AMAT 583 Homework 4. Due Thursday November 21, 2019

1. Which of the following subsets S of $\mathbb{R}^2$ are open?

a. $S = \{(1, 1)\}$,
b. S= the x-axis,
c. $S = \{(x, y) \mid y - 1 < x < y + 1\}$,
d. $S = \{(x, y) \mid y - 1 < x \leq y + 1\}$,
e. $S = \{(x, y) \mid y - 1 \leq x \leq y + 1\}$.

2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Give an explicit expression for $f^{-1}(U)$, for each of the following sets $U \subset \mathbb{R}$:

a. $U = [1, 4]$,
b. $U = [-4, -1]$,
c. $U = \{2\}$,
d. $U = (0, 1)$.
d. $U = [0, \infty)$.

3. Consider the function $f : [0, 1] \to \mathbb{R}^2$ given by $f(x) = (x, 2x)$. Sketch $\mathrm{im}(f)$ and give an explicit expression for $f^{-1}(U)$, for each of the following sets $U \subset \mathbb{R}$.

a. $U = [0, 1] \times [0, 2]$,
b. $U = \mathbb{R}^2$,
c. $U = \{(2, 3)\}$,
d. $U = [0, 1] \times [0, 1]$.

4. Give an example of a pair of metrics on $\mathbb{R}$ which are not topologically equivalent.

5. For each of the following graphs $G = (V, E)$, sketch the graph, and give an explicit expression for each connected component:

a. $V = \{a, b, c, d, e\}$, $E = \{[a, b], [a, c], [a, d], [a, e]\}$.
b. $V = \{a, b, c, d, e\}$, $E = \{[a, b], [b, c], [c, d], [d, a]\}$,
c. $V = \{a, b, c, d, e, f\}$, $E = \{[a, b], [b, c], [c, a], [d, e], [e, f]\}$,
d. $V = \{a, b, c, d, e, f\}$, $E = \{[a, b], [b, c], [c, a], [d, e], [e, f], [f, a]\}$,
e. $V = \{a, b, c\}$, $E = \{\}$.

6. For each of the following metric spaces (X, d) give an explicit expression for the neighborhood graph $N_z(X)$, for each $z \in \mathbb{N}$. Draw the single linkage dendrogram.

a. $X = \{(1,1),(0,0),(2,0),(0,2),(2,2)\} \subset \mathbb{R}^2$, $d = d_{\max}$.
b. $X = \{(1,1),(0,0),(2,0),(0,2),(2,2)\} \subset \mathbb{R}^2$, $d = d_1$.

7. Give an example of a metric space X, and a non-negative integer z such that $N_z(X) \neq N_{z+1}(X)$ but $P_z = P_{z+1}$, where P_y denotes the single-linkage clustering of X at scale y. [HINT: take X to have 3 elements.]

8. [Bonus] Let $O = \{\mathbb{R}, \emptyset\} \cup \{(-\infty, a] \mid a \in \mathbb{R}\}$. Is O a topology on $\mathbb{R}$? Explain your answer.