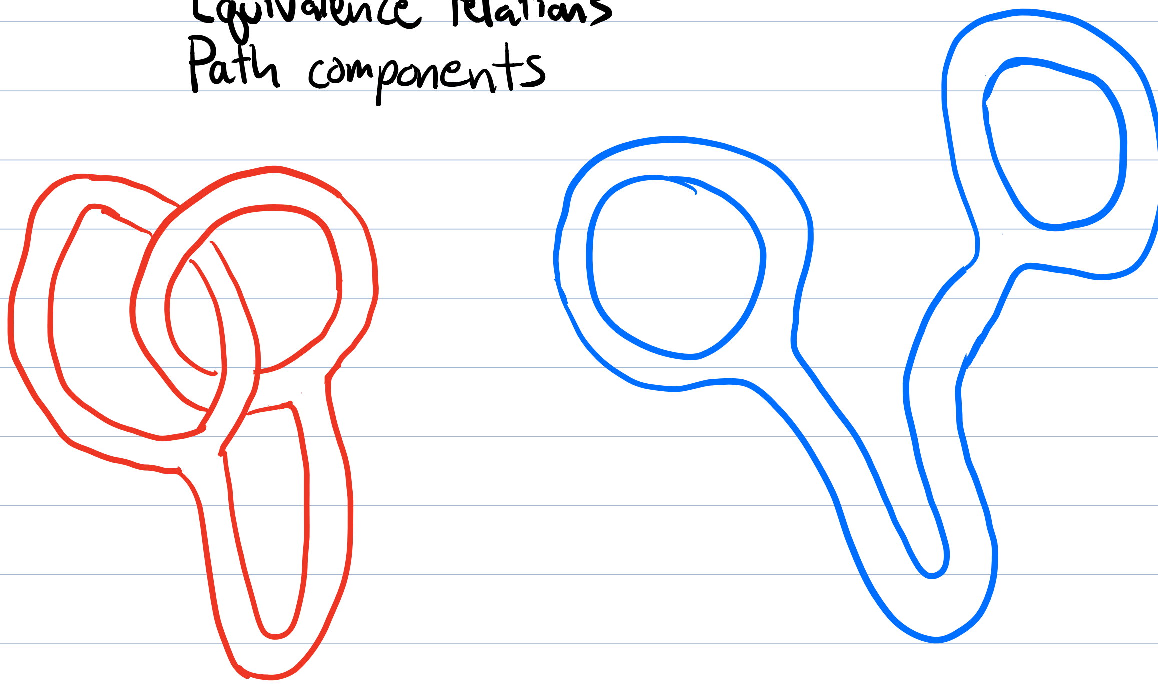
AMAT 342 Lecture 8, 9/19/81

<u>Today</u>: Surprising isotopies
Equivalence relations
Path components



Shown above are two embeddings in $\mathbb{R}^3$ of the surface of a donut with two holes.

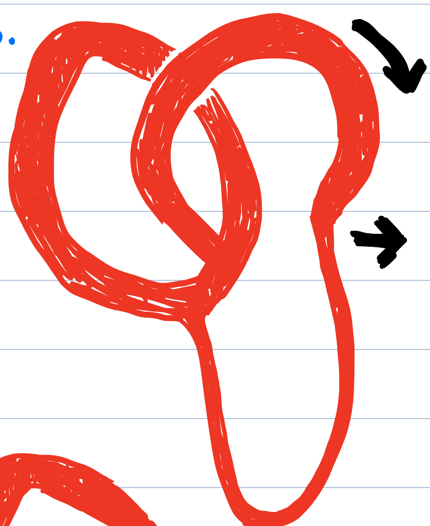
Perhaps surprisingly, these are isotopic. This was a homework problem. In fact it's a classic problem for an undergrad topology course.

Solution: (Adapted from "Intuitive topology" by V.V. Prasalov)

1.

2.

3.



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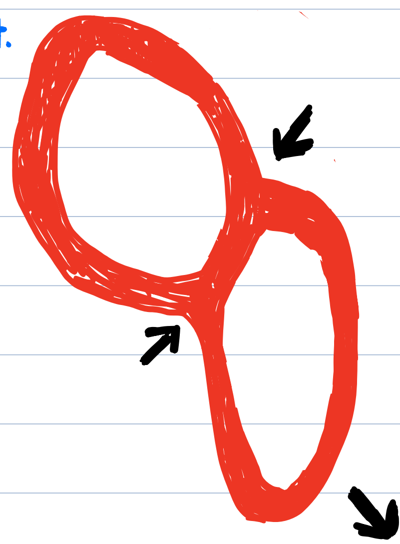

