AMAT 583 Practice Final Solutions

Name: ____________________________

Instructions:

- Please put write your answers on the exam. If you run out of space, you can hand in an extra sheet, if necessary. You do not need to hand in your scratch paper.
- You are allowed one page of handwritten notes, front and back.
- No calculators, phones, etc. are allowed.
- The exam ends at the end of class, 4:05. If you finish early, you may hand in the exam and leave.
- Since the room has no clock, I will periodically announce the time.

NOTE: Sketches are appended to the back of the solutions.

1.

a. Sketch the Cartesian product $[-1, 2] \times [0, 1]$. (This is a subset of $\mathbb{R}^2$.) Is this path connected? How many path components does it have? **Answer**: One path component; clearly there is a continuous path from any point to any other point which lies in the set.
b. Sketch the set $\{1, 2, 3\} \times [0, 1]$. Is this path connected? How many path components does it have? **Answer**: Not path connnected; three path components.
c. Sketch the set $[0, 1] \times [0, 1] \times \{0, 1\} \subset \mathbb{R}^3$.

2. For each of the following functions,

- State whether the function is an injection, surjection, or bijection.
- Give the image of the function.
- If the function is a bijection, give its inverse.

a. $S = \{1, 2, 3\}$, $T = \{A, B\}$, $f : S \to T$ given by $f(1) = A$, $f(2) = A$, $f(3) = B$. **Answer**: f is not injective because $f(1) = f(2)$. But it is surjective, because $\text{im}(f) = T$.
b. $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = \cos(x) + 1$, **Answer**: g is not injective. For example, $g(0, 0) = g(0, 1) = 2$. the image of $\cos : \mathbb{R} \to \mathbb{R}$ is $[-1, 1]$, so $\text{im}(g) = [0, 2] \neq \mathbb{R}$. Hence g is not surjective.
c. $h : \mathbb{R} \to [0, \infty)$, $h(x) = x^2 + 1$. **Answer**: h is not injective. For example $h(1) = h(-1) = 2$. $\text{im}(h) = [1, \infty)$, so h is not surjective.

3. Suppose S and T are sets, and $f : S \to T$ is a surjective function. What is $\mathrm{im}(f)$? **Answer**: $\mathrm{im}(f) = T$. A surjective function is one whose image is the entire codomain.

4. True or false:

a. The composition of two continuous functions is continuous. **Answer**: True (this and the other facts below were covered in class).
b. The composition of two bijective functions is bijective. [HINT: Think about inverses.] **Answer**: True. Consider the composition $g \circ f$, where both f and g are bijective. Then f and g are invertible, and $g \circ f$ has inverse $(f \circ g)^{-1} = f^{-1} \circ g^{-1}$. Hence $g \circ f$ is a bijection.
c. The composition of two homeomorphisms is a homeomorphism. **Answer**: True.

5. Give an example of a function which is an embedding, but not a homeomorphism.

Answer: There are lots of examples. A simple one would be $f : I \to \mathbb{R}$, given by $f(x) = x$.

Writing just this would be sufficient, but to help you review, I will give more detail on why this is an embedding bu not a homeomorphism. Clearly $\mathrm{im}(f) = I$. f isn't surjective, since for example $2 \notin I = \mathrm{im}(f)$. Hence f is not a homeomorphism. But the map $\tilde{f} : I \to I$, given by $\tilde{f}(x) = f(x) = x$ is the identity map on I. Hence $\tilde{f}$ is a homeomorphism, with inverse itself. So f is an embedding.

6. Let set S denote of capital letters A,B,C,D,E,F, where we regard each letter as a union of curves with no thickness. Consider the equivalence relation $\sim$ on S defined by $x \sim y$ if and only if x is homeomorphic to y. What is the set of equivalence classes of $\sim$? (Give an explicit description.)

