AMAT 342 Lecture 7

<u>Today</u>: Isotopy continued
- definition, revisited
- properties
- more examples.

<u>Review</u>:

<u>Definition</u>: For continuous maps $f, g: S \to T$ a <u>homotopy</u> from f to g is a continuous map

$$h: S \times I \to T$$

such that $h_0 = f$ and $h_1 = g$.

(<u>recall</u>: for $t \in I$, $h_t: S \to T$ is given by $h_t(x) = h(x,t)$.)

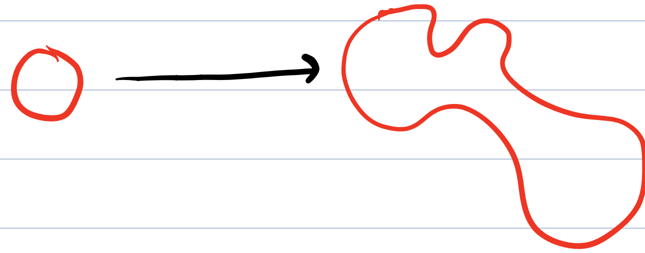
function obtained by fixing the 2^{nd} argument of h.

Note: Any continuous function $h: S \times I \to T$ is a homotopy from h_0 to h_1. In light of this, we sometimes refer to any continuous map $h: S \times I \to T$ as a homotopy, without explicitly mentioning what this homotopy is from and to.

<u>Embedding</u>: For $f: S \to T$ any function, define $\tilde{f}: S \to \mathrm{im}(f)$ by $\tilde{f}(x) = f(x)$ $\forall x \in S$.

A continuous map $f: S \to T$ is an <u>embedding</u> if $\tilde{f}$ is a homeomorphism
- An embedding is an injection, but the converse is not true

- Simple intuition: An embedding is like a homeomorphism, but the codomain can have extra points not in the image.



Note: If S has a property called compactness, then any continuous injection $f: S \to \mathbb{R}^n$ is an embedding.

e.g., S^1 is compact. We'll discuss compactness later.

Isotopy (Definition from last time)

Definition: For $S, T \subset \mathbb{R}^n$ an isotopy from S to T is a homotopy $h: X \times I \to \mathbb{R}^n$ such that

$$\operatorname{im}(h_0) = S, \quad \operatorname{im}(h_1) = T,$$

$$h_t: X \to \mathbb{R}^n \text{ is an embedding for all } t \in I.$$

Clarifications:

- In the above definition, homotopy just means "continuous map."
- The definition doesn't explictly put any requirements on X, but it follows from the definition that X has to be homeomorphic to both S and T:

$\tilde{h}_0: X \to \operatorname{im}(h_0)$ is a homeomorphism because h_0 is an embedding.

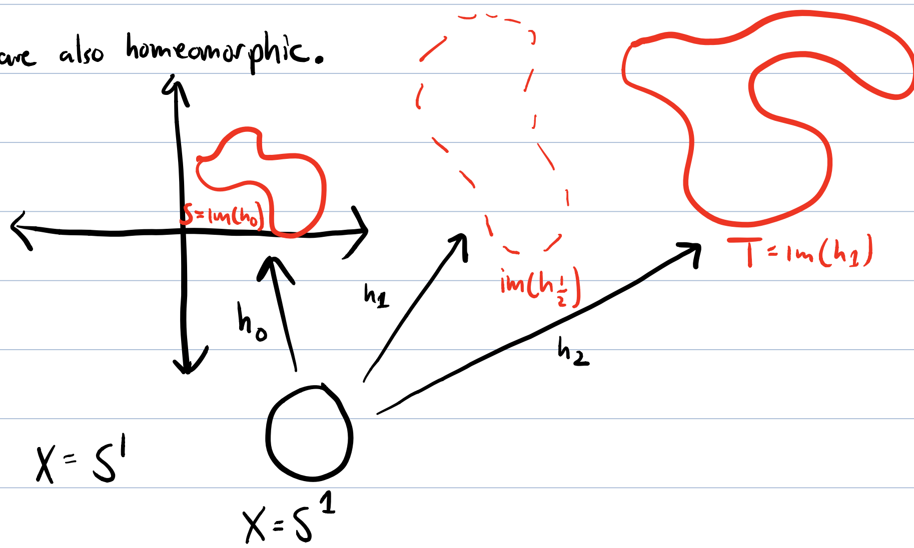
$\overset{S}{\shortparallel}$

$\tilde{h}_1 : X \to \mathrm{im}(h_1)$ is a homeomorphism because h_1 is an embedding.