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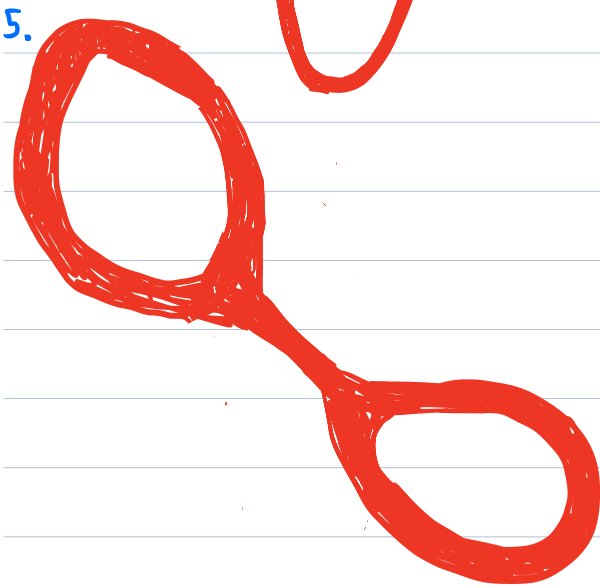
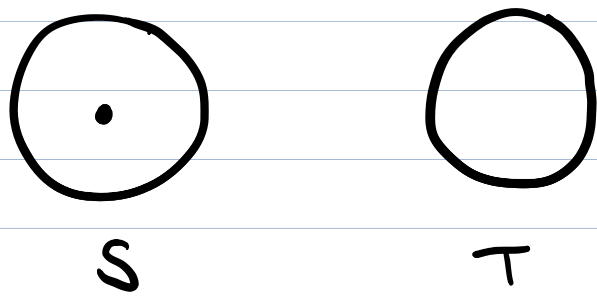


More surprising isotopies will be shown in class on my laptop. See posted slides.

Path components

In past lectures, we've mentioned in passing that the number of components a shape has is preserved by homeomorphism.

Illustration:



S and T are not homeomorphic because S has two components, while T has one.

Goal: Give a precise definition of "component" (path component) and prove that if $f: S \to T$ is a homeomorphism and S k path components, then T has k path components.

To define path component, I will introduce the fundamental notion of equivalence classes. This is itself an elegant + useful concept that every math major should know.

Equivalence relations

You may have not heard this term, but you know many examples of this.

Let S be any set. A relation on S is a function $R: S \times S \longrightarrow \{0, 1\}$.

"no" "yes"

Notation: Instead of writing $R(x,y) = 1$, we write xRy.
" " " $R(x,y) = 0$, we write $x \not R y$.

slash through the R.

Example: "Less than" $<$ is a relation on $\mathbb{Z}$.

That is, we can think of $<$ as a function

$$<: \mathbb{Z} \times \mathbb{Z} \longrightarrow \{0, 1\},$$

e.g. $<(a,b) = 1$ is written as $a < b$
$<(a,b) = 0$ is written as $a \not< b$.

Note: We pretty much never write $\underbrace{<(a,b)}_{\text{awkward!}}$ but the idea that $<$ is a function with domain $\mathbb{Z} \times \mathbb{Z}$ is useful.

Example $\leq$, $>$, and $\geq$ are also relations on $\mathbb{Z}$.

Example As in homework 1, let $P(\mathbb{Z})$ denote the power set of $\mathbb{Z}$ = set of all subsets of $\mathbb{Z}$. Then $\subset$ is a relation on $P(\mathbb{Z})$.

In fact, $\subset$ is a relation on $P(S)$ for any set S.

Equivalence relations (often denoted $\sim$)

A relation $\sim$ on S is an **equivalence relation** if

1) $x \sim x \ \forall \ x \in S$ [reflexivity]
2) $x \sim y$ iff $y \sim x$ [symmetry]
3) $x \sim y,\ y \sim z \Rightarrow x \sim z$ [transitivity]

if $x \sim y$, we say x is equivalent to y.

Example: The equivalence relation $<$ on $\mathbb{Z}$ satisfies only property 3, e.g. $2 \not< 2$, and $3 < 5$ but $5 \not< 3$.

Example: The relation $\leq$ on $\mathbb{Z}$ satisfies properties 1 and 3, but not 2.

Examples: 1) For any set S, the relation $\sim$ given by $x \sim y \ \forall \ x, y \in S$ is an equivalence relation.

2) Similarly, the relation $\sim$ given by $x \sim y$ only if $x = y$ is an equivalence relation.