Answer: $\{\{\mathsf{A}\}, \{\mathsf{B}\}, \{\mathsf{C}\}, \{\mathsf{D}\}, \{\mathsf{E}, \mathsf{F}\}\}$

7. Let $S = \{A, B, C, D\}$. For each of the following relations $\sim$ on S, say whether S is an equivalence relation.

a. $x \sim y$ iff $x = y$ OR both x and y lie in the subset $\{A, B, C\}$, **Answer**: Yes.
b. $x \sim y$ iff $x = y$ OR both x and y lie in $\{A, B\}$ OR both x and y lie in $\{B, C\}$. **Answer**: No. Transitivity fails. For example, $A \sim B$ and $B \sim C$ but $A \not\sim C$.

8. Define a relation on $\sim$ on $\mathbb{Z}$ by taking $x \sim y$ iff both x and y are positive. Is $\sim$ an equivalence relation? Explain your answer. **Answer**: No. Property 1) [reflexivity] fails, for example, $-1 \not\sim -1$.

9. For $r > 0$ and $x \in \mathbb{R}^2$, let $B(x, r) = \{y \in \mathbb{R}^2 \mid d(x, y) < r\}$, where d denotes the Euclidean distance. For each of the following sets $S \subset \mathbb{R}^2$, sketch S and give the number of path components of S.

a. $S = B((0, 0), 1) \cup B((1, 0), 1)$. **Answer**: One path component.

b. $S = B((0, 0), 1) \cup B((3, 3), 1) \cup B((6, 6), 1)$. **Answer**: Three path components.

10. Let $S = \{(x, y) \subset \mathbb{R}^2 \mid |y| > 2\}$. How many path components does y have? **Answer**: Two. There is a path connecting any two points in S with positive y-value, and a path connecting any two points in S with negative-y value, but no path in S connecting a point with positive y-value to one with negative y-value.

11. For each of the following functions $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$, say whether d is a metric. Briefly explain your reasoning.

a. $d(x, y) = \begin{cases} 0 & \text{if x=y,} \\ 2 & \text{otherwise.} \end{cases}$. **Answer**: d is a metric. Properties 1 and 2 are clearly satisfied. If $x = z$, then $d(x, z) = 0$ and the triangle inequality clearly holds. Otherwise, $d(x, z) = 2$, and either $y \neq z$, in which case $d(y, z) = 2$, or $y \neq x$ in which case $d(x, y) = 2$. In either case, it is clear that $2 \leq d(x, y) + d(y, z)$, which gives the triangle inequality.

d. $d(x, y) = |x| + |y|$. **Answer**: d is not a metric: $d(1, 1) = 2 \neq 0$, so property 1 does not hold.

12. Let d be an integer-valued metric on $\mathbb{Z}$. Suppose that $d(1, 2) = 1$ and $d(2, 3) = 1$.

a. What are the possible values of $d(3, 1)$? [Hint: There are exactly two.] **Answer**: By the triangle inequality, only values of 1 and 2 are possible. (A value of 0 is not possible by the first property of a metric space, since $1 \neq 3$).

b. [Bonus] For each possible value in the first part, give an example of a metric d on $\mathbb{Z}$ such that $d(1, 3)$ realizes this value, with $d(1, 2)$ and $d(2, 3)$ as specified above. **Answer**: For $d(x, y) = |x - y|$ (the usual Euclidean distance), we have $d(1, 3) = 2$.

For

$$d(x, y) = \begin{cases} 0 & \text{if x=y,} \\ 1 & \text{otherwise,} \end{cases}$$

we have $d(1, 3) = 1$.

13. Give the edit distance $d_{\text{edit}}(X, Y)$ for each of the following choices of X and Y:

a. $X = AAA$, $Y = ATT$, **Answer**: 2

b. $X = ATCG$, $Y = ATCG$, **Answer**: 0

c. $X = TTCC$, $Y = TATCC$. **Answer**: 1.

