AMAT 583 Midterm 1

Name: ____________________________

Instructions:

- Please put write your answers on the exam. If you run out of space, you can hand in an extra sheet, if necessary. You do not need to hand in your scratch paper.
- You are allowed one page of handwritten notes, front and back.
- No calculators, phones, etc. are allowed.
- The exam ends at the end of class, 4:05. If you finish early, you may hand in the exam and leave.
- Since the room has no clock, I will periodically announce the time.

1. [3 points]

- Sketch the Cartesian product $[1,2] \times [2,3]$. (This is a subset of $\mathbb{R}^2$.)
- Is this path connected?

Answer: This is a solid square with lower left corner (1,2) and upper right corner (2,3). The boundary of the square is included, so should be drawn with a solid line (not dashed). This is path connected; the path γ from x to y can be taken to be a straight line, i.e., $\gamma(t) = x(1-t) + yt$. [Giving this much detail about path connectedness was not necessary for full credit.]

2. [6 points] For each of the following functions,

- State whether the function is an injection, surjection, or bijection.
- Give the image of the function.
- If the function is a bijection, give its inverse.

a. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x - 1$. **Answer**: bijection, so $\text{im}(f) = \mathbb{R}$. $f^{-1}(x) = x + 1$.
b. $g : \mathbb{R}^2 \to \mathbb{R}$, $g((x, y)) = x - 1$. **Answer**: Surjection but not injection. For example, $g(1, 0) = g(1, 1)$. $\text{im}(f) = \mathbb{R}$.
c. $h : \mathbb{R} \to \mathbb{R}^2$, $h(x) = (0, x - 1)$. **Answer**: Injection but not surjection. $\text{im}(f)$ is the $y - axis$, i.e., $\text{im}(f) = \{(0, y) \mid y \in \mathbb{R}\}$.

3. [3 points] Give the definition of the image of a function $f : S \to T$, using the curly-bracket notation. **Answer**: $\text{im}(f) = \{y \in T \mid y = f(x) \text{ for some } x \in S\}$.

4. [3 points] Give the definition of a homeomorphism by completing the following sentence: "A function $f : S \to T$ is a *homeomorphism* if ________." **Answer**: f is a continuous bijection and f^{-1} is also continuous.

5. [3 points] Consider the continous map $f : [0, 2\pi) \to \mathbb{R}^2$ given by $f(x) = (\cos x, \sin x)$.

a. Is f a bijection? **Answer**: No, because it is not surjective. For example, $(2, 2) \notin \text{im}(f)$.
b. Is f an injection? **Answer**: Yes.
c. Is f an embedding? Explain you answer. **Answer**: No, because although the $\tilde{f} : [0, 2\pi) \to \text{im}(f) = S^1$ is a bijection, its inverse is not continuous at $(1, 0)$, so $\tilde{f}$ is not a homeomorphism.

6. [4 points] Consider the set of digits $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, where we regard each digit as a thickened subset of $\mathbb{R}^2$, as shown in the following picture:

Consider the equivalence relation $\sim$ on S defined by $x \sim y$ if and only if x is homeomorphic to y. What is the set of equivalence classes of $\sim$? (Give an explicit description.) **Answer**: $S/\sim = \{\{1, 2, 3, 5, 7\}, \{4, 6, 9\}, \{8\}\}$.

7. [5 points] Let

$$S = \{(x, 1) \mid x \in I\} \subset \mathbb{R}^2,$$
$$T = \{(x, 2) \mid x \in I\} \subset \mathbb{R}^2.$$

a. Sketch S and T.
b. Give an explicit homeomorphism $f : S \to T$. **Answer**: $f((x, 1) = (x, 2)$.
c. Give an explicit expression for f^{-1}. **Answer**: $f^1((x, 2) = (x, 1)$.
d. Give an explicit expression for an isotopy $h : S \times I \to \mathbb{R}^2$ from S to T. **Answer**: $h((x, 1), t) = (x, 1 + t)$.
e. Is the function h invertible? Explain your answer. [Hint: is h a surjection?]. **Answer**: No, because h is not surjective: $\text{im}(h) = [0, 1] \times [1, 2] \neq \mathbb{R}^2$.

8. [3 points] T_k denote a subset of the unit circle S^1 obtained by removing k distinct points from S^1. How many path components does T_k have? Explain your answer. [HINT: To get an idea of the answer, it may help to first think about this for small values of k, like $k = 1, 2, 3$.]. **Answer**: The answer is k. Removing 1 point from S^1 gives a space homemorphic to $(0, 1)$, which has a single path component. Removing a second point from S^1 gives a space homeomorphic to two copies of $(0, 1)$, which thus has two path components. Continuing like this, we see that removing k points from S^1 gives a space homeomorphic to k copies of $(0, 1)$, which has k path components.

9. [3 points] For each of the following functions $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$, say whether d is a metric. Briefly explain your reasoning.

a. $d(x, y) = \begin{cases} 0 & \text{if x=y,} \\ 1 & \text{otherwise.} \end{cases}$.

b. $d(x, y) = 2|x - y|$

c. $d(x, y) = \frac{|x|}{1+|y|}$.

Answer: See the solutions to homework #3 for the answers.

10. [3 points] For each of the following relations $\sim$ on the integers $\mathbb{Z}$, state whether $\sim$ is an equivalence relation. Explain your answer.

a. $x \sim y$ if and only if $x - y$ is odd. **Answer**: Not an equivalence relation. It's not reflexive.

b. $x \sim y$ if and only if $x - y = 1$. **Answer**: Not an equivalence relation. It's not symmetric.

c. $x \sim y$ if and only if $x - y \leq 0$. **Answer**: Not an equivalence relation. It's not symmetric.

11. [3 points] Compute the edit distance between each of the following pairs of sequences:

a. $x = AAA$, $y = CGT$, **Answer**: 3,

b. $x = AAA$, $y = ATA$, **Answer**: 1,

c. $x - AAA$, $y = AA$. **Answer**: 1.