AMAT 342 Lec 9 9/24/19

Today: Path components, continued
Metric spaces: Topology beyond Euclidean subspaces.

Recall from last time:
A relation $\sim$ on a set S is an equivalence relation if

1) $x \sim x \;\forall\, x \in S$ [reflexivity]
2) $x \sim y$ iff $y \sim x$ [symmetry]
3) $x \sim y,\ y \sim z \Rightarrow x \sim z$ [transitivity]

if $x \sim y$, we say x is equivalent to y.

review

Equivalence classes **Def:** For $\sim$ an equivalence relation on S and $x \in S$, let $[x]$ denote the set $\underbrace{\{y \in S \mid y \sim x\}}_{\text{set of all elements of } S \text{ equivalent to } x.} \subset S$.
We call $[x]$ an equivalence class of $\sim$.

Fact: For any equivalence relation $\sim$ on a set S, every element of S is contained in exactly one equivalence class of $\sim$.

Note: $[x] = [y]$ iff $x \sim y$.

[For the proof, see the notes from Lec. 8].

Notation: $S/\sim$ is the set of equivalence classes of $\sim$.

subset of Euclidean space

Recall from your homework: For a space S and $x, y \in S$, a path from x to y is a continuous function $\gamma : I \to S$ such that $\gamma(0) = x$, $\gamma(1) = y$.

Define a relation $\sim$ on S by $x \sim y$ iff $\exists$ a path from x to y.

more review

Proposition: $\sim$ is an equivalence relation.

(Proof was not covered in class last time.)

Pf: **Reflexivity:** For $x \in S$, the path $\gamma : I \to S$, given by $\gamma(t) = x$ $\forall t \in I$, is a path from x to itself.

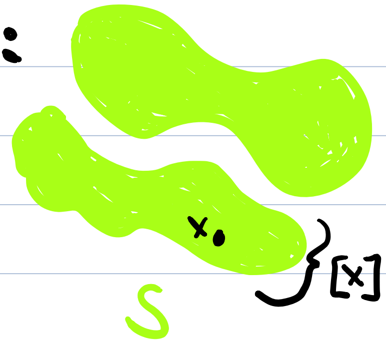
Symmetry: If γ is a path from x to y, then $\bar{\gamma} : I \to S$, $\bar{\gamma}(t) = \gamma(1-t)$ is a path from y to x.

Transitivity: If α is a path from x to y, and β is a path from y to z, then a path γ from x to z is given by

$$\gamma : I \to S, \quad \gamma(t) = \begin{cases} \alpha(2t) & \text{for } t \in [0, \frac{1}{2}] \\ \beta(2t-1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

Definition: A path component of S is an equivalence class of $\sim$, i.e. an element of $S/\sim$.

Illustration: The set $S \subset \mathbb{R}^2$ shown has two path components.



Notation: $S/\sim$ is written as $\pi(S)$.

The set of path components of S.

Definition: S is path connected if $\pi(S)$ contains exactly one element. **Note:** If S is non-empty, this is equivalent to the def. of path connected in HW #2.

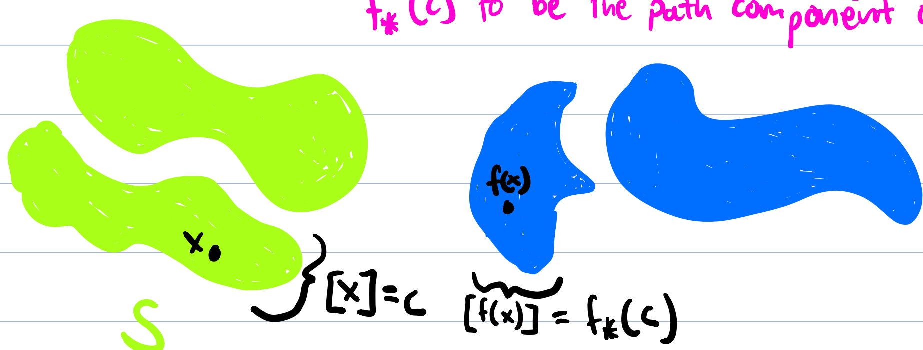
Proposition: If S and T are homeomorphic, then there is a bijection from $\pi(S)$ to $\pi(T)$.

Thus, if S has k path components, so does T.