$\hphantom{\tilde{h}_1 : X \to \mathrm{im}}\underset{T}{\shortparallel}$

hence, S and T are also homeomorphic.

Illustration:

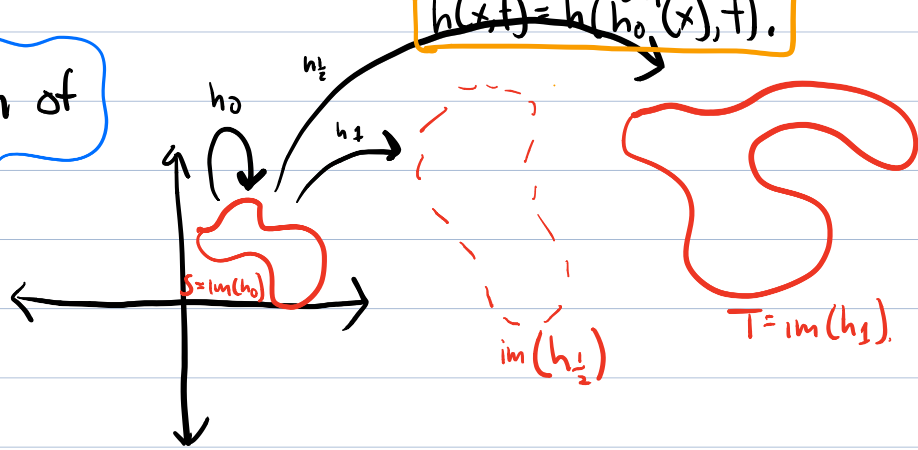


I can take $X = S^1$

Fact: If there exists an isotopy h from S to T then there exists an isotopy $\bar{h} : S \times I \to \mathbb{R}^n$ (that is, $X = S$!) with $h_0 : S \to \mathbb{R}^n$ the inclusion.

$\boxed{\text{Proof: define } \bar{h} \text{ by } \bar{h}(x,t) = h(\tilde{h}_0^{-1}(x), t).}$

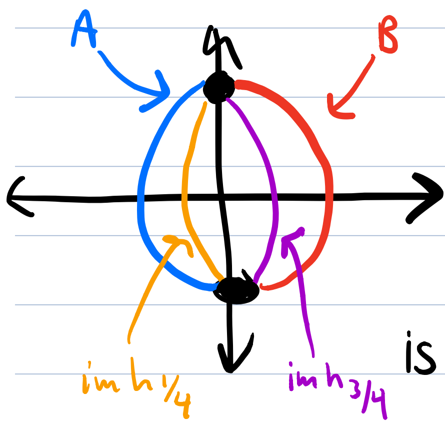
Illustration of fact



In practice, it will be fine for now to restrict attention to such isotopies.

Example: Let $A = \{(x,y) \in S^1 \mid x \leq 0\}$

$B = \{(x,y) \in S^1 \mid x \geq 0\}$.



$h : L \times I \to \mathbb{R}^2$,

$h((x,y), t) = ((1-2t)x, y)$

is an isotopy from A to B.

Explanation: $h_0(x,y) = ((1-0)x, y) = (x,y)$ so h_0 is the inclusion of A into $\mathbb{R}^2$.

$h_1(x,y) = (-x, y)$, so $\operatorname{im} h_1 = B$.

$h_t(x,y) = ((1-2t)x, y)$.

Not hard to check that each h_t is an embedding.

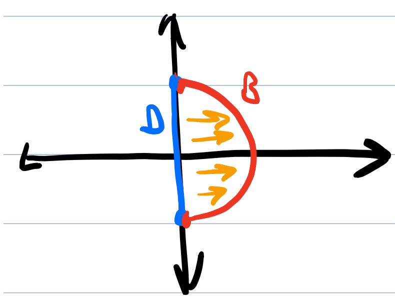
Example:

S and T are not isotopic.

Exercise: Let B be as in the last example.

Let $D = \{0\} \times [-1,1] = \{(0,y) \mid -1 \le y \le 1\}$.

a) Give a homeomorphism $f: D \to B$.