14. Let $f : I \times I \to I \times I$ be given by $f(x, y) = (0, 0)$ for all $(x, y) \in I \times I$.

a. Give a homotopy from the identity map on $I \times I$ (denoted $\mathrm{Id}_{I\times I}$) to f. **Answer**: Let us write $I \times I$ as I^2, as we sometimes have in class. Define the homotopy $h : I^2 \times I \to I^2$ by $h(\vec{x}, t) = (1 - t)\vec{x}$. This can be checked to be continuous, and it's clear that $h_0 = \mathrm{Id}_{I\times I}$, while $h_1 = f$.

b. Give a homotopy from f to $\mathrm{Id}_{I\times I}$. **Answer**: This is similar to the above. Define the homotopy $h : I^2 \times I \to I^2$ by $h(\vec{x}, t) = t\vec{x}$.

15. True or false:

a. If X if homotopy equivalent to Y, then Y is homotopy equivalent to X. **Answer**: true; this is clear from the definitions.

b. If X if homotopy equivalent to Y, and Y is homotopy equivalent to Z, then X is homotopy equivalent to Z. **Answer**: true; this was mentioned in class, but not proven. (The proof is not that hard, though.)

c. If X if homotopy equivalent to Y, then X and Y are homeomorphic, **Answer**: false; you were asked to provide a couple of counter examples on homework 5.

d. If X and Y are homeomorphic, then X and Y are homotopy equivalent. **Answer**: true; this follows from the fact that if two maps are equal, then they are homotopic.

16. Let set S denote of capital letters A,B,C,D,E,F, where we regard each letter as a union of curves with no thickness. Consider the equivalence relation $\sim$ on S defined by $x \sim y$ if and only if x is homotopy equivalent to y. What is the set of equivalence classes of $\sim$? (Give an explicit description.)

Answer: A letter with 0 holes is homotopy equivalent to a single point. A letter with 1 hole is homotopy equivalent to a circle. A letter with 2 holes is homotopy equivalent to an eight. Hence, the answer is: $\{\{\mathsf{A}, \mathsf{D}\}, \{\mathsf{B}\}, \{\mathsf{C}, \mathsf{E}, \mathsf{F}\}\}$

17.

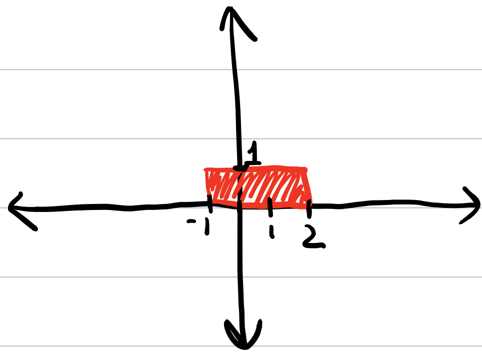
a. Is the circle S^1 homotopy equivalent to the disc $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$? **Answer**: No. Intuitively, this is because he disc has no holes, but the circle has one hole. (We will talk about technology for proving such a statement in the second semester.)

b. Is the disc D homotopy equivalent to a single point? **Answer**: Yes. We proved this in class.

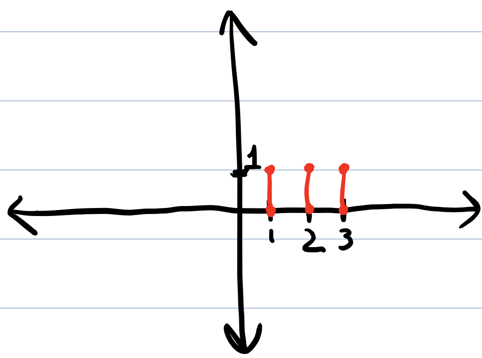
18. Let $G = (V, E)$, be given by $V = \{a, b, c, d, e\}$, $E = \{[a, b], [c, d], [d, e]\}$. Sketch the graph G, and give an explicit expression for each connected component of G: **Answer**: There are to connected components: $G^1 = (\{a, b\}, \{[a, b]\})$, and $G^2 = (\{c, d, e\}, \{[c, d], [d, e]\})$

