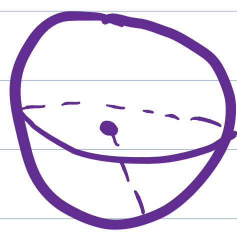
AMAT 342, Sept. 12, 2018

Today: Homotopy
Embeddings
Isotopy

As mentioned in the last lecture, isotopy is a formal notion continuous deformation that models the temporal evolution of a geometric object as it is deformed.

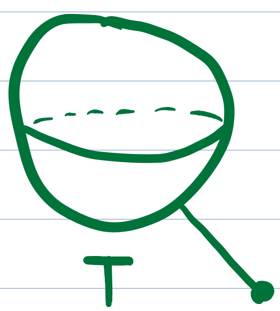
Motivating example (review from last time)

Let $S, T \subset \mathbb{R}^2$ be as illustrated:



S

S is a unit circle with a line segment attached to one point. The line segment points inward.



T

T is also a unit circle with a line segment attached to the same point, but now line segment points outward.

review

S and T are homeomorphic.

However, if S and T were made of rubber, we couldn't deform S into T without tearing. The line segment would have to pass through the sphere.

Formally, we express this idea using <u>isotopy</u>.

To define isotopy, we need to first define <u>homotopies</u> and <u>embeddings</u>

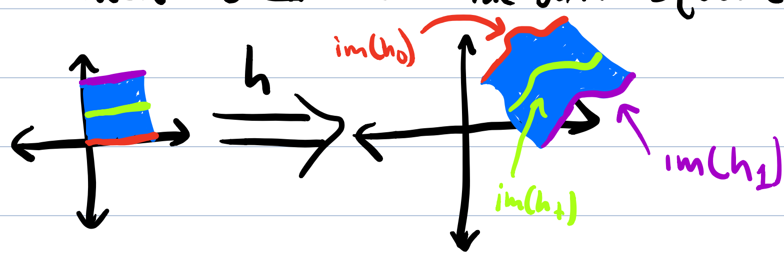
Homotopy is a notion of continuous deformation for <u>functions</u> (rather than spaces).

For $S \subset \mathbb{R}^n$, $h : \overbrace{S \times I}^{\text{thickening of } S} \to T$ a continuous function and $t \in I$, let $h_t : S \to T$ be given by $h_t(x) = h(x,t)$.

<u>Interpretation</u>: we can think of h as a family of continuous functions $\{h_t \mid t \in I\}$ from S to T evolving in time. (We interpret t as time.) The continuity of h means that h_t "evolves continuously" as t changes.

Example: $S = I$, $T = \mathbb{R}^2$.

Then $S \times I = I^2$. The unit square.



Each $h_t : I \to \mathbb{R}^2$ species a curve in $\mathbb{R}^2$.
As t increases, these cuves evolve continuously

Definition: For continuous maps $f, g : S \to T$ a <u>homotopy</u> from f to g is a continuous map

$$h : S \times I \to T$$

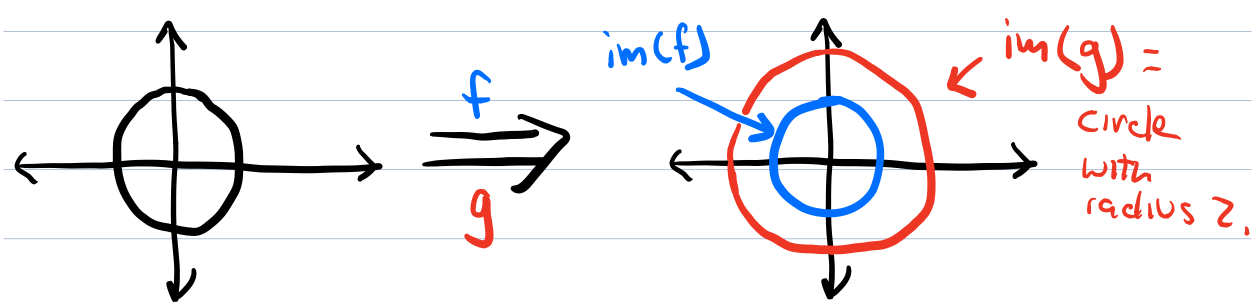
such that $h_0 = f$ and $h_1 = g$.

<u>Note</u>: We will see that isotopy is a special kind of homotopy!

