AMAT 342

Review of Metric Spaces and Topological Spaces

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1 Metric Spaces

Definition 1 (metric space)**.** A *metric* on a set S is a function $d : S \times S \to [0, \infty)$ satisfying the following three properties:

- $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$ for all $x, y \in S$,
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in S$.

A *metric space* is a pair (S, d), where S is a set and d is a metric on S.

Example 1.1. We define three metrics d_2, d_1, and $d_{\max}$ on $\mathbb{R}^n$:

$$d_2(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|,$$

$$d_{\max}(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|.$$

d_2 is often called the *Euclidean distance*, and d_1 is often called the *Manhattan distance*, or the *taxicab metric.*

Remark 1.2. If $M = (S, d)$ is a metric space, we often abuse notation/language slightly and conflate M with its underlying set S. For example,

- "an element of M" means "an element of S",
- a "subset of M" means "a subset of S,"

- if $M = (S, d)$ and $M' = (S', d')$ are metric spaces, a function $f : M \to M'$ is understood to be a function $f : S \to S'$.

This convention sometimes allows us to simplify notation when working with metric spaces.

Remark 1.3 (Subspaces of metric spaces)**.** If $M = (S, d)$ is a metric space and $S' \subset S$, then d induces a metric d' on S', given by $d'(x, y) = d(x, y)$ for all $x, y \in S'$. Thus, we can always regard a subset of a metric space as a metric space. We call (S', d') a *subspace* of M.

The definition of continuity of functions between subsets of Euclidean spaces generalizes to functions between metric spaces, as follows:

Definition 2 (Continuity)**.** Let M and N be metric spaces with metrics d_M and d_N, respectively. A function $f : M \to N$ is said to be continuous at $x \in M$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $d_M(x, y) < \delta$ implies $d_N(x, y) < \epsilon$. f is said to be continuous if it is continuous at all $x \in M$.

Remark 1.4. Given Definition 2, the definition of homeomorphism that we gave in class for subsets of Euclidean spaces extends to metric spaces, as do a number of other basic topology definitions we gave earlier in the semester, like path components and embeddings. In fact, as a rule, all the basic concepts in topology can formulated for metric spaces, and even more generally for abstract topological spaces.

1.1 Open Sets

Let M be a metric space with metric d_M. For $r > 0$ and $x \in M$, the open ball in M of radius r centered at x is the subset of M given by

$$B(x, r) = \{y \in M \mid d_M(x, y) < r\}.$$

More succinctly, $B(x, r)$ is called an *open ball.*

Definition 3 (Unions)**.** Let A be any set, and suppose that for each $x \in A$ we have a set S_x. Then the *union* of the sets S_x, denoted $\bigcup_{x \in A} S_x$, is the set containing an element y if and only if $y \in S_x$ for some $x \in A$.

If $S = \{1, 2\}$ this is just the usual union $S_1 \cup S_2$. But our definition also allows for infinite unions.

Definition 4 (Open sets)**.** For M a metric space, an *open set* in M is a union of open balls. (More formally, a subset $U \subset M$ is open if there exists a set $A \subset M$ and a number $r_x > 0$ for each $x \in A$ such that $U = \bigcup_{x \in A} B(x, r_x)$.)

Remark 1.5. Taking $A = \emptyset$, we see that $\emptyset$ is an open set of M. Moreover, it is easily checked that M is an open set of M.

This following useful proposition was proven in class (see the notes for Lecture 17, page 4):

Proposition 1.6. *If U is an open set of M, then for each $x \in U$, there is a ball centered at x that lies entirely in U. That is, there exists $r > 0$ such that $B(x, r) \subset U$.*

Remark 1.7. In view of the last proposition, in the formal definition of an open set above (Definition 4), we can always choose A to be U.

Remark 1.8 (Open sets of metric subspaces)**.** If N is a metric space, $U \subset M \subset N$, and we regard M as a metric space by restriction of the metric on N, as in Remark 1.3, then it is possible for U to be an open set in M but not in N. For example, if $N = \mathbb{R}$ with the Euclidean metric d_2, $M = [1, 1]$, and $U = [0, 1)$, then U is open in M but not in N. (This example appeared in homework 6, problem 2.) On the other hand, it can be checked that if U is open in N, then U is open in M.

Exercise 1. Prove the claim in the last sentence of the remark above.

1.2 Boundaries

Definition 5 (Boundary points)**.** Let U be a subset of a metric space M. $x \in M$ is a *boundary point* if every open ball centered at x contains at least one point in U and one point not in U.

