AMAT 583 Practice Final

Name: ____________________________

Instructions:

- Please put write your answers on the exam. If you run out of space, you can hand in an extra sheet, if necessary. You do not need to hand in your scratch paper.
- You are allowed one page of handwritten notes, front and back.
- No calculators, phones, etc. are allowed.
- The exam ends at the end of class, 4:05. If you finish early, you may hand in the exam and leave.
- Since the room has no clock, I will periodically announce the time.

1.

a. Sketch the Cartesian product $[-1, 2] \times [0, 1]$. (This is a subset of $\mathbb{R}^2$.) Is this path connected? How many path components does it have?
b. Sketch the set $\{1, 2, 3\} \times [0, 1]$. Is this path connected? How many path components does it have?
c. Sketch the set $[0, 1] \times [0, 1] \times \{0, 1\} \subset \mathbb{R}^3$.

2. For each of the following functions,

- State whether the function is an injection, surjection, or bijection.
- Give the image of the function.
- If the function is a bijection, give its inverse.

a. $S = \{1, 2, 3\}$, $T = \{A, B\}$, $f : S \to T$ given by $f(1) = A$, $f(2) = A$, $f(3) = B$.
b. $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = \cos(x) + 1$,
c. $h : \mathbb{R} \to [0, \infty)$, $h(x) = x^2 + 1$.

3. Suppose S and T are sets, and $f : S \to T$ is a surjective function. What is $\mathrm{im}(f)$?

4. True or false:

a. The composition of two continuous functions is continuous.
b. The composition of two bijective functions is bijective. [HINT: Think about inverses.]
c. The composition of two homeomorphisms is a homeomorphism.

5. Give an example of a function which is an embedding, but not a homeomorphism.

6. Let set S denote of capital letters A,B,C,D,E,F, where we regard each letter as a union of curves with no thickness. Consider the equivalence relation $\sim$ on S defined by $x \sim y$ if and only if x is homeomorphic to y. What is the set of equivalence classes of $\sim$? (Give an explicit description.)

7. Let $S = \{A, B, C, D\}$. For each of the following relations $\sim$ on S, say whether S is an equivalence relation,

a. $x \sim y$ iff $x = y$ OR both x and y lie in the subset $\{A, B, C\}$,
b. $x \sim y$ iff $x = y$ OR both x and y lie in $\{A, B\}$ OR both x and y lie in $\{B, C\}$.

8. Define a relation on $\sim$ on $\mathbb{Z}$ by taking $x \sim y$ iff both x and y are positive. Is $\sim$ an equivalence relation? Explain your answer.

9. For $r > 0$ and $x \in \mathbb{R}^2$, let $B(x, r) = \{y \in \mathbb{R}^2 \mid d(x, y) < r\}$, where d denotes the Euclidean distance. For each of the following sets $S \subset \mathbb{R}^2$, sketch S and give the number of path components of S.

a. $S = B((0, 0), 1) \cup B((1, 0), 1)$.
b. $S = B((0, 0), 1) \cup B((3, 3), 1) \cup B((6, 6), 1)$.

10. Let $S = \{(x, y) \subset \mathbb{R}^2 \mid |y| > 2\}$. How many path components does y have?

11. For each of the following functions $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$, say whether d is a metric. Briefly explain your reasoning.

a. $d(x,y) = \begin{cases} 0 & \text{if x=y,} \\ 2 & \text{otherwise.} \end{cases}$.
d. $d(x, y) = |x| + |y|$.

12. Let d be an integer-valued metric on $\mathbb{Z}$. Suppose that $d(1, 2) = 1$ and $d(2, 3) = 1$.

a. What are the possible values of $d(3, 1)$? [Hint: There are exactly two.]
b. [Bonus] For each possible value in the first part, give an example of a metric d on $\mathbb{Z}$ such that $d(1, 3)$ realizes this value, with $d(1, 2)$ and $d(2, 3)$ as specified above.

13. Give the edit distance $d_{\text{edit}}(X, Y)$ for each of the following choices of X and Y:

a. $X = AAA$, $Y = ATT$,
b. $X = ATCG$, $Y = ATCG$,
c. $X = TTCC$, $Y = TATCC$.

14. Let $f : I \times I \to I \times I$ be given by $f(x, y) = (0, 0)$ for all $(x, y) \in I \times I$.

a. Give a homotopy from the identity map on $I \times I$ (denoted $\mathrm{Id}_{I \times I}$) to f.
b. Give a homotopy from f to $\mathrm{Id}_{I \times I}$.

15. True or false:

   a. If X if homotopy equivalent to Y, then Y is homotopy equivalent to X.
   b. If X if homotopy equivalent to Y, and Y is homotopy equivalent to Z, then X is homotopy equivalent to Z.
   c. If X if homotopy equivalent to Y, then X and Y are homeomorphic,
   d. If X and Y are homeomorphic, then X and Y are homotopy equivalent.

16. Let set S denote of capital letters A,B,C,D,E,F, where we regard each letter as a union of curves with no thickness. Consider the equivalence relation $\sim$ on S defined by $x \sim y$ if and only if x is homotopy equivalent to y. What is the set of equivalence classes of $\sim$? (Give an explicit description.)

17.

   a. Is the circle S^1 homotopy equivalent to the disc $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$?
   b. Is the disc D homotopy equivalent to a single point?

18. Let $G = (V, E)$, be given by $V = \{a, b, c, d, e\}$, $E = \{[a, b], [c, d], [d, e]\}$. Sketch the graph G, and give an explicit expression for each connected component of G:

19. Let $X = \{(0, 0), (1, 1), (2, 3)\} \subset \mathbb{R}^2$, with the Euclidean metric d_2.

   a. Sketch the neighborhood graph $N_r(X)$ for $r = 2$,
   b. What is the smallest value of r such that $N_r(X)$ has just one connected component.
   c. What is the smallest value of r such that $N_r(X)$ has a cycle?
   d. Draw the (trimmed) single-linkage dengrogram of X,
   e. [Bonus, something like this may appear as extra credit on the exam] What is the barcode of this single linkage dendrogram?