Interesting example: Let $\sim$ be the relation on $\mathbb{Z}$ defined by $a \sim b$ iff $a-b$ is even.

This is an equivalence relation:

Succinct proof:

1) $a-a$ is 0, which is even, $\forall a \in \mathbb{Z}$.
2) $a-b$ is even iff $b-a = -(a-b)$ is even.
3) if $a-b$ is even and $b-c$ is even, then $a-c = (a-b)+(b-c)$ is even, because the sum of two even #'s is even.

Equivalence classes Def: For $\sim$ an equivalence relation on S and $x \in S$, let $[x]$ denote the set $\underbrace{\{y \in S \mid y \sim x\}}_{\text{set of all elements of } S \text{ equivalent to } x.} \subset S$. We call $[x]$ an equivalence class of $\sim$.

Example: Let $\sim$ be the equivalence relation on $\mathbb{Z}$ given in the previous example.

Q: What is $[0]$? A: $z \sim 0$ iff $z-0$ is even iff z is even.
So $[0] =$ the even integers $:= E$

Q: What is $[2]$? A: $z \sim 2$ if $z-2$ is even iff z is even.
So $[2] = E$.
In fact, for any even number z, $[z] = E$.

Q: What is $[1]$? **A:** $z \sim 1$ iff $z-1$ is even iff z is odd.
So $[1]$ = the odd integers $:= O$.
Similarly, for any odd z, $[z] = O$.

So there are just two equivalence classes for this relation, E and O.

__Fact__: For any equivalence relation $\sim$ on a set S, every element of S is contained in exactly one equivalence class of $\sim$.

__Pf__: For $x \in S$, $x \in [x]$ because $\sim$ is reflexive.
Suppose $x \in [z]$. $[z] = \{y \in S \mid y \sim z\}$. So $x \sim z$, and thus $z \sim x$. If $y \in [z]$, then $y \sim z$. By transitivity then, $y \sim x$, so $y \in [x]$. This shows that $[z] \subset [x]$. A very similar little argument shows that $[x] \subset [z]$. Thus $[x] = [z]$. This shows that x belongs to exactly one equivalence class, namely $[x]$. $\blacksquare$

__Notation__: $S/\sim$ denotes the set of equivalence classes of of $\sim$.

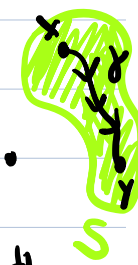
__Example__: Let $\sim$ be the equivalence relation on $\mathbb{Z}$ of the previous examples.

Then $\mathbb{Z} = \{E, O\}$.

Path components

subset of Euclidean space

Recall from your homework: For a space S and $x, y \in S$, a path from x to y is a continuous function $\gamma : I \to S$ such that $\gamma(0) = x$, $\gamma(1) = y$.



Define a relation $\sim$ on S by $x \sim y$ iff $\exists$ a path from x to y.

Proposition: $\sim$ is an equivalence relation.

Pf: Reflexivity: For $x \in S$, the path $\gamma : I \to S$, given by $\gamma(t) = x$ $\forall t \in I$, is a path from x to itself.

Symmetry: If γ is a path from x to y, then $\bar{\gamma} : I \to S$, $\bar{\gamma}(t) = \gamma(1-t)$ is a path from y to x.

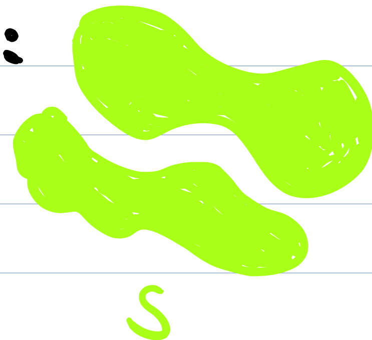
Transitivity: If α is a path from x to y, and β is a path from y to z, then a path γ from x to z is given by

$$\gamma : I \to S, \quad \gamma(t) = \begin{cases} \alpha(2t) & \text{for } t \in [0, \frac{1}{2}] \\ \beta(2t-1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

Definition: A <u>path component</u> of S is an equivalence class of $\sim$, i.e. an element of $S/\sim$.

Illustration:

The set $S \subset \mathbb{R}^2$ shown has two path components.



Definition: S is <u>path connected</u> if $S/\sim$ contains exactly one element. Note: If S is non-empty, this is equivalent to the def. of path connected in HW #2.

[Lecture ended here]

Proposition: If S and T are homeomorphic, then there is a bijection from $S/\sim$ to $T/\sim$.
Thus, if S has k path components, so does T.

Proof: Next time...