Proposition 1.9. *For any metric space M, $U \subset M$ is open if and only if U contains none of its boundary points.*

Proposition 1.9 is an easy consequence of Proposition 1.6. For the proof, see the solutions to homework 5, where this was given as an exercise. Proposition 1.9 can be useful for determining whether a subset of a metric space is open, as in homeworks 5 and 6.

1.3 Characterization of Continuity in terms of Open Sets

Definition 6 (Inverse images)**.** For any function $f : S \to T$ and $U \subset T$, the *inverse image* of U under f (also called the preimage), denoted $f^{-1}(U)$, is the subset of S given by

$$f^{-1}(U) = \{x \in S \mid f(x) \in U\}.$$

Proposition 1.10. *A function of metric spaces $f : M \to N$ is continuous iff $f^{-1}(U)$ is open whenever $U \subset N$ is open.*

The proof of Proposition 1.10 was not given in class, but is in the notes for Lecture 17 (see page 6).

Remark 1.11. Proposition 1.10 tells us that from the standpoint of continuity, the metric on a metric space matters only via open sets it generates. Thus, since topology is at its heart a study of continuous functions, the choice of metric also matters from the vantage point of topology only via the open sets it generates.

1.4 Topological equivalence of metrics

For M a metric space, let $\mathcal{U}(M)$ denote the set of all open sets of M. With Remark 1.11 in mind, we make the following definition:

Definition 7 (Topological Equivalence)**.** Let d and d' be metrics on the same set S. d and d' are said to be *topologically equivalent* if the set of open sets of $\mathcal{U}((S,d)) = \mathcal{U}((S,d'))$.

The following gives a sufficient condition for two metrics to be topologically equivalent.

Proposition 1.12. *If there exist positive constants α, β such that*

$$\alpha d(x,y) \leq d'(x,y) \leq \beta d(x,y)$$

for all $x, y \in S$, then d and d' are topologically equivalent.

The proof of Proposition 1.12 was assigned as problem 3 in homework 6; see the solutions.

One encounters topologically equivalent metrics quite often in practice. The following gives one important example:

Proposition 1.13. *The metrics d_1, d_2, and $d_{\max}$ on $\mathbb{R}^n$ are all topologically equivalent.*

The proof of this is an application of Proposition 1.12. For the (easy) proof of equivalence of d_1 and $d_{\max}$ see the solution to problem 4 from homework 6. For the proof of equivalence of d_1 and d_2, see the notes for lecture 18. (This uses the Cauchy-Schwarz inequality from linear algebra).

Remark 1.14. It is easy to check that if d and d' are topologically equivalent metrics on a set S, then the metric spaces (S, d) and (S, d') are homeomorphic. In fact, the identity map on S is a homeomorphism between the two metric spaces.

2 Topological Spaces

As Remark 1.11 indicates, the open sets of a metric space are what matter in topology. In fact, it turns out to sometimes be a hindrance in topology to worry about the extra data of the metric, when all that really is needed is the open sets. This suggests that we should try to develop the basic theory in an abstract, metric-free way that explicitly puts open sets at the center of the formalism. In fact, such an abstract approach to topology is standard; the key definitions are Definitions 8 and 9 below:

Definition 8 (Topological Space)**.** A topological space is a pair $T = (S, O^S)$, where S is any set and O^S is any set of subsets of S satisfying the following four properties:

1. Unions of (possibly infinitely many) elements of O^S are in O^S. (More precisely, for A any set, if we are given given a set $U_x \in O^S$ for each $x \in A$, then $\bigcup_{x \in A} U_x \in O^S$.)

2. Intersections of finitely many elements of O^S are in O^S. (That is, if $U_1, U_2, \ldots, U_n \in O^s$, then $U_1 \cap U_2, \cap \ldots \cap U_n \in O^S$.)

3. $S \in O^S$.

4. $\emptyset \in O^S$.

O^S is called a *topology* on S. Elements of O^S are called *open sets* of T.

Remark 2.1. We now make a remark in very much the same spirit as Remark 1.2: If $T = (S, O^S)$ is a metric space, we often abuse notation/language slightly and conflate T with its underlying set S. For example,

- "an element of T" means "an element of S",
- a "subset of T" means "a subset of S,"
- if $T = (S, O^S)$ and $T' = (S', O^{S'})$ are metric spaces, a function $f : T \to T'$ is understood to be a function $f : S \to S'$.