19. Let $X = \{(0,0), (1,1), (2,3)\} \subset \mathbb{R}^2$, with the Euclidean metric d_2.
    a. Sketch the neighborhood graph $N_r(X)$ for $r = 2$.
    b. What is the smallest value of r such that $N_r(X)$ has just one connected component. **Answer**: $\sqrt{5}$.
    c. What is the smallest value of r such that $N_r(X)$ has a cycle? **Answer**: $\sqrt{13}$.
    d. Draw the (trimmed) single-linkage dendrogram of X,
    e. [Bonus, something like this may appear as extra credit on the exam] What is the barcode of this single linkage dendrogram? **Answer**: $\{[0,\infty), [0,\sqrt{2}), [0,\sqrt{5})\}$.

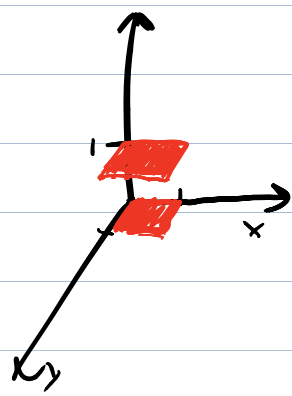
1a.



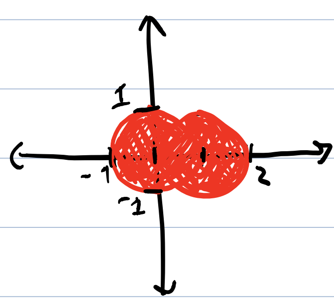
1b.



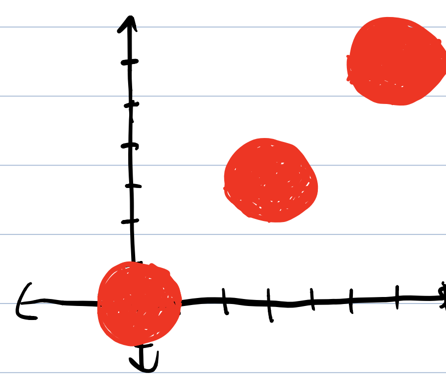
1c.



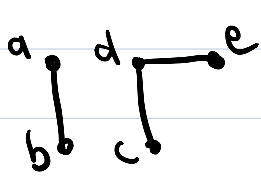
9a.



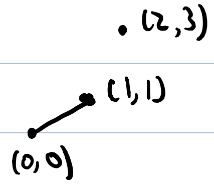
9b



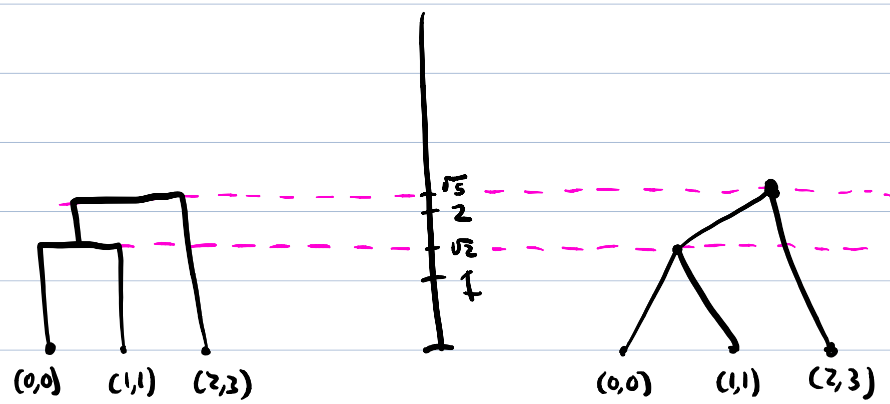
18.



19 a.



d.



Option 1

Option 2

[either answer is fine]