[<u>Note</u>: Any continuous map $h : S \times I \to T$ is a homotopy from h_0 to h_1.]

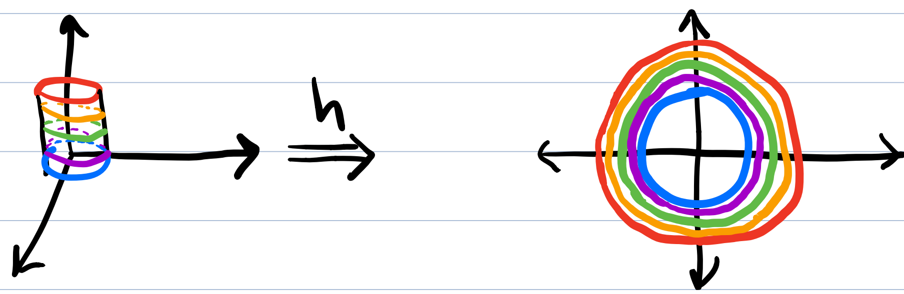
Example $f, g : S^1 \to \mathbb{R}^2$ $\quad f(\vec{x}) = \vec{x}$ (f is the inclusion map.)

unit circle in $\mathbb{R}^2$. $\quad g(\vec{x}) = 2\vec{x}$



Let $h: S^1 \times I \longrightarrow \mathbb{R}^2$ be given by
$h(\vec{x}, t) = (1+t)\vec{x}$.
Then $h_t: S^1 \to \mathbb{R}^2$ is given by $h_t(\vec{x}) = (1+t)\vec{x}$,
and clearly $h_0 = f$, $h_1 = g$.

$S^1 \subset \mathbb{R}^2$ and $I \subset \mathbb{R}$, so $S^1 \times I \subset \mathbb{R}^3$.
In fact, $S^1 \times I$ is a cylinder, and the following illustrates h:



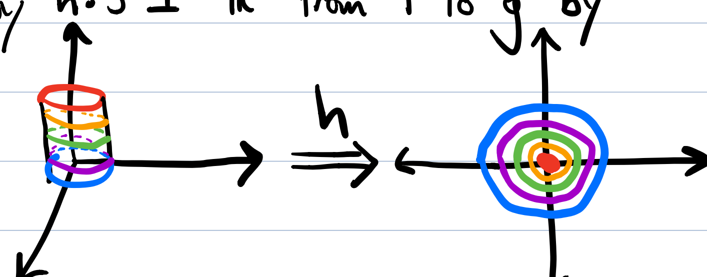
$im(h_t)$ is shown above for $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

Example: Let $f: S^1 \longrightarrow \mathbb{R}^2$ be the inclusion map, and

This example is similar to the last one and will be skipped in class.

let $g: S^1 \to \mathbb{R}^2$ be given by
$g(x) = (0,0)$ for all $x \in S^1$.

We specify a homotopy $h: S^1 \times I \to \mathbb{R}^2$ from f to g by
$h(\vec{x}, t) = t\vec{x}$.



Note that $\mathrm{im}(h_t)$ is a circle for $t<1$ and a point for $t=1$. as above, $\mathrm{im}(h_t)$ is shown for $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

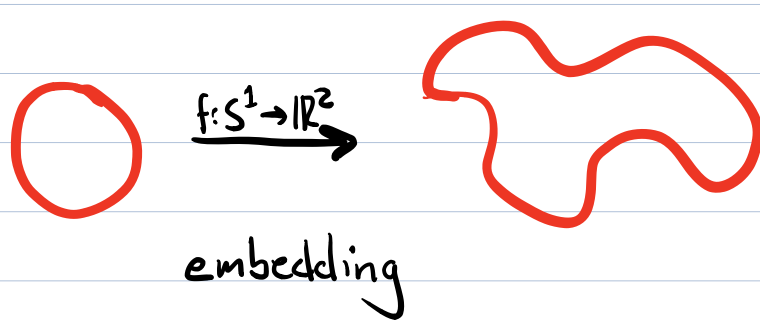
Embeddings

Recall: for any function $f: S \to T$, there is an associated function onto the image of f, namely

$$\tilde{f}: S \to \mathrm{im}(f)$$

given by $\tilde{f}(x) = f(x)$. That is f and $\tilde{f}$ are given by the same rule, but the codomain of $\tilde{f}$ is as small as possible.