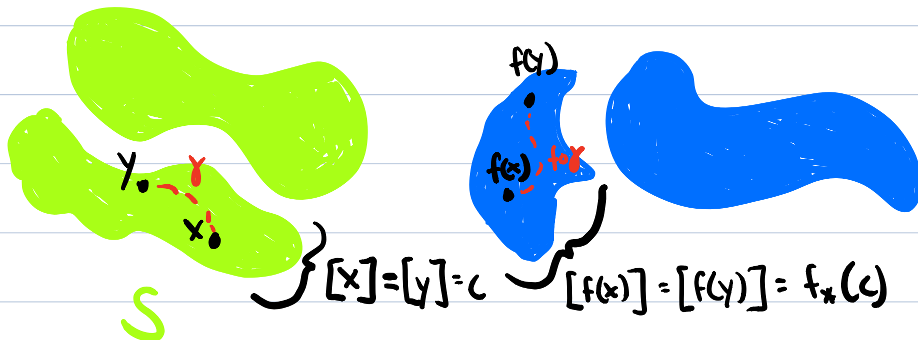
Proof: For any continuous function $f: S \to T$, we define a function: $f_*: \pi(S) \to \pi(T)$ by

$$f_*([x]) = [f(x)]$$

In other words, for c a path component of S, define $f_*(c)$ by choosing $x \in c$ and taking $f_*(c)$ to be the path component of T containing $f(x)$.



Note: We need to check that this definition doesn't depend on the choice of $x \in C$.



That is, we need to check that if $[x]=[y]$ then $[f(x)]=[f(y)]$.

If $[x]=[y]$, then $x \sim y$, i.e., there is a path $\gamma: I \to S$ from x to y. $f \circ \gamma : I \to T$ is a path from $f(x)$ to $f(y)$, so $f(x) \sim f(y)$, which implies $[f(x)]=[f(y)]$. ✓

We'll show that f_* is invertible, hence a bijection, when f is a homeomorphism.

For this, we need two facts:

1) For any $S \subseteq \mathbb{R}^n$, and $Id^S : S \to S$ the identity map, (i.e., $Id^S(x)=x \ \forall x$),

$Id^S_* = Id^{\pi(S)} : \pi(S) \to \pi(S)$.

Pf: $Id^S_*([x]) = [Id(x)] = [x]$.

2) For any continuous maps $f: S \to T$, $g: T \to U$,
$(g\circ f)_* = g_* \circ f_* : \pi(S) \to \pi(U)$

<u>Pf</u>: $(g\circ f)_*([x]) = [g\circ f[x]] = [g(f(x))] = g_*([f(x)]) = g_*(f_*[x]) = g_* \circ f_*([x])$.

[lecture ended here]

Now assume $f: S \to T$ is a homeomorphism.
Then f, f^{-1} are both continuous, and we have

$f^{-1}\circ f = \mathrm{Id}^S$

$f \circ f^{-1} \ \mathrm{Id}^T$

$$\text{Thus, } (f^{-1}\circ f)_* = \overset{\mathrm{Id}^{\pi(S)}}{\overset{\shortparallel}{\mathrm{Id}^S_*}} \Rightarrow f^{-1}_* \circ f_* = \mathrm{Id}^{\pi(S)}$$

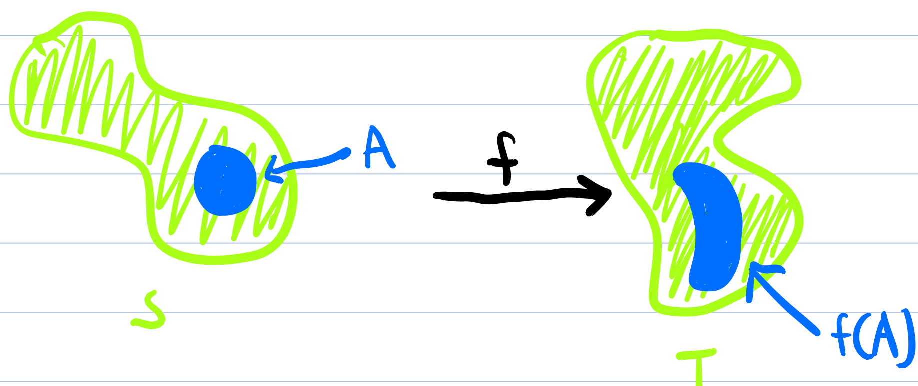
$$(f\circ f^{-1})_* = \underset{\mathrm{Id}^{\pi(T)}}{\underset{\shortparallel}{\mathrm{Id}^T_*}} \Rightarrow f_* \circ f^{-1}_* = \mathrm{Id}^{\pi(T)}.$$