We have now defined open sets in two settings, first, for metric spaces (Definition 4), and then again in the abstract setting of topological spaces (Definition 8). As one would hope, the latter case generalizes the former; this is made precise by the following result:

Proposition 2.2. *Let $M = (S, d)$ be any metric space. $\mathcal{U}(M)$, the set of open sets of M, is a topology on S, which we call the* metric topology. *Thus $(S, \mathcal{U}(M))$ is a topological space.*

Proof. To prove the proposition, one has to check that properties 1-4 of a topological space are satisfied. The only property that is not immediate is property 2. This follows from Proposition 1.6 via a short argument that was given in class. (See the notes for Lecture 18, page 6.) $\square$

Most of the interesting examples of topological spaces arise from metric spaces, as in Proposition 2.2.

In view of Proposition 1.10, the following definition is natural:

Definition 9 (continuous maps between topological spaces)**.** A function $f : T \to T'$ of topological spaces is said to be *continuous* if $f^{-1}(U)$ is open whenever U is open.

Example 2.3. For any set S, let $\mathcal{P}(S)$ denote the set of all subsets of S. Then $(S, \mathcal{P}(S))$ is a topological space. $\mathcal{P}(S)$ is called the *discrete topology* on S, and we say that the topological space $(S, \mathcal{P}(S))$ is *discrete*.

Proposition 2.4. *A topological space T is discrete if and only if $\{x\}$ is open for all $x \in T$.*

Proposition 2.5. *For any finite metric space M, the metric topology is discrete.*

Proof. Let $r > 0$ be the minimum distance between any two elements of M. Then for any $x \in M$, $B(x, r/2) = \{x\}$, so $\{x\}$ is open. The result now follows from Proposition 2.4. $\square$

Remark 2.6. The discrete topology always arises as the metric topology of some metric d on S. For example, we may take $d(x,y) = \begin{cases} 0 & \text{if x=y,} \\ 1 & \text{if } x \neq y. \end{cases}$. For any $x \in S$, $B(x, 1/2) = \{x\}$, so $\{x\}$ is open, and it follows from Proposition 2.4 that the metric topology is discrete.

Example 2.7. For any set S, $\{S, \emptyset\}$ is a topology on S, called the *trivial topology*. We showed in class that if S has at least two elements, then the trivial topology is not the metric topology of any metric on S.

Definition 10 (subspace topology)**.** For $T = (S, O^S)$ a topological space and $X \subset S$ any subset, we define a topology O^X on X by

$$O^X = \{U \cap X \mid U \in O^S\}.$$

We call O^X the *subspace topology*. As a rule, we regard any subset of a topological space as a topological space with the subspace topology.

The following shows that in the case of a subspace of a metric space, the subspace topology coincides with the metric topology, as one would hope:

Proposition 2.8. *For N a metric space and $M \subset N$ a subspace, as defined in Remark 1.3, the metric topology on M is the same as the subspace topology for the metric topology on N.*

The proof of this was omitted both in class and in the notes.

3 Quotient Spaces

Recall that for any equivalence relation $\sim$ on a set S, we let $S/\!\sim$ denote the set of equivalence classes of $\sim$. We have a surjection $\pi : S \to S/\!\sim$, given by $\pi(x) = [x]$.

Given a topological space $T = (S, O^S)$ and an equivalence relation $\sim$ on S, we define the *quotient space* $T/\!\sim$ to be the topological space

$$T/\!\sim \; = \; (S/\!\sim, O^{\sim}),$$

where $O^{\sim}$ is defined as follows:

$$O^{\sim} = \{U \subset S/{\sim} \mid \pi^{-1}(U) \in O^S\}.$$

Thus, a set U is open in $T/{\sim}$ if and only if $\pi^{-1}(U)$ is open in T.

Intuitively, the $T/\sim$ is the topological space obtained by gluing together points $x, y \in T$ if and only $x \sim y$. As we discussed in class, this definition of quotient space in fact gives a topological space that glues stuff together as little as possible, subject to the constraint that if $x \sim y$, then x gets glued to y. This is made precise by the following proposition, which is proved in the lecture notes (and was stated but not proved in class):

Proposition 3.1. *For any topological space $T = (S, O^S)$ and equivalence relation $\sim$ on S,*

(i) The surjection $\pi : T \to T/{\sim}$ is continuous.

(ii) For any other continuous surjection $f : T \to X$ such that $f(x) = f(y)$ whenever $x \sim y$, there exists a unique continuous function $\bar{f} : T/{\sim} \to X$ such that $f = \bar{f} \circ \pi$.