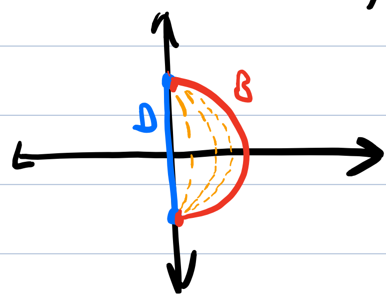
Answer: $f(0,y) = (\sqrt{1-y^2}, y)$

note: $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$
$f(x,y) \in S^1$ because
$(\sqrt{1-y^2})^2 + y^2 = 1$.

b) Given an explicit expression for f^{-1}.

$f^{-1}(x,y) = (0,y)$.

c) Give an isotopy from D to B.



$h: D \times I \to \mathbb{R}^2,$

$h((0,y),t) = (t\sqrt{1-y^2}, y)$

$h_0 = (0,y),\quad im(h_0) = y.$

$h_1 = (\sqrt{1-y^2}, y)$, so $im(h_1) = im(f) = B$

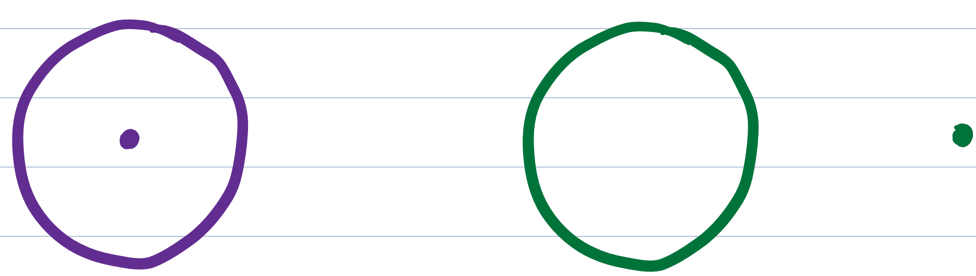
(easy to check that each h_t is an embedding).

d) Give an isotopy from B to D.

$h: B \times I \to \mathbb{R}^2;\quad h((x,y),t) = (x(1-t), y)$

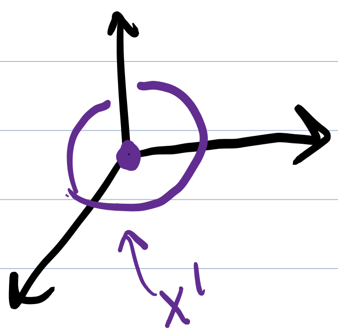
<u>Note</u>: Whether S and T are isotopic depends on where S and T are embedded. (That's not true for homeomorphism!)

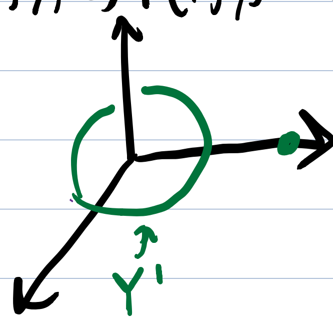
<u>Example</u> $X = S^1 \cup \{0\} \subset \mathbb{R}^2 \quad Y = S^1 \cup \{(3,0)\}.$

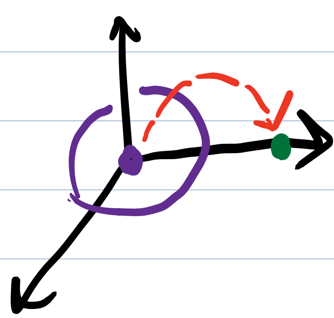


X and Y homeomorphic, not isotopic.
But if we embed X, Y in $\mathbb{R}^3$, then they are isotopic there.

That is, let $X' = \{(x,y,0) \mid (x,y) \in X\} \subset \mathbb{R}^3$
$Y' = \{(x,y,0) \mid (x,y) \in Y\} \subset \mathbb{R}^3$







There's an isotopy $h: X' \times I \to \mathbb{R}^3$ which moves the extra point as shown in red.

Similarly, if we embed S and T of the previous example into $\mathbb{R}^4$, they are isotopic there.

Facts about isotopies:

Symmetry: If there exists an isotopy from S to T, then there exists an isotopy from T to S.

"Isotopies can be reversed"

Pf: If $h: X \times I \to \mathbb{R}^n$ is an isotopy from S to T then $\bar{h}: X \times I \to T$, given by $\bar{h}(x,t) = h(x, 1-t)$ is an isotopy from T to S.