Def: A continuous map $f: S \to T$ is an <u>embedding</u> if f is a $\underbrace{\text{homeomorphism onto its image}}_{\text{i.e., } \tilde{f} \text{ is a homeomorphism}}$. (for concreteness, think of T as $\mathbb{R}^n$)



embedding

not an embedding

Fact: Any embedding is an injection but not every continuous injection is an embedding.

__Proof of injectivity:__ If $\tilde{f}$ is a homeomorphism then it is bijective, hence injective. $f = j \circ \tilde{f}$, where $j: \operatorname{im}(f) \to T$ is the inclusion map. j is injective. The composition of two injective functions is injective, so f is injective. $\blacksquare$

__Example:__ The following illustrates that a continuous injection is not necessarily an embedding

Consider $f: [0, 2\pi) \to \mathbb{R}^2$, $f(x) = (\cos x, \sin x)$.
We seen above that $\tilde{f}$ is a continuous bijection but not a homeomorphism.

Isotopy

Definition: For $S, T \subset \mathbb{R}^n$ an <u>isotopy</u> from S to T is a homotopy $h: X \times I \to \mathbb{R}^n$ such that

$$\operatorname{im}(h_0) = S, \quad \operatorname{im}(h_1) = T,$$

in this context, "homotopy" means "continuous map."

$$h_t: X \to \mathbb{R}^n \text{ is an embedding for all } t \in I.$$

If there exists an isotopy from S to T, we say S and T are <u>isotopic</u>. <u>Note:</u> It follows from the definition that X is homeomorphic to both S and T.

See next lecture for an explanation

<u>Interpretation:</u>
- $\operatorname{im}(h_t)$ is the snapshot at time t of a continuous deformation from S to T.
- Continuity of h ensures that these "snapshots" evolve continuously in time.
- Fact that h_t is an embedding ensures all $\operatorname{im}(h_t)$ are homeomorphic.

<u>Example</u>: Let $T \subset \mathbb{R}^2$ be the circle of radius 2 centered at the origin.

The homotopy $h: S^1 \times I \to \mathbb{R}^2$, $h(\vec{x}, t) = (1+t)\vec{x}$ in the example above is an isotopy from S^1 to T. ← circle of radius 2.

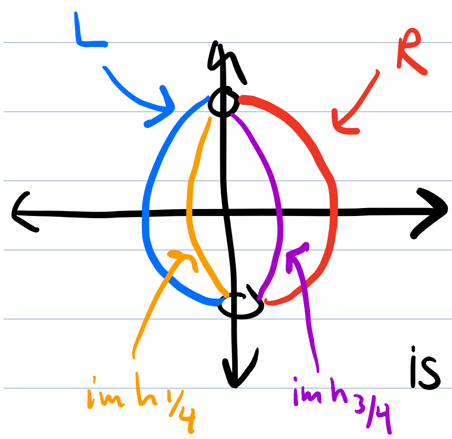
<u>Note</u>: If S and T are isotopic, then they are homeomorphic; for h any isotopy from S to T, $\tilde{h}_1 \circ \tilde{h}_0^{-1}$ is a homeomorphism from S to T.

Explanation: $h_1 : S \to \mathbb{R}^n$ is an embedding, hence a homeomorphism onto its image. But $\mathrm{im}(h_1) = T$.

[Lecture ended here. A lot of this was reviewed in lecture 7]

Example: Let $L = \{(x,y) \in S^1 \mid x < 0\}$

$R = \{(x,y) \in S^1 \mid x > 0\}$.



$h : L \times I \to \mathbb{R}^2$,

$h((x,y), t) = ((1-2t)x, y)$

is an isotopy from L to R.

Explanation: $h_0(x,y) = ((1-0)x, y) = (x,y)$ so $h_0 = \mathrm{Id}_L$.

$h_1(x,y) = (-x, y)$, so $\mathrm{im}\, h_1 = R$.

$h_t(x,y) = ((1-2t)x, y)$.