Thus, $f_*: \pi(S) \to \pi(T)$ is invertible, with inverse f^{-1}_*. $\blacksquare$

<u>Application</u>: Consider the symbols $+$, $=$, and $\div$ as subsets of $\mathbb{R}^2$.
$|\pi(+)| = 1$, $|\pi(=)| = 2$, $|\pi(\div)| = 3$. Thus none is homeomorphic to any other.

<u>Application</u>: We prove that as unions of curves w/ no thickness, X and Y are not homeomorphic.

<u>Fact</u>: If $f: S \to T$ is a homeomorphism and $A \subset S$, then A and $f(A)$ are homeomorphic, where $f(A) = \{y \in T \mid y = f(x) \text{ for some } x \in A\}$.



<u>proof of fact</u>: (to be skipped in class) Let $j: A \to S$ be the inclusion. $\text{im}(f \circ j) = f(A)$. Since f is a bijection, so $\widetilde{f \circ j}: A \to f(A)$. It follows from the facts about continuity stated in an earlier lecture that $\widetilde{f \circ j}$ is continuous. Moreover, if $j': f(A) \to T$ is the inclusion, $(\widetilde{f \circ j})^{-1} = \widetilde{f^{-1} \circ j'}$, and this is continuous by the same reasoning.

<u>Proof that X and Y are not homeomorphic</u>:
Let $X' \subset X$ be obtained by removing the center point p. $|\pi(X')| = 4$. Note that there

no way to remove a single point from Y to get $Y' \subset Y$ with $|\pi(Y)| = 4$

If we have a homeomorphism $f: X \to Y$, then $f(X')$ is obtained from Y by removing $f(p)$, and $|\pi(f(X'))| = |\pi(X')| = 4$ by the prop., which is impossible. Thus, no homeomorphism $f: X \to Y$ can exist.

Topology Beyond Subsets of Euclidean Space

So far in this course, we've only considered continuity of functions $f: S \to T$ where S and T subsets of Euclidean spaces.

we sometimes use the word "subspace"

Hence, all the topological concepts we've introduced so far, e.g.,

- homeomorphism
- isotopy
- path components

have been defined in class only for Euclidean subspaces.

However, these ideas make sense in much more generality, and that extra generality can be extremely useful.

In fact, there are two levels to this extra generality. We discuss first level now.

Recall our definition of a continuous function between Euclidean subspaces:

<u>Formal Definition of Continuity</u>
We say $f: S \to T$ is <u>continuous</u> at $x \in S$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in S$ and $d(x, y) < \delta$, then $d(f(x), f(y)) < \epsilon$.

We say f is <u>continuous</u> if it is continuous at all $x \in S$.

from Lec. 3

<u>Important observation</u>: The only way we are using the fact that S and T are Euclidean subspaces is through their distance functions.

$\Rightarrow$ Continuity should make sense for any functions between sets endowed with "distance functions".

There are many extremely important examples, beyond the Euclidean subspaces we've already seen.

To explain this formally, we introduce <u>metric spaces</u>.

A <u>metric space</u> is a set S, together with a function $d: S \times S \to [0,\infty)$ satisfying:

1) $d(x,y) = 0$ if and only if $x = y$.
2) $d(x,y) = d(y,x)$ [symmetry]
3) $d(x,z) \leq d(x,y) + d(y,z)$ $\forall$ $x,y,z \in S$ [triangle inequality].

We call d a <u>metric</u>.

<u>Examples:</u>

- The familar example: $S = \mathbb{R}^n$, $d_2 : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$,

$$d_2(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2}$$

- $S = \mathbb{R}^n$, $d_1 : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$

$$d_1(x,y) = |x_1-y_1| + |x_2-y_2| + \cdots + |x_n-y_n|$$

- $S = \mathbb{R}^n$, $d_{max}: \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$,

$$d_{max}(x,y) = \max\left(|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\right)$$