Transitivity: If S, T are isotopic and T, U are isotopic, so are S, U.

(The proof takes just a few lines.)

Example: Consider the thick capital letters

X Y

Both are isotopic to the disc $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

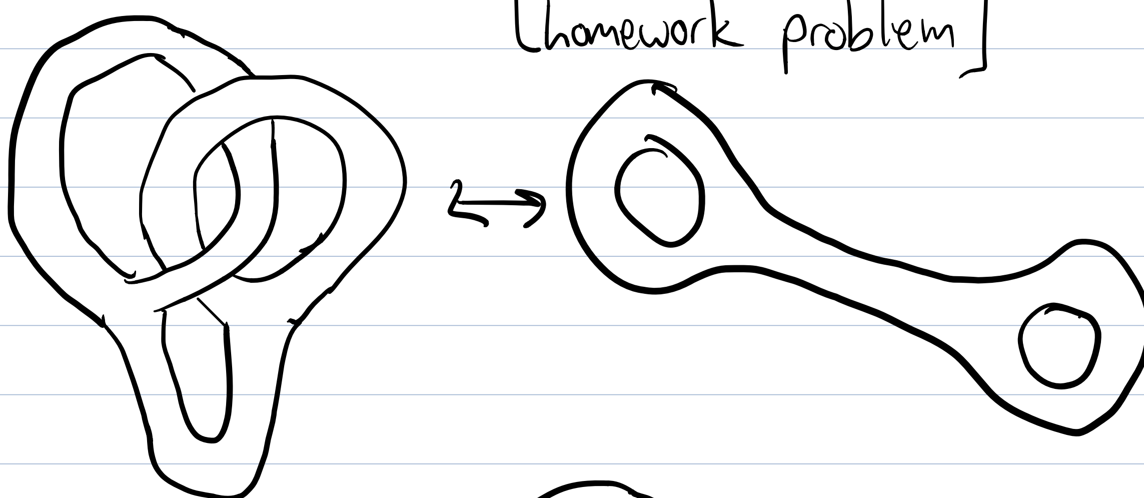
Isotopy from D to **X**

Hence, by transitivity, **X** and **Y** are isotopic.
In particular, they are homeomorphic.

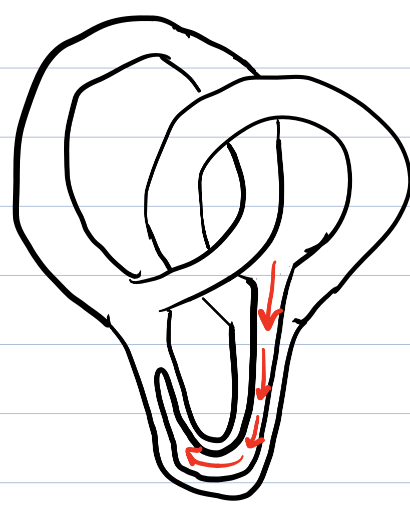
Thus we see that whether two letters are homeomorphic depends on whether we consider the thin or thick versions.

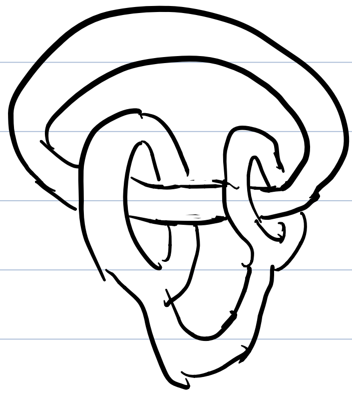
<u>Unintuitive isotopies</u>.

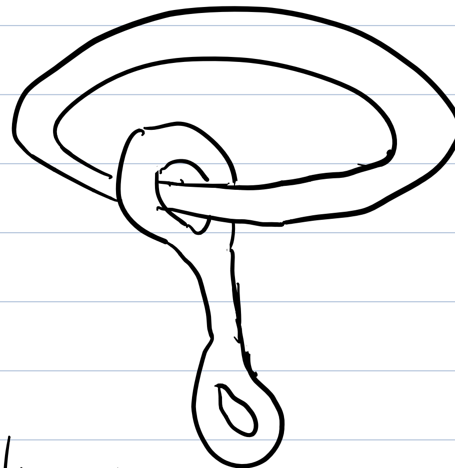
[homework problem]



<u>hint:</u>







Another well-known
example