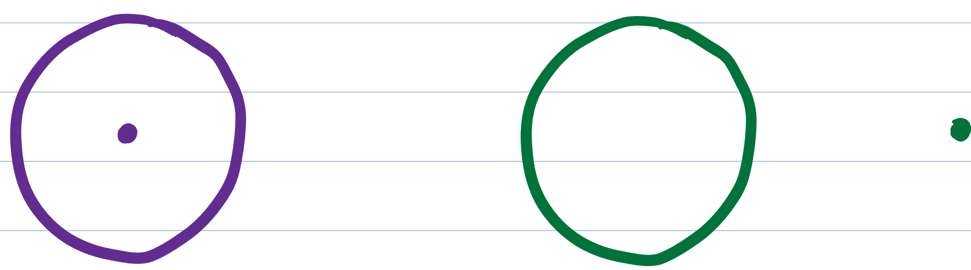
Not hard to check that each h_t is an embedding.

Example:

S and T are not isotopic.

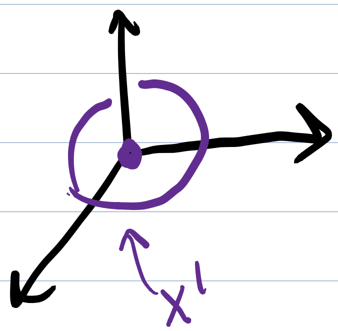
Note: Whether S and T are isotopic depends on where S and T are embedded. (That's not true for homeomorphism!)

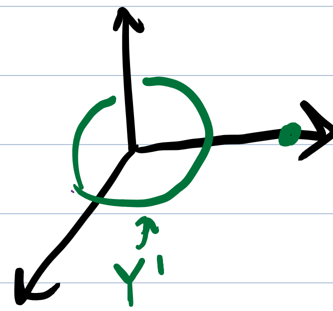
Example $X = S^1 \cup \{0\} \subset \mathbb{R}^2$ $Y = S^1 \cup \{(3,0)\}$.

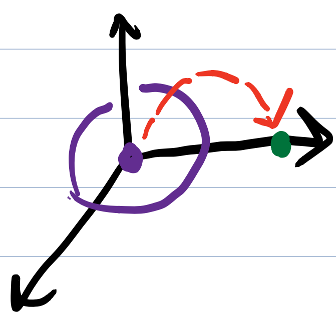


X and Y homeomorphic, not isotopic.
But if we embed X, Y in $\mathbb{R}^3$, then they are isotopic there.

That is, let $X' = \{(x,y,0) \mid (x,y) \in X\} \subset \mathbb{R}^3$
$Y' = \{(x,y,0) \mid (x,y) \in Y\} \subset \mathbb{R}^3$







There's an isotopy $h: X' \times I \to \mathbb{R}^3$ which moves the extra point as shown in red.

Similarly, if we embed S and T of the previous example into $\mathbb{R}^4$, they are isotopic there.

__Facts about isotopies:__ The same properties of

__Symmetry:__ If there exists an isotopy from S to T, then there exists an isotopy from T to S.

"Isotopies can be reversed"

__Pf:__ If $h: X \times I \to \mathbb{R}^n$ is an isotopy from S to T then $\bar{h}: X \times I \to T$, given by $\bar{h}(x,t) = h(x, 1-t)$ is an isotopy from T to S.

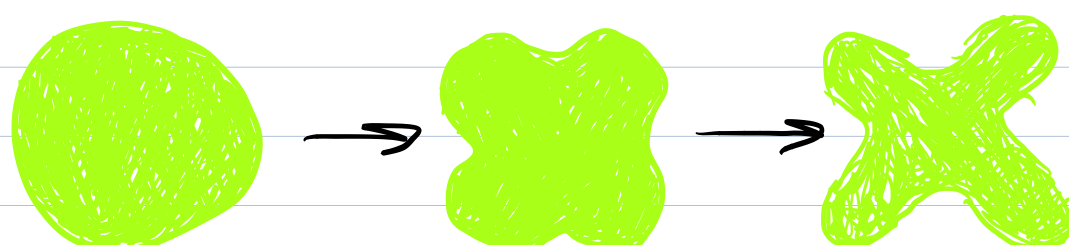
Transitivity: If S, T are isotopic and T, U are isotopic, so are S, U.

(The proof takes just a few lines.)

Example: Consider the thick capital letters

X **Y**

Both are isotopic to the disc $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$.



Isotopy from D to **X**

Hence, by transitivity, **X** and **Y** are isotopic. In particular, they are homeomorphic.

Thus we see that whether two letters are homeomorphic depends on whether we consider the thin or thick versions.